

## A $B_p$ CONDITION FOR THE STRONG MAXIMAL FUNCTION

LIGUANG LIU AND TERESA LUQUE

ABSTRACT. A strong version of the Orlicz maximal operator is introduced and a natural  $B_p$  condition for the rectangle case is defined to characterize its boundedness. This fact let us to describe a sufficient condition for the two weight inequalities of the strong maximal function in terms of power and logarithmic bumps. Results for the multilinear version of this operator and for others multi(sub)linear maximal functions associated with bases of open sets are also studied.

### 1. INTRODUCTION

As it is well-known, Sawyer ([28]) characterized the pair of weights  $(u, v)$  for which the Hardy-Littlewood maximal operator,  $M$ , is a bounded operator from  $L^p(v)$  to  $L^p(u)$  for  $1 < p < \infty$ . He showed that  $M : L^p(v^p) \rightarrow L^p(u^p)$  if and only if  $(u, v)$  satisfies the testing condition

$$(1.1) \quad \sup_Q \frac{\int_Q (uM(\chi_Q v^{-p'}))^p dx}{v^{-p'}(Q)} < \infty.$$

On the other hand, it is also known that the two weight Muckenhoupt condition  $A_p$ ,

$$\sup_Q \left( \frac{1}{|Q|} \int_Q u^p dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q v^{-p'} dx \right)^{1/p'} < \infty,$$

is necessary but not sufficient for the maximal operator to be strong type  $(p, p)$ . The fact that Sawyer's condition involves the maximal operator itself makes it often difficult to test in practice. Therefore, though this condition characterizes completely the two weight problem, it would be more useful to look for sufficient conditions close in form to the  $A_p$  condition. The first step in this direction was due to Neugebauer ([22]): he noticed that if the pair of weights  $(u, v)$  is such that for  $r > 1$ ,

$$\sup_Q \left( \frac{1}{|Q|} \int_Q u^{pr} dx \right)^{1/pr} \left( \frac{1}{|Q|} \int_Q v^{-p'r} dx \right)^{1/p'r} < \infty$$

for all cubes, then

$$(1.2) \quad \int_{\mathbf{R}^n} (uMf)^p dx \leq C \int_{\mathbf{R}^n} (vf)^p dx$$

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for all nonnegative functions  $f$  and some positive constant  $C$  independent of  $f$ . If for a given cube  $Q$  we define the normalized  $L^p$  norm by

$$\|u\|_{p,Q} := \left( \frac{1}{|Q|} \int_Q u^p dx \right)^{1/p},$$

then Neugebauer's condition can be restated in terms of a normalized  $L^p$  norm as follows:

$$(1.3) \quad \|u\|_{pr,Q} \|v^{-1}\|_{p'r,Q} < \infty.$$

Notice that the  $A_p$  condition can also be rewritten as

$$(1.4) \quad \|u\|_{p,Q} \|v^{-1}\|_{p',Q} < \infty.$$

This tells us that if we replace the average  $L^p$  and  $L^{p'}$  norms in (1.4) by some stronger ones, then we can get a condition that is sufficient for (1.2) to hold. At the same time, this new condition preserves the geometric structure of the classical  $A_p$ . Conditions like (1.3) are known as power bump conditions because the norms involved in the two weight  $A_p$  condition are "bumped up" in the Lebesgue scale.

Motivated by [22], [9] and [10], Pérez ([25], [23]) generalized these last conditions replacing the localized norms in (1.4) by some other larger than the  $L^p$  one, but not as big as the  $L^{pr}$ . Indeed, he proved that it was enough to substitute only the norm associated to the weight  $v^{-1}$  by a stronger one defined in terms of certain Banach function spaces  $X$  with an appropriate boundedness property.

To be more precise, we let  $\Phi$  be a Young function (cf. section 2) and define the normalized Luxemburg norm on a cube  $Q$  by

$$\|u\|_{\Phi,Q} := \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left( \frac{u}{\lambda} \right) dx \leq 1 \right\}.$$

Associated with each Young function  $\Phi$ , one can define a complementary function

$$(1.5) \quad \bar{\Phi}(s) := \sup_{t>0} \{st - \Phi(t)\}$$

for  $s \geq 0$ . Such  $\bar{\Phi}$  is also a Young function and it plays, together with the class  $B_p$ , an important role in the generalization of these bump conditions. Recall that a Young function  $\Phi \in B_p$  if there is a positive constant  $c$  for which

$$(1.6) \quad \int_c^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} < \infty.$$

This growth condition was introduced in [25] where it was proved that if  $\Phi$  is a Young function such that  $\bar{\Phi} \in B_p$ , and  $(u, v)$  is a pair of weights such that

$$(1.7) \quad \sup_Q \|u\|_{p,Q} \|v^{-1}\|_{\Phi,Q} < \infty$$

for every cube  $Q$ , then (1.2) holds. Moreover the  $B_p$  condition is sharp in the sense that: if  $M$  is strong  $(p, p)$  and  $(u, v)$  satisfy (1.7), then  $\bar{\Phi} \in B_p$ . This result has been generalized very recently to the more general context of Banach function spaces by Pérez and Mastyo ([21]). For a more complete account of all this we refer to the recent book [6].

Besides its inherent significance for this problem, for many other operators conditions like (1.7) have resulted in optimal sufficient conditions for weak and strong

type inequalities. In general, given any pair of weights  $(u, v)$  we will define the  $A_p$  bump condition as

$$(1.8) \quad \sup_Q \|u\|_{\Psi, Q} \|v^{-1}\|_{\Phi, Q} < \infty$$

where  $\Psi$  and  $\Phi$  are Young functions such that  $\bar{\Psi} \in B_{p'}$  and/or  $\bar{\Phi} \in B_p$  and  $Q$  is any cube  $\mathbf{R}^n$ . This or related conditions are being used in the study of operators more singular than the Hardy-Littlewood maximal function. The first case was considered by Pérez in [26] for fractional integral operators where a fractional version of (1.8) was used to obtain a two weight  $L^p$  estimate. The same problem for the Hilbert transform was proved in [5] and by different methods in [7] for any Calderón-Zygmund operator with  $C^1$  kernel. Very recently the solution was extended in [8] to the Lipschitz case, proved full in generality in [18] and further improved in [15] with a better control on the bounds.

The main goal of this paper is to study the two weight norm inequalities for the boundedness of the strong maximal function using an appropriate  $B_p$  condition. We define this operator as

$$(1.9) \quad M_{\mathcal{R}} f(x) := \sup_{R \ni x, R \in \mathcal{R}} \frac{1}{|R|} \int_R |f(y)| dy.$$

where  $f$  is a locally integrable function and the supremum is taken over all rectangles  $R$  with sides parallel to the coordinate axes. The corresponding two weight problem for the strong maximal function was characterized by Jawerth (see [16]) in terms of a testing condition that it is even harder to verify than Sawyer's condition (1.1). The problem was also solved in [24] with a more similar approach to the one that we study here. It was proved that if  $(u, v)$  is a couple of weights satisfying the power condition for some  $r > 1$

$$(1.10) \quad \left( \frac{1}{|R|} \int_R u^p dx \right)^{1/p} \left( \frac{1}{|R|} \int_R v^{-p'r} dx \right)^{1/p'} < \infty$$

for every rectangle  $R$ , and suppose that  $u^p$  satisfies the condition (A): there are constants  $0 < \lambda < 1$  and  $0 < c(\lambda) < \infty$  such that

$$(A) \quad u^p(\{x \in \mathbf{R}^n : M_{\mathcal{R}}(\chi_E)(x) > \lambda\}) \leq c(\lambda) u^p(E)$$

for all measurable sets  $E$ , then  $M_{\mathcal{R}} : L^p(v^p) \rightarrow L^p(u^p)$ . In this case, the strong weighted estimate is obtained from weak type ones using interpolation and the fact that there is a reverse Hölder's property for the weights that verifies (1.10). However, this good property disappears if we substitute the  $L^{p'}$ -norm associated to the weight  $v^{-1}$  by a weaker one. Therefore, we will need a different approach to solve the two weight problem with general bump conditions.

To state the main result of this article, we define the class of Young functions that enable us to obtain bump conditions in the rectangle case.

**Definition 1.1.** Let  $1 < p < \infty$ . A Young function  $\Phi$  is said to satisfy the strong  $B_p^*$  condition, if there is a positive constant  $c$  such that

$$(1.11) \quad \int_c^\infty \frac{\Phi_n(\Phi(t))}{t^p} \frac{dt}{t} < \infty,$$

where  $\Phi_n(t) := t[\log(e+t)]^{n-1} \sim t[1 + (\log^+ t)^{n-1}]$  for all  $t > 0$ . In this case, we say that  $\Phi \in B_p^*$ .

Then we have the following result.

**Theorem 1.2.** *Let  $1 < p < \infty$ , and let  $\Phi$  be a Young function such that the complementary Young function  $\bar{\Phi}$  satisfies the condition (1.11).*

(i) *Let  $(u, v)$  be a couple of weights such that  $u^p$  satisfies the condition (A) and*

$$(1.12) \quad \left( \frac{1}{|R|} \int_R u^p \right)^{1/p} \|v^{-1}\|_{\Phi, R} \leq K,$$

*for some positive constant  $K$  and for all rectangles  $R$ . Then there is a constant  $C$  such that*

$$\int_{\mathbb{R}^n} (uM_{\mathcal{R}}f)^p dx \leq C \int_{\mathbb{R}^n} (vf)^p dx,$$

*for all non-negative functions  $f$ .*

(ii) *Condition (1.11) is also necessary. That is, suppose that  $\Phi$  has the property that  $M_{\mathcal{R}} : L^p(v^p) \rightarrow L^p(u^p)$  whenever the couple of weights  $(u, v)$  satisfies*

$$\left( \frac{1}{|R|} \int_R u^p \right)^{1/p} \|v^{-1}\|_{\Phi, R} \leq K,$$

*for some positive constant  $K$  and for all rectangles  $R$ ; then  $\bar{\Phi} \in B_p^*$ .*

If this result is compared with the analogous one for the Hardy-Littlewood operator ([25, Theorem 1.5]), then there is a key difference between them. For the former not only do we need a more restrictive class of young functions (the class  $B_p^*$ ), but also it is necessary to ask for an extra condition (A) on the weight  $u$ . To understand the role of this extra condition (A), we should keep in mind the next relevant fact. The study of the boundedness properties of a maximal operator with respect to a family of bounded measurable sets is closely connected to studying the covering properties of that family (cf. [12]). But since the geometry of rectangles in  $\mathbf{R}^n$  is much more intricate than that of cubes in  $\mathbf{R}^n$ , the classical covering lemmas don't work in the rectangle case. Particularly, the Calderón-Zygmund decomposition that is strongly used in the cube case cannot be used here. In this sense, the condition (A) is necessary to deal with rectangles and with their covering properties (see Lemma 4.2 below). The problem with this condition is that, as happened with Sawyer's condition, it involves itself the operator and therefore it would be useful to have a replacement that did not. The  $A_\infty$  condition would be a good candidate since it is simpler than the (A) condition; however, it is also stronger (see for example [24, p. 1123]). Unfortunately, repeated efforts to get a weaker condition and a simpler covering argument have failed, but we believe that such a result would be very interesting and would provide new insights about the study of the strong maximal operator.

In this article, we also address similar questions involving the multilinear version of the strong maximal function and some other more general maximal functions. We define the strong multilinear maximal function as

$$(1.13) \quad \mathcal{M}_{\mathcal{R}}(\vec{f})(x) := \sup_{R \ni x} \prod_{j=1}^m \frac{1}{|R|} \int_R |f_j(y)| dy, \quad x \in \mathbf{R}^n$$

where  $\vec{f} = (f_1, \dots, f_m)$  is an  $m$ -dimensional vector of locally integrable functions and where the supremum is taken over all rectangles with sides parallel to the coordinate axes. This multilinear maximal operator was defined for first time by

Lerner et al. in [19] with the usual cubes instead of rectangles. In that paper it is shown that this operator plays a central role in the theory of multilinear Calderón-Zygmund operators. The operator (1.13) as well as its version for a general basis  $\mathcal{B}$  (cf. section 3) was introduced and studied in [14]. In particular, it was shown the weak boundedness of  $\mathcal{M}_{\mathcal{R}}$  whenever the weights satisfy a certain power bump variant of the multilinear version of the  $A_p$  condition. That is, for  $1 < p_1, \dots, p_m < \infty$  and  $0 < p < \infty$  such that  $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$ , the multilinear strong maximal function maps

$$L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m) \rightarrow L^{p,\infty}(\nu)$$

provided that  $(\nu, \vec{w}) = (\nu, w_1, \dots, w_m)$  are weights that satisfy the power bump condition

$$(1.14) \quad \sup_{R \in \mathcal{R}} \left( \frac{1}{|R|} \int_R \nu(x) dx \right) \prod_{j=1}^m \left( \frac{1}{|R|} \int_R w_j^{(1-p'_j)r} dx \right)^{\frac{p}{p'_j r}} < \infty$$

for some  $r > 1$ . In the case that  $\nu = \prod_{j=1}^m w_j^{p/p_j}$  the strong boundedness of  $\mathcal{M}_{\mathcal{R}}$  is also characterized; see [14, Corollary 2.4 and Theorem 2.5]. Multiple weight theory that adapted to the basis  $\mathcal{B} = \mathcal{Q}$  and to its multilinear operator associated,  $\mathcal{M}_{\mathcal{Q}}$ , has been fully developed by Lerner et al. [19] and generalized very recently by Moen [20].

Inspired by these previous works, we will introduce the multilinear version of (1.7) and (1.12) for weights  $(\nu, \vec{w})$  associated with general basis. Then, the  $L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m) \rightarrow L^p(\nu)$  boundedness of  $\mathcal{M}_{\mathcal{B}}$  (cf. section 3) will be proved whenever  $\nu$  is any arbitrary weight such that  $\nu^p$  satisfies condition (A). This result is given in Theorem 3.1. As an application of this theorem, we will obtain the strong version of [14, Theorem 2.3] and we will deduce the analogous result of [20, Theorem 6.6] for the strong multilinear maximal function. See Corollary 3.3 and Corollary 3.4.

The general organization of this paper is as follows. Section 2 contains some preliminaries about Orlicz spaces and a characterization of the strong  $B_p^*$  condition. Section 3 presents some definitions about general basis and the statement of the main strong weight result for a general multilinear operator (Theorem 3.1). Also, we give the proofs of Corollaries 3.3 and 3.4, and we will deduce the proof of Theorem 1.2 by applying Corollary 3.4. Finally, the last section shows the proof of Theorem 3.1.

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## 2. CHARACTERIZATION OF THE $B_p^*$ CONDITION

To present this characterization we need to recall a few facts about Orlicz spaces and we shall refer the reader to [6, Chapter 5] and [27] for complete details. A function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  is a Young function if it is continuous, convex and strictly increasing satisfying  $\Phi(0) = 0$  and  $\Phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . A Young function  $\Phi$  is said to be doubling if there exists a positive constant  $\alpha$  such that

$$\Phi(2t) \leq \alpha \Phi(t)$$

for all  $t \geq 0$ . The normalized  $\Phi$ -norm of a function  $f$  over a set  $E$  with finite measure is defined by

$$\|f\|_{\Phi, E} := \inf \left\{ \lambda > 0 : \frac{1}{|E|} \int_E \Phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

The complementary of the Young function (1.5) has properties

$$(2.1) \quad \Phi^{-1}(t)\bar{\Phi}^{-1}(t) \sim t \quad \text{for all } t \in (0, \infty)$$

and

$$st \leq C \left[ \Phi(t) + \bar{\Phi}(s) \right]$$

for all  $s, t \geq 0$ . Also the  $\bar{\Phi}$ -norms are related to the  $L_\Phi$ -norms via the *the generalized Hölder's inequality*, namely

$$(2.2) \quad \frac{1}{|E|} \int_E |f(x)g(x)| dx \leq 2 \|f\|_{\Phi, E} \|g\|_{\bar{\Phi}, E}.$$

Consider the Orlicz maximal operator

$$M_\Phi^\mathcal{Q} f(x) := \sup_{Q \ni x, Q \in \mathcal{Q}} \|f\|_{\Phi, Q},$$

where the supremum is taken over all cubes containing  $x$ . Pérez [25, Theorem 1.7] proved the following key observation: when  $1 < p < \infty$  and  $\Phi$  is a doubling Young function, then

$$M_\Phi^\mathcal{Q} : L^p(\mathbf{R}^n) \longrightarrow L^p(\mathbf{R}^n) \quad \text{if and only if } \Phi \text{ satisfies (1.6).}$$

Here we remark that the hypothesis of  $\Phi$  being doubling was only used to prove the necessity of the  $B_p$  condition but we show now that can be removed. Indeed, if we assume that for any non-negative function  $f$  the operator  $M_\Phi^\mathcal{Q}$  is bounded on  $L^p(\mathbf{R}^n)$  and we take  $f = \chi_{[0,1]^n}$ , we have

$$\int_{\mathbf{R}^n} M_\Phi^\mathcal{Q}(\chi_{[0,1]^n})(y)^p dy < \infty.$$

Now, it is easy to see that there exists a positive dimensional constant  $b$  such that whenever  $|y| > 1$

$$M_\Phi^\mathcal{Q}(\chi_{[0,1]^n})(y) = \frac{1}{\Phi^{-1}\left(\frac{|y|^n}{b}\right)}.$$

Hence

$$\begin{aligned} \int_{\mathbf{R}^n} M_\Phi^\mathcal{Q}(\chi_{[0,1]^n})(y)^p dy &\geq p \int_0^\infty t^p \left| \left\{ y \in \mathbf{R}^n : |y| > 1, \frac{1}{\Phi^{-1}\left(\frac{|y|^n}{b}\right)} > t \right\} \right| \frac{dt}{t} \\ &= p \int_0^\infty t^p \left| \left\{ y \in \mathbf{R}^n : 1 < |y| < \Phi\left(\frac{1}{t}\right)^{1/n} b^{1/n} \right\} \right| \frac{dt}{t} \\ &= c_n p \int_0^\infty t^p \left( b\Phi\left(\frac{1}{t}\right) - 1 \right) \frac{dt}{t}, \end{aligned}$$

where  $c_n$  is a positive constant depending only on  $n$ . Since  $\Phi$  is increasing and  $\Phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , we can choose some  $t_0 > 0$  such that for every  $t \leq t_0$ ,

$$b\Phi\left(\frac{1}{t}\right) - 1 \geq \frac{b}{2}\Phi\left(\frac{1}{t}\right).$$

Then

$$\begin{aligned} \infty &> c_n p \int_0^\infty t^{p-1} \left( b\Phi\left(\frac{1}{t}\right) - 1 \right) dt \\ &\geq \frac{c_n p b}{2} \int_0^{t_0} t^{p-1} \Phi\left(\frac{1}{t}\right) dt = \frac{c_n p b}{2} \int_{1/t_0}^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t}. \end{aligned}$$

Motivated by Pérez [25, Theorem 1.7] and the previous observation, in this section we consider the Orlicz maximal operator  $M_\Phi^{\mathcal{R}}$  associated with rectangles rather than cubes. Precisely, for each locally integrable function  $f$  and Young function  $\Phi$  we define the *Orlicz maximal operator*  $M_\Phi^{\mathcal{R}}$  by

$$M_\Phi^{\mathcal{R}} f(x) := \sup_{R \ni x, R \in \mathcal{R}} \|f\|_{B, R}$$

where the supremum is taken over all rectangles with sides parallel to the coordinate axes containing  $x$ . In particular, when  $\Phi(t) = t$  the maximal operator  $M_\Phi^{\mathcal{R}}$  is exactly the classical *strong maximal function* (1.9).

The next characterization shows that the boundedness of  $M_\Phi^{\mathcal{R}}$  is closely connected with the class  $B_p^*$ .

**Theorem 2.1.** *Let  $1 < p < \infty$ . Suppose that  $\Phi$  is a Young function. Then the following statements are equivalent:*

- (i)  $\Phi \in B_p^*$ ;
- (ii) the operator  $M_\Phi^{\mathcal{R}}$  is bounded on  $L^p(\mathbf{R}^n)$ ;
- (iii) there exists a positive constant  $C$  such that

$$(2.3) \quad \int_{\mathbf{R}^n} [M_{\mathcal{R}}(f)(y)]^p \frac{1}{[M_\Phi^{\mathcal{R}}(u^{1/p})(y)]^p} dy \leq C \int_{\mathbf{R}^n} f(y)^p \frac{1}{u(y)} dy$$

for all non-negative functions  $f$  and  $u$ ;

- (iv) there exists a positive constant  $C$  such that for all non-negative functions  $f$  and all  $w$  satisfying the condition (A),

$$(2.4) \quad \int_{\mathbf{R}^n} [M_\Phi^{\mathcal{R}}(f)(y)]^p w(y) dy \leq C \int_{\mathbf{R}^n} f(y)^p M_{\mathcal{R}} w(y) dy.$$

As particular examples of Young functions  $\Phi \in B_p^*$ , one can easily check that a Young function  $\Phi$  satisfies the condition (1.11) if

$$\Phi(t) \sim t^\alpha \log^{-\beta}(e+t) \quad -\infty < \alpha < p, \beta \in \mathbf{R};$$

$$\Phi(t) \sim t^p \log^{-\beta}(e+t) \quad \beta > n;$$

or the weaker one

$$\Phi(t) \sim t^p \log^{-n}(e+t) [\log \log(e+t)]^{-\gamma} \quad \gamma > 1.$$

*Proof of Theorem 2.1.* We assume that (i) holds and show (ii). To this end, for each  $t > 0$ , we split the function  $f$  into  $f = f_t + f^t$  with  $f_t := f\chi_{|f|>t/2}$  and  $f^t := f\chi_{|f|\leq t/2}$ . Then,

$$M_\Phi^{\mathcal{R}} f \leq M_\Phi^{\mathcal{R}}(f_t) + M_\Phi^{\mathcal{R}}(f^t) \leq M_\Phi^{\mathcal{R}}(f_t) + t/2$$

and

$$\{x \in \mathbf{R}^n : M_\Phi^{\mathcal{R}} f(x) > t\} \subset \{x \in \mathbf{R}^n : M_\Phi^{\mathcal{R}}(f_t)(x) > t/2\}.$$

Set

$$\Omega_t := \{x \in \mathbf{R}^n : M_\Phi^{\mathcal{R}}(f_t)(x) > t/2\}.$$

Choose a compact set  $K \subset \Omega_t$  such that  $|\Omega_t|/2 \leq |K| \leq |\Omega_t|$ . There exists a sequence of rectangles  $\{R_j\}_{j=1}^N$  such that  $K \subset \cup_{j=1}^N R_j$  and

$$\|f_t\|_{\Phi, R_j} > t \quad \forall j \in \{1, \dots, N\}.$$

By [14, Lemma 6.1], the condition  $\|f_t\|_{\Phi, R_j} > t$  implies that

$$1 < \left\| \frac{f_t}{t} \right\|_{\Phi, R_j} \leq \frac{1}{|R_j|} \int_{R_j} \Phi \left( \frac{|f_t(y)|}{t} \right) dy.$$

Applying now the covering lemma from [3] (see also [1, Theorem 4.1 (C)]), there are dimensional positive constants  $\delta, c$  and a subfamily  $\{\tilde{R}_j\}_{j=1}^\ell$  of  $\{R_j\}_{j=1}^N$  satisfying

$$\left| \bigcup_{j=1}^N R_j \right| \leq c \left| \bigcup_{j=1}^\ell \tilde{R}_j \right|,$$

and

$$\int_{\bigcup_{j=1}^\ell \tilde{R}_j} \exp \left( \delta \sum_{j=1}^\ell \chi_{\tilde{R}_j}(x) \right)^{\frac{1}{n-1}} dx \leq 2 \left| \bigcup_{j=1}^\ell \tilde{R}_j \right|.$$

Let  $\tilde{E} := \bigcup_{j=1}^\ell \tilde{R}_j$ . Recall that for each  $\theta > 0$ , there exists a constant  $C_\theta$  such that for all  $a, b \geq 0$ ,

$$ab \leq C_\theta (e^{\theta a} - 1 + b[1 + (\log_+ b)^{n-1}]) = C_\theta (e^{\theta a} - 1 + \Phi_n(b));$$

see [1, p. 887]. Then, for all  $\epsilon > 0$ ,

$$\begin{aligned} |\tilde{E}| &\leq \sum_{j=1}^\ell |\tilde{R}_j| \\ &\leq \sum_{j=1}^\ell \int_{\tilde{R}_j} \Phi \left( \frac{|f_t(y)|}{t} \right) dy \\ &= \int_{\bigcup_{j=1}^\ell \tilde{R}_j} \sum_{j=1}^\ell \chi_{\tilde{R}_j}(y) \Phi \left( \frac{|f_t(y)|}{t} \right) dy \\ &\leq \epsilon C_\delta \int_{\bigcup_{j=1}^\ell \tilde{R}_j} \left[ \exp \left( \delta \sum_{j=1}^\ell \chi_{\tilde{R}_j}(y) \right) - 1 + \Phi_n \left( \frac{1}{\epsilon} \Phi \left( \frac{|f_t(y)|}{t} \right) \right) \right] dy \\ &\leq \epsilon C_\delta \left\{ |\tilde{E}| + \Phi_n(1/\epsilon) \int_{\tilde{E}} \Phi_n \left( \Phi \left( \frac{|f_t(y)|}{t} \right) \right) dy \right\}. \end{aligned}$$

Choosing  $\epsilon > 0$  small enough we obtain

$$|\tilde{E}| \leq C \int_{\tilde{E}} \Phi_n \left( \Phi \left( \frac{|f_t(y)|}{t} \right) \right) dy.$$

Since  $|\Omega_t| \sim |K|$  and  $|K| \leq c|\tilde{E}|$  and  $\Phi_n(\Phi(0)) = 0$ , it follows that

$$|\Omega_t| \leq C \int_{\tilde{E}} \Phi_n \left( \Phi \left( \frac{|f_t(y)|}{t} \right) \right) dy \leq C \int_{\{y \in \mathbf{R}^n : |f(y)| > t/2\}} \Phi_n \left( \Phi \left( \frac{|f(y)|}{t} \right) \right) dy.$$



This inequality and the fact  $\{x \in \mathbf{R}^n : M_{\bar{\Phi}}^{\mathcal{R}} f(x) > t\} \subset \Omega_t$ , together with the change of variable  $s = |f(y)|/t$ , yields

$$\begin{aligned}
\|M_{\bar{\Phi}}^{\mathcal{R}} f\|_{L^p(\mathbf{R}^n)}^p &= p \int_0^\infty t^p |\{x \in \mathbf{R}^n : M_{\bar{\Phi}}^{\mathcal{R}} f(x) > t\}| \frac{dt}{t} \\
&\leq p \int_0^\infty t^p |\Omega_t| \frac{dt}{t} \\
&\leq C \int_0^\infty \int_{\{y \in \mathbf{R}^n : |f(y)| > t/2\}} t^p \bar{\Phi}_n \left( \Phi \left( \frac{|f(y)|}{t} \right) \right) dy \frac{dt}{t} \\
&= C \int_{\mathbf{R}^n} \int_0^{2|f(y)|} t^p \bar{\Phi}_n \left( \Phi \left( \frac{|f(y)|}{t} \right) \right) \frac{dt}{t} dy \\
&\leq C \int_{\mathbf{R}^n} \int_{1/2}^\infty |f(y)|^p \frac{\bar{\Phi}_n(\Phi(s))}{s^p} \frac{ds}{s} dy \\
&\leq C \|f\|_{L^p(\mathbf{R}^n)}^p,
\end{aligned}$$

where in the last step we use the hypothesis  $\bar{\Phi} \in B_p^*$ . This proves (ii).

Let us assume that (ii) holds. Using the generalized Hölder's inequality (2.2) we obtain

$$M_{\mathcal{R}}(hg)(y) \leq M_{\bar{\Phi}}^{\mathcal{R}} h(y) M_{\bar{\Phi}}^{\mathcal{R}} g(y),$$

which together with the boundedness of  $M_{\bar{\Phi}}^{\mathcal{R}}$  on  $L^p(\mathbf{R}^n)$  implies that

$$\begin{aligned}
\int_{\mathbf{R}^n} [M_{\mathcal{R}}(hg)(y)]^p \frac{1}{[M_{\bar{\Phi}}^{\mathcal{R}} g(y)]^p} dy &\leq \int_{\mathbf{R}^n} [M_{\bar{\Phi}}^{\mathcal{R}} h(y)]^p dy \\
&\leq \|M_{\bar{\Phi}}^{\mathcal{R}}\|_{L^p(\mathbf{R}^n) \rightarrow L^p(\mathbf{R}^n)}^p \int_{\mathbf{R}^n} h(y)^p dy,
\end{aligned}$$

from which we obtain the claim (iii) by taking  $h = fu^{-1/p}$  and  $g = u^{1/p}$ .

To prove that (iii) implies (i), for any  $N \in \mathbb{N}$ , we let  $f := \chi_{[0,1]^n}$  and  $u_N := \chi_{[0,1]^n} + \frac{\chi_{\mathbf{R}^n \setminus [0,1]^n}}{N}$  in (2.3). Hence we get

$$\int_{\mathbf{R}^n} \left( \frac{M_{\mathcal{R}}(\chi_{[0,1]^n})(y)}{M_{\bar{\Phi}}^{\mathcal{R}}(\chi_{[0,1]^n} + \frac{\chi_{\mathbf{R}^n \setminus [0,1]^n}}{N})(y)} \right)^p dy \leq C.$$

Observing that  $M_{\bar{\Phi}}^{\mathcal{R}}(f+g) \leq M_{\bar{\Phi}}^{\mathcal{R}} f + M_{\bar{\Phi}}^{\mathcal{R}} g$  and using the monotone convergence lemma, we deduce

$$(2.5) \quad \int_{\mathbf{R}^n} \left( \frac{M_{\mathcal{R}}(\chi_{[0,1]^n})(y)}{M_{\bar{\Phi}}^{\mathcal{R}}(\chi_{[0,1]^n})(y)} \right)^p dy \leq C.$$

It is easy to see that for any point  $(y_1, \dots, y_n) \in \mathbf{R}^n$  such that  $y_j > 1$  for all  $j \in \{1, \dots, n\}$ , we have

$$M_{\mathcal{R}}(\chi_{[0,1]^n})(y) = \sup_{R \ni y, R \in \mathcal{R}} \frac{|R \cap [0,1]^n|}{|R|} = \frac{1}{y_1 y_2 \cdots y_n},$$

and

$$\begin{aligned}
M_{\bar{\Phi}}^{\mathcal{R}}(\chi_{[0,1]^n})(y) &= \sup_{R \ni y, R \in \mathcal{R}} \inf \left\{ \lambda > 0 : \bar{\Phi}(\lambda^{-1}) \leq \frac{|R|}{|R \cap [0,1]^n|} \right\} \\
&= \sup_{R \ni y, R \in \mathcal{R}} \frac{1}{\bar{\Phi}^{-1} \left( \frac{|R|}{|R \cap [0,1]^n|} \right)}
\end{aligned}$$

$$= \frac{1}{\bar{\Phi}^{-1}(y_1 y_2 \cdots y_n)}.$$

Inserting these two estimates into (2.5) and using (2.1), we deduce that

$$\begin{aligned} \infty &> \int_1^\infty \cdots \int_1^\infty \left( \frac{\bar{\Phi}^{-1}(y_1 y_2 \cdots y_n)}{y_1 y_2 \cdots y_n} \right)^p dy_n \cdots dy_1 \\ &\sim \int_1^\infty \cdots \int_1^\infty \left( \frac{1}{\Phi^{-1}(y_1 y_2 \cdots y_n)} \right)^p dy_n \cdots dy_1. \end{aligned}$$

Then it follows that

$$\begin{aligned} \int_1^\infty \left( \frac{1}{\Phi^{-1}(y_1 y_2 \cdots y_n)} \right)^p dy_n &= \frac{1}{y_1 \cdots y_{n-1}} \int_{\Phi^{-1}(y_1 \cdots y_{n-1})}^\infty \frac{\Phi'(z)}{z^p} dz \\ &\geq \frac{1}{y_1 \cdots y_{n-1}} \int_{\Phi^{-1}(y_1 \cdots y_{n-1})}^\infty \frac{\Phi(z)}{z^{p+1}} dz, \end{aligned}$$

where we have used the fact that  $\Phi'(t) \geq \frac{\Phi(t)}{t}$  for any Young function  $\Phi$ . Now we take the integral in the variable  $y_{n-1}$  and we obtain

$$\begin{aligned} &\int_1^\infty \int_1^\infty \left( \frac{1}{\Phi^{-1}(y_1 y_2 \cdots y_n)} \right)^p dy_n dy_{n-1} \\ &\geq \int_1^\infty \frac{1}{y_1 \cdots y_{n-1}} \int_{\Phi^{-1}(y_1 \cdots y_{n-1})}^\infty \frac{\Phi(z)}{z^{p+1}} dz dy_{n-1} \\ &= \int_{\Phi^{-1}(y_1 \cdots y_{n-2})}^\infty \int_1^{\frac{\Phi(z)}{y_1 \cdots y_{n-2}}} \frac{1}{y_1 \cdots y_{n-1}} dy_{n-1} \frac{\Phi(z)}{z^{p+1}} dz \\ &= \frac{1}{y_1 \cdots y_{n-2}} \int_{\Phi^{-1}(y_1 \cdots y_{n-2})}^\infty \ln \left( \frac{\Phi(z)}{y_1 \cdots y_{n-2}} \right) \frac{\Phi(z)}{z^{p+1}} dz. \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_1^\infty \frac{1}{y_1 \cdots y_{n-2}} \int_{\Phi^{-1}(y_1 \cdots y_{n-2})}^\infty \ln \left( \frac{\Phi(z)}{y_1 \cdots y_{n-2}} \right) \frac{\Phi(z)}{z^{p+1}} dz dy_{n-2} \\ &= \frac{1}{y_1 \cdots y_{n-3}} \int_{\Phi^{-1}(y_1 \cdots y_{n-3})}^\infty \int_1^{\frac{\Phi(z)}{y_1 \cdots y_{n-3}}} \frac{1}{y_{n-2}} \ln \left( \frac{\Phi(z)}{y_1 \cdots y_{n-2}} \right) dy_{n-2} \frac{\Phi(z)}{z^{p+1}} dz \\ &= \frac{1}{y_1 \cdots y_{n-3}} \int_{\Phi^{-1}(y_1 \cdots y_{n-3})}^\infty \left( \ln \left( \frac{\Phi(z)}{y_1 \cdots y_{n-3}} \right) \right)^2 \frac{\Phi(z)}{z^{p+1}} dz. \end{aligned}$$

We iterate this process by integrating over the next variables  $y_{n-3}, \dots, y_1$  in turn and we obtain

$$\begin{aligned} \infty &> \int_1^\infty \cdots \int_1^\infty \left( \frac{1}{\Phi^{-1}(y_1 y_2 \cdots y_n)} \right)^p dy_n \cdots dy_1 \\ &\geq \int_{\Phi^{-1}(1)}^\infty (\ln(\Phi(z)))^{n-1} \frac{\Phi(z)}{z^{p+1}} dz \\ &\geq \int_{\Phi^{-1}(e)}^\infty \frac{\Phi_n(\Phi(z))}{z^{p+1}} dz, \end{aligned}$$

which proves (i).

To conclude the proof of this theorem, note that (iv) implies (ii) by choosing  $w = 1$  in the right side of (2.4). In order to prove that (ii) implies (iv) we will

proceed using an argument very similar to the one presented in the proof of Theorem 3.1. For this reason, we will give the details to complete this proof in the fourth section, Remark 4.3. Here we point out that the proof of Theorem 3.1 below is independent of Theorem 2.1.  $\square$

It should be remarked that the classical  $B_p$  condition (1.6) is not sufficient for the  $L^p(\mathbf{R}^n)$ -boundedness of  $M_\Phi^{\mathcal{R}}$ . For this, we consider for example the function  $f = \chi_{[0,1]^n}$ , and

$$\Phi(t) = \frac{t^p}{(\log(1+t))^{1+\delta}} \quad \text{for all } t \in (0, \infty)$$

with  $0 < \delta < 1$ . It is easy to verify that such a function  $\Phi$  satisfies (1.6) but fails for (1.11). For simplicity, we consider only the case when  $n = 2$ . If  $|x_i| \gg 1$  (we can take  $|x_i| > 4$ ) for  $i = 1, 2$ , then

$$M_\Phi^{\mathcal{R}} f(x_1, x_2) = \frac{1}{\Phi^{-1}(|x_1||x_2|)} \sim \frac{1}{|x_1||x_2|^{1/p} (\log(1+|x_1||x_2|))^{(1+\delta)/p}},$$

by terms of the fact that  $\Phi^{-1}(t) \sim t^{1/p} (\log(1+t))^{(1+\delta)/p}$  for all  $t \in (0, \infty)$ . Using Fubini's theorem, we have

$$\begin{aligned} \int_{\mathbf{R}^2} (M_\Phi^{\mathcal{R}} f(x))^p dx &\geq \int_4^\infty \int_4^\infty \left( \frac{1}{\Phi^{-1}(|x_1||x_2|)} \right)^p dx_2 dx_1 \\ &\geq \int_4^\infty \int_4^\infty \frac{1}{x_1 x_2 (\log(1+x_1 x_2))^{1+\delta}} dx_2 dx_1 \\ &\geq \int_4^\infty \int_4^\infty \frac{1}{(1+x_1 x_2) (\log(1+x_1 x_2))^{1+\delta}} dx_2 dx_1 \\ &\sim \frac{1}{\delta} \int_4^\infty \frac{1}{x_1 (\log(1+4x_1))^\delta} dx_1 \\ &\geq \frac{1}{\delta} \int_{16}^\infty \frac{1}{(1+x_1) (\log(1+x_1))^\delta} dx_1 = \infty. \end{aligned}$$

However,

$$\|f\|_{L^p(\mathbf{R}^2)} = \|\chi_{[0,1]^2}\|_{L^p(\mathbf{R}^2)} = 1.$$

Hence,  $M_\Phi^{\mathcal{R}}$  is not bounded on  $L^p(\mathbf{R}^2)$ . The general case for  $n > 2$  is similar and we omit the details.

Though the  $B_p$  condition (1.6) is not sufficient for the  $L^p(\mathbf{R}^n)$ -boundedness of  $M_\Phi^{\mathcal{R}}$ , we can remedy this situation if we restrict to those Young functions that are submultiplicative; see Proposition 2.2 below. We say that a Young function  $\Phi$  is submultiplicative if for each  $t, s > 0$ ,

$$\Phi(ts) \leq \Phi(t)\Phi(s).$$

**Proposition 2.2.** *Let  $1 < p < \infty$ . Assume that  $\Phi$  is a submultiplicative Young function such that  $\Phi \in B_p$ . Then the operator  $M_\Phi^{\mathcal{R}}$  is bounded on  $L^p(\mathbf{R}^n)$ .*

*Proof.* This is a simple consequence of the fact that for a submultiplicative Young function such that  $\Phi \in B_p$ , there exists  $\epsilon > 0$  for which  $\Phi \in B_{p-\epsilon}$  ([25, Lemma 4.3]). Indeed, using the previous theorem we only need to prove that  $\Phi \in \Phi_p^*$ . Note that

$$(2.6) \quad \int_c^\infty \frac{\Phi_n(\Phi(s)) ds}{s^p} \frac{1}{s} = \int_c^\infty \frac{\Phi(s) ds}{s^p} \frac{1}{s} + \int_c^\infty \frac{\Phi(s) (\log^+ \Phi(s))^{n-1} ds}{s^{p-\epsilon}} \frac{1}{s^\epsilon} \frac{1}{s}.$$

It is clear that the first term in the right hand of (2.6) is bounded. On the other hand,

$$\frac{(\log^+ \Phi(s))^{n-1}}{s^\epsilon} \leq \frac{\Phi(s)^{\delta(n-1)}}{s^\epsilon \delta^{(n-1)}},$$

with  $\delta > 0$ . Since  $\Phi$  is in the class  $B_p$ , it follows that  $\Phi(t) \lesssim t^p$  for  $t \geq 1$  and hence for  $\delta = \frac{\epsilon}{p(n-1)}$  the above term is bounded. This further implies that the second term of (2.6) is bounded by a constant multiple of

$$\int_c^\infty \frac{\Phi(s) ds}{s^{p-\epsilon} s},$$

which together with the aforementioned fact that  $\Phi \in B_{p-\epsilon}$  gives the boundedness of the second term of (2.6).  $\square$

*Remark 2.3.* We observe that a typical Young function that belongs to the class  $B_p$  and that it is also submultiplicative is  $\Phi(t) = t^r$  with  $1 \leq r < p$ . Another more interesting example is the function  $\Phi$  given by  $\Phi(t) = t^r(1 + \log_+ t)^\alpha$  with  $1 \leq r < p$  and  $\alpha > 0$ . It is not difficult to see that such functions are submultiplicative and they are in the  $B_p$  class. See Cruz-Urbe and Fiorenza [4] for related discussions on the topic of submultiplicative Young functions.

### 3. WEIGHTED THEORY FOR GENERAL BASES AND PROOF OF THEOREM 1.2

We start by introducing some notation that we will use through this section. By a *basis*  $\mathcal{B}$  in  $\mathbf{R}^n$  we mean a collection of open sets in  $\mathbf{R}^n$ . The most important examples of bases arise by taking  $\mathcal{B} = \mathcal{Q}$  the family of all open cubes in  $\mathbf{R}^n$  with sides parallel to the axes,  $\mathcal{B} = \mathcal{D}$  the family of all open dyadic cubes in  $\mathbf{R}^n$ , and  $\mathcal{B} = \mathcal{R}$  the family of all open rectangles in  $\mathbf{R}^n$  with sides parallel to the axes. Another interesting example is the set  $\mathfrak{R}$  of all rectangles in  $\mathbb{R}^3$  with sides parallel to the coordinate axes whose side lengths are  $s$ ,  $t$ , and  $st$ , for some  $t, s > 0$ .

Assume that  $\mathcal{B}$  is a basis and that  $\{\Psi_j\}_{j=1}^m$  is a sequence of Young functions, we define the multi(sub)linear Orlicz maximal function by

$$\mathcal{M}_{\Psi}^{\mathcal{B}}(\vec{f})(x) := \sup_{B \in \mathcal{B}, B \ni x} \prod_{j=1}^m \|f_j\|_{\Psi_j, B}.$$

In particular, when  $\Psi_j(t) = t$  for all  $t \in (0, \infty)$  and all  $j \in \{1, \dots, m\}$ , we simply write  $\mathcal{M}_{\Psi}^{\mathcal{B}}$  as  $\mathcal{M}_{\mathcal{B}}$ ; that is,

$$\mathcal{M}_{\mathcal{B}}(\vec{f})(x) = \sup_{B \in \mathcal{B}, B \ni x} \prod_{j=1}^m \frac{1}{|B|} \int_B |f_j(y)| dy.$$

When  $m = 1$ , we use  $M_{\Psi}^{\mathcal{B}}$  and  $M_{\mathcal{B}}$  to respectively denote  $\mathcal{M}_{\Psi}^{\mathcal{B}}$  and  $\mathcal{M}_{\mathcal{B}}$ .

We say that  $w$  is a *weight* associated with the basis  $\mathcal{B}$  if  $w$  is a non-negative measurable function in  $\mathbf{R}^n$  such that  $w(B) = \int_B w(y) dy < \infty$  for each  $B$  in  $\mathcal{B}$ . A weight  $w$  associated with  $\mathcal{B}$  is said to satisfy the  $A_{p, \mathcal{B}}$  condition,  $1 < p < \infty$ , if

$$\sup_{B \in \mathcal{B}} \left( \frac{1}{|B|} \int_B w dx \right) \left( \frac{1}{|B|} \int_B w^{1-p'} dx \right)^{\frac{p}{p'}} < \infty.$$

In the limiting case  $p = 1$  we say that  $w$  satisfies the  $A_{1, \mathcal{B}}$  if

$$\left( \frac{1}{|B|} \int_B w(y) dy \right) \operatorname{esssup}_B w^{-1} \leq c$$

for all  $B \in \mathcal{B}$ ; this is equivalent to  $M_{\mathcal{B}}w(x) \leq cw(x)$  for almost all  $x \in \mathbf{R}^n$ . It follows from these definitions and Hölder's inequality that  $A_{p,\mathcal{B}} \subset A_{q,\mathcal{B}}$  if  $1 \leq p \leq q \leq \infty$ . Then it is natural to define the class  $A_{\infty,\mathcal{B}}$  by setting

$$A_{\infty,\mathcal{B}} := \bigcup_{p>1} A_{p,\mathcal{B}}.$$

For a general basis  $\mathcal{B}$  we obtain the following strong type result.

**Theorem 3.1.** *Let  $1 < p_1, \dots, p_m < \infty$  and  $0 < p < \infty$  such that  $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$ . Assume that  $\mathcal{B}$  is a basis and that  $\{\Psi_j\}_{j=1}^m$  is a sequence of Young functions such that*

$$M_{\vec{\Psi}}^{\mathcal{B}}(\vec{f})(x) := \sup_{B \in \mathcal{B}, B \ni x} \prod_{j=1}^m \|f_j\|_{\Psi_j, B}$$

is bounded from  $L^{p_1}(\mathbf{R}^n) \times \dots \times L^{p_m}(\mathbf{R}^n)$  to  $L^p(\mathbf{R}^n)$ . Let  $(\nu, \vec{w}) = (\nu, w_1, \dots, w_m)$  be weights such that  $\nu^p$  satisfies condition (A), and that

$$(3.1) \quad \sup_{B \in \mathcal{B}} \left( \frac{1}{|B|} \int_B \nu(x)^p dx \right)^{1/p} \prod_{j=1}^m \|w_j^{-1}\|_{\Psi_j, B} < \infty.$$

Then  $\mathcal{M}_{\mathcal{B}}$  is bounded from  $L^{p_1}(w_1^{p_1}) \times \dots \times L^{p_m}(w_m^{p_m})$  to  $L^p(\nu^p)$ .

*Remark 3.2.* We observe that for all  $x \in \mathbf{R}^n$  and for all non-negative functions  $\vec{f} = (f_1, \dots, f_m)$ ,

$$\mathcal{M}_{\vec{\Psi}}^{\mathcal{B}}(\vec{f})(x) \leq \prod_{j=1}^m M_{\Psi_j}^{\mathcal{B}}(f_j)(x).$$

Thus, if we assume that each  $M_{\Psi_j}^{\mathcal{B}}$  is bounded on  $L^{p_j}(\mathbf{R}^n)$ , then  $\mathcal{M}_{\vec{\Psi}}^{\mathcal{B}}$  is bounded from  $L^{p_1}(\mathbf{R}^n) \times \dots \times L^{p_m}(\mathbf{R}^n)$  to  $L^p(\mathbf{R}^n)$ , and consequently, the conclusion of Theorem 3.1 gives us that  $\mathcal{M}_{\mathcal{B}}$  is bounded from  $L^{p_1}(w_1^{p_1}) \times \dots \times L^{p_m}(w_m^{p_m})$  to  $L^p(\nu^p)$  when  $(\nu, \vec{w})$  satisfies (3.1).

Between these general bases, we are particularly interested in Muckenhoupt basis introduced in [23]. We say that  $\mathcal{B}$  is a *Muckenhoupt basis* if for any  $1 < p < \infty$  and for any  $w \in A_{p,\mathcal{B}}$ ,  $M_{\mathcal{B}}$  is bounded in  $L^p(w)$ . Most of the important bases are in this class and, in particular, those mentioned above:  $\mathcal{Q}, \mathcal{D}, \mathcal{R}$ . The fact that  $\mathcal{R}$  is a Muckenhoupt basis can be found in [13]. The basis  $\mathcal{R}$  is also a Muckenhoupt basis as shown by R. Fefferman [11].

For Muckenhoupt bases, the generalization of the power bump condition (1.14) assures the boundedness of  $M_{\vec{\Psi}}^{\mathcal{B}}$ . Therefore we can deduce the following result.

**Corollary 3.3.** *Let  $\mathcal{B}$  be a Muckenhoupt basis. Let  $\frac{1}{m} < p < \infty$  and  $1 < p_1, \dots, p_m < \infty$  such that  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ . If the weights  $(\nu, \vec{w}) = (\nu, w_1, \dots, w_m)$  satisfy the power bump condition*

$$(3.2) \quad \sup_{B \in \mathcal{B}} \left( \frac{1}{|B|} \int_B \nu(x) dx \right) \prod_{j=1}^m \left( \frac{1}{|B|} \int_B w_j^{(1-p'_j)r} dx \right)^{\frac{p}{p'_j r}} < \infty$$

for some  $r > 1$  and  $\nu$  satisfies condition (A), then  $\mathcal{M}_{\mathcal{B}}$  is bounded from  $L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m)$  to  $L^p(\nu)$ .

*Proof.* For each  $j \in \{1, \dots, m\}$ , we set  $\tilde{w}_j := w_j^{1/p_j}$  and  $\Psi_j(t) := t^{p'_j r}$  for all  $t \in (0, \infty)$ . Set  $\tilde{\nu} := \nu^{1/p}$ . Then the power bump condition (3.2) can be rewritten as

$$\sup_{B \in \mathcal{B}} \left\{ \frac{1}{|B|} \int_B \tilde{\nu}^p dx \right\}^{1/p} \prod_{j=1}^m \|\tilde{w}_j^{-1}\|_{\Psi_j, B} < \infty.$$

In this case, for all  $x \in \mathbf{R}^n$ ,

$$M_{\tilde{\Psi}_j}^{\mathcal{B}} f(x) = \sup_{B \in \mathcal{B}, B \ni x} \|f\|_{\tilde{\Psi}_j, B} = \sup_{B \in \mathcal{B}, B \ni x} \left\{ \frac{1}{|B|} \int_B |f(y)|^{(p'_j r)'} dy \right\}^{1/(p'_j r)'}$$

Since  $\mathcal{B}$  is a Muckenhoupt basis and  $(p'_j r)' < p_j$ , every  $M_{\tilde{\Psi}_j}^{\mathcal{B}}$  is bounded on  $L^{p_j}(\mathbf{R}^n)$ . By Remark 3.2 this implies that  $\mathcal{M}_{\tilde{\Psi}}^{\mathcal{B}}$  is bounded from  $L^{p_1}(\mathbf{R}^n) \times \dots \times L^{p_m}(\mathbf{R}^n)$  to  $L^p(\mathbf{R}^n)$ . Thus, by Theorem 3.1

$$\mathcal{M}_{\mathcal{B}} : L^{p_1}(\tilde{w}_1^{p_1}) \times \dots \times L^{p_m}(\tilde{w}_m^{p_m}) \rightarrow L^p(\tilde{\nu}^p),$$

which completes the proof.  $\square$

A result stronger than Corollary 3.3 is the following boundedness of the multilinear strong maximal function, where  $(\nu, \vec{w})$  satisfy some logarithmic type condition.

**Corollary 3.4.** *Let  $1 < p_1, \dots, p_m < \infty$  and  $\frac{1}{m} < p < \infty$  such that  $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$ . Let  $(\nu, \vec{w}) = (\nu, w_1, \dots, w_m)$  such that  $\nu$  and all the  $w_j$ 's are weights, and  $\nu^p$  satisfies condition (A). If there is a positive constant  $K$  such that for all rectangles  $R$ ,*

$$\left( \frac{1}{|R|} \int_R \nu(x)^p dx \right)^{1/p} \prod_{j=1}^m \|w_j^{-1}\|_{\Psi_j, R} < \infty,$$

where every  $\Psi_j$  is a Young function such that  $\bar{\Psi}_j \in B_{p_j}^*$ . Then  $\mathcal{M}_{\mathcal{R}}$  is bounded from  $L^{p_1}(w_1^{p_1}) \times \dots \times L^{p_m}(w_m^{p_m})$  to  $L^p(\nu^p)$ .

*Proof.* From Theorem 2.1 and the assumption that each  $\bar{\Psi}_j$  is a Young function satisfying the condition (1.11), it follows that every  $M_{\bar{\Psi}_j}^{\mathcal{R}}$  is bounded on  $L^{p_j}(\mathbf{R}^n)$ . Then, applying Remark 3.2 and Theorem 3.1 with  $\mathcal{B} = \mathcal{R}$ , we obtain the desired conclusion.  $\square$

We end this section with the proof of Theorem 1.2 that is a straight consequence of Corollary 3.4.

*Proof of Theorem 1.2.* We notice first that (i) is the linear case ( $m = 1$ ) of Corollary 3.4. For the proof of (ii) we proceed as in [25, Proposition 3.2]. That is, consider any non-negative function  $g$  and define the couple of weights  $(u, v) = (M_{\Phi}^{\mathcal{R}}(g^{1/p})^{-1}, g^{-1/p})$ . Obviously,  $(u, v)$  satisfies condition (1.12) with constant  $K = 1$ . Hence, by hypothesis there is a constant  $C$  such that

$$\int_{\mathbf{R}^n} [M_{\mathcal{R}}(f)(y)]^p \frac{1}{[M_{\Phi}^{\mathcal{R}}(g^{1/p})(y)]^p} dy \leq C \int_{\mathbf{R}^n} f(y)^p \frac{1}{g(y)} dy.$$

Finally, by Theorem 2.1, this inequality implies that  $\Phi \in B_p^*$ , which completes the proof.  $\square$

4. PROOF OF THE STRONG TYPE ESTIMATE IN THE  $(m + 1)$ -WEIGHT CASE

To prove Theorem 3.1, we use an argument that combines ideas from [14, Theorem 2.5], the second proof of Theorem 3.7 in [19], and some other tools from [16] and [25]. First, we will recall an additional definition for general bases and a special case of a lemma from [16] that we will need for the proof of Theorem 3.1. The following definition concerns the concept of  $\alpha$ -scattered families, which was considered in the works [17] and [16] and implicitly in [2] and [3].

**Definition 4.1.** Let  $\mathcal{B}$  be a basis and  $0 < \alpha < 1$ . A finite sequence  $\{\tilde{A}_i\}_{i=1}^M \subset \mathcal{B}$  of sets of finite Lebesgue measure is called  $\alpha$ -scattered with respect to the Lebesgue measure if for all  $1 < i \leq M$ ,

$$\left| \tilde{A}_i \cap \bigcup_{s < i} \tilde{A}_s \right| \leq \alpha |\tilde{A}_i|.$$

The proof of the following lemma is in [16, p. 370, Lemma 1.5]; see also [14].

**Lemma 4.2.** Let  $\mathcal{B}$  be a basis and let  $w$  be a weight associated to this basis. Suppose further that  $w$  satisfies condition (A) for some  $0 < \lambda < 1$  and  $0 < c(\lambda) < \infty$ . Then given any finite sequence  $\{A_i\}_{i=1}^M$  of sets  $A_i \in \mathcal{B}$ ,

- (a) there exists a subsequence  $\{\tilde{A}_i\}_{i \in I}$  of  $\{A_i\}_{i=1}^M$  which is  $\lambda$ -scattered with respect to the Lebesgue measure;
- (b)  $\tilde{A}_i = A_i$ ,  $i \in I$ ;
- (c) for any  $1 \leq i < j \leq M + 1$ ,

$$w\left(\bigcup_{s < j} A_s\right) \leq c(\lambda) \left[ w\left(\bigcup_{s < i} A_s\right) + w\left(\bigcup_{i \leq s < j} \tilde{A}_s\right) \right],$$

where  $\tilde{A}_s = \emptyset$  when  $s \notin I$ .

*Proof of Theorem 3.1.* Let  $N > 0$  be a large integer. We will prove the required estimate for the quantity

$$\int_{2^{-N} < \mathcal{M}_{\mathcal{B}}(\vec{f}) \leq 2^{N+1}} \mathcal{M}_{\mathcal{B}}(\vec{f})(x)^p \nu(x)^p dx$$

with a bound independent of  $N$ . We claim that for each integer  $k$  with  $|k| \leq N$ , there exist a compact set  $K_k$  and a finite sequence  $b_k = \{B_{\alpha}^k\}_{\alpha \geq 1}$  of sets  $B_{\alpha}^k \in \mathcal{B}$  such that

$$\nu^p(K_k) \leq \nu^p(\{\mathcal{M}_{\mathcal{B}}(\vec{f}) > 2^k\}) \leq 2 \nu^p(K_k)$$

The sequence of sets  $\{\cup_{B \in b_k} B\}_{k=-N}^N$  is decreasing. Moreover,

$$\bigcup_{B \in b_k} B \subset K_k \subset \{\mathcal{M}_{\mathcal{B}}(\vec{f}) > 2^k\},$$

and

$$(4.1) \quad \prod_{j=1}^m \frac{1}{|B_{\alpha}^k|} \int_{B_{\alpha}^k} |f_j(y)| dy > 2^k, \quad \alpha \geq 1,$$

To see the claim, for each  $k$  we choose a compact set  $\tilde{K}_k \subset \{\mathcal{M}_{\mathcal{B}}(\vec{f}) > 2^k\}$  such that

$$\nu^p(\tilde{K}_k) \leq \nu^p(\{\mathcal{M}_{\mathcal{B}}(\vec{f}) > 2^k\}) \leq 2 \nu^p(\tilde{K}_k).$$

For this  $\tilde{K}_k$ , there exists a finite sequence  $b_k = \{B_\alpha^k\}_{\alpha \geq 1}$  of sets  $B_\alpha^k \in \mathcal{B}$  such that every  $B_\alpha^k$  satisfies (4.1) and such that  $\tilde{K}_k \subset \cup_{B \in b_k} B \subset \{\mathcal{M}_{\mathcal{B}}(\vec{f}) > 2^k\}$ . Now, we take a compact set  $K_k$  such that  $\cup_{B \in b_k} B \subset K_k \subset \{\mathcal{M}_{\mathcal{B}}(\vec{f}) > 2^k\}$ . Finally, to ensure that  $\{\cup_{B \in b_k} B\}_{k=-N}^N$  is decreasing, we begin the above selection from  $k = N$  and once a selection is done for  $k$  we do the selection for  $k - 1$  with the next additional requirement

$$\tilde{K}_{k-1} \supset K_k.$$

This proves the claim. Since  $\{\cup_{B \in b_k} B\}_{k=-N}^N$  is a sequence of decreasing sets, we set

$$\Omega_k = \begin{cases} \bigcup_{\alpha} B_{\alpha}^k = \bigcup_{B \in b_k} B & \text{when } |k| \leq N, \\ \emptyset & \text{when } |k| > N. \end{cases}$$

Observe that these sets are decreasing in  $k$ , i.e.,  $\Omega_{k+1} \subset \Omega_k$  when  $-N < k \leq N$ .

We now distribute the sets in  $\bigcup_k b_k$  over  $\mu$  sequences  $\{A_i(l)\}_{i \geq 1}$ ,  $0 \leq l \leq \mu - 1$ , where  $\mu$  will be chosen momentarily to be an appropriately large natural number. Set  $i_0(0) = 1$ . In the first  $i_1(0) - i_0(0)$  entries of  $\{A_i(0)\}_{i \geq 1}$ , i.e., for

$$i_0(0) \leq i < i_1(0),$$

we place the elements of the sequence  $b_N = \{B_{\alpha}^N\}_{\alpha \geq 1}$  in the order indicated by the index  $\alpha$ . For the next  $i_2(0) - i_1(0)$  entries of  $\{A_i(0)\}_{i \geq 1}$ , i.e., for

$$i_1(0) \leq i < i_2(0),$$

we place the elements of the sequence  $b_{N-\mu}$ . We continue in this way until we reach the first integer  $m_0$  such that  $N - m_0\mu \geq -N$ , when we stop. For indices  $i$  satisfying

$$i_{m_0}(0) \leq i < i_{m_0+1}(0),$$

we place in the sequence  $\{A_i(0)\}_{i \geq 1}$  the elements of  $b_{N-m_0\mu}$ . The sequences  $\{A_i(l)\}_{i \geq 1}$ ,  $1 \leq l \leq \mu - 1$ , are defined similarly, starting from  $b_{N-l}$  and using the families  $b_{N-l-s\mu}$ ,  $s = 0, 1, \dots, m_l$ , where  $m_l$  is chosen to be the biggest integer such that  $N - l - m_l\mu \geq -N$ .

Since  $\nu^p$  is a weight associated to  $\mathcal{B}$  and it satisfies condition (A), we can apply Lemma 4.2 to each  $\{A_i(l)\}_{i \geq 1}$  for some fixed  $0 < \lambda < 1$ . Then we obtain sequences

$$\{\tilde{A}_i(l)\}_{i \geq 1} \subset \{A_i(l)\}_{i \geq 1}, \quad 0 \leq l \leq \mu - 1,$$

which are  $\lambda$ -scattered with respect to the Lebesgue measure. In view of the definition of the set  $\Omega_k$  and the construction of the families  $\{A_i(l)\}_{i \geq 1}$ , we can use assertion (c) of Lemma 4.2 to obtain that for any  $k = N - l - s\mu$  with  $0 \leq l \leq \mu - 1$  and  $1 \leq s \leq m_l$ ,

$$\begin{aligned} \nu^p(\Omega_k) &= \mu^p(\Omega_{N-l-s\mu}) \leq c \left[ \nu^p(\Omega_{k+\mu}) + \nu^p \left( \bigcup_{i_s(l) \leq i < i_{s+1}(l)} \tilde{A}_i(l) \right) \right] \\ &\leq c \nu^p(\Omega_{k+\mu}) + c \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \nu^p(\tilde{A}_i(l)). \end{aligned}$$

For the case  $s = 0$ , we have  $k = N - l$  and

$$\nu^p(\Omega_k) = \nu^p(\Omega_{N-l}) \leq c \sum_{i=i_0(l)}^{i_1(l)-1} \nu^p(\tilde{A}_i(l)).$$



Now, all these sets  $\{\tilde{A}_i(l)\}_{i=i_s(l)}^{i_{s+1}(l)-1}$  belong to  $b_k$  with  $k = N - l - s\mu$  and therefore

$$(4.2) \quad \prod_{\alpha=1}^m \frac{1}{|\tilde{A}_i(l)|} \int_{\tilde{A}_i(l)} |f_j(x)| dx > 2^k.$$

It now readily follows that

$$\int_{2^{-N} < \mathcal{M}_{\mathcal{B}}(\vec{f}) \leq 2^{N+1}} \mathcal{M}_{\mathcal{B}}(\vec{f})(x)^p \nu^p(x) dx \leq 2^p \sum_{k=-N}^{N-1} 2^{kp} \nu^p(\Omega_k)$$

and then

$$(4.3) \quad \begin{aligned} \sum_{k=-N}^{N-1} 2^{kp} \nu^p(\Omega_k) &= \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} 2^{p(N-l-s\mu)} \nu^p(\Omega_{N-l-s\mu}) \\ &= c \sum_{\ell=0}^{\mu-1} \sum_{1 \leq s \leq m_\ell} 2^{p(N-l-s\mu)} \nu^p(\Omega_{N-l-s\mu+\mu}) \\ &\quad + c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} 2^{p(N-l-s\mu)} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \nu^p(\tilde{A}_i(l)). \end{aligned}$$

Observe that the first term in the last equality of (4.3) is equal to

$$c 2^{-p\mu} \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell-1} 2^{p(N-l-s\mu)} \nu^p(\Omega_{N-l-s\mu}) \leq c 2^{-p\mu} \sum_{k=-N}^{N-1} 2^{kp} \nu^p(\Omega_k).$$

If we choose  $\mu$  so large that  $c 2^{-p\mu} \leq \frac{1}{2}$  and since everything involved is finite the first term on the right hand side of (4.3) can be subtracted from the left hand side of (4.3). This yields

$$\int_{2^{-N} < \mathcal{M}_{\mathcal{R}}(\vec{f}) \leq 2^{N+1}} \mathcal{M}_{\mathcal{R}}(\vec{f})^p \nu dx \leq 2^{p+1} c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} 2^{p(N-l-s\mu)} \nu^p(\tilde{A}_i(l)).$$

By (4.2) and the generalized Hölder's inequality (2.2) we obtain

$$(4.4) \quad \begin{aligned} &\sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} 2^{p(N-l-s\mu)} \nu^p(\tilde{A}_i(l)) \\ &\leq c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \nu^p(\tilde{A}_i(l)) \left[ \prod_{j=1}^m \frac{1}{|\tilde{A}_i(l)|} \int_{\tilde{A}_i(l)} |f_j| dx \right]^p \\ &\leq c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \nu^p(\tilde{A}_i(l)) \left[ \prod_{j=1}^m \|f_j w_j\|_{\tilde{\Psi}_j, \tilde{A}_i(l)} \|w_j^{-1}\|_{\Psi_j, \tilde{A}_i(l)} \right]^p \\ &\leq c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \left[ \prod_{j=1}^m \|f_j w_j\|_{\tilde{\Psi}_j, \tilde{A}_i(l)} \right]^p |\tilde{A}_i(l)|, \end{aligned}$$

where in the last step we use the assumption (3.1).

For each  $l$  we let  $I(l)$  be the index set of  $\{\tilde{A}_i(l)\}_{0 \leq s \leq m_\ell, i_s(l) \leq i < i_{s+1}(l)}$ , and

$$E_1(l) = \tilde{A}_1(l) \quad \& \quad E_i(l) = \tilde{A}_i(l) \setminus \bigcup_{s < i} \tilde{A}_s(l) \quad \forall i \in I(l).$$

Since the sequences  $\{\tilde{A}_i(l)\}_{i \in I(l)}$  are  $\lambda$ -scattered with respect to the Lebesgue measure, for each  $i$   $|\tilde{A}_i(l)| \leq \frac{1}{1-\lambda}|E_i(l)|$ . Then we have the following estimate for (4.4)

$$(4.5) \quad \frac{C}{1-\lambda} c \sum_{l=0}^{\mu-1} \sum_{i \in I(l)} \left[ \prod_{j=1}^m \|f_j w_j\|_{\tilde{\Psi}_j, \tilde{A}_i(l)} \right]^p |E_i(l)|.$$

The collection  $\{E_i(l)\}_{i \in I(l)}$  is a disjoint family, we can therefore use the fact that  $\mathcal{M}_{\tilde{\Psi}, \mathcal{B}}$  is bounded from  $L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n)$  to  $L^p(\mathbf{R}^n)$  so as to estimate this last equation (4.5). Then

$$\begin{aligned} & \sum_{l=0}^{\mu-1} \sum_{i \in I(l)} \int_{E_i(l)} \left[ \mathcal{M}_{\tilde{\Psi}, \mathcal{B}}((f_1 w_1, \dots, f_m w_m))(x) \right]^p dx \\ & \leq c_\mu \int_{\mathbf{R}^n} \left[ \mathcal{M}_{\tilde{\Psi}, \mathcal{B}}((f_1 w_1, \dots, f_m w_m))(x) \right]^p dx \\ & \leq C \prod_{j=1}^m \|f_j w_j\|_{L^{p_j}(\mathbf{R}^n)}^p. \end{aligned}$$

Letting  $N \rightarrow \infty$  yields the desired assertion of the theorem.  $\square$

*Remark 4.3.* We point out that the fact that Theorem 2.1(ii) implies Theorem 2.1(iv) can be deduced by proceeding as in the proof of Theorem 3.1. Indeed, we will prove the required estimate for the quantity

$$\int_{2^{-N} < M_{\Phi}^{\mathcal{R}}(f) \leq 2^{N+1}} M_{\Phi}^{\mathcal{R}}(f)(x)^p w(x) dx \lesssim \int_{\mathbf{R}^n} f(y)^p M_{\mathcal{R}} w(y) dy,$$

where  $N$  is a large integer. We use the same covering argument of Theorem 3.1 replacing (4.1) by

$$\frac{1}{|R_\alpha^k|} \int_{R_\alpha^k} |f(y)| dy > 2^k.$$

Repeating equations (4.2) and (4.3), we will get

$$\begin{aligned} & \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} 2^{p(N-l-s\mu)} w(\tilde{A}_i(l)) \\ & \leq c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} w(\tilde{A}_i(l)) \|f\|_{\Phi, \tilde{A}_i(l)}^p \\ & \leq c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \left\| f \left( \frac{w(\tilde{A}_i(l))}{|\tilde{A}_i(l)|} \right)^{1/p} \right\|_{\Phi, \tilde{A}_i(l)}^p |\tilde{A}_i(l)| \\ & \leq c \sum_{\ell=0}^{\mu-1} \sum_{0 \leq s \leq m_\ell} \sum_{i=i_s(l)}^{i_{s+1}(l)-1} \left\| f(M_{\mathcal{R}} w)^{1/p} \right\|_{\Phi, \tilde{A}_i(l)}^p |\tilde{A}_i(l)|, \end{aligned}$$

where in the penultimate step we used the generalized Hölder's inequality (2.2). Finally, we will obtain the claimed conclusion using the fact that the operator  $M_{\Phi}^{\mathcal{R}}$  is bounded on  $L^p(\mathbf{R}^n)$ . The details are left to the reader.

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LIGUANG LIU, DEPARTMENT OF MATHEMATICS, SCHOOL OF INFORMATION, RENMIN UNIVERSITY OF CHINA, BEIJING 100872, P. R. CHINA  
*E-mail address:* liuliguang@ruc.edu.cn

TERESA LUQUE, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SEVILLA, 41080 SEVILLA, SPAIN.  
*E-mail address:* tluquem@us.es