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ON THE EXISTENCE AND UNIQUENESS OF LIMIT CYCLES IN PLANAR PIECEWISE LINEAR SYSTEMS WITHOUT SYMMETRY

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Abstract. Some techniques to show the existence and uniqueness of limit cycles, typically stated for smooth vector fields, are extended to continuous piecewise-linear differential systems. New results are obtained for systems with three linearity zones without symmetry and having one equilibrium point in the central region. We also revisit the case of systems with only two linear zones giving shorter proofs of known results.

1. Introduction and statement of main results

For planar differential systems, the analysis of the possible existence of limit cycles and their uniqueness is a problem which has attracted the interest of many works in the past. For smooth systems, good classical references in the field are the books [24, 25]. The restriction of this problem to polynomial differential equations is the well-known 16th problem Hilbert's problem [14]. In this context, a celebrated and rather general canonical form is the Liénard equation. Since Hilbert's problem turns out to be a strongly difficult one, Smale [22] has particularized it to Liénard differential systems in his list of problems for the present century. For just continuous or even smooth Liénard systems there are many results on the non-existence, existence and uniqueness of limit cycles, see for instance [1, 4, 6, 18, 23, 25]. Going beyond the smooth case, the first natural step is to allow non-smooothness while keeping the continuity, as has been done in some recent works [9, 15, 16]. In a further step, other authors have considered a line of discontinuity in the vector field defining the planar system, see [12, 26].

In this paper, we adapt some techniques from the smooth case to continuous piecewise linear differential systems, obtaining new results for systems without symmetry. We also revisit the case of systems with only two linear zones giving shorter proofs of known results.

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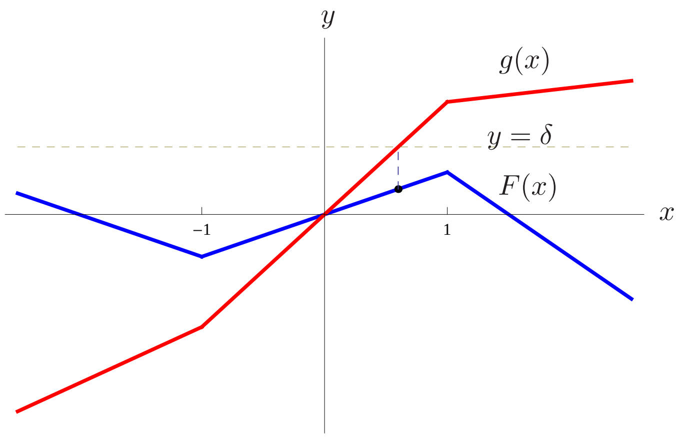

FIGURE 1. The graphs of F and g considered in the paper have no symmetries. The point indicates the only equilibrium point, which is initially not at the origin.

In most of interesting applications, piecewise linear differential systems have two or three different linearity regions separated by parallel straight lines, see [2]. For such systems we assume without loss of generality that the lines separating these regions are $x = -1$ and $x = 1$. Furthermore, it is rather usual for these systems to exhibit only one equilibrium point, which is supposed to be in the central linearity region when the system has three linear zones, to be denoted in the sequel as L (left), C (central), and R (right). If the system has only two zones, we assume that the left and the central zones are in fact only one. Then, under these generic assumptions, continuous piecewise linear differential systems (CPWL, for short) can be written in the Liénard form

$$\begin{aligned} \dot{x} &= F(x) - y, \\ \dot{y} &= g(x) - \delta, \end{aligned} \tag{1}$$

where

$$F(x) = \begin{cases} t_R(x-1) + t_C, & \text{if } x \geq 1, \\ t_C x, & \text{if } |x| \leq 1, \\ t_L(x+1) - t_C, & \text{if } x \leq -1, \end{cases} \tag{2}$$

and

$$g(x) = \begin{cases} d_R(x-1) + d_C, & \text{if } x \geq 1, \\ d_C x, & \text{if } |x| \leq 1, \\ d_L(x+1) - d_C, & \text{if } x \leq -1. \end{cases} \tag{3}$$

Our assumption on the equilibrium point requires that the determinant in central region be positive, that is, $d_C > 0$, and also $-d_C < \delta < d_C$, along with $d_L, d_R \geq 0$. This means that the only equilibrium point is located at the line $x = \bar{x} = \delta/d_C \in (-1, 1)$. The corresponding traces t_L, t_C, t_R could be arbitrary, but we know from Bendixson theory that they cannot have the same sign for the existence of limit cycles.

Remark 1. *The above formulation includes as particular cases the following ones. If $t_C = t_L$ and $d_C = d_L$ then we have a system with only two different linearity zones, thoroughly analyzed in* [9]. *If $t_R = t_L$, $d_R = d_L$ and $\delta = 0$, then we have a symmetric system with three different linearity zones, thoroughly analyzed in* [10]. *The case $t_R = t_L$, $d_R = d_L$ and $\delta \neq 0$ was considered in* [19]. *Examples of this last situation appeared in* [3].

Under suitable hypotheses that include systems without any symmetries, see Fig. 1, our main results are the following.

Theorem 1. *Consider the differential system* (1)–(3) *with only one equilibrium point in the central zone, i.e. $d_C > 0$, $-d_C < \delta < d_C$, and $d_L, d_R \geq 0$. If the external traces satisfy $t_L, t_R < 0$, while the central trace is positive, that is $t_C > 0$, then the equilibrium point is surrounded by a limit cycle which is unique and stable.*

Theorem 1 is shown in Section 5, and of course, by using the opposite sign distribution for the traces, we could state a similar theorem on existence and uniqueness of an unstable limit cycle.

For the case of only two linearity zones the corresponding result is a bit more involved.

Theorem 2. *Consider the differential system* (1)–(3) *with only two linearity zones, more specifically, under the assumptions $t_C = t_L$ and $d_C = d_L > 0$, and one equilibrium point in the left zone, i.e. $\delta < d_C$, and $d_R \geq 0$. Assume also that the left trace satisfy $t_L > 0$, while the right trace is negative, that is $t_R < 0$. The following statements hold.*

- **(a)** *A necessary condition for the existence of periodic orbits is that the equilibrium point be a topological focus, that is $t_L^2 - 4d_L < 0$.*
- **(b)** *If $t_L^2 - 4d_L < 0$ then the system has periodic orbits if and only if the following condition holds*

$$\frac{t_L}{\sqrt{d_L}} + \frac{t_R}{\sqrt{d_R}} < 0.$$

In such case, the equilibrium point is surrounded by a limit cycle which is unique and stable.

Theorem 2 is not new. It was stated in [8] in a equivalent way without an explicit proof and indicating how several existing results could be concatenated to get such theorem. It can also be considered as a byproduct of the case-by-case study made in [9]. We include here to emphasize how the useful techniques introduced in proving Theorem 1 allow us to obtain a much shorter proof of Theorem 2.

The rest of the paper is organized s follows. First, in Section 2 we illustrate through some relevant examples the usefulness of the achieved results. Next section is devoted to review the ideas underlying the Massera's approach for uniqueness of limit cycles. Finally, our main results are shown in Section 5.

2. Examples

We analyze here a celebrated piecewise linear model in mathematical biology. Following [3], the equations for the two-dimensional McKean model of a single neuron take the form

$$\begin{aligned} C\dot{v} &= f(v) - w + I, \\ \dot{w} &= v - \gamma w, \end{aligned} \tag{4}$$

where v stands for the voltage, w is the gating variable and

$$f(v) = \begin{cases} -v, & \text{if } v < a/2, \\ v - a, & \text{if } a/2 \leq v \leq (1+a)/2, \\ 1 - v, & \text{if } v > (1+a)/2. \end{cases} \tag{5}$$

Here, $C > 0$, $\gamma > 0$, I is a constant drive, and $f(v)$ is a PWL caricature of the cubic FitzHugh-Nagumo nonlinearity $f(v) = v(1-v)(v-a)$, provided that $0 < a < 1$, see [3] for more details.

First, we put system (4) in the form given in (1). We start by using instead of w a new variable u such that $w = C(u + \gamma v)$, so that we get a new system where the dynamics of the second variable depends only on the first one, namely

$$\begin{aligned} C\dot{v} &= f(v) - C\gamma - Cu + I, \\ \dot{u} &= v - \gamma f(v) - \gamma I. \end{aligned} \tag{6}$$

An appropriate translation and scaling of variables suffices now to arrive at the form (1). We take

$$\begin{aligned} x &= 4v - 2a - 1, \\ Cy &= 4Cu - C\gamma(2a+1) - 4I + 2a - 1, \end{aligned} \tag{7}$$

and the computations give

$$t_L = t_R = -\frac{1}{C} - \gamma, \quad t_C = \frac{1}{C} - \gamma, \quad d_L = d_R = \frac{1+\gamma}{C}, \quad d_C = \frac{1-\gamma}{C}, \tag{8}$$

and

$$\delta = \frac{\gamma(1 - 2a + 4I) - 2a - 1}{C}. \tag{9}$$

Since $d_L = d_R > 0$, it is now direct to conclude that to have always only one equilibrium point we need $\gamma < 1$, for then $d_C > 0$. Furthermore, to be able of having oscillations in the model, we also need that $\gamma < 1/C$, since then $t_C > 0$. Finally, in order to apply Theorem 1, we translate the condition $-d_C < \delta < d_C$, getting the equivalent inequalities

$$\frac{1+\gamma}{2\gamma} a < I < \frac{1+\gamma}{2\gamma} a + \frac{1-\gamma}{2\gamma},$$

so that, under these conditions we can guarantee a unique stable limit cycle in the model. The uniqueness of the limit cycle and this quantitative information on the admissible range for the drive I, deduced from Theorem 1, is an interesting information which complements the study made in [3].

Note that our results also apply to more general versions of the PWL function $f(v)$, when the slopes of external pieces are different. This is the case for instance of the PWL models considered in [7] to study the oscillations in an electronic circuit whose nonlinearity comes from an Esaki diode.

3. The Massera's method for uniqueness of limit cycles

We review in this section a geometrical argument which is usually known as the Massera's method; it will allow us, after adequate adaptations, to show the uniqueness of limit cycles in the CPWL differential systems considered in this paper, when they satisfy certain hypotheses. Uniqueness results for limit cycles are typically rather involved; see [24, 25], for a review in the subject. Here we reformulate in a specific way the simple and elegant idea proposed by J.L. Massera in his brief note extending a previous result of G. Sansone, see [20] and the recent study on the legacy of the latter author in [21] .

First, we recall some notions and introduce some definitions. A *period annulus* is a region in the plane completely filled by non-isolated periodic orbits. We say that a vector field has the *non-negative rotation property* whenever along any half-ray starting from the origin the angle of the vector field measured with respect the positive direction of the x-axis does not decrease as long as one moves far from the origin. For a closed orbit surrounding the origin, we say that it is *star-like with respect to the origin* when any segment joining the origin and a point of the closed orbit has no other points in common with the closed orbit, and consequently such segments are in the interior of the closed orbit. The following result can be stated.

Lemma 1. (Massera's method) *Consider a Liénard system with a continuous vector field given by* $\dot{x} = F(x) - y$, $\dot{y} = g(x)$, *and assume that* $xg(x) > 0$ *for* $x \neq 0$, *and that* $F(0) = 0$, *so that the only equilibrium point is at the origin. Assume that the system has the non-negative rotation property and that period annuli are not possible. If the system has a closed orbit then it is star-like with respect to the origin and it is a limit cycle which is unique and stable.*

Proof. First, we will show that if the system has a closed orbit then it is star-like with respect to the origin. Obviously, since the vector field is continuous the periodic orbit must surround the origin, see Theorem 3.1 in [13]. Suppose that such an orbit is not star-like with respect to the origin. Then there must be a half-ray that starting from the origin intersects the closed orbit in more than one point; in fact such half-ray must have at least three points in common with the closed orbit, see Fig. 2. It is easy to conclude that the angle of the vector field measured with respect the positive direction of the x-axis cannot be monotone, that is, first decreases to increase later or vice versa. This is not compatible with the non-negative rotation property, getting the desired contradiction.

We now assume that there exists a closed orbit Γ that surrounds the origin, which must be star-like with respect to it by the above argument, see Fig. 2 (right). Then, using Γ as starting point, one can build a geodesic

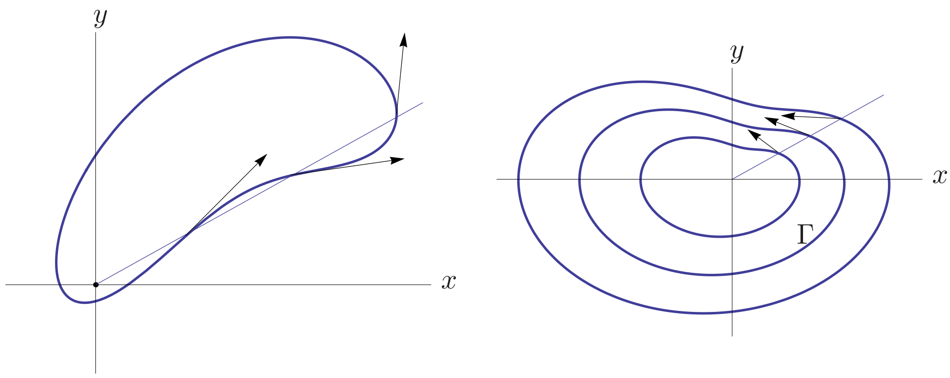

FIGURE 2. (Left) Closed orbits that are not star-like with respect to the origin cannot appear in systems with the non-negative rotation property. (Right) If Γ is a star-like closed orbit, one can build a geodesic system of closed curves by homotetical transformations. Then, the non-negative rotation property assures the stability of the closed orbit Γ and hence its uniqueness.

system of closed curves by homotetical transformations, foliating the entire plane by the curves $k\Gamma$ for all $k > 0$. Consider now a half-ray starting from the origin and take into account the non-negative rotation property. Of course the vector field is tangent to Γ at the point where the half-ray intersects Γ, see Figure 2 (right). Now the non-negative rotation property assures that in the points where the half-ray intersects the closed curves of the geodesic system near Γ the vector field points in such a direction that it is guaranteed the stability of the periodic orbit, even in the case the periodic orbit considered is not isolated. Since we exclude the possibility of any period annulus and there cannot be consecutive nested stable periodic orbits, if there exists such an orbit then it must be isolated and stable, that is, it should be the unique stable limit cycle. The conclusion follows. $\square$

4. PRELIMINARY RESULTS AND UNIQUENESS OF LIMIT CYCLES

In this section we give some preliminary results and also include uniqueness results for possible periodic orbits. Note that if we make in system (1) the change $X = -x, Y = -y$ we get the system

$$\begin{array}{rcl} \dot{X} & = & \widetilde{F}(X) - Y, \\ \dot{Y} & = & \widetilde{g}(X) + \delta, \end{array} \tag{10}$$

where the new functions $\widetilde{F}$ and $\widetilde{g}$ are obtained from the given in (2) by interchanging the subscripts L and R. Thus, there is no loss of generality in assuming $\delta \geq 0$ hereafter.

Our first result is just a preparation lemma, reducing by one the number of parameters and looking for a more compact equivalent expression for system (1)–(3).

Lemma 2. *System* (1)–(3) *with* $0 \leq \delta < d_C$ *is topologically equivalent to the system*

$$\begin{aligned} \dot{x} &= F_n(x) - y, \\ \dot{y} &= g_n(x), \end{aligned} \tag{11}$$

where

$$F_n(x) = \begin{cases} a_R(x - x_R) + a_C x_R, & \text{if } x \geq x_R, \\ a_C x, & \text{if } x_L \leq x \leq x_R, \\ a_L(x - x_L) + a_C x_L, & \text{if } x \leq x_L, \end{cases} \tag{12}$$

and

$$g_n(x) = \begin{cases} b_R(x - x_R) + x_R, & \text{if } x \geq x_R, \\ x, & \text{if } x_L \leq x \leq x_R, \\ b_L(x - x_L) + x_L, & \text{if } x \leq x_L, \end{cases} \tag{13}$$

with $x_L = -1 - \bar{x}$, $x_R = 1 - \bar{x}$, *for* $0 \leq \bar{x} = \delta/d_C < 1$, *and the new piecewise slopes satisfy* $a_Z\sqrt{d_C} = t_Z, b_Z d_C = d_Z$, *for each* $Z \in \{L, C, R\}$.

Proof. First, we put the equilibrium point at the origin by the translation $\widetilde{x} = x - \bar{x}$, $\widetilde{y} = y - t_C\bar{x}$. This makes that the new vertical lines separating the zones be $\widetilde{x} = x_L$ and $\widetilde{x} = x_R$ and the δ-term in the second equation disappears. Next, we make the change of variables and time defined by $X = \widetilde{x}$, $\omega Y = \widetilde{y}$, and $\tau = \omega t$, with $\omega^2 = d_C$. We obtain

$$\frac{dX}{d\tau} = \frac{F(X + \bar{x})}{\omega} - Y, \quad \frac{dY}{d\tau} = \frac{g(X + \bar{x})}{\omega^2},$$

so that the conclusion follows from the two obvious equalities $F(X + \bar{x}) = \omega F_n(X)$ and $g(X + \bar{x}) = \omega^2 g_n(X)$. □

The following remark will be useful to split the analysis of system with three zones into two different subcases with only two zones.

Remark 2. *System* (11)-(13) *is invariant under the following symmetry*

$$(x, y, t, a_C, a_L, a_R, b_L, b_R, x_L, x_R) \to (-x, -y, t, a_C, a_R, a_L, b_R, b_L, -x_R, -x_L).$$

Now, we consider a system with only two linearity zones which can be obtained from system (11)-(13) by suppressing the left zone and extending the central zone to the left, which is equivalent to assume $a_L = a_C$ and $b_L = 1$.

Proposition 1. *Consider the two-zones piecewise linear differential system* $\dot{x} = F(x) - y$, $\dot{y} = g(x)$, *where*

$$g(x) = \begin{cases} x, & \text{if } x < x_R, \\ b_R(x - x_R) + x_R, & \text{if } x \geq x_R, \end{cases}$$

and

$$F(x) = \begin{cases} a_C x, & \text{if } x < x_R, \\ a_R(x - x_R) + a_C x_R, & \text{if } x \geq x_R, \end{cases}$$

with $a_C > 0$, $a_R < 0$, $b_R \geq 0$ and $x_R > 0$. The following statements hold.

(a) *If $b_R = 1$ then the system has the non-negative rotation property.*

(b) *If $0 \leq b_R \neq 1$ then the system can be transformed in an equivalent system with the non-negative rotation property.*

Proof. To show the non-negative rotation property we will compute the slope of the vector field along half-rays of the form $y = \lambda x$. In the following computations there naturally appears the expression $F(x) - \lambda x$ in some denominators; obviously, we can disregard the points of vertical slope in which $F(x) - \lambda x = 0$.

If $b_R = 1$ then $g(x) = x$ for all $x \in \mathbb{R}$. In this case, the slope of the vector field along the half-rays $y = \lambda x$ is given by

$$m_\lambda(x) = \left.\frac{dy}{dx}\right|_{y=\lambda x} = \frac{x}{F(x) - \lambda x},$$

which is constant for $x \leq x_R$. For $x > x_R$, it has the derivative

$$\frac{dm_\lambda(x)}{dx} = \frac{F(x) - \lambda x - x(a_R - \lambda)}{[F(x) - \lambda x]^2} = \frac{x_R(a_C - a_R)}{[F(x) - \lambda x]^2},$$

which is always positive. The non-negative rotation property is concluded for this simple case.

If $b_R \neq 1$, the numerator in the computation of the derivative of $m_\lambda(x)$ turns out to be dependent on λ and the sign of numerator could change. However, we can transform the system by introducing a new first variable $u = u(x)$ so that the new second equation become $\dot{y} = u$ for all u. For that, it suffices to write $u = \operatorname{sgn}(x)\sqrt{2G(x)}$, where $G(x) = \int_0^x g(s)ds$. Note that $u = x$ if $x \leq x_R$ and then the slope of the vector field in this case is not altered. Now, we study its slope for $u > x_R$. Clearly, from $u^2(x) = 2G(x)$ we have $u(x)u'(x) = g(x)$ for all x, and so

$$\frac{du}{dx} = \frac{g(x)}{u}. \tag{14}$$

Therefore,

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = \frac{g(x)}{u}\left[F(x) - y\right],$$

and in the new variables the system is equivalent to the equation

$$\frac{dy}{du} = \frac{\frac{dy}{dt}}{\frac{du}{dt}} = \frac{u}{F(x(u)) - y},$$

that is, to the system $\dot{u} = F(x(u)) - y$, $\dot{y} = u$. As in the previous case $g(x) = x$, we can write

$$m_\lambda(u) = \left.\frac{dy}{du}\right|_{y=\lambda u} = \frac{u}{F(x(u)) - \lambda u},$$

and also

$$\frac{dm_\lambda(u)}{du} = \frac{F(x(u)) - \lambda u - u\,[F'(x(u))x'(u) - \lambda]}{[F(x(u)) - \lambda u]^2} = \frac{F(x(u)) - a_R u\, x'(u)}{[F(x(u)) - \lambda u]^2},$$

finally arriving at

$$\frac{dm_\lambda(u)}{du} = \frac{x_R(a_C - a_R) + a_R[x(u) - u\, x'(u)]}{[F(x(u)) - \lambda u]^2}. \tag{15}$$

We will study the sign of $x(u) - u\, x'(u)$ for $x > x_R$. From (14) and the equality $u^2 = 2G(x) = b_R(x - x_R)^2 + 2x_R x - x_R^2$ for $x > x_R$, we have

$$x(u) - u\, x'(u) = \frac{xg(x) - u^2}{g(x)} =$$

$$= \frac{x[b_R(x - x_R) + x_R] - [b_R(x - x_R)^2 + 2x_R x - x_R^2]}{g(x)},$$

and after obvious simplifications, we get

$$x(u) - u\, x'(u) = \frac{(b_R - 1)x_R(x - x_R)}{g(x)}.$$

Now, if $b_R \leq 1$ the above expression is non-positive for $x > x_R$, and then the expression in (15) is obviously positive. The non-negative rotation property follows.

The remaining case $b_R > 1$ can be also managed by noticing that, when $x > x_R$, we have

$$0 < \frac{(b_R - 1)x_R(x - x_R)}{g(x)} = \frac{(b_R - 1)x_R(x - x_R)}{b_R(x - x_R) + x_R} < \frac{(b_R - 1)x_R}{b_R},$$

so that

$$x_R(a_C - a_R) + a_R[x(u) - u\, x'(u)] >$$

$$> x_R(a_C - a_R) + a_R\frac{(b_R - 1)x_R}{b_R} =$$

$$= x_R\left(a_C - \frac{a_R}{b_R}\right) > 0,$$

and since (15) is again positive, the conclusion follows. □

From the above result it should be noticed that some systems originally not having the non-negative rotation property can be transformed in equivalent systems satisfying such property. Possible periodic orbits can be deformed in shape by the transformation given in the above proof, but stability and uniqueness results for periodic orbits can be translated between such equivalent systems. More precisely, the following remarks can be stated.

Remark 3. *From Lemma 1, we can conclude for the systems with two linearity zones given in Proposition 1 that if there is a closed orbit, then it surrounds the origin and it is a limit cycle which is unique and stable.*

Remark 4. *If we consider systems with three linearity zones given in* (11)-(13) *with* $x_L < 0 < x_R$ *and satisfying the conditions* $a_L, a_R < 0$, $a_C > 0$ *and* $b_L, b_R \geq 0$, *then from Proposition 1 we can deduce that such systems can be transformed in equivalent systems having the non-negative rotation property for all the half-rays contained in the half-plane* $x \geq 0$. *By using the symmetry given in Remark 2, and applying again Proposition 1, we can deduce that such systems can also be transformed in equivalent systems having the non-negative rotation property for all the half-rays contained in the half-plane* $x \leq 0$. *In short, systems with three linearity zones given in* (11)-(13) *with* $x_L < 0 < x_R$ *and satisfying the conditions* $a_L, a_R < 0$, $a_C > 0$ *and* $b_L, b_R \geq 0$, *can be transformed in equivalent systems having the non-negative rotation property in the whole plane. Consequently, from Lemma 1 we can also conclude for such systems that if there is a closed orbit, then it surrounds the origin and it is a limit cycle which is unique and stable.*

5. Existence of limit cycles and proof of main results

In this section, we will use as main tools the Poincaré return map to show the existence of periodic orbits. We start by considering systems with three linearity zones given in (11)-(13) with $x_L < 0 < x_R$ for which the conditions $a_L, a_R < 0$, $a_C > 0$ and $b_L, b_R \geq 0$ hold. We will use the positive and negative parts of the y-axis as domain and range for defining two different half-return maps, namely a right half-return map P_R and a left half-return map P_L.

We start by studying the qualitative properties of the right half-return map P_R defined in the whole negative y-axis, by taking the orbit starting at the point $(0, -y)$, with $y > 0$, and coming back to the positive y-axis at the point $(0, P_R(y))$. The following lemma, proved here for sake of completeness, is a modification of a classical result, see for instance the proof of Theorem 11.4 in [17]. It assures, under certain hypotheses, the existence of such a map for all $y > 0$ and gives its asymptotic behavior as $y \to \infty$.

Lemma 3. *Consider a Liénard system with a continuous vector field given by* $\dot{x} = F(x) - y$, $\dot{y} = g(x)$. *Assume that* $F(x)$ *is positive and increasing for small positive values of* x, *it has a positive zero only at* $x = x_1 > 0$, *and it is decreasing to* $-\infty$ *as* $x \to \infty$ *monotonically for* $x > x_1$. *Assuming also that* $g(0) = 0$, *and* $g(x) > 0$ *for all* $x > 0$, *the following statements hold.*

The orbits starting at the point $(0, -y)$, *with* $y > 0$, *enter the half-plane* $x > 0$ *and go around the origin in an counterclockwise path, coming back to the* y*-axis at the point* $(0, P_R(y))$, *with* $P_R(y)) > 0$. *The difference* $P_R(y) - y$ *is positive for small values of* y, *but eventually becomes negative, tending to* $-\infty$ *as* $y \to \infty$.

Proof. Clearly, the unique equilibrium of the system in the half-plane $x \geq 0$ is the origin. From the hypotheses, any orbit starting at the point $(0, -y)$, with $y > 0$, enters the half-plane $x > 0$ with null slope, to have positive slope while $y < F(x)$. The slope of the orbit becomes infinite when $y = F(x)$ and

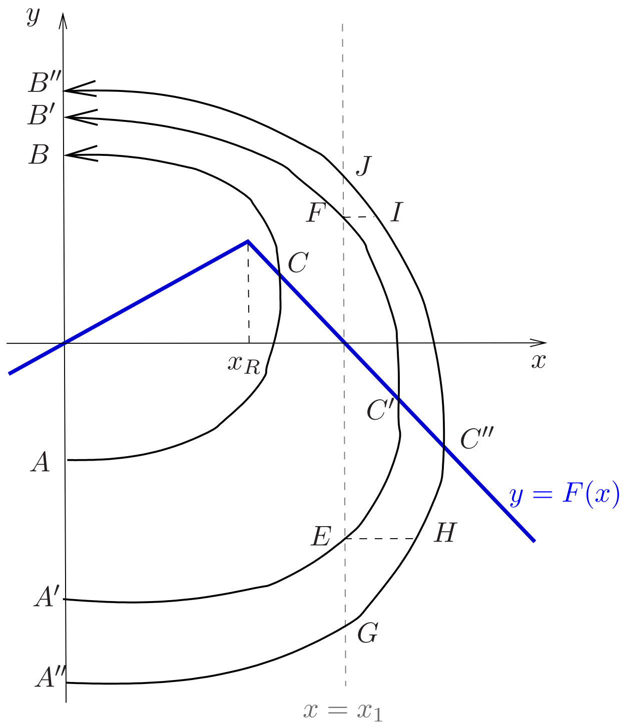

FIGURE 3. Three typical orbits of a Liénard system with a piecewise linear function $F(x)$ satisfying hypotheses of Lemma 3.

eventually becomes negative, finally arriving again to the y-axis with zero slope, at the point $(0, P_R(y))$ after making a half turn around the origin. We will study how much changes along such a half-turn the function

$$V(x,y) = G(x) + \frac{y^2}{2}$$

where

$$G(x) = \int_0^x g(u)du.$$

Note that $\dot{V}(x,y) = g(x)[F(x) - y] + y\,g(x) = F(x)\,g(x)$ and that $G(0) = 0$. Assume three nested arcs ACB, $A'C'B'$ and $A''C''B''$, see Figure 3, corresponding to orbits of the system. Suppose that the first orbit ACB is contained in the strip $0 < x < x_1$, where $F(x) > 0$ and $dy > 0$. Thus $F(x)dy > 0$ along such arc, and consequently

$$V(B) - V(A) = \int_A^B dV = \int_{y_A}^{y_B} F(x)dy > 0.$$

Therefore, since $2(V(B) - V(A)) = y_B^2 - y_A^2$, we have

$$y_B - |y_A| = P_R(|y_A|) - |y_A| > 0. \tag{16}$$

Consider now the arcs of orbits $A'C'B'$ and $A''C''B''$ not completely contained in the strip $0 < x < x_1$, see Fig. 3. Considering the parts of the arcs in such a strip where $F(x) > 0$, and since $F(x) - y$ along $A''G$ is greater than along $A'E$, where G and E are the points of the arcs which $x = x_1$, we have

$$\begin{aligned} V(G) - V(A'') = \int_{A''}^{G} dV = \int_{A''G} \frac{F(x)g(x)}{F(x)-y} dx < \\ < \int_{A'E} \frac{F(x)g(x)}{F(x)-y} dx = \int_{A'}^{E} dV = V(E) - V(A') \end{aligned} \tag{17}$$

Let H, I the points where the parallel lines to the x-axis passing through E and F intersect the arc $A''C''B''$. Since $F(x) < 0$ along GH and $dy > 0$ for $x > 0$, we obtain

$$V(H) - V(G) = \int_{G}^{H} dV = \int_{GH} F(x)dy < 0. \tag{18}$$

Now, since $F(x)$ along HI is negative and exceeds in absolute value $F(x)$ along EF for the same value of y, it follows that

$$\begin{aligned} V(I) - V(H) = \int_{H}^{I} dV = \int_{HI} F(x)dy < \\ < \int_{EF} F(x)dy = \int_{E}^{F} dV = V(F) - V(E). \end{aligned} \tag{19}$$

Along IJ, as in the study made along GH, it holds that

$$V(J) - V(I) < 0 \tag{20}$$

As in (17), we obtain

$$V(B'') - V(J) < V(B') - V(F) \tag{21}$$

Adding inequalities (17)–(21), we obtain

$$V(B'') - V(A'') < V(B') - V(A'),$$

that is, $y_{B''}^2 - y_{A''}^2 < y_{B'}^2 - y_{A'}^2$, or equivalently,

$$P_R(|y_{A''}|) - |y_{A''}| < P_R(|y_{A'}|) - |y_{A'}|. \tag{22}$$

We conclude that for the orbits starting at $(0, -y)$ and crossing the graph $y = F(x)$ for $x > x_1$ the difference $P_R(y) - y$ is monotonically decreasing. It remains to show that it tends to $-\infty$ when $y \to \infty$. Of course, if $P_R(y)$ turns to be bounded then the conclusion is trivial. In any case, it suffices to observe that from the first part of (17) we have that $V(G) - V(A'') > 0$ but decreasing to 0 as the point C'' goes far from the origin; the same is true for $V(B'') - V(J)$. However the contribution of the difference $V(J) - V(G)$

is negative and unbounded as the point C'' goes far from the origin. The conclusion follows. □

Clearly, we can do a similar study for the left half-return map P_L defined in the positive y-axis, by taking the orbit starting at the point $(0, y)$, with $y > 0$, and coming back to the negative y-axis at the point $(0, -P_L(y))$. In fact, by using Remark 2, the following result is straightforward.

Lemma 4. *Consider systems with three linearity zones given in* (11)-(13) *with* $x_L < 0 < x_R$ *and satisfying the conditions* $a_L, a_R < 0$*,* $a_C > 0$ *and* $b_L, b_R \geq 0$*. The following statements hold.*

- **(a)** *The orbits starting at the point* $(0, -y)$*, with* $y > 0$*, enter the half-plane* $x > 0$ *and go around the origin in an counterclockwise path, coming back to the* y*-axis at the point* $(0, P_R(y))$*, with* $P_R(y) > 0$*. The difference* $P_R(y) - y$ *is positive for small values of* y*, but eventually becomes negative, tending to* $-\infty$ *as* $y \to \infty$*.*
- **(b)** *The orbits starting at the point* $(0, y)$*, with* $y > 0$*, enter the half-plane* $x < 0$ *and go around the origin in an counterclockwise path, coming back to the* y*-axis at the point* $(0, -P_L(y))$*, with* $P_L(y) > 0$*. The difference* $P_L(y) - y$ *is positive for small values of* y*, but eventually becomes negative, tending to* $-\infty$ *as* $y \to \infty$*.*

With these results, it is easy to give a proof of Theorem 1.

Proof of Theorem 1. From Lemma 2 we can pass to an equivalent system in the form (11)-(13) and satisfying all the hypotheses of Lemma 4. We start by studying the existence of periodic orbits.

Clearly the existence of periodic orbits is equivalent to the existence of two positive values y_L and y_R such that

$$\begin{array}{rcl} P_R(y_R) & = & y_L, \\ y_R & = & P_L(y_L). \end{array} \tag{23}$$

Adding and subtracting the above equations we get an equivalent system of sufficient and necessary conditions for existence of periodic orbits, namely

$$\begin{array}{rcl} P_R(y_R) + y_R & = & P_L(y_L) + y_L, \\ P_R(y_R) - y_R & = & -\left[P_L(y_L) - y_L\right]. \end{array} \tag{24}$$

Since by standard results on uniqueness of solutions we know that P_R and P_L are monotone increasing functions, see Proposition 1.21 in [5], we can define two new functions $\widehat{P}_R$ and $\widehat{P}_L$ such that for each $Y = P_Z(y) + y > 0$ we take $\widehat{P}_Z(Y) = P_Z(y) - y$, where $y > 0$ and $Z \in \{L, R\}$. These new functions represent a different parameterization of the graphs of P_R and P_L and have the same qualitative behavior, that is, both are positive for sufficiently small $Y > 0$ and eventually become negative, tending to $-\infty$ as $Y \to \infty$. Furthermore, the conditions (24) for existence of periodic orbits translate now to the existence of a value $Y > 0$ being solution of the single equation $\widehat{P}_R(Y) = -\widehat{P}_L(Y)$, that is of

$$\widehat{P}_R(Y) + \widehat{P}_L(Y) = 0.$$

The lefthand side of above equation is positive for sufficiently small $Y > 0$ and eventually become negative for sufficiently big Y. It suffices now to apply the intermediate value theorem for continuous functions to conclude the existence of at least a solution, and so a periodic orbit of the system.

The uniqueness and stability of periodic orbits come from Remark 4 and the conclusion follows. □

The situation is slightly different in the case of systems with only two zones. In fact, for such systems the equilibrium point cannot be a node since its invariant manifolds are straight lines that should extend to infinity, precluding so the existence of periodic orbits. Anyway, the proof of Theorem 2 can be stated shortly as follows.

Proof of Theorem 2. As indicated above, the equilibrium point must be a focus and statement (a) follows.

From Lemma 2 we can pass to the corresponding system in the form (11)-(13), now with $a_L = a_C$ and $b_L = 1$, and satisfying all the hypotheses of Proposition 3. We start again by studying the existence of periodic orbits.

The unstable focus condition in the left zone is equivalent to

$$0 < \frac{t_L}{\sqrt{d_L}} < 2$$

and translates to the condition $0 < a_L < 2$ and now the system becomes purely linear for $x < x_R$ with the unstable focus at the origin. Then it is easy to see, see for instance [11], that P_L is a linear function given by $P_L(y) = e^{\pi\gamma_L} y$ where

$$\gamma_L = \frac{a_L}{\sqrt{4 - a_L^2}} < 0.$$

Regarding P_R we know that it behaves qualitatively as indicated in Proposition 3. Taking now the functions $\widehat{P}_R(Y)$ and $\widehat{P}_L(Y)$ introduced in the proof of Theorem 1 we will determine when there is an intersection between the graphs of $\widehat{P}_R$ and $-\widehat{P}_L$.

We note first that the slope of the graph of $\widehat{P}_L$ is a positive constant and equal to

$$\frac{P_L(y) - y}{P_L(y) + y} = \frac{e^{\pi\gamma_L} y - y}{e^{\pi\gamma_L} y + y} = \frac{e^{\pi\gamma_L} - 1}{e^{\pi\gamma_L} + 1} < 1.$$

If the dynamics on the right zone is of node type, we know that P_R is bounded by the invariant manifolds of the virtual node. Then the slope of the graph of $\widehat{P}_R$ as $Y \to \infty$ tends to

$$\lim_{y\to\infty} \frac{P_R(y) - y}{P_R(y) + y} = -1,$$

and therefore the graphs of $\widehat{P}_R$ and $-\widehat{P}_L$ must intersect at least once. Note that in this case we have in terms of original parameters

$$\frac{t_R}{\sqrt{d_R}} \leq -2$$

so that the condition in the theorem always holds.

If the dynamics on the right zone is of focus type and we define

$$\gamma_R = \frac{a_R}{\sqrt{4b_R - a_R^2}} < 0,$$

the slope of the graph of $\widehat{P}_R$ as $Y \to \infty$ tends to

$$\lim_{y\to\infty} \frac{P_R(y) - y}{P_R(y) + y} = \lim_{y\to\infty} \frac{e^{\pi\gamma_R} y - y}{e^{\pi\gamma_R} y + y} = \frac{e^{\pi\gamma_R} - 1}{e^{\pi\gamma_R} + 1} < 0,$$

since $a_R < 0$. Now the intersection of the graphs of $\widehat{P}_R$ and $-\widehat{P}_L$ is assured if and only if

$$-\frac{e^{\pi\gamma_L} - 1}{e^{\pi\gamma_L} + 1} > \frac{e^{\pi\gamma_R} - 1}{e^{\pi\gamma_R} + 1},$$

which is equivalent after standard algebraic manipulations to

$$\gamma_L + \gamma_R < 0.$$

This condition reads, in terms of the original parameters, as

$$\gamma_L + \gamma_R = \frac{t_L/\sqrt{d_L}}{\sqrt{4 - t_L^2/d_L}} + \frac{t_R/\sqrt{d_L}}{\sqrt{4d_R/d_L - t_R^2/d_L}} = \frac{t_L}{\sqrt{4d_L - t_L^2}} + \frac{t_R}{\sqrt{4d_R - t_R^2}} < 0.$$

Other standard algebraic manipulations now show that such condition for existence of periodic orbits is equivalent to

$$\frac{t_L}{\sqrt{d_L}} + \frac{t_R}{\sqrt{d_R}} < 0,$$

as stated in the theorem.

The uniqueness of periodic orbits comes directly from Remark 3 and the conclusion follows. □

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