Presentations for the monoids of singular braids on closed surfaces

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Abstract

We give presentations, in terms of generators and relations, for the monoids $SB_n(M)$ of singular braids on closed surfaces. The proof of the validity of these presentations can also be applied to verify, in a new way, the presentations given by Birman for the monoids of Singular Artin braids.

1 Introduction

In this paper we deal with the braid groups of a closed surface M. These groups are a natural generalization of Artin braid groups [A] and of the fundamental group of M. They are also subgroups of some *Mapping Class groups* of M, and finally they are fundamental groups of the so called *Configuration spaces* of M (see [B] for a general exposition).

They can be defined as follows. Fix n ($n \geq 1$) distinct points $\{P_1, \ldots, P_n\} \in M$. A *$n$-braid* on M is an n-tuple $b = (b_1, \ldots, b_n)$ of disjoint smooth paths b_i in $M \times [0, 1]$, such that for all i, the path b_i runs, monotonically on $t \in [0, 1]$, from $(P_i, 0)$ to some $(P_j, 1)$. These n-braids are considered modulo *isotopy* (deformation of braids fixing the ends), and there exists a multiplication of braids, given by concatenation of paths. The set of isotopy classes of n-braids on M, along with this multiplication, forms the *braid group with n strings on M*, denoted by $B_n(M)$.

The following is a simple presentation of $B_n(M)$, in terms of generators and relations, where M is a closed, orientable surface of genus g [G-M]:

- Generators: $\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_{2g}$.
- Relations:

$$
\begin{array}{lll}
\text{(R1)} & \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| \geq 2) \\
\text{(R2)} & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & (1 \leq i \leq n-2) \\
\text{(R3)} & a_1 \cdots a_{2g} a_1^{-1} \cdots a_{2g}^{-1} = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 & \\
\text{(R4)} & a_r A_{2,s} = A_{2,s} a_r & (1 \leq r, s \leq 2g;\;\; r \neq s) \\
\text{(R5)} & (a_1 \cdots a_r)\, A_{2,r} = \sigma_1^2 A_{2,r}\, (a_1 \cdots a_r) & (1 \leq r \leq 2g) \\
\text{(R6)} & a_r\sigma_i = \sigma_i a_r & (1 \leq r \leq 2g;\;\; i \geq 2)
\end{array}
$$

where

$$A_{2,r} = \sigma_1^{-1}\left(a_1 \cdots a_{r-1} a_{r+1}^{-1} \cdots a_{2g}^{-1}\right)\sigma_1^{-1}.$$

The generators are represented in Figure 1, where we have drawn the the canonical projections on M of the considered braids, and M is represented as a polygon of $4g$ sides, pairwise identified.

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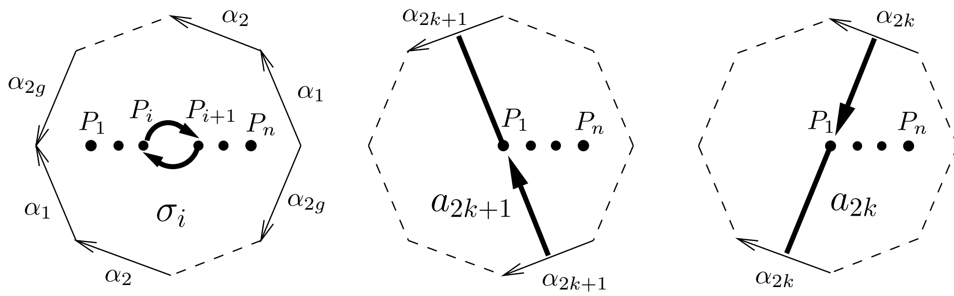


Figure 1: The generators of $B_n(M)$.

We can also find in [G-M] a similar presentation, when M is a non-orientable, closed surface.

In the same way that singular Artin braids were defined (see [B2]) to study Vassiliev invariants for these braids, we can define *singular braids* on M. Their definition is the same that the one of non-singular braids, but this time we allow a finite number of *singular points* (transverse intersection of two strings). The isotopy classes of these singular braids, with the analogous multiplication, form the *monoid of singular braids with n strings on M*, denoted by $SB_n(M)$. This monoid is used in [G-MP] to define the Vassiliev invariants of braids on closed, orientable surfaces, proving, among other results, that these invariants classify these braids.

In [B2] we can find presentations for SB_n, the monoids of singular Artin braids, in terms of generators and relations. The main result of this paper is to give presentations for $SB_n(M)$. We will see, as well, that the proof of this results furnishes a new proof of the validity of the presentations in [B2].

2 Presentation of $SB_n(M)$

We shall now give a presentation of $B_n(M)$, when M is a closed, orientable surface of genus $g \geq 0$. The non-orientable case is completely analogous, and is treated in a final remark at the end of this paper. We define, for all $i = 1, \ldots, n-1$, the singular braid τ_i as in Figure 2, where the only non-trivial strings are the i-th and the $(i+1)$-th ones, which intersect to form a singular point. The result is the following:

Theorem 2.1. *The monoid $SB_n(M)$ admits the following presentation:*

- *Generators:* $\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_{2g}, \tau_1, \ldots, \tau_{n-1}$.
- *Relations:*

(R1-R6) Relations of $B_n(M)$

(R7) $\sigma_i \tau_j = \tau_j \sigma_i \qquad (|i-j| \geq 2)$

(R8) $\tau_i \tau_j = \tau_j \tau_i \qquad (|i-j| \geq 2)$

(R9) $\sigma_i \tau_i = \tau_i \sigma_i \qquad (i = 1, \ldots, n-1)$

(R10) $\sigma_i \sigma_j \tau_i = \tau_j \sigma_i \sigma_j \qquad (|i-j| = 1)$

(R11) $(a_{i,r} a_{i+1,r}) \tau_i (a_{i+1,r}^{-1} a_{i,r}^{-1}) = \tau_i \qquad (i = 1, \ldots, n-1;\ r = 1, \ldots, 2g)$

(R12) $\tau_i a_{j,r} = a_{j,r} \tau_i \qquad (j \neq i, i+1;\ r = 1, \ldots, 2g)$

where

$$a_{i,r} = \begin{cases} (\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}) a_r (\sigma_1^{-1} \cdots \sigma_{i-1}^{-1}) & \textit{if } r \textit{ is odd,} \\ (\sigma_{i-1} \cdots \sigma_1) a_r (\sigma_1 \cdots \sigma_{i-1}) & \textit{if } r \textit{ is even.} \end{cases}$$

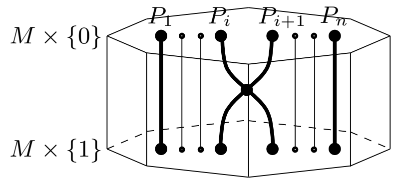


Figure 2: The singular braid τ_i.

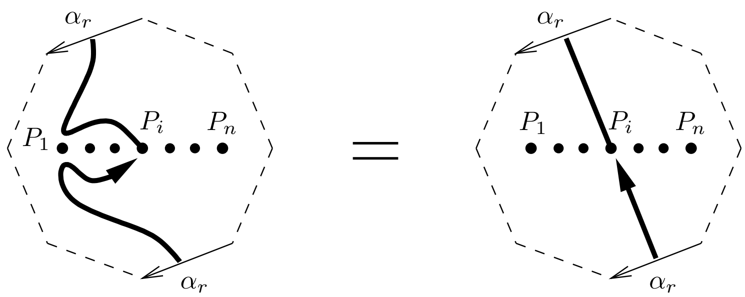


Figure 3: The braid $a_{i,r} = (\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}) a_r (\sigma_1^{-1} \cdots \sigma_{i-1}^{-1})$, when r is odd.

Remark that $a_{i,r}$ can be thought of as the i-th string crossing the "wall" α_r, as we can see in Figure 3 for the case when r is odd.

Proof of Theorem 2.1: First, it is evident that $\{\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_{2g}, \tau_1, \ldots, \tau_{n-1}\}$ is a set of generators of $SB_n(M)$, once that we know (by [G-M]) that $\{\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_{2g}\}$ generates $B_n(M)$.

It is also easy to prove that the proposed relations hold: (R1-R6) hold in $B_n(M)$, which is a sub-monoid of $SB_n(M)$. (R7-R10) are known to hold in SB_n, so they hold in a cylinder $D \times [0,1]$, where D is a disk containing the n points $P_1, \ldots, P_n$. We have just to extend the corresponding isotopy to all $M \times [0,1]$ by the identity. (R11) can be seen to hold in Figure 4, and finally (R12) is clear, since the only nontrivial strings of τ_i and $a_{j,r}$ can be isotoped to have disjoint projections on M, so these braids commute.

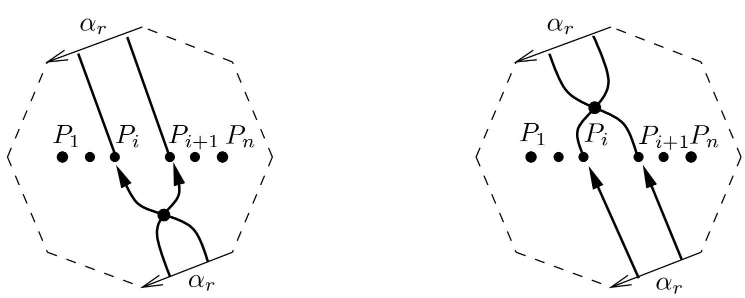


Figure 4: The braids $a_{i,r} a_{i+1,r} \tau_i$ and $\tau_i a_{i,r} a_{i+1,r}$ are isotopic, when r is odd.

In order to show that the relations are sufficient, we need the following lemma:

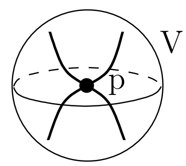
Lemma 2.2. *The monoid $SB_n(M)$ is left-cancelative. That is, for all $a, b, c \in SB_n(M)$, one has: $c\,a = c\,b \Rightarrow a = b$.*

Proof: Since $\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_{2g}$ are invertible, for they belong to $B_n(M)$, we just need to prove that $\tau_i a = \tau_i b \Rightarrow a = b$ for all $a, b \in SB_n(M)$ and all $i = 1, \ldots, n-1$.

Thus, let us suppose that there exists an isotopy H_t of $M \times [0,1]$, such that $H_0 = \mathrm{id}_{M\times[0,1]}$

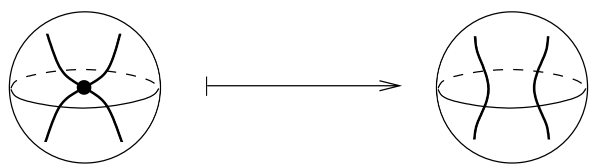
and $H_1(\tau_i a) = \tau_i b$. Call p the first singular point of $\tau_i a$ (the one corresponding to τ_i), and let $p_t = H_t(p)$. One has $p_0 = p_1 = p$.

Let V be the interior of a sphere of radius ε centered at p. We take ε small enough, such that $V \cap (\tau_i a)$ is as follows:



Denote $s_t = H_t(\tau_i a)$ and $V_t = H_t(V)$. We can suppose, without loss of generality, that V_t is the interior of the sphere of radius ε centered at p_t, and that $V_t \cup s_t$ is as in the above picture.

Now, for $t \in [0,1]$, denote by $\widetilde{s}_t$ the braid which is obtained by modifying s_t, only inside V_t, as follows:



We observe that $\widetilde{s}_0 = a$ and $\widetilde{s}_1 = b$, so H_t is an isotopy which transforms a into b. Therefore, $a = b$. $\square$

Let us then show that Relations (R1-R12) are sufficient. Let $b, b' \in SB_n(M)$ be two isotopic singular braids, written in the generators of Theorem 2.1. We must show that we can transform b into b' by using Relations (R1-R12).

Being isotopic, both braids have the same number of singular points, say k. If $k = 0$, the result follows from [G-M], since (R1-R6) are sufficient relations for $B_n(M)$.

Suppose that $k > 0$, and the result holds for braids with less than k singular points. We can assume that the first letter of b is τ_i, for some i (otherwise we can multiply b and b' on the left by the greatest nonsingular "prefix" of b). We will show that, using (R1-R12), we can transform b' into a braid whose first letter is τ_i. The result then follows from Lemma 2.2, and by induction hypothesis.

Let p be the point of b' corresponding (via isotopy) to the first singular point of b. This point p must correspond to some τ_j, letter of b'. By (R10) and the braid relations (R1-R2), we can easily deduce the following:

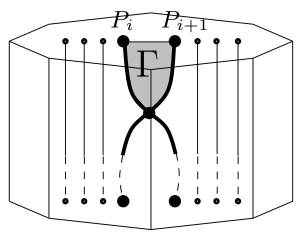
$$\tau_j = \begin{cases} (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_i)(\sigma_j\sigma_{j-1}\cdots\sigma_{i+1})\tau_i(\sigma_{i+1}^{-1}\cdots\sigma_j^{-1})(\sigma_i^{-1}\cdots\sigma_{j+1}^{-1}) & \text{if } i < j, \\ (\sigma_{j+1}\sigma_{j+2}\cdots\sigma_i)(\sigma_j\sigma_{j+1}\cdots\sigma_{i-1})\tau_i(\sigma_{i-1}^{-1}\cdots\sigma_j^{-1})(\sigma_i^{-1}\cdots\sigma_{j+1}^{-1}) & \text{if } i > j. \end{cases}$$

Hence, we can assume that the letter corresponding to p is τ_i.

Let us then write $b' = u\,\tau_i\,v$, where $u, v \in SB_n(M)$ and τ_i is the above letter. Since b is isotopic to b', we can assume, up to replacing τ_i by $\sigma_i\tau_i\sigma_i^{-1}$ (using (R9)), that the i-th string of u ends at the point (P_i, s), for some $s \in [0,1]$. Hence, its canonical projection on M is a loop in M based at P_i, which induces an element $\mu \in \pi_1(M, P_1)$. This element can be modified as desired: it suffices to use (R11), replacing τ_i by $a_{i,r}^{\varepsilon} a_{i+1,r}^{\varepsilon} \tau_i\, a_{i+1,r}^{-\varepsilon} a_{i,r}^{-\varepsilon}$ ($\varepsilon = \pm 1$), to have μ transformed into $\mu\,\overline{a}_{i,r}^{\varepsilon}$, where $\overline{a}_{i,r}^{\varepsilon}$ is the projection on M of the i-th string of $a_{i,r}^{\varepsilon}$. Since $\{\overline{a}_{i,1}, \ldots, \overline{a}_{i,2g}\}$ is a set of generators of $\pi_1(M, P_i)$, we can assume that $\mu = 1$.

Now notice that u has less than k singular points, hence any braid isotopic to u can be obtained from it by applying (R1-R12), by induction hypothesis. We can then deform u in such a way that its i-th string will not go through the "walls" $\alpha_1, \ldots, \alpha_{2g}$ (we can do this since $\mu = 1$).

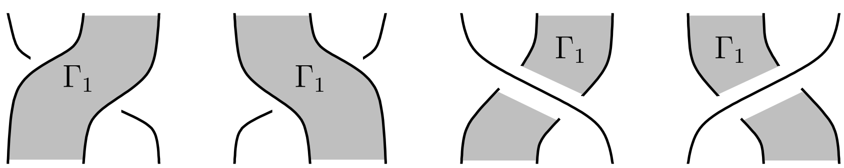
Let us go back to b, and consider a "band" Γ, determined by the i-th and the $(i+1)$-th strings of b, and which goes from $s = 0$ to the first singular point of b, as in the figure below.



b

Consider also an isotopy H_t which transforms b into b'. Recall that $b' = u\,\tau_i v$, where the i-th string of u does not go through the walls. We can now consider $\Gamma_1 = H_1(\Gamma)$, and deform the $(i+1)$-th string of u along this band, in such a way that it will be as close to the i-th string as desired (recall that we are allowed to deform u). We can then assume that neither the i-th nor the $(i+1)$-th string of u goes through the "walls" of the cylinder $M \times [0,1]$. Moreover, using (R9) we can modify the number of crossings of these two strings, as desired (just replacing τ_i by $\sigma_i^r \tau_i \sigma_i^{-r}$, $r \in \mathbb{Z}$). Therefore, we can assume that they do not cross, i.e. there is no σ_j in u involving the $i-th$ and the $(i+1)$-th strings.

We can also assume that these two strings are so close that one has the following property: if there is a letter σ_j^ε ($\varepsilon = \pm 1$) of u which involves the i-th or the $(i+1)$-th string, then this letter, together with either the previous or the following one, forms a sub-word of u of one of the following four types:



$\sigma_j \sigma_{j+1}$ $\qquad$ $\sigma_{j+1}^{-1} \sigma_j^{-1}$ $\qquad$ $\sigma_j^{-1} \sigma_{j+1}^{-1}$ $\qquad$ $\sigma_{j+1} \sigma_j$

But in this case it is easy to see that, using relations (R7), (R8), (R10) and (R12), we can "raise" the point p, until we get τ_i as the first letter of b'. So by Lemma 2.2 we can cancel τ_i, and by induction hypothesis the resulting braids are equivalent by means of Relations (R1-R12). This ends the proof of Theorem 2.1 $\square$

Remark 2.3. There is an analogous presentation of $SB_n(M)$, when M is a non-orientable, closed surface. We just need to consider the presentation given in [G-M] for $B_n(M)$. Then replace, in the presentation of Theorem 2.1, the generators $a_1, \ldots, a_{2g}$ by the corresponding generators on the non-orientable surface, and Relations (R1-R6) by the relations given in [G-M]. The same proof remains valid.

Remark 2.4. The presentation given in Theorem 2.1 can be easily simplified. It suffices to eliminate the generators $\tau_2, \ldots, \tau_{n-1}$, replacing in the relations τ_3 by $(\sigma_2\sigma_1\sigma_3\sigma_2)\tau_1(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})$, and eliminating all relations containing some τ_j ($j \neq 1, 3$), since they are obtained from the remaining ones. We proposed the presentation above since it is more useful for handling singular braids.

Remark 2.5. We can also replace (R1-R6) by any other set of sufficient relations for the given generators of $B_n(M)$.

Remark 2.6. The above proof of Theorem 2.1, after eliminating every allusion to $\pi_1(M)$, is a new proof of the validity of the presentation for SB_n proposed in [B2].

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