ATTRACTORS FOR IMPULSIVE NON-AUTONOMOUS DYNAMICAL SYSTEMS AND THEIR RELATIONS

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Abstract. In this work, we deal with several different notions of attractors that may appear in the impulsive non-autonomous case and we explore their relationships to obtain properties regarding the different scenarios of asymptotic dynamics, such as the cocycle attractor, the uniform attractor and the global attractor for the impulsive skew-product semiflow. Lastly, we illustrate our theory by exhibiting an example of a non-classical non-autonomous parabolic equation with subcritical nonlinearity and impulses.

Keywords: non-autonomous dynamical system; impulses; skew-product semiflow; global attractor; c-global attractor; uniform attractor; cocycle attractor.

2010 Mathematics Subject Classification: Primary 35B41. Secondary 34A37; 35R12.

1. Introduction

What are the differences that appear when we change from an autonomous equation to an non-autonomous one? Does the asymptotic behavior of the solutions become different?

This change may be very underrated and our first answer may be negative. We might believe that there are not many changes in the behavior of the solutions of autonomous and non-autonomous equations. As one may see in [10, 11], this is not the case. In fact, there are infinitely many differences between these two cases. To illustrate this difference, let us consider a general differential equation of the form

$$\begin{cases} \dot{u} = f(t,u), \ t > s, \\ u(s) = u_0 \in X, \end{cases} \tag{1.1}$$

where X is a Banach space and $f : \mathbb{R} \times D \subset \mathbb{R} \times X \to X$ is a map belonging to some metric space $\mathscr{C}$. Assume that there exists a unique solution $[s, +\infty) \ni t \mapsto u(t, s, f, u_0) \in X$ of (1.1) defined for all times $t \geqslant s$, for each $f \in \mathscr{C}$, $u_0 \in X$ and $s \in \mathbb{R}$.

[1] Partially supported by FAPESP grant 2014/25970-5 and CNPq grant 307317/2013-7, Brazil.

[2] Supported by Partially supported by FEDER and Ministerio de Economía y Competitividad (Spain) under grant MTM2011-22411, and Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía) under Proyecto de Excelencia P12-FQM-1492.

[3] Supported by FAPESP grant 2014/20691-0, Brazil.

Thanks to the uniqueness of solution, one can see that when f is time-independent, that is, $f(t,x) = f(x)$ for all $t \in \mathbb{R}$, we have $u(t,s,f,u_0) = u(t-s,0,f,u_0)$ and the asymptotic behavior of solutions can be studied when $t \to +\infty$ (that is, considering the evolution of the solution as the final time evolves) or making $s \to -\infty$ (which is equivalent to consider the behavior of the solution as we take earlier and earlier initial times). In this case, these two scenarios coincide and give us the same description.

However, if f is time dependent then these two scenarios give rise to completely different behaviors. We may study the asymptotic behavior with respect to the elapsed time $t - s$ or with respect to s (when $s \to -\infty$ and t is arbitrary but fixed). These are called, respectively, forwards and pullback dynamics and are, in general, unrelated. It is natural that they are unrelated, for instance, the set of vector fields driving the solution may be completely different. We have one vector field $f(t,\cdot)$ for each time $t \in \mathbb{R}$.

There is no reason for this to be different in the impulsive case. We know now, after the previous discussion in [4], that the behavior of impulsive solutions in the non-autonomous case is much richer (and harder to analyze) than in the autonomous case. Hence, bearing this in mind, we may wonder about the relationships amongst the several different scenarios that appear in the non-autonomous impulsive case.

Note that the theory described in [10, 11, 14, 15] has, so far, no analogous when it comes to the impulsive framework. So, this paper shall be devoted to relate the several different kinds of attractors that come to play when dealing with non-autonomous impulsive dynamical systems.

Moreover, the results presented in this paper are totally different from the results which deal with random dynamical systems, where the impulses occur in time. Indeed, the results of this paper concern with impulses at variable times that depend on the phase space (impulses "occur" in space). Impulses that vary in time are more attractive due to their complexity and applicability in real world problems, see for instance [5, 6, 7]. As an example, we may cite the billiard-type system which can be modeled by differential systems with impulses acting on the first derivatives of the solutions. Indeed, the positions of the colliding balls do not change at the moments of impact (impulse), but their velocities gain finite increments (the velocity will change according to the position of the ball). The reader may consult [27] for the study of pullback attractors of non-autonomous random dynamical systems.

In the next lines we describe the organization of the paper and the main results.

In Section 2, we present the continuous non-autonomous dynamical systems theory. We remind the reader that the notion of attractors in the non-autonomous framework can have several interpretations. For a more careful description, the reader may consult [11].

In Section 3 we present, also briefly, the theory of impulsive non-autonomous dynamical systems which was first developed in [4]. The results of this section, of course, include the theory of autonomous dynamical systems in [5], but with some differences. In Section 4, we are concerned with such differences and we present new results for the existence of global attractors in the impulsive autonomous case. More precisely, in this section, we introduce the notion of *c-global attractors* (see Definition 4.5) and we exhibit a characterization result to ensure the existence of c-global attractors for impulsive autonomous dynamical systems. We point out that our result for c-global attractors (see Theorem 4.7) possesses simpler hypotheses than the results in [5]. Also, we present an alternate result to obtain global attractors as in [5] (see Theorem 4.9).

In Section 5, we use the continuous theory presented in [11] to define different notions for attractors in the impulsive non-autonomous case. The relationships among these attractors are considered in this section.

Finally, we apply our theory to fully describe the dynamics of an impulsive non-autonomous non-classical parabolic equation in Section 6.

2. Non-autonomous dynamical systems

We begin our study by brief recalling the theory of continuous non-autonomous dynamical systems. For more details on this topic, the reader may consult [1, 2, 3, 11, 13, 16, 23].

Let $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all natural numbers. Let Σ be a complete metric space and $\{\theta_t \colon t \geqslant 0\}$ be a **semigroup** in Σ, that is, it is a family of continuous maps from Σ into itself, satisfying the following conditions: $\theta_0\sigma = \sigma$ for all $\sigma \in \Sigma$, $\theta_{t+s} = \theta_t\theta_s$ for all $t, s \in \mathbb{R}_+$ and the map $\mathbb{R}_+ \times \Sigma \ni (t, \sigma) \mapsto \theta_t\sigma$ is continuous.

Also, let us consider another complete metric space (X, d) and for each pair $(t, \sigma) \in \mathbb{R}_+ \times \Sigma$, let $\varphi(t, \sigma) \colon X \to X$ be a map satisfying the following properties:

- **(i)** $\varphi(0, \sigma)x = x$ for all $x \in X$ and $\sigma \in \Sigma$;
- **(ii)** $\varphi(t + s, \sigma) = \varphi(t, \theta_s\sigma)\varphi(s, \sigma)$ for all $t, s \in \mathbb{R}_+$ and $\sigma \in \Sigma$;
- **(iii)** the map $\mathbb{R}_+ \times \Sigma \times X \ni (t, \sigma, x) \mapsto \varphi(t, \sigma)x \in X$ is continuous.

Definition 2.1. With the previous definitions and relations, $(\varphi, \theta)_{(X,\Sigma)}$ is said to be a **non-autonomous dynamical system**, or simply a **NDS**.

The semigroup $\{\theta_t \colon t \geqslant 0\}$ in this context is called **driving semigroup**, the map φ is called **cocycle** and the property **(ii)** above is commonly known as the **cocycle property**.

A **non-autonomous set** is a family $\hat{D} = \{D(\sigma)\}_{\sigma\in\Sigma}$ of subsets of X indexed in Σ. We say that $\hat{D}$ is an **open** (**closed, compact**) non-autonomous set if each **fiber** $D(\sigma)$ is an open (closed, compact) subset of X. A non-autonomous set $\hat{D}$ is called $\varphi-$**invariant** if

$$\varphi(t, \sigma)D(\sigma) = D(\theta_t\sigma) \quad \text{for all} \quad t \geqslant 0 \ \text{ and each } \ \sigma \in \Sigma.$$

A non-autonomous set $\hat{A}$ $\varphi-$**pullback attracts** $\hat{D}$ if $\lim_{t\to+\infty} \mathrm{dist}(\varphi(t,\theta_{-t}\sigma)D(\theta_{-t}\sigma), A(\sigma)) = 0$ for all $\sigma \in \Sigma$. Here, we use the **Hausdorff semidistance** between two sets, that is, for $A, B \subset X$ nonempty we denote $\mathrm{dist}(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b)$.

A **universe** $\mathfrak{D}$ is a collection of non-autonomous sets which is closed with respect to inclusion, that is, if $\hat{D}_1 \in \mathfrak{D}$ and $D_2(\sigma) \subset D_1(\sigma)$ for all $\sigma \in \Sigma$, then $\hat{D}_2 \in \mathfrak{D}$.

Definition 2.2. Given a NDS $(\varphi,\theta)_{(X,\Sigma)}$ and a universe $\mathfrak{D}$, a compact non-autonomous set $\hat{A}$ is called the $(\varphi,\mathfrak{D})-$**cocycle attractor** if:

- **(i)** $\hat{A}$ is $\varphi-$invariant;
- **(ii)** $\hat{A}$ $\varphi-$pullback attracts all non-autonomous sets in $\mathfrak{D}$;
- **(iii)** $\hat{A}$ is the minimal among all closed non-autonomous sets with property **(ii)**.

An important notion that relates the different aspects of the non-autonomous framework, and is vastly used in [10, 11], is the *skew-product semiflow*, which we recall next.

Definition 2.3. Given a NDS $(\varphi,\theta)_{(X,\Sigma)}$, the semigroup $\{\Pi(t)\colon t \geqslant 0\}$ in $\mathbb{X} = X \times \Sigma$ given by

$$\Pi(t)(x,\sigma) = (\varphi(t,\sigma)x, \theta_t\sigma) \quad \text{for all} \quad (x,\sigma) \in \mathbb{X} \text{ and } t \geqslant 0, \tag{2.1}$$

is said to be a **skew-product semiflow**. We say that $\{\Pi(t)\colon t \geqslant 0\}$ is the skew-product semiflow **associated** with the NDS $(\varphi,\theta)_{(X,\Sigma)}$.

Remark 2.4.

1. It is easy to see that if $\Sigma = \{\sigma_0\}$, then the NDS is, in fact, autonomous. Defining $\pi(t) = \varphi(t,\sigma_0)$ for all $t \geqslant 0$, we conclude that $\{\pi(t)\colon t \geqslant 0\}$ defines a semigroup in X.
2. Although the general theory of non-autonomous dynamical systems can be developed with a semigroup $\{\theta_t\colon t \geqslant 0\}$, in order to simplify the notation we will consider only the case where $\{\theta_t\colon t \in \mathbb{R}\}$ is a **group**.

Recall that an **evolution process** in X is a family of continuous maps $\{T(t,s)\colon t \geqslant s\}$ from X to itself, satisfying the following conditions:

- **(a)** $T(t,t)x = x$ for all $x \in X$ and $t \in \mathbb{R}$;
- **(b)** $T(t,s) = T(t,\tau)T(\tau,s)$ for all $t \geqslant \tau \geqslant s$;
- **(c)** the map $\mathcal{P} \times X \ni (t,s,x) \mapsto T(t,s)x \in X$ is continuous, where $\mathcal{P} = \{(t,s) \in \mathbb{R}^2\colon t \geqslant s\}$.

It is not difficult to verify that given an evolution process $\{T(t,s)\colon t \geqslant s\}$ in X, we can define a NDS by the map $(t,s,x) \in \mathbb{R}_+ \times \mathbb{R} \times X \mapsto \varphi(t,s)x \in X$ given by

$$\varphi(t,s)x = T(t+s,s)x, \quad \text{for each } t \geqslant 0,\ s \in \mathbb{R} \text{ and } x \in X,$$

where we take $\Sigma = \mathbb{R}$ and $\theta_t s = t + s$ for all $t, s \in \mathbb{R}$.

2.1. **Attractors of non-autonomous dynamical systems and their relations.** We have two important approaches that were developed in order to study the asymptotic behavior of non-autonomous differential equations as (1.1):

(1) the theory of uniform attractors, a minimal compact (not invariant) set that forwards attracts bounded sets uniformly with respect to the initial time;

(2) the theory of pullback attractors, a family of compact sets which is invariant and pull-back (but, in general, not forwards) attracts bounded sets.

These two approaches were treated, at first, as unrelated notions. However, in [10] the authors explore both notions, and using the skew-product semiflow (2.1) associated with equation (1.1), important relationships between **(1)** and **(2)** were proved in [10].

To give an idea of their results, we consider $f \in C_b(\mathbb{R} \times X, X)$, the set of all continuous functions from $\mathbb{R} \times X$ into X such that given $B \subset X$ bounded and $J \subset \mathbb{R}$, $f(J \times B)$ is bounded in X with a suitable metric ϱ. Denote by Σ_0 the set of all translates of f in the first variable,

$$\Sigma_0(f) = \{f(s + \cdot, \cdot) : \ s \in \mathbb{R}\},$$

and define the shift operator $\theta_t : C_b(\mathbb{R} \times X, X) \to C_b(\mathbb{R} \times X, X)$ by $\theta_t f(\cdot, \cdot) = f(t + \cdot, \cdot)$.

Remark 2.5. Note that in this case, since f is defined for all times $t \in \mathbb{R}$, θ_t is in fact a group.

If f is autonomous, that is, f does not depend on the time variable, or if f is periodic on time, then the set Σ_0 is a closed space. However, if f is more general (for instance, quasiperiodic in time) then Σ_0 is not closed, so it is convenient to consider its closure in the metric ϱ of $C_b(\mathbb{R} \times X, X)$:

$$\Sigma = \text{closure of } \Sigma_0(f) \text{ in } C_b(\mathbb{R} \times X, X) \text{ in the metric } \varrho.$$

The set Σ is commonly known as the **hull** of f in $(C_b(\mathbb{R} \times X, X), \varrho)$ and it is also denoted by $H(f)$, see [16, 26]. It is clear that the continuity of θ_t in Σ_0 extends to the continuity of θ_t in Σ.

Remark 2.6. We could also consider $f \in C_b(\mathbb{R}_+ \times X, X)$, that is, f defined only for positive times (which happens in general, when dealing with real world phenomena). In this case Σ is the closure of the set $\{f(s + \cdot, \cdot) \colon s \geqslant 0\}$, known as the **positive hull** of f, and θ_t defines a semigroup on Σ.

We may now study the differential equation as the combination of a base flow $\{\theta_t\}_{t \in \mathbb{R}}$ on Σ and, for each $\sigma \in \Sigma$, the map $\mathbb{R}_+ \times X \ni (t, u_0) \mapsto \varphi(t, \sigma)u_0 \in X$ where, for each $u_0 \in X$, $\mathbb{R}_+ \ni t \mapsto \varphi(t, \sigma)u_0 \in X$ is the solution of the initial value problem

$$\begin{cases} \dot{u} = \sigma(t, u), \ t > 0, \\ u(0) = u_0 \in X. \end{cases} \tag{2.2}$$

Thus, given a non-autonomous differential equation such as (2.2), we have three different systems to consider:

- the *evolution process* $T_\sigma(t,s)u_0 = \varphi(t-s,\theta_s\sigma)u_0$, for each $\sigma \in \Sigma$;
- the *non-autonomous dynamical system* $(\varphi,\theta)_{(X,\Sigma)}$;
- the *skew-product semiflow* $\{\Pi(t)\colon t \geqslant 0\}$ associated with $(\varphi,\theta)_{(X,\Sigma)}$ defined on the product space $\mathbb{X} = X \times \Sigma$.

Each of these dynamical systems yields different notions of attractors:

- a *pullback attractor* $\{A_\sigma(t)\}_{t\in\mathbb{R}}$ for the evolution process $\{T_\sigma(t,s)\colon t \geqslant s\}$, for each $\sigma \in \Sigma$;
- a *cocycle attractor* $\{A(\sigma)\}_{\sigma\in\Sigma}$ for $(\varphi,\theta)_{(X,\Sigma)}$;
- a *uniform attractor* $\mathcal{A}$ for $(\varphi,\theta)_{(X,\Sigma)}$;
- a *global attractor* $\mathbb{A}$ for the skew-product semiflow $\{\Pi(t)\colon t \geqslant 0\}$.

The reader may consult [10] which presents the relations between the skew-product semiflow and the uniform attractor, between the skew-product semiflow and the cocycle attractor and the relation between the skew-product semiflow and the pullback attractor. These results will be provided in the context of impulsive systems in Section 5.

3. Impulsive non-autonomous dynamical systems

In this section, we present the theory of impulsive non-autonomous dynamical systems, which was first presented in [4]. To this end, let $(\varphi,\theta)_{(X,\Sigma)}$ be a NDS and for each $D \subset X$, $J \subset \mathbb{R}_+$ and $\sigma \in \Sigma$, we define

$$F_\varphi(D,J,\sigma) = \{x \in X\colon \varphi(t,\sigma)x \in D \text{ for some } t \in J\},$$

and also, if $\mathbb{D} \subset \mathbb{X} = X \times \Sigma$ we define

$$F_\Pi(\mathbb{D},J) = \{(x,\sigma) \in \mathbb{X}\colon \Pi(t)(x,\sigma) \in \mathbb{D} \text{ for some } t \in J\}.$$

A point $x \in X$ is said to be an **initial point** if $F_\varphi(x,\tau,\sigma) = \varnothing$ for all $\tau > 0$ and for all $\sigma \in \Sigma$.

Definition 3.1. An **impulsive non-autonomous dynamical system**, or simply an **INDS**, $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ consists of a NDS $(\varphi,\theta)_{(X,\Sigma)}$, a nonempty closed subset $M \subset X$ such that for each $x \in M$ and each $\sigma \in \Sigma$ there exists $\epsilon_{x,\sigma} > 0$ such that

$$\bigcup_{t\in(0,\epsilon_{x,\sigma})} F_\varphi(x,t,\theta_{-t}\sigma) \cap M = \varnothing \quad \text{and} \quad \{\varphi(s,\sigma)x\colon s \in (0,\epsilon_{x,\sigma})\} \cap M = \varnothing, \tag{3.1}$$

and a continuous function $I\colon M \to X$ whose action will be specified later.

The set M is called the **impulsive set** and the function I is called **impulse function**. We also define $M^+_\varphi(x,\sigma) = \{\varphi(\tau,\sigma)x\colon \tau > 0\} \cap M$.

One important property which may be observed is that if $M^+_\varphi(x,\sigma) \neq \varnothing$, then there exists $t > 0$ such that $\varphi(t,\sigma)x \in M$ and $\varphi(\tau,\sigma)x \notin M$ for $0 < \tau < t$, and as a consequence, for each $\sigma \in \Sigma$, we are able to define the function $\phi(\cdot,\sigma)\colon X \to (0,+\infty]$ by

$$\phi(x,\sigma) = \begin{cases} s, & \text{if } \varphi(s,\sigma)x \in M \text{ and } \varphi(t,\sigma)x \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } \varphi(t,\sigma)x \notin M \text{ for all } t > 0. \end{cases} \tag{3.2}$$

In the first case, the value $\phi(x,\sigma)$ represents the smallest positive time such that the positive semitrajectory of x in the fiber σ meets M and we say that the point $\varphi(\phi(x,\sigma),\sigma)x$ is the **impulsive point** of x in the fiber σ.

Definition 3.2. Given $\sigma \in \Sigma$, the **impulsive positive semitrajectory** of $x \in X$ starting at σ by the INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ is a map $\tilde{\varphi}(\cdot,\sigma)x$ defined in an interval $J_{(x,\sigma)} \subset \mathbb{R}_+$, $0 \in J_{(x,\sigma)}$, with values in X given inductively by the following rule: if $M^+_\varphi(x,\sigma) = \varnothing$, then $\tilde{\varphi}(t,\sigma)x = \varphi(t,\sigma)x$ for all $t \in [0,+\infty)$ and in this case $\phi(x,\sigma) = +\infty$. However, if $M^+_\varphi(x,\sigma) \neq \varnothing$ then we denote $x = x_0^+$ and we define $\tilde{\varphi}(\cdot,\sigma)x$ on $[0,\phi(x_0^+,\sigma)]$ by

$$\tilde{\varphi}(t,\sigma)x = \begin{cases} \varphi(t,\sigma)x_0^+, & \text{if} \quad 0 \leqslant t < \phi(x_0^+,\sigma), \\ I(\varphi(\phi(x_0^+,\sigma),\sigma)x_0^+), & \text{if} \quad t = \phi(x_0^+,\sigma). \end{cases}$$

Now let $s_0 = \phi(x_0^+,\sigma)$, $x_1 = \varphi(s_0,\sigma)x_0^+$ and $x_1^+ = I(\varphi(s_0,\sigma)x_0^+)$. In this case $s_0 < +\infty$ and the process can go on, but now starting at x_1^+. If $M^+_\varphi(x_1^+,\theta_{s_0}\sigma) = \varnothing$ then we define $\tilde{\varphi}(t,\sigma)x = \varphi(t - s_0,\theta_{s_0}\sigma)x_1^+$ for $s_0 \leqslant t < +\infty$ and in this case $\phi(x_1^+,\theta_{s_0}\sigma) = +\infty$. However, if $M^+_\varphi(x_1^+,\theta_{s_0}\sigma) \neq \varnothing$, then we define $\tilde{\varphi}(\cdot,\sigma)x$ on $[s_0, s_0 + \phi(x_1^+,\theta_{s_0}\sigma)]$ by

$$\tilde{\varphi}(t,\sigma)x = \begin{cases} \varphi(t - s_0,\theta_{s_0}\sigma)x_1^+, & \text{if} \quad s_0 \leqslant t < s_0 + \phi(x_1^+,\theta_{s_0}\sigma), \\ I(\varphi(\phi(x_1^+,\theta_{s_0}\sigma),\theta_{s_0}\sigma)x_1^+), & \text{if} \quad t = s_0 + \phi(x_1^+,\theta_{s_0}\sigma). \end{cases}$$

Now let $s_1 = \phi(x_1^+,\theta_{s_0}\sigma)$, $x_2 = \varphi(s_1,\theta_{s_0}\sigma)x_1^+$ and $x_2^+ = I(\varphi(s_1,\theta_{s_0}\sigma)x_1^+)$, and so on. This process ends after a finite number of steps if $M^+_\varphi(x_n^+,\theta_{t_n}\sigma) = \varnothing$ for some $n \in \mathbb{N} \cup \{0\}$, or it may proceed indefinitely, if $M^+_\varphi(x_n^+,\theta_{t_n}\sigma) \neq \varnothing$ for all $n \in \mathbb{N} \cup \{0\}$ and in this case $\tilde{\varphi}(\cdot,\sigma)x$ is defined in the interval $[0, T(x,\sigma))$, where $T(x,\sigma) = \sum_{i=0}^{+\infty} s_i$.

As in [4], we assume hereon the following assumption:

$$T(x,\sigma) = +\infty \quad \text{for all} \quad x \in X \text{ and } \sigma \in \Sigma. \tag{\textbf{H0}}$$

Remark 3.3. In the particular case when $\Sigma = \{\sigma_0\}$, these previous definitions reduce to the case of *autonomous impulsive dynamical systems*. The theory of autonomous impulsive dynamical systems and their attractors, may be found, for instance, in [5, 6, 7, 8, 9, 17, 18, 19, 21, 22, 24].

The construction of the function ϕ and the impulsive positive semitrajectory $\tilde{\varphi}$ allows us to state the following important relationship, whose proof may be found in [4]. Let $(\varphi, \theta)_{(X,\Sigma)}$ be a NDS and $\{\Pi(t)\colon t \geqslant 0\}$ be its associated skew-product semiflow in $\mathbb{X}$. Define $\tilde{\Pi}^*$ by

$$\tilde{\Pi}^*(t)(x,\sigma) = (\tilde{\varphi}(t,\sigma)x, \theta_t\sigma) \quad \text{for all} \quad (x,\sigma) \in \mathbb{X} \text{ and } t \geqslant 0,$$

and also let $\{\tilde{\Pi}(t)\colon t \geqslant 0\}$ be the impulsive dynamical system $(\mathbb{X}, \Pi, \mathbb{M}, \mathbb{I})$, where $\mathbb{M} = M \times \Sigma$ and $\mathbb{I} : \mathbb{M} \to \mathbb{X}$ is given by $\mathbb{I}(x,\sigma) = (I(x), \sigma)$, for $x \in M$ and $\sigma \in \Sigma$. Then

$$\tilde{\Pi}^*(t) = \tilde{\Pi}(t) \quad \text{for all} \quad t \geqslant 0.$$

Moreover, if ϕ is the function defined in (3.2), then it coincides with the function used to define the impulsive positive semitrajectory $\{\tilde{\Pi}(t)\colon t \geqslant 0\}$. Also, for each $\sigma \in \Sigma$ and $t, s \in \mathbb{R}_+$, we have

$$\tilde{\varphi}(t+s,\sigma) = \tilde{\varphi}(t,\theta_s\sigma)\tilde{\varphi}(s,\sigma)$$

that is, $\tilde{\varphi}$ satisfies the cocycle property.

A key property that the above relation provides, as seen in [4], is that the following diagram is commutative:

$$\begin{array}{ccc} (\varphi,\theta)_{(X,\Sigma)} & \longrightarrow & \{\Pi(t)\colon t \geqslant 0\} \\ \downarrow & \circlearrowleft & \downarrow \\ [(\varphi,\theta)_{(X,\Sigma)}, M, I] & \longrightarrow & (\mathbb{X}, \Pi, \mathbb{M}, \mathbb{I}) \end{array}$$

that is, given a NDS $(\varphi, \theta)_{(X,\Sigma)}$, if we construct the INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ and we consider the impulsive skew-product semiflow associated with $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$, then we obtain the same object as if we first constructed the skew-product semiflow $\{\Pi(t)\colon t \geqslant 0\}$ associated with $(\varphi,\theta)_{(X,\Sigma)}$ and then using this skew-product to construct the impulsive dynamical system $(\mathbb{X}, \Pi, \mathbb{M}, \mathbb{I})$.

This is essential to the work that is about to be presented, relating several different notions of asymptotic behavior in the impulsive non-autonomous case.

3.1. **Tube conditions.** The so called "tube conditions" are very important for the theory of impulsive dynamical systems. Here, we briefly present the results of [4] (which uses the results of [20] and the above diagram) for tube conditions of impulsive non-autonomous dynamical systems. Recall that $\mathbb{X} = X \times \Sigma$ and $\mathbb{M} = M \times \Sigma$.

Definition 3.4. *A closed set $\mathbb{S}$ containing $(x,\sigma) \in \mathbb{X}$ is called a* **section** *through (x,σ) if there exist $\lambda > 0$ and a closed subset $\mathbb{L}$ of $\mathbb{X}$ such that:*

(a) $\mathbb{F}_\Pi(\mathbb{L}, \lambda) = \mathbb{S}$;
(b) $\mathbb{F}_\Pi(\mathbb{L}, [0, 2\lambda])$ *contains a neighborhood of (x,σ);*

(c) $\mathbb{F}_{\Pi}(\mathbb{L},\nu)\bigcap\mathbb{F}_{\Pi}(\mathbb{L},\zeta)=\varnothing$, *if* $0\leqslant\nu<\zeta\leqslant 2\lambda$.

We say that the set $\mathbb{F}_{\Pi}(\mathbb{L},[0,2\lambda])$ *is a* $\lambda-$**tube** (*or simply* **tube**) *and the set* $\mathbb{L}$ *is a* **bar**.

The Definition 3.4 is the same definition of tube for general impulsive systems (X,π,M,I), see [20].

Definition 3.5. *A point* $(x,\sigma)\in\mathbb{M}$ *satisfies the* **strong tube condition** (**STC**)*, if there exists a section* $\mathbb{S}$ *through* (x,σ) *such that* $\mathbb{S}=\mathbb{F}_{\Pi}(\mathbb{L},[0,2\lambda])\bigcap\mathbb{M}$. *Also, we say that a point* $(x,\sigma)\in\mathbb{M}$ *satisfies the* **special strong tube condition** (**SSTC**) *if it satisfies STC and the* $\lambda-$*tube* $\mathbb{F}(\mathbb{L},[0,2\lambda])$ *is such that* $\mathbb{F}(\mathbb{L},[0,\lambda])\bigcap\mathbb{I}(\mathbb{M})=\varnothing$.

Now, we introduce the concepts of STC and SSTC in the context of INDS.

Definition 3.6. Let $[(\varphi,\theta)_{(X,\Sigma)},M,I]$ be an INDS. We say that a point $x\in M$ satisfies the $\varphi-$**strong tube condition** ($\varphi-$**STC**), if for each $\sigma\in\Sigma$, the pair (x,σ) satisfies STC with respect to the impulsive skew-product $(\mathbb{X},\Pi,\mathbb{M},\mathbb{I})$. Also, we say that a point $x\in M$ satisfies the $\varphi-$**special strong tube condition** ($\varphi-$**SSTC**), if for each $\sigma\in\Sigma$, the pair (x,σ) satisfies SSTC with respect to the impulsive skew-product $(\mathbb{X},\Pi,\mathbb{M},\mathbb{I})$.

Theorem 3.7. [4, Theorem 3.5] *Let* $[(\varphi,\theta)_{(X,\Sigma)},M,I]$ *be an INDS such that each point of* M *satisfies* $\varphi-$*STC. Then* ϕ *is upper semicontinuous in* $X\times\Sigma$ *and it is continuous in* $(X\backslash M)\times\Sigma$. *Moreover, if there are no initial points in* M *and* ϕ *is continuous at* (x,σ) *for some* $\sigma\in\Sigma$, *then* $x\notin M$.

Proposition 3.8. [4, Proposition 3.7] *Let* $[(\varphi,\theta)_{(X,\Sigma)},M,I]$ *be an INDS such that* $I(M)\cap M=\varnothing$ *and let* $y\in M$ *satisfy* $\varphi-$*SSTC. Then, for each* $\sigma\in\Sigma$, *the point* (y,σ) *satisfies SSTC with a* $\lambda-$*tube* $\mathbb{F}_{\Pi}(\mathbb{L},[0,2\lambda])$ *such that* $\tilde{\Pi}(t)(X\times\Sigma)\bigcap F_{\Pi}(\mathbb{L},[0,\lambda])=\varnothing$ *for all* $t>\lambda$.

3.2. **Existence of impulsive cocycle attractors.** In [4], the authors introduce the definition of impulsive non-autonomous dynamical systems and also find sufficient conditions to ensure the existence of an impulsive cocycle attractor. In this subsection, we present their main results.

The definition of $\tilde{\varphi}-$invariance is analogous to the notion of $\varphi-$invariance simply replacing φ by $\tilde{\varphi}$.

Definition 3.9. Given an INDS $[(\varphi,\theta)_{(X,\Sigma)},M,I]$, a non-autonomous set $\hat{B}$ is said to be **pullback** $(\tilde{\varphi},\mathfrak{D})-$**attracting**, if for each $\sigma\in\Sigma$ and $\hat{D}\in\mathfrak{D}$ we have

$$\lim_{t\to+\infty}\operatorname{dist}(\tilde{\varphi}(t,\theta_{-t}\sigma)D(\theta_{-t}\sigma),B(\sigma))=0.$$

Definition 3.10. Given a universe $\mathfrak{D}$ and an INDS $[(\varphi,\theta)_{(X,\Sigma)},M,I]$, a compact non-autonomous set $\hat{A}$ is called the $(\tilde{\varphi},\mathfrak{D})-$**impulsive cocycle attractor** if:

(i) $\hat{A}\backslash M=\{A(\sigma)\backslash M\}_{\sigma\in\Sigma}$ is $\tilde{\varphi}-$invariant;

(ii) $\hat{A}$ is pullback $(\tilde{\varphi}, \mathfrak{D})-$attracting;
(iii) $\hat{A}$ is minimal among the closed non-autonomous sets satisfying **(ii)**.

Remark 3.11. If $\hat{A}_1$ and $\hat{A}_2$ are two $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractors then

$$A_1(\sigma)\backslash M = A_2(\sigma)\backslash M \quad \text{for all} \quad \sigma \in \Sigma.$$

In order to find sufficient conditions to ensure the existence of an impulsive cocycle attractor for an INDS, the key role is played by the *impulsive pullback ω-limit set.*

Definition 3.12. Given a non-autonomous set $\hat{B}$ and $\sigma \in \Sigma$, we define the **impulsive pullback ω-limit of $\hat{B}$ at** σ as the set

$$\tilde{\omega}(\hat{B}, \sigma) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \bigcup_{\epsilon \in [0, s^{-1})} \tilde{\varphi}(t + \epsilon, \theta_{-t}\sigma) B(\theta_{-t}\sigma)}$$

and the **impulsive pullback ω-limit** of $\hat{B}$ as the non-autonomous set $\tilde{\omega}(\hat{B}) = \{\tilde{\omega}(\hat{B}, \sigma)\}_{\sigma \in \Sigma}$.

The following characterization is crucial for the theory, and the proof (analogous to the continuos case) can be found in [4, Lemma 4.2].

Lemma 3.13. *We have*

$$\tilde{\omega}(\hat{B}, \sigma) = \{x \in X : \text{ there exist sequences } \{t_n\}_{n \in \mathbb{N}}, \{\epsilon_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \text{ and } \{x_n\}_{n \in \mathbb{N}} \subseteq B(\theta_{-t_n}\sigma)$$
$$\text{with } t_n \overset{n \to +\infty}{\longrightarrow} +\infty, \ \epsilon_n \overset{n \to +\infty}{\longrightarrow} 0 \text{ such that } \tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma) x_n \overset{n \to +\infty}{\longrightarrow} x\}$$

and $\tilde{\omega}(\hat{B}, \sigma)$ is closed.

It is clear that, if we are in the continuous case, that is, $M = \varnothing$, then the impulsive pullback ω-limit coincides with the pullback ω-limit. Now, for the results that follow, we fix a universe $\mathfrak{D}$.

Definition 3.14. An INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ is said to be **pullback $\mathfrak{D}-$asymptotically compact**, if for any $\sigma \in \Sigma$, $\hat{D} \in \mathfrak{D}$ and sequences $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$, $\{x_n\}_{n \in \mathbb{N}} \subset X$ with $t_n \overset{n \to +\infty}{\longrightarrow} +\infty$ and $x_n \in D(\theta_{-t_n}\sigma)$ for $n \in \mathbb{N}$, then the sequence $\{\tilde{\varphi}(t_n, \theta_{-t_n}\sigma)x_n\}_{n \in \mathbb{N}}$ possesses a convergent subsequence.

The main result of [4] is stated next. The only difference is that we replace the condition "there is a pullback $\mathfrak{D}-$absorbing non-autonomous set $\hat{K} \in \mathfrak{D}$" by "there is a pullback $(\tilde{\varphi}, \mathfrak{D})-$attracting non-autonomous set $\hat{K} \in \mathfrak{D}$". The proof is the same, see [4, Theorem 5.1].

Theorem 3.15. [4, Theorem 5.1] *Let $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ be an INDS pullback $\mathfrak{D}-$asymptotically compact such that $I(M) \cap M = \varnothing$ and every point from M satisfies $\varphi-$SSTC. Assume that there exists a pullback $(\tilde{\varphi}, \mathfrak{D})-$attracting non-autonomous set $\hat{K} \in \mathfrak{D}$. Then, the non-autonomous set $\hat{A}$, given by $A(\sigma) = \tilde{\omega}(\hat{K}, \sigma)$, is the $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractor.*

4. Impulsive dynamical systems: autonomous vs. non-autonomous

In [5], the authors developed the theory of global attractors for the autonomous impulsive scenario. In their work, they established sufficient conditions for the existence of the impulsive global attractor for a given *autonomous* impulsive system (X, π, M, I), with the following definition:

Definition 4.1. A subset $\mathcal{A} \subset X$ will be called a **global attractor** for the IDS (X, π, M, I) if it satisfies the following conditions:

- **(i)** $\mathcal{A}$ is precompact and $\mathcal{A} = \overline{\mathcal{A}} \backslash M$;
- **(ii)** $\mathcal{A}$ is $\tilde{\pi}-$invariant;
- **(iii)** $\mathcal{A}$ $\tilde{\pi}-$attracts all bounded subsets of X.

We recall in Definition 4.1 that a subset $A \subset X$ is $\tilde{\pi}-$invariant if $\pi(t)A = A$ for all $t \geqslant 0$ and $A \subset X$ $\tilde{\pi}-$attracts all bounded subsets of X if $\lim\limits_{t \to +\infty} \operatorname{dist}(\tilde{\pi}(t)B, A) = 0$ for all bounded $B \subset X$. With this definition, the authors present their main result:

Theorem 4.2. [5, Theorem 4.7] *Let (X, π, M, I) be an IDS such that $I(M) \cap M = \varnothing$, every point from M satisfies SSTC, there exists a precompact set K, with $K \cap M = \varnothing$, such that K $\tilde{\pi}-$absorbs all bounded subsets of X (for any bounded subset B of X there is $t_B \geqslant 0$ such that $\tilde{\pi}(t)B \subset K$ for all $t \geqslant t_B$) and there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. Then (X, π, M, I) has a global attractor $\mathcal{A}$ and we have $\mathcal{A} = \tilde{\omega}(K) \backslash M$.*

In the non-autonomous case, even when we impose the same hypotheses, we were not able to prove the existence of an object that generalizes naturally the concept of global attractor of Definition 4.1. The crucial result that is used in [5] is Proposition 3.14, which ensures that given a set B, if $\tilde{\omega}(B)$ $\tilde{\pi}-$attracts B, then so does $\tilde{\omega}(B) \backslash M$. Recall that $\tilde{\omega}(B) = \{x \in X :$ there exist sequences $\{x_n\}_{n \in \mathbb{N}} \subseteq B$ and $\{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+$ with $t_n \overset{n \to +\infty}{\longrightarrow} \infty$ such that $\tilde{\pi}(t_n)x_n \overset{n \to +\infty}{\longrightarrow} x\}$. With this result, they are able to construct a precompact set, disjoint from M, that $\tilde{\pi}-$attracts all bounded subsets of X. The proof of this result is a direct consequence of Lemma 3.13 of their paper. We present here the non-autonomous version of this lemma.

Lemma 4.3. (Non-autonomous version of [5, Lemma 3.13]) *Let $[(\varphi, \theta)_{(X, \Sigma)}, M, I]$ be a pullback $\mathfrak{D}-$asymptotically compact INDS, with $I(M) \cap M = \varnothing$ and every point from M satisfies $\varphi-$SSTC. Let $\sigma \in \Sigma$ and assume that there exists $\xi > 0$ such that $\phi(z, \omega) \geqslant \xi$ for all $z \in I(M)$ and ω in a neighborhood $\Sigma_\sigma \subset \Sigma$ of σ. If $\hat{B} \in \mathfrak{D}$ is a nonempty non-autonomous set and $x \in \tilde{\omega}(\hat{B}, \sigma) \cap M$, then there exists a sequence $\{y_m\}_{m \in \mathbb{N}}$ such that for each $m \in \mathbb{N}$ we have $y_m \in \tilde{\omega}(\hat{B}, \theta_{-\frac{1}{m}}\sigma) \backslash M$, $\varphi(1/m, \theta_{-\frac{1}{m}}\sigma)y_m = x$ and $y_m \overset{m \to +\infty}{\longrightarrow} x$.*

Proof: Let $x \in \tilde{\omega}(\hat{B}, \sigma) \cap M$. Then there exist positive sequences $t_n \overset{n \to +\infty}{\longrightarrow} +\infty$, $\epsilon_n \overset{n \to +\infty}{\longrightarrow} 0$ and $x_n \in B(\theta_{-t_n}\sigma)$ such that $z_n \doteq \tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n \overset{n \to +\infty}{\longrightarrow} x$. By Proposition 3.8, the point (x, σ)

satisfies SSTC with a $\lambda-$tube $\mathbb{F}_\Pi(\mathbb{L},[0,2\lambda])$ such that $\tilde{\Pi}(t)(X\times\Sigma)\bigcap F_\Pi(\mathbb{L},[0,\lambda])=\varnothing$ for all $t>\lambda$. Since

$$\tilde{\Pi}(t_n+\epsilon_n)(x_n,\theta_{-t_n}\sigma)=(z_n,\theta_{\epsilon_n}\sigma)\overset{n\to+\infty}{\longrightarrow}(x,\sigma),$$

we may assume that $(z_n,\theta_{\epsilon_n}\sigma)\in F_\Pi(\mathbb{L},(\lambda,2\lambda])$ for all $n\in\mathbb{N}$.

We may choose a subsequence if necessary, which we will call the same, and a sequence $\eta_n\overset{n\to+\infty}{\longrightarrow}0$, $\eta_n>0$, such that $\Pi(\eta_n)(z_n,\theta_{\epsilon_n}\sigma)\in\mathbb{M}$, that is, $\varphi(\eta_n,\theta_{\epsilon_n}\sigma)z_n\in M$ for all $n\in\mathbb{N}$. We may also assume that $\eta_n<\frac{\xi}{2}$ for all $n\in\mathbb{N}$.

Let $\zeta>0$ be such that $\theta_t\sigma\in\Sigma_\sigma$ for all $t\in(-\zeta,\zeta)$. Recall that there exists $\epsilon_{x,\sigma}>0$ such that $\bigcup_{t\in(0,\epsilon_{x,\sigma})}F_\varphi(x,t,\theta_{-t}\sigma)\cap M=\varnothing$. Let $m_0\in\mathbb{N}$ be such that $\frac{1}{m_0}<\min\{\epsilon_{x,\sigma},\frac{\xi}{2},\zeta\}$ and $n_0\in\mathbb{N}$ be such that $\epsilon_n<\zeta$ for all $n\geqslant n_0$.

For each integer $m\geqslant m_0$, we consider the sequence $w_n^m=\tilde{\varphi}(t_n-\frac{1}{m}+\epsilon_n,\theta_{-t_n}\sigma)x_n$, $n\in\mathbb{N}$. By the pullback $\mathfrak{D}-$asymptotic compactness and the fact that $\hat{B}\in\mathfrak{D}$, we may assume that $w_n^m\overset{n\to+\infty}{\longrightarrow}y_m\in\tilde{\omega}(\hat{B},\theta_{-\frac{1}{m}}\sigma)$, for each $m\geqslant m_0$.

We claim that $s_n^m\doteq\phi(w_n^m,\theta_{\epsilon_n-\frac{1}{m}}\sigma)>\frac{1}{m}$ for all $n\geqslant n_0$ and $m\geqslant m_0$. Indeed, suppose to the contrary that $s_n^m\leqslant\frac{1}{m}$ for some $n\geqslant n_0$ and $m\geqslant m_0$. We have $\varphi(s_n^m,\theta_{\epsilon_n-\frac{1}{m}}\sigma)w_n^m\in M$ and $v_n^m=\tilde{\varphi}(s_n^m,\theta_{\epsilon_n-\frac{1}{m}}\sigma)w_n^m\in I(M)$. Now note that

$$\begin{aligned}\varphi(\eta_n+1/m-s_n^m,\theta_{s_n^m+\epsilon_n-\frac{1}{m}}\sigma)v_n^m&=\varphi(\eta_n,\theta_{\epsilon_n}\sigma)\varphi(1/m-s_n^m,\theta_{s_n^m+\epsilon_n-\frac{1}{m}}\sigma)v_n^m=\\&=\varphi(\eta_n,\theta_{\epsilon_n}\sigma)z_n\in M\end{aligned}$$

since $\frac{1}{m}-s_n^m<\frac{1}{m}<\xi$ and $\theta_{s_n^m+\epsilon_n-\frac{1}{m}}\sigma\in\Sigma_\sigma$. But it is a contradiction, since $0<\eta_n+\frac{1}{m}-s_n^m<\eta_n+\frac{1}{m}<\xi$ and $v_n^m\in I(M)$. This shows that for $n\geqslant n_0$ and $m\geqslant m_0$, we have

$$\varphi(1/m,\theta_{\epsilon_n-\frac{1}{m}}\sigma)w_n^m=\tilde{\varphi}(1/m,\theta_{\epsilon_n-\frac{1}{m}}\sigma)w_n^m=z_n.$$

By the continuity of φ, as $n\to+\infty$, we get $\varphi(1/m,\theta_{-\frac{1}{m}}\sigma)y_m=x\in M$. Since $1/m<\epsilon_{x,\sigma}$, we obtain $y_m\in\tilde{\omega}(\hat{B},\theta_{-\frac{1}{m}}\sigma)\backslash M$.

If $\{y_m\}_{m\in\mathbb{N}}$ does not converge to x, then we can choose a convergent subsequence $\{y_{m_l}\}_{l\in\mathbb{N}}$ to a point $x_0\neq x$, but $x=\varphi(1/m_l,\theta_{-\frac{1}{m_l}}\sigma)y_{m_l}\overset{l\to+\infty}{\longrightarrow}\varphi(0,\sigma)x_0=x_0$, which gives us a contradiction and proves that $y_m\overset{m\to+\infty}{\longrightarrow}x$. □

As an immediate consequence of this result we obtain:

Corollary 4.4. *With the conditions of Lemma 4.3, given $\epsilon>0$ we have*

$$\tilde{\omega}(\hat{B},\sigma)\cap M\subset\overline{\bigcup_{s\in[0,\epsilon]}\tilde{\omega}(\hat{B},\theta_{-s}\sigma)\backslash M}\quad\text{for all }\ \sigma\in\Sigma.$$

We can easily check that Corollary 4.4 is not enough to prove a result as Proposition 3.14 of [5] for the non-autonomous case. It was expected that a result as in [5] would not be natural, since as in the non-autonomous case, we are constantly changing the fibers σ.

Also, we can see that Definition 3.10 is *not* a natural extension of Definition 4.1 to the non-autonomous case. With this idea in mind, we introduce the following definition:

Definition 4.5. A subset $\mathcal{A} \subset X$ is called a $c-$**global attractor** for the IDS (X, π, M, I) if:

(i) $\mathcal{A}$ is compact;
(ii) $\mathcal{A} \backslash M$ is $\tilde{\pi}-$invariant;
(iii) $\mathcal{A}$ $\tilde{\pi}-$attracts all bounded subsets of X.

In order to obtain a simpler result to the autonomous case, we first need to the following definition:

Definition 4.6. An IDS (X, π, M, I) is called **asymptotically compact** if for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and any sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ is bounded, then the set $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ is precompact.

The results of [4] can now be applied to the autonomous case, see the next theorem.

Theorem 4.7. *Let (X, π, M, I) be an asymptotically compact IDS such that $I(M) \cap M = \varnothing$, every point from M satisfies SSTC and there exists a bounded set K which $\tilde{\pi}-$attracts bounded subsets from X. Then (X, π, M, I) has a $c-$global attractor $\mathcal{A}$ and we have $\mathcal{A} = \tilde{\omega}(K)$.*

Proof: Let $\Sigma = \{\sigma\}$ and $\mathfrak{D}$ be the universe of all bounded subsets of X. Let $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ be an INDS, where

$$\varphi(t, \sigma)x = \pi(t)x$$

for all $t \in \mathbb{R}_+$ and $x \in X$, and $\theta_t\sigma = \sigma$ for all $t \in \mathbb{R}$. Note that $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ is pullback $\mathfrak{D}-$asymptotically compact and $\hat{K} = \{K(\sigma)\}$, with $K(\sigma) = K$, is a pullback $(\tilde{\varphi}, \mathfrak{D})-$attracting non-autonomous set. Define $\mathbb{X} = X \times \{\sigma\}$, $\mathbb{M} = M \times \{\sigma\}$, $\mathbb{I}(x, \sigma) = (I(x), \sigma)$ for all $x \in X$ and $\Pi(t)(x, \sigma) = (\varphi(t, \sigma)x, \theta_t\sigma)$ for all $t \in \mathbb{R}_+$ and $x \in X$. We claim that M satisfies $\varphi-$SSTC. Indeed, let $x \in M$ be arbitrary. Since x satisfies SSTC there exist a $\lambda-$section S through x and a bar L such that $F(L, \lambda) = S$, $F(L, [0, 2\lambda])$ is a neighborhood of x, $F(L, \mu) \cap F(L, \nu) = \varnothing$ for all $0 \leqslant \mu < \nu \leqslant 2\lambda$ and $F(L, [0, \lambda]) \cap I(M) = \varnothing$. Now, we define $\mathbb{S} = S \times \{\sigma\}$ and $\mathbb{L} = L \times \{\sigma\}$. It is not difficult to see that $\mathbb{F}_\Pi(\mathbb{L}, [0, 2\lambda])$ is a $\lambda-$tube through (x, σ) with section $\mathbb{S}$ satisfying $\mathbb{F}_\Pi(\mathbb{L}, [0, \lambda]) \cap \mathbb{I}(\mathbb{M}) = \varnothing$. Thus, the claim is proved.

By Theorem 3.15, the the non-autonomous set $\hat{A}$, given by $A(\sigma) = \tilde{\omega}(K, \sigma)$, is the $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractor of $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$. Note that $\tilde{\omega}(K, \sigma) = \tilde{\omega}(K)$ and it satisfies the conditions (i), (ii) and (iii) from Definition 4.5. Consequently, $\tilde{\omega}(K)$ is the $c-$global attractor of (X, π, M, I). □

It is clear that $c-$global attractors extend Definition 4.1. Thus, we have a straightforward relationship between these two object, given by the following result.

Proposition 4.8. *Let (X, π, M, I) be an IDS. If $\mathcal{A}$ is its global attractor, then $\mathcal{A}_1 = \bar{\mathcal{A}}$ is its $c-$global attractor.*

However, the other implication is not true, that is, if $\mathcal{A} \subset X$ is a $c-$global attractor for the IDS (X, π, M, I), then $\mathcal{A}_2 = \mathcal{A} \backslash M$ may not be an impulsive attractor for the IDS (X, π, M, I) in general, since property **(iii)** of Definition 4.1 may not be satisfied. Nevertheless, using Theorem 4.7 and Corollary 4.4, we are able to state one more result:

Theorem 4.9. *Let (X, π, M, I) be an asymptotically compact IDS such that $I(M) \cap M = \varnothing$, every point from M satisfies SSTC, there exists a bounded set K which $\tilde{\pi}-$attracts bounded subsets from X and there exists $\xi > 0$ such that $\phi(z) \geqslant \xi$ for all $z \in I(M)$. Then (X, π, M, I) possesses a global attractor $\mathcal{A}$ and we have $\mathcal{A} = \tilde{\omega}(K) \backslash M$.*

Note that this is the same result as Theorem 4.2, without the assumption that $K \cap M = \varnothing$ and with the condition of $\tilde{\pi}-$absorbing sets replaced by $\tilde{\pi}-$attraction. One can see that the notion of $c-$global attractor is more natural to deal with in the autonomous framework if we want to consider the impulsive cocycle attractors in the non-autonomous case, since the latter is a natural extension of the first. That being said, throughout the paper, we shall use the notion of $c-$*global attractors* for the autonomous case.

4.1. **Asymptotic compactness.** In this subsection, we shall explore the property of *asymptotic compactness* for an impulsive autonomous dynamical system, given in Definition 4.6. The definition of asymptotic compact for a semigroup $\{\pi(t) \colon t \geqslant 0\}$ in X is analogous, just replacing $\tilde{\pi}$ by π.

First, we shall prove that, if we do not assume any additional hypothesis on the impulsive set M and the impulsive function I, these two concepts are not equivalent.

Example 4.10. Consider the ordinary differential equation in $\mathbb{R}$ given by

$$\dot{x} = |x| \tag{4.1}$$

and let $\pi(t)x_0$ denote the solution of (4.1) for $t \geqslant 0$ with initial condition x_0. We have $\pi(t)x_0 = x_0 e^{-t}$ for $x_0 < 0$ and $\pi(t)x_0 = x_0 e^{t}$ for $x_0 \geqslant 0$. The semigroup $\{\pi(t) \colon t \geqslant 0\}$ is <u>not</u> asymptotically compact.

Now, consider the set $M \doteq \mathbb{N} = \{1, 2, 3, \cdots\}$ and the impulsive function given by $I(n) = -1$, for all integers $n \geqslant 1$. It is simple to see that $(\mathbb{R}, \pi, M, I)$ is an asymptotically compact autonomous impulsive dynamical system.

Example 4.11. Consider the ordinary differential equation in $\mathbb{R}$ given by

$$\dot{x} = -x \tag{4.2}$$

and let $\pi(t)x_0$ denote the solution of (4.2) for $t \geqslant 0$ with initial condition x_0. We have $\pi(t)x_0 = x_0 e^{-t}$ for all $x_0 \in \mathbb{R}$. Then this semigroup has a global attractor, namely the set $\{0\}$ and, hence, it is asymptotically compact.

Now, define $M \doteq \{n - \frac{1}{n} : n \in \mathbb{N} \text{ and } n \geqslant 2\}$ and $I(n - \frac{1}{n}) = n + 1$ for each natural $n \geqslant 2$. It is not difficult to see that $(\mathbb{R}, \pi, M, I)$ is an autonomous impulsive dynamical system. For any $x_0 > \frac{3}{2}$, we can check that $\tilde{\pi}(t)x_0 \to +\infty$ as $t \to +\infty$, which means that $(\mathbb{R}, \pi, M, I)$ is not asymptotically compact.

The conclusion is that the asymptotic compactness of $\{\pi(t) : t \geqslant 0\}$ does not imply the asymptotic compactness of $(\mathbb{R}, \pi, M, I)$. Moreover, the system $(\mathbb{R}, \pi, M, I)$ can be asymptotically compact even when $\{\pi(t) : t \geqslant 0\}$ is not. So the natural question is: may we impose conditions on $\{\pi(t) : t \geqslant 0\}$, M and I to ensure the asymptotic compactness of (X, π, M, I)?

Our next result provides a positive answer to this question, with a fairly simple condition.

Proposition 4.12. *Assume that (X, π, M, I) is an IDS such that $\{\pi(t) : t \geqslant 0\}$ is asymptotically compact and $I(M)$ is precompact. Then (X, π, M, I) is asymptotically compact.*

Proof: Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence with $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a bounded sequence such that $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ is bounded. In the sequel, we consider some cases:

- **(i)** $t_n < \phi(x_n)$ for all $n \in \mathbb{N}$. In this case, we have $\tilde{\pi}(t_n)x_n = \pi(t_n)x_n$ and every subsequence of $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ possess a convergent subsequence, by the asymptotic compactness of $\{\pi(t) : t \geqslant 0\}$.
- **(ii)** $t_n = \phi(x_n)$ for all $n \in \mathbb{N}$. Clearly $\tilde{\pi}(t_n)x_n = I(\pi(t_n)x_n)$ and every subsequence of $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, since $I(M)$ is precompact.
- **(iii)** $t_n > \phi(x_n)$ for all $n \in \mathbb{N}$. In this case, there exist sequences $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{z_n\}_{n\in\mathbb{N}} \subset I(M)$ such that $\tilde{\pi}(t_n)x_n = \pi(s_n)z_n$. If $\{s_n\}_{n\in\mathbb{N}}$ is bounded, the precompactness of $I(M)$ and the continuity of the map $\mathbb{R}_+ \times X \ni (t, x) \mapsto \pi(t)x \in X$ shows that every subsequence of $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. On the other hand, if $s_n \overset{n\to+\infty}{\longrightarrow} +\infty$, then every subsequence of $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, by the asymptotic compactness of $\{\pi(t) : t \geqslant 0\}$.

Lastly, note that considering subsequences if necessary, we can always assume that one of conditions **(i)**, **(ii)** or **(iii)** holds, which proves that $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ is precompact. Therefore, (X, π, M, I) is asymptotically compact. □

Remark 4.13. Proposition 4.12 can be easily extended to the non-autonomous case, when $\mathfrak{D}$ is the universe of non-autonomous sets with bounded union. Namely, in this case, if $(\varphi, \theta)_{(X,\Sigma)}$ is pullback $\mathfrak{D}-$asymptotically compact and $I(M)$ is precompact, then $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ is also pullback $\mathfrak{D}-$asymptotically compact. In this case, the asymptotic compactness of $(\varphi, \theta)_{(X,\Sigma)}$ is defined as in Definition 3.14, with $\tilde{\varphi}$ replaced by φ.

5. Relationships among attractors

In order to reach the full depth of properties of an impulsive non-autonomous dynamical system, we must be able to relate all possible different frameworks that one can obtain when dealing with a non-autonomous impulsive problem. We will explore each framework in detail to obtain the relations among all the different scenarios. To this end, we will assume from now on the following assumption:

$$\Sigma \text{ is compact and invariant under the action of the driving group } \{\theta_t\colon t \in \mathbb{R}\}. \tag{5.1}$$

5.1. **The impulsive uniform attractor and the impulsive skew-product semiflow.** In what follows, we present the definition of an *impulsive uniform attractor* for an impulsive non-autonomous dynamical system. Besides, we derive its relationship with the global attractor of the associated impulsive skew-product semiflow.

To begin, we present a result that relates impulsive attraction of the impulsive non-autonomous dynamical system with attration of the impulsive skew-product semiflow. For that, let d be a metric in X and ρ be a metric in Σ. We consider the space $X \times \Sigma$ with metric

$$d_{X\times\Sigma}((x_1,\sigma_1),(x_2,\sigma_2)) = d(x_1,x_2) + \rho(\sigma_1,\sigma_2) \quad \text{for all} \quad x_1, x_2 \in X \text{ and } \sigma_1,\sigma_2 \in \Sigma.$$

Thus, for $A, B \subset X$ and $\Sigma_1, \Sigma_2 \subset \Sigma$, we have $\operatorname{dist}(A,B) = \sup\limits_{a\in A}\inf\limits_{b\in B} d(a,b)$ and

$$\operatorname{Dist}(A\times\Sigma_1, B\times\Sigma_2) = \sup_{(a,\sigma_1)\in A\times\Sigma_1}\ \inf_{(b,\sigma_2)\in B\times\Sigma_2} d_{X\times\Sigma}((a,\sigma_1),(b,\sigma_2)).$$

Proposition 5.1. *Let $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ be an INDS, $\{\tilde{\Pi}(t) : t \geqslant 0\}$ be its associated skew-product semiflow on $X\times\Sigma$ and assume that (5.1) holds. Then the following two properties are equivalent:*

- **(i)** *there exists a compact subset $\mathbb{K}$ of $X\times\Sigma$ such that for every bounded subset $\mathbb{B}$ of $X\times\Sigma$*
$$\lim_{t\to+\infty} \operatorname{Dist}(\tilde{\Pi}(t)\mathbb{B},\mathbb{K}) = 0;$$
- **(ii)** *there exists a compact subset K of X such that for every bounded subset B of X*
$$\lim_{t\to+\infty}\sup_{\sigma\in\Sigma} \operatorname{dist}(\tilde{\varphi}(t,\sigma)B, K) = 0.$$

Proof: Suppose that **(i)** holds. Let $K = P_X\mathbb{K}$ (the canonical projection of the first coordinate), B be a bounded subset of X and $\mathbb{B} := B\times\Sigma$. Then $\mathbb{B}$ is bounded in $X\times\Sigma$ and $\lim\limits_{t\to+\infty}\operatorname{Dist}(\tilde{\Pi}(t)\mathbb{B},\mathbb{K}) = 0$. Since

$$\operatorname{dist}(\tilde{\varphi}(t,\sigma)B, K) \leqslant \operatorname{Dist}(\tilde{\Pi}(t)\mathbb{B},\mathbb{K}),$$

for all $\sigma\in\Sigma$, then **(ii)** follows.

Now, let us assume that **(ii)** holds. Take $\mathbb{K} = K \times \Sigma$, which is compact since K and Σ are compact. Since any bounded subset $\mathbb{B}$ of $X \times \Sigma$ is contained in a set of the form $B \times \Sigma$, where B is a bounded subset of X, and

$$\tilde{\Pi}(t)\mathbb{B} \subset \tilde{\Pi}(t)[B \times \Sigma] \subset \bigcup_{\sigma \in \Sigma} \tilde{\varphi}(t,\sigma)B \times \Sigma,$$

if follows that

$$\mathrm{Dist}(\tilde{\Pi}(t)\mathbb{B}, \mathbb{K}) \leqslant \mathrm{Dist}\left(\bigcup_{\sigma \in \Sigma} \tilde{\varphi}(t,\sigma)B \times \Sigma, K \times \Sigma\right) \leqslant \sup_{\sigma \in \Sigma} \mathrm{dist}(\tilde{\varphi}(t,\sigma)B, K).$$

Therefore, **(i)** follows. □

This result forms the basis of the following definition.

Definition 5.2. An INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ is said to be **uniformly asymptotically compact**, if there exists a compact set $K \subset X$ such that for every bounded subset B of X

$$\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \mathrm{dist}(\tilde{\varphi}(t,\sigma)B, K) = 0. \tag{5.2}$$

We have just shown that, if the INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ is uniformly asymptotically compact, $I(M) \cap M = \varnothing$ and each point of M satisfies $\varphi-$SSTC, then the associated skew-product semiflow $\{\tilde{\Pi}(t) : t \geqslant 0\}$ has a $c-$global attractor $\mathbb{A}$ (see Theorem 4.7). Note that the attracting property of $\mathbb{A}$ for $\{\tilde{\Pi}(t) \colon t \geqslant 0\}$ implies the attracting property of the set $\mathcal{A} = P_X\mathbb{A}$ for $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$.

One can see that the property of minimality is preserved, that is, the global attractor $\mathbb{A}$ is the minimal closed set in $X \times \Sigma$ that attracts all bounded sets and its projection $\mathcal{A}$ is the minimal closed subset of X that is uniformly attracting (in the sense of (5.2)), for all bounded subsets B of X, because if $\tilde{\mathcal{A}} \subset X$ is uniformly attracting then $\tilde{\mathcal{A}} \times \Sigma$ is attracting for $\{\tilde{\Pi}(t) : t \geqslant 0\}$, from whence $\mathbb{A} \subset \tilde{\mathcal{A}} \times \Sigma$ and thus $\mathcal{A} \subset \tilde{\mathcal{A}}$. This remark thus yields the definition of the *impulsive uniform attractor*.

Definition 5.3. The **impulsive uniform attractor** $\mathcal{A}$ of the INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ is a compact subset of X such that given $B \subset X$ bounded, we have

$$\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \mathrm{dist}(\tilde{\varphi}(t,\sigma)B, \mathcal{A}) = 0 \tag{5.3}$$

and $\mathcal{A}$ is minimal among all closed sets with property (5.3).

We have therefore the following result.

Theorem 5.4. *Let $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ be an INDS and $\{\tilde{\Pi}(t) : t \geqslant 0\}$ be its associated skew-product semiflow on $X \times \Sigma$. Assume that $I(M) \cap M = \varnothing$ and every point from M satisfies $\varphi-$SSTC. Then $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ has an impulsive uniform attractor $\mathcal{A}_1$ if and only if $\{\tilde{\Pi}(t) :$*

$t \geqslant 0\}$ has a $c-$global attractor $\mathbb{A}$. Moreover, when these attractors exist, if we define $\mathcal{A} = P_X\mathbb{A}$, then

$$\mathcal{A}\backslash M \subset \mathcal{A}_1 \subset \mathcal{A}.$$

Proof: Note that the existence of each one of them implies the existence of the other, see Proposition 5.1, Theorem 4.7 and the comments after Definition 5.2. When these attractors exist, since $\mathcal{A}$ clearly uniformly attracts bounded sets, the minimality condition of $\mathcal{A}_1$ ensures that $\mathcal{A}_1 \subset \mathcal{A}$. To see the second inclusion, note that $\mathcal{A}_1 \times \Sigma$ attracts bounded sets of $X \times \Sigma$ under $\tilde{\Pi}$. This fact and the invariance of $\mathbb{A}\backslash(M \times \Sigma)$ show that $\mathbb{A}\backslash(M \times \Sigma) \subset \mathcal{A}_1 \times \Sigma$, and therefore $\mathcal{A}\backslash M \subset \mathcal{A}_1$. □

5.2. **The impulsive uniform attractor, the impulsive skew-product semiflow and the impulsive cocycle attractor.** In this subsection, we dedicate ourselves to describe the relations between the previously defined impulsive uniform attractor and the impulsive cocycle attractor defined in Section 3. To this end, we also use the impulsive skew-product semiflow, as we will see in the two following results. The first one ensures the existence of an impulsive cocycle attractor, known as the existence of the impulsive uniform attractor.

Theorem 5.5. *Let $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ be an INDS, $I(M) \cap M = \varnothing$, each point of M satisfies $\varphi-$SSTC, $\{\tilde{\Pi}(t) : t \geqslant 0\}$ be its associated skew-product semiflow on $X \times \Sigma$ with a $c-$global attractor $\mathbb{A}$ and $\mathfrak{D}$ be the universe of all non-autonomous sets $\hat{D}$ with $\bigcup_{\sigma\in\Sigma} D(\sigma)$ bounded in X. Then there exists a $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractor $\hat{A}_1$ of $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$. Moreover, the non-autonomous set $\hat{A}$ with $A(\sigma) = \{x \in X : (x, \sigma) \in \mathbb{A}\}$ is such that $\hat{A} \in \mathfrak{D}$ is compact, $\{A(\sigma)\backslash M\}_{\sigma\in\Sigma}$ is $\tilde{\varphi}-$invariant, $A(\sigma)\backslash M \subset A_1(\sigma) \subset A(\sigma)$ and*

$$\mathbb{A}\backslash(M \times \Sigma) \subset \left(\bigcup_{\sigma\in\Sigma} [A(\sigma) \times \{\sigma\}]\right)\backslash(M \times \Sigma) \subset \bigcup_{\sigma\in\Sigma} [A_1(\sigma) \times \{\sigma\}] \subset \bigcup_{\sigma\in\Sigma} [A(\sigma) \times \{\sigma\}] \subset \mathbb{A}.$$

In particular, $\mathbb{A}\backslash(M \times \Sigma) = \left(\bigcup_{\sigma\in\Sigma} [A(\sigma) \times \{\sigma\}]\right)\backslash(M \times \Sigma) = \left(\bigcup_{\sigma\in\Sigma} [A_1(\sigma) \times \{\sigma\}]\right)\backslash(M \times \Sigma).$

Proof: Since $\mathbb{A}$ is the $c-$global attractor of $\{\tilde{\Pi}(t) : t \geqslant 0\}$, it follows that $\hat{A} \in \mathfrak{D}$, $A(\sigma) = \{x \in X : (x, \sigma) \in \mathbb{A}\}$ is compact and $\{A(\sigma)\backslash M\}_{\sigma\in\Sigma}$ is $\tilde{\varphi}-$invariant for each $\sigma \in \Sigma$.

By Theorem 5.4 and Theorem 3.15, the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ admits an impulsive uniform attractor K such that the $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractor is given by the non-autonomous set $\{\tilde{\omega}(\hat{K}, \sigma)\}_{\sigma\in\Sigma}$, where $K(\sigma) = K$ for each $\sigma \in \Sigma$. By the $\tilde{\varphi}-$invariance of $\{A(\sigma)\backslash M\}_{\sigma\in\Sigma}$ we have

$$\text{dist}(A(\sigma)\backslash M, \tilde{\omega}(\hat{K}, \sigma)) = \text{dist}(\tilde{\varphi}(t, \theta_{-t}\sigma)A(\theta_{-t}\sigma)\backslash M, \tilde{\omega}(\hat{K}, \sigma)) \to 0, \text{ as } t \to +\infty,$$

that is, $A(\sigma)\backslash M \subset \tilde{\omega}(\hat{K}, \sigma)$ for all $\sigma \in \Sigma$.

Now, let $x \in \tilde{\omega}(\hat{K}, \sigma)$. Then there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{x_n\}_{n\in\mathbb{N}} \subset K$ with $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, $\epsilon_n \overset{n\to+\infty}{\longrightarrow} 0$ such that $\tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n \overset{n\to+\infty}{\longrightarrow} x$. By the proof of Proposition 5.1 and by Theorem 4.7, if we put $\mathbb{K} = K \times \Sigma$, then the $c-$global attractor of the associated skew-product semiflow $\{\tilde{\Pi}(t) : t \geqslant 0\}$ is given by $\mathbb{A} = \tilde{\omega}(\mathbb{K})$. If we consider the sequence $(x_n, \theta_{-t_n}\sigma) \in \mathbb{K}$, $n \in \mathbb{N}$, then we get

$$\tilde{\Pi}(t_n + \epsilon_n)(x_n, \theta_{-t_n}\sigma) = (\tilde{\varphi}(t_n + \epsilon_n, \theta_{-t_n}\sigma)x_n, \theta_{\epsilon_n}\sigma) \overset{n\to+\infty}{\longrightarrow} (x, \sigma).$$

Then $(x, \sigma) \in \mathbb{A}$ which implies that $\tilde{\omega}(\hat{K}, \sigma) \subset A(\sigma)$. Consequenlty,

$$\mathbb{A}\backslash(M \times \Sigma) \subset \left(\bigcup_{\sigma\in\Sigma} [A(\sigma) \times \{\sigma\}]\right) \backslash M \times \Sigma \subset \bigcup_{\sigma\in\Sigma} \left[\tilde{\omega}(\hat{K}, \sigma) \times \{\sigma\}\right] \subset \bigcup_{\sigma\in\Sigma} [A(\sigma) \times \{\sigma\}] \subset \mathbb{A}.$$

□

With this theorem, we get a direct result relating the impulsive uniform attractor and the impulsive cocycle attractor.

Corollary 5.6. *If $I(M) \cap M = \varnothing$, each point of M satisfies $\varphi-$SSTC, $\mathfrak{D}$ is the universe of all non-autonomous sets $\hat{D}$ with $\bigcup_{\sigma\in\Sigma} D(\sigma)$ bounded in X and the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ has an impulsive uniform attractor $\mathcal{A}$, then it possesses a $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractor $\hat{A}_1$ and we have*

$$\bigcup_{\sigma\in\Sigma} A_1(\sigma)\backslash M = \mathcal{A}\backslash M. \tag{5.4}$$

Proof: By Theorem 5.4, the impulsive skew-product semiflow $\{\tilde{\Pi}(t)\colon t \geqslant 0\}$ has a $c-$global attractor, which we shall denote by $\mathbb{A}_2$, and if $\mathcal{A}_2 = P_X\mathbb{A}_2$ then

$$\mathcal{A}_2\backslash M \subset \mathcal{A} \subset \mathcal{A}_2. \tag{5.5}$$

Now, using Theorem 5.5, the $c-$global attractor $\mathbb{A}_2$ of $\tilde{\Pi}$ implies the existence of a $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractor $\hat{A}_1$ of $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ such that $A_2(\sigma)\backslash M \subset A_1(\sigma) \subset A_2(\sigma)$, for each $\sigma \in \Sigma$, where $A_2(\sigma) = \{x \in X\colon (x, \sigma) \in \mathbb{A}_2\}$. Clearly we have $\mathcal{A}_2 = \bigcup_{\sigma\in\Sigma} A_2(\sigma)$, which proves that

$$\mathcal{A}_2\backslash M \subset \bigcup_{\sigma\in\Sigma} A_1(\sigma) \subset \mathcal{A}_2. \tag{5.6}$$

Thus, equations (5.5) and (5.6) prove the result. □

To obtain the converse result, that is, to ensure the existence of the impulsive uniform attractor using the impulsive cocycle attractor, we need some additional hypothesis of uniform attraction, as present the next result.

Theorem 5.7. *Let $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ be an INDS, $I(M)\cap M = \varnothing$, each point of M satisfies $\varphi-$SSTC, $\{\tilde{\Pi}(t) : t \geqslant 0\}$ be the associated skew-product semiflow on $X\times\Sigma$ and $\mathfrak{D}$ be the universe of all non-autonomous sets $\hat{D}$ with $\bigcup_{\sigma\in\Sigma} D(\sigma)$ bounded in X. Assume that $\hat{A}$ is the $(\tilde{\varphi},\mathfrak{D})-$impulsive cocycle attractor of $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$, $\bigcup_{\sigma\in\Sigma} A(\sigma)$ is precompact in X and $\bigcup_{\sigma\in\Sigma} A(\sigma)$ is uniformly attracting, that is, for each bounded subset B of X we have*

$$\lim_{t\to+\infty}\sup_{\omega\in\Sigma}\operatorname{dist}\left(\tilde{\varphi}(t,\omega)B, \bigcup_{\sigma\in\Sigma} A(\sigma)\right) = 0.$$

Then the impulsive skew-product semiflow $\{\tilde{\Pi}(t) : t \geqslant 0\}$ has a $c-$global attractor $\mathbb{A}_1$, and defining $\mathbb{A} = \bigcup_{\sigma\in\Sigma}[A(\sigma)\times\{\sigma\}]$, we have

$$\mathbb{A}\backslash(M\times\Sigma) = \mathbb{A}_1\backslash(M\times\Sigma).$$

Proof: Define $K \doteq \overline{\bigcup_{\sigma\in\Sigma} A(\sigma)}$. We know that K is compact by hypothesis and K $\tilde{\varphi}-$uniformly attracts bounded sets. Hence, the impulsive skew-product semiflow $\{\tilde{\Pi}(t) : t \geqslant 0\}$ has a $c-$global attractor $\mathbb{A}_1$ by Proposition 5.1 and Theorem 4.7.

The invariance of $\mathbb{A}\backslash(M\times\Sigma)$ follows from the $\tilde{\varphi}-$invariance of $\{A(\sigma)\backslash M\}_{\sigma\in\Sigma}$ and it shows that $\mathbb{A}\backslash(M\times\Sigma)\subset\mathbb{A}_1\backslash(M\times\Sigma)$.

We can apply now Theorem 5.5 to ensure the existence of a $(\tilde{\varphi},\mathfrak{D})-$impulsive cocycle attractor $\hat{A}_2$ such that, if $A_1(\sigma) = \{x\in X : (x,\sigma)\in\mathbb{A}_1\}$, we have

$$A_1(\sigma)\backslash M \subset A_2(\sigma)\subset A_1(\sigma).$$

Since $A_2(\sigma)\backslash M = A(\sigma)\backslash M$, by Remark 3.11, we have $A_1(\sigma)\backslash M \subset A(\sigma)\backslash M$ and hence $\mathbb{A}_1\backslash(M\times\Sigma)\subset\mathbb{A}\backslash(M\times\Sigma)$. □

5.3. **The impulsive uniform attractor, the impulsive pullback attractor and the impulsive skew-product semiflow.** To begin this subsection, we present some definitions. A **non-autonomous set**, in this context, is a family $\hat{D} = \{D(t)\}_{t\in\mathbb{R}}$ of subsets of X indexed in $\mathbb{R}$. We say that $\hat{D}$ is an **open** (**closed, compact**) non-autonomous set if each **fiber** $D(t)$ is an open (closed, compact) subset of X. A **universe** $\mathfrak{D}$ is a collection of non-autonomous sets such that, if $\hat{D}_1\in\mathfrak{D}$ and $D_2(t)\subset D_1(t)$ for all $t\in\mathbb{R}$, then $\hat{D}_2\in\mathfrak{D}$.

Definition 5.8. Given an INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ and $\sigma\in\Sigma$, we define the **impulsive evolution process** associated to σ as the two-parameter family $\{\tilde{T}_\sigma(t,s) : t\geqslant s\}$ given by

$$\tilde{T}_\sigma(t,s)x = \tilde{\varphi}(t-s,\theta_s\sigma)x \quad \text{for all} \quad x\in X. \tag{5.7}$$

It is fairly easy to verify that $\tilde{T}_\sigma(t,t)x = x$ for all $t \in \mathbb{R}$ and $x \in X$ and $\tilde{T}_\sigma(t,\tau)\tilde{T}_\sigma(\tau,s) = \tilde{T}_\sigma(t,s)$ for all $t \geqslant \tau \geqslant s$. Moreover, we say that a non-autonomous set $\hat{D}$ is called $\tilde{T}_\sigma-$**invariant** if $\tilde{T}_\sigma(t,s)D(s) = D(t)$ for all $t \geqslant s$, and a non-autonomous set $\hat{A}$ $\tilde{T}_\sigma-$**pullback attracts** $\hat{D}$ if

$$\lim_{s\to-\infty} \operatorname{dist}(\tilde{T}_\sigma(t,s)D(s), A(t)) = 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

Definition 5.9. Given an impulsive evolution process $\{\tilde{T}_\sigma(t,s)\colon t \geqslant s\}$ associated to σ as in (5.7) and a universe $\mathfrak{D}$, a compact non-autonomous set $\hat{A}$ is called the $(\tilde{T}_\sigma, \mathfrak{D})-$**pullback attractor** if:

(i) $\hat{A}$ is $\tilde{T}_\sigma-$invariant;
(ii) $\hat{A}$ $\tilde{T}_\sigma-$pullback attracts all non-autonomous sets in $\mathfrak{D}$;
(iii) $\hat{A}$ is the minimal among all closed non-autonomous sets with property **(ii)**.

We may now present our result for impulsive evolution processes.

Theorem 5.10. *Let $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ be an INDS, $I(M) \cap M = \varnothing$, each point of M satisfies $\varphi-$SSTC, $\{\tilde{\Pi}(t) : t \geqslant 0\}$ be its associated skew-product semiflow and $\mathfrak{D}$ be the universe of all non-autonomous sets $\hat{D}$ with $\bigcup_{t\in\mathbb{R}} D(t)$ bounded in X. Assume that $\{\tilde{\Pi}(t) : t \geqslant 0\}$ has a $c-$global attractor $\mathbb{A}$ and let $\mathcal{A} = P_X\mathbb{A}$. Then, for each $\sigma \in \Sigma$, the impulsive evolution process $\{\tilde{T}_\sigma(t,s)\colon t \geqslant s\}$ given by*

$$\tilde{T}_\sigma(t,s)x = \tilde{\varphi}(t-s, \theta_s\sigma)x, \ x \in X,$$

possesses a $(\tilde{T}_\sigma, \mathfrak{D})-$pullback attractor $\hat{A}_\sigma = \{A_\sigma(t)\}_{t\in\mathbb{R}}$. Moreover,

$$\mathbb{A}\backslash(M\times\Sigma) = \left(\bigcup_{\sigma\in\Sigma}\bigcup_{t\in\mathbb{R}} A_\sigma(t)\times\{\theta_t\sigma\}\right)\backslash(M\times\Sigma) \quad \textit{and} \quad \bigcup_{t\in\mathbb{R}} A_\sigma(t)\backslash M \subset \mathcal{A}\backslash M.$$

Proof: Let $\mathfrak{D}_1$ be the universe of all non-autonomous sets $\hat{D}$ with $\bigcup_{\sigma\in\Sigma} D(\sigma)$ bounded in X. By Theorem 5.4 and Corollary 5.6, there exists a $(\tilde{\varphi}, \mathfrak{D}_1)-$impulsive cocycle attractor $\hat{A}_1$ of $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$ such that

$$\bigcup_{\sigma\in\Sigma} A_1(\sigma)\backslash M = \mathcal{A}\backslash M \quad \text{and} \quad \mathbb{A}\backslash(M\times\Sigma) = \bigcup_{\sigma\in\Sigma}\Big[A_1(\sigma)\times\{\sigma\}\Big]\backslash(M\times\Sigma). \tag{5.8}$$

Let $\sigma \in \Sigma$ be arbitrary and define $A_\sigma(t) = A_1(\theta_t\sigma)$ for all $t \in \mathbb{R}$. Then $\hat{A}_\sigma$ is a $(\tilde{T}_\sigma, \mathfrak{D})-$pullback attractor with $\bigcup_{t\in\mathbb{R}} A_\sigma(t)\backslash M \subset \bigcup_{\omega\in\Sigma} A_1(\omega)\backslash M = \mathcal{A}\backslash M$. Moreover, by (5.8), we get

$$\begin{aligned}\mathbb{A}\backslash(M\times\Sigma) &= \bigcup_{\sigma\in\Sigma}\Big[A_1(\sigma)\times\{\sigma\}\Big]\backslash(M\times\Sigma) = \bigcup_{\sigma\in\Sigma}\Big[A_\sigma(0)\times\{\sigma\}\Big]\backslash(M\times\Sigma)\\ &= \left(\bigcup_{\sigma\in\Sigma}\bigcup_{t\in\mathbb{R}} A_\sigma(t)\times\{\theta_t\sigma\}\right)\backslash(M\times\Sigma),\end{aligned}$$

which concludes the proof. □

6. Application

We will consider the following non-classical non-autonomous parabolic equation

$$\begin{cases} u_t - \gamma(t)\Delta u_t - \Delta u = f(u), \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \\ u(0,x) = u_0(x), \text{ in } \Omega, \end{cases} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, for some $n \geqslant 3$, with f and γ satisfying some suitable conditions. A detailed study of this equation and some non-autonomous perturbations can be found, for instance, in [12, 25].

Assume that $\gamma : \mathbb{R} \to (0, +\infty)$ is a uniformly continuous function which satisfies $0 < \gamma_0 \leqslant \gamma(t) \leqslant \gamma_1 < +\infty$ and f is a twice continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$|f(s_1) - f(s_2)| \leqslant c|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \ s_1, s_2 \in \mathbb{R}, \tag{\textbf{H1}}$$

and

$$\limsup_{|s|\to+\infty} \frac{f(s)}{s} \leqslant \delta < \lambda_1, \tag{\textbf{H2}}$$

where λ_1 is the first eigenvalue of $A = -\Delta$ with Dirichlet boundary condition, for some $c > 0$ and $1 \leqslant \rho < \frac{n+2}{n-2}$.

Let us consider the $L^2(\Omega)$-bounded operators

$$B_\gamma(t) = (I + \gamma(t)A)^{-1} \text{ and } \tilde{A}_\gamma(t) = AB_\gamma(t),$$

whose domains do not depend on time and the operators $\mathbb{R} \ni t \mapsto B_\gamma(t)$ and $\mathbb{R} \ni t \mapsto \tilde{A}_\gamma(t)$ are absolutely continuous functions.

Then, we can rewrite equation (6.1) as

$$\begin{cases} u_t = -\tilde{A}_\gamma(t)u + B_\gamma(t)f(u), \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \\ u(0,x) = u_0(x), \text{ in } \Omega. \end{cases} \tag{6.2}$$

It is a well known result that, since f satisfies (**H1**), its associated Nemytskii operator f^e defines a map from $H^1_0(\Omega)$ into $L^{\frac{2n}{n+2}}(\Omega)$ which is Lipschitz continuous in bounded subsets of $H^1_0(\Omega)$. Moreover f^e takes bounded subsets of $H^1_0(\Omega)$ in precompact sets of $H^{-1} = (H^1_0(\Omega))'$.

Now, let $C_b(\mathbb{R})$ be the space of continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}$. Define the *shift operator* $\theta_s\colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$ by $\theta_s g(\cdot) = g(s + \cdot)$ and consider the set

$$\Gamma = \overline{\{\theta_s\gamma\}_{s\in\mathbb{R}}},$$

where the closure is taken in $C_b(\mathbb{R})$. Let $Z_1 = C^0_b(\mathbb{R} \times H^1_0(\Omega), H^1_0(\Omega))$ be the set of continuous functions which are bounded in the variable in the first coordinate and bounded in bounded subsets of $H_0^1(\Omega)$ in the second variable. Define

$$\Sigma = \overline{\{\theta_s(B_\gamma f^e - \tilde{A}_\gamma)\}_{s\in\mathbb{R}}}^{Z_1}.$$

Then, using [12, Lemma 5.2], we have $\Sigma = \{B_\sigma f^e - \tilde{A}_\sigma\}_{\sigma\in\Gamma}$ and clearly Σ is compact.

We can use the results of [12] to obtain the following properties for (6.1):

Theorem 6.1. *Assume that* (**H1**)-(**H2**) *hold. Then:*

1. *equation* (6.1) *generates a non-autonomous dynamical system in* $(\varphi,\theta)_{(H_0^1(\Omega),\Sigma)}$*;*
2. *there exist constants* $\alpha, k, \beta > 0$*, independent of* $\sigma \in \Gamma$*, such that*
$$\|\varphi(t,\sigma)u_0\|_{H_0^1}^2 \leqslant \alpha\|u_0\|_{H_0^1}^2 e^{-kt} + \beta, \tag{6.3}$$
where $\varphi(t,\sigma)$ *is the solution of* (6.1) *at time* t*, with* $\sigma\in\Gamma$ *replacing* γ*;*
3. *the skew-product semiflow* $\{\Pi(t)\colon t \geqslant 0\}$ *associated with the non-autonomous dynamical system given in the previous item possesses a global attractor* $\mathbb{A}$ *in* $H_0^1(\Omega)\times\Sigma$.

Now, we assume that M is an impulsive set for the NDS $(\varphi,\theta)_{(H_0^1(\Omega),\Sigma)}$ generated by (6.1) such that every point of M satisfies $\varphi-$SSTC, $I\colon M \to H_0^1(\Omega)$ is a continuous function such that $I(M)\cap M = \varnothing$, $I(M)$ is compact in $H_0^1(\Omega)$ and for a given $\sigma\in\Sigma$ there is $\delta=\delta(\sigma)>0$ such that $\phi(x,\omega) \geqslant \delta$ for all $x \in I(M)$ and $\omega \in \{\theta_t\sigma : t\in\mathbb{R}\}$ (this hypothesis also ensures condition (**H0**)).

Defining $\mathfrak{D}$ as the universe of all non-autonomous sets indexed in Σ with bounded union, we can use Theorem 6.1 (condition 2) and Proposition 4.12 to see that the impulsive skew-product semiflow $(H_0^1(\Omega)\times\Sigma, \Pi, M\times\Sigma, \mathbb{I})$ is asymptotically compact, where $\mathbb{I}(x,\lambda) = (I(x),\lambda)$ for each $(x,\lambda)\in M\times\Sigma$.

Using similar results as in [4], one can show the existence of a uniform attracting bounded set K for the INDS $[(\varphi,\theta)_{(X,\Sigma)}, M, I]$.

Proposition 6.2. *There exists a bounded set* K *such for any bounded subset* B *of* $H_0^1(\Omega)$ *there exists* $t_0 = t_0(B) \geqslant 0$ *with*

$$\tilde{\varphi}(t,\sigma)B \subset K \quad \text{for all } \; t\geqslant t_0 \text{ and } \sigma\in\Sigma.$$

Proof: Since $I(M)$ is compact, there exists $\mu > 0$ such that $\|u\|_{H_0^1(\Omega)} \leqslant \mu$, for all $u\in I(M)$. Define K as the ball in $H_0^1(\Omega)$ centered in zero of radius $2(\alpha\mu+\beta)$, where α and β are taken from the assertion of Theorem 6.1. The result now follows as in [4, Proposition 6.4]. □

This previous proposition and Proposition 5.1 imply that there exists a bounded set $\mathbb{K} = K\times\Sigma \subset H_0^1(\Omega)$ which $\tilde{\Pi}-$attracts bounded subsets of $H_0^1(\Omega)$. This, together with the asymptotical compactness of $\{\tilde{\Pi}(t)\colon t\geqslant 0\}$ ensures the existence of a $c-$global attractor $\tilde{\mathbb{A}}$ for $(H_0^1(\Omega)\times\Sigma, \Pi, M\times\Sigma, \mathbb{I})$. Thus, using Theorem 5.4, Corollary 5.6 and Theorem 5.10 we have:

(a) the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ has an impulsive uniform attractor $\mathcal{A}_1$ and if $\mathcal{A} = P_{H_0^1(\Omega)}\tilde{\mathbb{A}}$ we get

$$\mathcal{A}_1 \backslash M = \mathcal{A} \backslash M;$$

(b) the INDS $[(\varphi, \theta)_{(X,\Sigma)}, M, I]$ has a $(\tilde{\varphi}, \mathfrak{D})-$impulsive cocycle attractor $\{A(\lambda)\}_{\lambda \in \Sigma}$ and

$$\mathcal{A} \backslash M = \bigcup_{\lambda \in \Sigma} A(\lambda) \backslash M;$$

(c) for each $\sigma \in \Gamma$, if $\lambda = B_\sigma f^e - \tilde{A}_\sigma$, the impulsive evolution process $\tilde{T}_\lambda(t,s) = \tilde{\varphi}(t-s, \theta_s \lambda)$ possesses a pullback $(\tilde{T}_\lambda, \mathfrak{D})-$attractor $\{A_\lambda(t)\}_{t \in \mathbb{R}}$ and

$$\mathcal{A} \backslash M = \bigcup_{\lambda \in \Sigma} \bigcup_{t \in \mathbb{R}} A_\lambda(t) \backslash M.$$

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