A note on an ergodic theorem in weakly uniformly convex geodesic spaces

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Abstract

Karlsson and Margulis [10] proved in the setting of uniformly convex geodesic spaces, which additionally satisfy a nonpositive curvature condition, an ergodic theorem that focuses on the asymptotic behavior of integrable cocycles of nonexpansive mappings over an ergodic measure-preserving transformation. In this note we show that this result holds true when assuming a weaker notion of uniform convexity.

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1 Introduction

One recent research direction in ergodic theory consists in generalizing classical ergodic theorems to the setting of geodesic spaces with a sufficiently rich geometry. Let Y be a geodesic space and $D \subseteq Y$. Consider S a semigroup of nonexpansive (i.e. 1-Lipschitz) self-mappings defined on D and endow S with the Borel σ-algebra induced by the compact-open topology on S. Assume that (X, μ) is a probability measure space, $T : X \to X$ is an ergodic measure-preserving transformation and $w : X \to S$ is a measurable map. Define the cocycle

$$a_n(x) = w(x)w(Tx)\cdots w(T^{n-1}x).$$

Fix $y \in D$ and suppose that $\int_X d(y, a_1(x)y)d\mu(x) < \infty$. One can easily see that the sequence $\left(\int_X d(y, a_n(x)y)d\mu(x)\right)$ is subadditive and so, by Fekete's subadditive lemma [4], the following limit exists

$$0 \le A = \lim_{n\to\infty} \frac{1}{n}\int_X d(y, a_n(x)y)d\mu(x) = \inf_n \frac{1}{n}\int_X d(y, a_n(x)y)d\mu(x) < \infty.$$

As an immediate application of Kingman's subadditive ergodic theorem [12] one gets that for almost every $x \in X$, $\lim_{n\to\infty} \frac{1}{n} d(y, a_n(x)y) = A$.

Karlsson and Margulis [10] proved that if Y is a complete Busemann convex geodesic space satisfying a uniform convexity condition (see Section 2, (i) for the precise definition), the following holds: if $A > 0$, then for almost every $x \in X$, there exists a unique geodesic ray γ in Y starting at y and depending on x such that $\lim_{n\to\infty} \frac{1}{n} d(\gamma(An), a_n(x)y) = 0$. Thus, instead of the convergence of averages as in classical ergodic results, one basically obtains for almost every $x \in X$ the existence of a geodesic ray that issues at y such that, as $n \to \infty$, the values of the cocycles $a_n(x)$ at y are 'close' to this geodesic ray, a property which is referred to in [9] as ray approximation.

This result generalizes the multiplicative ergodic theorem of Oseledec [18] (see also [10, 9, 8]). A discussion on the asymptotic behavior of ergodic products of nonexpansive mappings and isometries of a (proper) metric space and applications thereof can be found in [9]. In a related line, ergodic-theoretic results in CAT(0) spaces were proved by Es-Sahib and Heinich [3] and Sturm [20] using barycenter techniques. Considering a different notion of barycenter map, Austin [1] proved an extension of the pointwise ergodic theorem for mappings with values in complete separable CAT(0) spaces. Following a similar proof strategy, but defining another suitable notion of barycenter (that actually recovers the one from [3]), Navas [16] gave an ergodic theorem for mappings taking values in Busemann spaces.

In this paper we show that the ergodic theorem given in [10] holds true for a more general class of geodesic spaces, by generalizing its proof to these spaces. More precisely, we prove that we can relax the uniform convexity assumption on Y used in [10] as follows: Y is said to be *weakly uniformly convex* if for any $a \in Y$, $r > 0$ and $\varepsilon \in (0, 2]$,

$$\delta_Y(a, r, \varepsilon) = \inf\{1 - d(a, m(x, y))/r : d(a, x), d(a, y) \le r, d(x, y) \ge \varepsilon r\} > 0,$$

where $m(x, y)$ is a midpoint of a geodesic segment from x to y. We refer to the mapping δ_Y as the *the modulus of convexity* of Y. This notion was used by Reich and Shafrir [19] in the setting of hyperbolic spaces. In addition, we assume that for every $a \in Y$ and $\varepsilon > 0$ there exists $s > 0$ such that

$$\inf_{r\ge s} \delta_Y(a, r, \varepsilon) > 0. \tag{1}$$

The main result of the paper is the following.

Theorem 1.1. *Assume that Y is a complete Busemann convex geodesic space that is weakly uniformly convex and satisfies (1). If $A > 0$, then for almost every $x \in X$, there exists a unique geodesic ray γ in Y that issues at y and depends on x such that* $\lim_{n\to\infty} \frac{1}{n} d(\gamma(An), a_n(x)y) = 0$.

In Section 2 we recall different notions of uniform convexity used in nonlinear settings, all of them fitting within the class of weakly uniform convex geodesic spaces defined as above and satisfying (1). We also discuss a convexity condition that we actually use in the proof and provide an example of a weakly uniformly convex geodesic space satisfying (1) which is not uniformly convex in the sense of [10]. Section 3 contains the proof of our main result.

2 Weakly uniformly convex geodesic spaces

Let (Y, d) be a metric space. Y admits midpoints if for every $x, y \in Y$ there exists a point $m(x, y) \in Y$ (called a *midpoint* of x and y) such that $d(x, m(x, y)) = d(y, m(x, y)) = d(x, y)/2$. A *geodesic path* (resp. *geodesic ray*) in Y is a distance-preserving map $\gamma : [0, l] \subseteq \mathbb{R} \to Y$ (resp. $\gamma : [0, \infty) \to Y$). If γ is a geodesic path such that $\gamma(0) = x$, $\gamma(l) = y$, we say that γ joins x, y and the image of γ is a *geodesic segment* from x to y, denoted by $[x, y]$ if there is no confusion. Y is a *(uniquely) geodesic space* if every two points in Y are joined by a (unique) geodesic path. Any complete metric space that admits midpoints is a geodesic space. A point $z \in Y$ belongs to a geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$ and we write $z = (1-t)x+ty$. For $t = 1/2$ we get a midpoint $m(x, y)$. See, for instance, [2] for details on geodesic spaces.

Following [14], we can define weak uniform convexity in an alternative way: a geodesic space Y is weakly uniformly convex if there exists a mapping $\eta : Y \times (0, \infty) \times (0, 2] \to (0, 1]$ such that for any $a \in Y, r > 0, \varepsilon \in (0, 2]$, every $x, y \in Y$ and all geodesic segments $[x, y]$ we have that,

$$\left.\begin{array}{l} d(a,x) \le r \\ d(a,y) \le r \\ d(x,y) \ge \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d\left(a, m(x,y)\right) \le (1 - \eta(a, r, \varepsilon))r. \tag{2}$$

Such a mapping η is referred to as *a modulus of weak uniform convexity*. This definition is equivalent to the previous one, with the modulus of convexity δ_Y giving the largest modulus of weak uniform convexity. A modulus η is said to be *monotone* if it is nonincreasing in the second argument.

A weakly uniformly convex geodesic space Y is strictly convex (i.e., for all $a, x, y \in Y$ with $x \neq y$, $d\left(a, m(x, y)\right) < \max\{d(a, x), d(a, y)\}$), hence uniquely geodesic. Likewise, if $d(a, x), d(a, y) \le r$, then $d(a, p) \le r$ for all $p \in [x, y]$.

If one can find a modulus $\eta = \eta(r, \varepsilon)$ that does not depend on a, we get the stronger notion of *uniform convexity*, first considered in a nonlinear setting in [7, 6], and η is called a *modulus of uniform convexity*.

We include next some particular notions of (weak) uniform convexity:

(i) Karlsson and Margulis [10] define uniform convexity in a metric space Y that admits midpoints as follows: Y is said to be uniformly convex if there exists a strictly decreasing and continuous function $g : [0,1] \to [0,1]$ with $g(0) = 1$, so that for any $a, x, y \in Y$ and any midpoint $m(x, y)$,

$$\frac{d(a, m(x,y))}{M} \leq g\left(\frac{d(x,y)}{2M}\right), \quad \text{where } M = \max\{d(x,a), d(y,a)\}. \tag{3}$$

Any geodesic space satisfying the above condition is uniformly convex in our sense. Indeed, given $r > 0$, $\varepsilon \in (0, 2]$, let $a, x, y \in Y$ satisfy $d(x, a) \leq r, d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$. Then $d(a, m(x, y)) \leq g(d(x, y)/(2M))M \leq g(\varepsilon/2)r$, so $\eta(\varepsilon) = 1 - g(\varepsilon/2) > 0$ is a modulus of uniform convexity that only depends on ε.

(ii) Gelander, Karlsson and Margulis [5] consider a strictly convex geodesic space Y to be uniformly convex if it is weakly uniformly convex and for all $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $\delta_Y(a, r, \varepsilon) \geq \eta(\varepsilon)$ for all $r > 0, a \in Y$.

(iii) In [11], a metric space Y that admits midpoints is called uniformly p-convex (where $p \in [1, \infty]$) if for every $\varepsilon > 0$ there exists $\rho_p(\varepsilon) \in (0, 1)$ such that for all $a, x, y \in Y$ with $d(x, y) > \varepsilon\mathcal{M}^p(d(a, x), d(a, y))$ for $p > 1$ and $d(x, y) > |d(a, x) - d(a, y)| + \varepsilon\mathcal{M}^1(d(a, x), d(a, y))$ for $p = 1$ we have that

$$d(a, m(x, y)) \leq (1 - \rho_p(\varepsilon))\mathcal{M}^p(d(a, x), d(a, y)), \tag{4}$$

where $\mathcal{M}^p(\alpha, \beta) = (\alpha^p/2 + \beta^p/2)^{1/p}$ and $\mathcal{M}^\infty(\alpha, \beta) = \max\{\alpha, \beta\}$.
By [11, Lemma 1.4], any uniformly p-convex space is uniformly ∞-convex which, in turn, is uniformly convex in our sense. Let $r > 0$, $\varepsilon \in (0, 2]$ and $a, x, y \in Y$ with $d(x, a) \leq r, d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$. Then $d(x, y) > (\varepsilon M)/2$, where M is as in (3). This yields $d(a, m(x, y)) \leq (1 - \rho_\infty(\varepsilon/2))M \leq (1 - \rho_\infty(\varepsilon/2))r$. Hence, Y has a modulus of uniform convexity $\eta(\varepsilon) = \rho_\infty(\varepsilon/2) > 0$ that only depends on ε.

Geodesic spaces which are p-uniformly convex in the sense of Naor and Silberman [15] (see also [13, 17]) satisfy all three definitions (i)-(iii). These spaces are defined as follows: for a fixed $1 < p < \infty$, a geodesic space Y is called *p-uniformly convex* with parameter $k > 0$ if for every $a, x, y \in Y$ and $t \in [0, 1]$,

$$d(a, (1-t)x + ty)^p \leq (1-t)d(a, x)^p + td(a, y)^p - \frac{k}{2}t(1-t)d(x, y)^p.$$

We see below that p-uniformly convex spaces indeed satisfy (i)-(iii). Remark first that, by [13, Proposition 2.5, (1)], $k \leq c_p$ with

$$c_p = \begin{cases} 2(p-1) & \text{if } p \in (1, 2) \\ 8/2^p & \text{if } p \geq 2. \end{cases}$$

Let $a \in Y$, $r > 0$, $\varepsilon \in (0, 2]$. Take $x, y \in Y$ such that $d(a, x) \leq r$, $d(a, y) \leq r$ and $d(x, y) \geq \varepsilon r$. Then $1 - d(a, m(x, y))/r \geq 1 - (1 - k\varepsilon^p/8)^{1/p}$. This implies (ii). To see that (i) is satisfied, consider $a, x, y \in Y$ and M as in (3). Then $d(a, m(x, y))/M \leq (1 - (k/8)(d(x, y)/M)^p)^{1/p}$. For $t \in [0, 1]$, take $g(t) = (1-k(2t)^p/8)^{1/p}$. Then (i) holds. Let $\varepsilon > 0$ and $\rho_p(\varepsilon) = 1-(1-k\varepsilon^p/8)^{1/p}$. One gets that (4) holds, hence (iii) is satisfied (see also [11, Example, p. 361]).

For the rest of this paper we assume that any weakly uniformly convex geodesic space also satisfies condition (1). From the above arguments it is easy to see that all uniformly convex spaces described in (i)-(iii) satisfy (1). Although we state our main result in the setting of weakly uniformly convex geodesic spaces that additionally satisfy a nonpositive curvature assumption, we remark that what we actually use in the proof is the following convexity condition which holds in any weakly uniformly convex geodesic space.

2.1 A convexity assumption

We say that a geodesic space Y has *property* (C) if there exists a mapping $\Psi : Y \times (0, \infty) \times (0, 2] \to (0, 1]$ satisfying:

(C1) for all $y \in Y, r > 0, \varepsilon \in (0, 2]$, every $x, z \in Y$ with $d(x, y) = r$, $d(y, z) \geq r$, if w belongs to a geodesic segment $[y, z]$ and $d(y, w) = r$, then

$$r + d(x, z) \leq d(y, z) + \Psi(y, r, \varepsilon)r \quad \text{implies} \quad d(w, x) \leq \varepsilon r.$$

(C2) for all $y \in Y, \varepsilon \in (0, 2]$, there exists $s > 0$ such that $\inf_{r \geq s} \Psi(y, r, \varepsilon) > 0$.

Lemma 2.1. *Any weakly uniformly convex geodesic space* Y *has property* (C) *with* $\Psi(y, r, \varepsilon) = \delta_Y(y, r, \varepsilon)$ *for all* $y \in Y$, $r > 0$ *and* $\varepsilon \in (0, 2]$.

Proof. Define $\Psi := \delta_Y$ and let $y, r, \varepsilon, x, z, w$ as in (C1). We may assume that $w \neq x$. By hypothesis, $d(x, z) \leq d(y, z) - r + \delta_Y(y, r, \varepsilon)r = d(z, w) + \delta_Y(y, r, \varepsilon)r$. Denote $m(w, x)$ by m. Then $d(m, z) < \max\{d(z, x), d(z, w)\} \leq d(z, w) + \delta_Y(y, r, \varepsilon)r$, hence $d(y, z) - d(y, m) < d(z, w) + \delta_Y(y, r, \varepsilon)r$, from where $d(y, m) > d(y, w) - \delta_Y(y, r, \varepsilon)r = (1 - \delta_Y(y, r, \varepsilon))r$. By weak uniform convexity, $d(y, m) \leq (1 - \delta_Y(y, r, d(x, w)/r))\, r$, hence $\delta_Y(y, r, d(x, w)/r) < \delta_Y(y, r, \varepsilon)$. As δ_Y is nondecreasing in the third argument, we get $d(x, w) \leq \varepsilon r$.
(C2) follows immediately from (1). ∎

2.2 An example

In the sequel we give an example of a weakly uniformly convex space satisfying (1) which is not uniformly convex in the sense of Karlsson and Margulis. We refer to [2] for all the undefined notions.

In the construction of our example, we glue spaces through points. Assume that (X, d_X) and (Y, d_Y) are geodesic spaces, $\theta \in X$ and $\tau \in Y$. Let Z be the quotient of the disjoint union $X \sqcup Y$ by the equivalence relation generated by $[\theta \sim \tau]$. We identify X and Y with their images in Z, write $Z = X \sqcup_\theta Y$ and refer to it as the *gluing* of X and Y obtained by identifying the points θ, τ. This is a geodesic metric space with the gluing metric

$$d(x,y) = \begin{cases} d_X(x,y), & \text{if } x, y \in X \\ d_Y(x,y), & \text{if } x, y \in Y \\ d_X(x,\theta) + d_Y(\tau, y), & \text{if } x \in X, y \in Y. \end{cases}$$

Lemma 2.2. *Assume that (X, d_X) and (Y, d_Y) are weakly uniformly convex geodesic spaces with monotone moduli η_X and η_Y, respectively. For every $\theta \in X$, $\tau \in Y$, $X \sqcup_\theta Y$ is weakly uniformly convex with a modulus*

$$\eta(a,r,\varepsilon) = \begin{cases} \min\{\eta_X(a,r,\varepsilon), \eta_Y(\tau,r,\varepsilon)\varepsilon/2, \varepsilon/4, \eta_X(a,r,\varepsilon/4)\}, & \text{if } a \in X \\ \min\{\eta_Y(a,r,\varepsilon), \eta_X(\theta,r,\varepsilon)\varepsilon/2, \varepsilon/4, \eta_Y(a,r,\varepsilon/4)\}, & \text{if } a \in Y. \end{cases}$$

Proof. Let $a \in X \sqcup_\theta Y$, $\varepsilon \in (0,2]$, $r > 0$ and $x, y \in X \sqcup_\theta Y$ be such that $d(a,x), d(a,y) \le r$ and $d(x,y) \ge \varepsilon r$. Let us denote $m(x,y)$ with m, for simplicity. We assume that $a \in X$, the case $a \in Y$ being similar. We distinguish the following cases:

(i) $x, y \in X$. Then, since η_X is a modulus of weak uniform convexity for X, $d(a,m) \le (1 - \eta_X(a,r,\varepsilon))r$.

(ii) $x, y \in Y$. Let $M := \max\{d(\theta,x), d(\theta,y)\}$. One can easily see that $\varepsilon r/2 \le M \le r$ and that $d(a,\theta) + M \le r$. By the weak uniform convexity of Y, $d(\theta,m) = d_Y(\tau,m) \le (1 - \eta_Y(\tau,r,\varepsilon))M$. It follows then that $d(a,m) \le (1 - \eta_Y(\tau,r,\varepsilon)\varepsilon/2)r$.

(iii) $x \in X$, $y \in Y$. If $m \in Y$, then $m \in [\theta, y]$, so $d(\theta,y) = d(\theta,m) + d(m,y)$. It follows that $d(a,m) = d(a,\theta) + d(\theta,m) = d(a,y) - d(m,y) \le (1 - \varepsilon/2)r$. If $m \in X$, so $m \in [x,\theta]$, we have two cases. If $d(m,\theta) < \varepsilon r/8$, then $d(\theta,y) > 3\varepsilon r/8$, so $d(a,\theta) = d(a,y) - d(\theta,y) \le (1 - 3\varepsilon/8)r$. Thus, $d(a,m) \le (1 - \varepsilon/4)r$. Suppose now that $d(m,\theta) \ge \varepsilon r/8$ and let $p \in [x,m]$ be such that m is the midpoint of the segment $[p,\theta]$. Then $d(a,p) \le r$ and, since $d(a,\theta) \le r$ and $d(p,\theta) \ge \varepsilon r/4$, apply the fact that X is weakly uniformly convex to get that $d(a,m) \le (1 - \eta_X(a,r,\varepsilon/4))r$.

□

As a consequence, if X and Y have moduli η_X, η_Y which depend only on ε, then $X \sqcup_\theta Y$ has a modulus η with the same property.

Consider the 2-dimensional sphere $\mathbb{S}^2$ which is a geodesic space when endowed with the distance $d_{\mathbb{S}^2}(x,y) = \arccos(x \mid y)$, where $(\cdot \mid \cdot)$ is the Euclidean scalar product. Let

$$b = \left(\sqrt{3}/2, 1/2, 0\right), c = (0,1,0), a_n = \left(1/(2n), \sqrt{3}/(2n), \sqrt{1 - 1/n^2}\right).$$

Then $m(b,c) = \left(1/2, \sqrt{3}/2, 0\right)$, $d_{\mathbb{S}^2}(b,c) = \pi/3$, $d_{\mathbb{S}^2}(a_n, b) = d_{\mathbb{S}^2}(a_n, c) = \arccos\left(\sqrt{3}/(2n)\right)$ and $d_{\mathbb{S}^2}(a_n, m(b,c)) = \arccos(1/n)$ for all $n \in \mathbb{N}, n \ge 2$.

Denote by Δ_n the geodesic triangle $\Delta(a_n, b, c)$. Since we will glue these triangles, we denote the points b and c by b_n and c_n, respectively. Note that, for each $n \geq 2$, Δ_n is a 2-uniformly convex geodesic space. In fact, Ohta proved in [17] that, for $\kappa > 0$, any CAT(κ) space X with $\text{diam}(X) < \pi/(2\sqrt{\kappa})$ is 2-uniformly convex with parameter $k = (\pi - 2\sqrt{\kappa}\,\sigma)\tan(\sqrt{\kappa}\,\sigma)$ for any $0 < \sigma \leq \pi/(2\sqrt{\kappa}) - \text{diam}(X)$. In particular, since Δ_n is a CAT(1) space and $\text{diam}(\Delta_n) < \pi/2$ it follows, by the argument given in Section 2, that it has a modulus of uniform convexity given by $\delta_{\Delta_n}(\varepsilon) = \sqrt{1 - (1 - k\varepsilon^2/8)}$.

Glue Δ_n and Δ_{n+1} successively through a point by identifying b_n with a_{n+1} in order to obtain a chain of triangles $Y = \sum_{n\geq 2} \Delta_n$. More precisely, one considers first the gluing $Y_1 := \Delta_2 \sqcup_{b_2} \Delta_3$, then one forms $Y_2 := Y_1 \sqcup_{b_3} \Delta_4$ and so on, for general $n \geq 2$, $Y_{n+1} := Y_n \sqcup_{b_{n+2}} \Delta_{n+3}$. One glues in this way the sequence of triangles (Δ_n) obtaining the unbounded geodesic space Y. Applying repeatedly Lemma 2.2, we get that for every $n \geq 1$, Y_n is uniformly convex, with a modulus of uniform convexity η_{Y_n} which depends only on ε.

Proposition 2.3. *Y is a weakly uniformly convex geodesic space satisfying* (1).

Proof. Let $a \in Y$, $\varepsilon \in (0, 2]$, $r > 0$ and $x, y \in Y$ be such that $d(a, x), d(a, y) \leq r$ and $d(x, y) \geq \varepsilon r$. If i is such that $a \in Y_i$, then $a, x, y \in Y_N$, with $N = i + \lceil r/\arccos\left(\sqrt{3}/(2i+4)\right)\rceil$. Thus, $d(a, m(x, y)) \leq (1 - \eta_{Y_N}(\varepsilon))r$.

We prove now that (1) holds. Take $s = (i+1)\pi/\varepsilon$ and $r \geq s$. Note that since $d(x, y) \geq \varepsilon r$, it follows that $d(x, y) \geq (i+1)\pi$. At least one of the points x and y does not belong to Y_i and we may assume that this point is y. One can also easily see that $m(x, y)$ does not belong to Y_i either and that $m(x, y) \in [a, y]$. Thus, as in the proof of Lemma 2.2, case (iii), we have that $d(a, m(x, y)) \leq (1 - \varepsilon/2)r$. Therefore, $\inf_{r\geq s} \delta_Y(a, r, \varepsilon) \geq \varepsilon/2$. ∎

Proposition 2.4. *Y does not admit a modulus of uniform convexity that does not depend on the center of balls.*

Proof. Let $r = \pi/2, \varepsilon = 2/3$ and $\delta \in (0, 1]$ be arbitrary. Take n sufficiently large such that $\arccos(1/n) > (1 - \delta)\pi/2$. Then $d(b_n, c_n) = \varepsilon r$, $d(a_n, b_n) = d(a_n, c_n) \leq r$, while $d(a_n, m(b_n, c_n)) > (1 - \delta)r$. ∎

3 Proof of the main theorem

We denote $d(y, a_n(x)y)$ by $D_n(x)$ and $d(y, a_{n-k}(T^k x)y)$ by $D_n(x, k)$.
Let E be the set of points $x \in X$ for which taking any $\varepsilon > 0$ there exist $M \in \mathbb{N}$ and infinitely many $n \in \mathbb{N}$ such that for every $k \in [M, n]$,

$$D_n(x) - D_n(x, k) \geq (A - \varepsilon)k.$$

By [10, Proposition 4.2] we know that $\mu(E) = 1$.

Lemma 3.1. *Let $x \in E$ such that $\lim_{n\to\infty} D_n(x)/n = A > 0$. Then for all sequences $(\alpha_i) \subseteq (0, 1]$, $(p_i) \subseteq \mathbb{N}$, there exist sequences $(K_i) \subseteq \mathbb{N}$, $(n_i) \subseteq \mathbb{N}$ satisfying for all $i \geq 1$:*

(i) $p_i \le K_i$, $n_i \in (K_{i+1}, n_{i+1})$ *and* $D_{n_{i+1}}(x) \ge \max\{D_{n_i}(x), An_i\}$*;*

(ii) for all $k \in [K_i, n_i]$, $|D_k(x) - Ak| \le Ak/2^i$ *and*

$$\left(1 - \min\{1/2^i, \alpha_i\}\right) D_k(x) + d(a_k(x)y, a_{n_i}(x)y) \le D_{n_i}(x). \tag{5}$$

Proof. For $i \ge 1$, take $\varepsilon_i = \min\{A/(1+2^{i+1}), A\alpha_i/(2+\alpha_i)\}$. Then

$$\frac{2\varepsilon_i}{A - \varepsilon_i} \le \min\{1/2^i, \alpha_i\}. \tag{6}$$

Since $x \in E$, there exist M_i and infinitely many n satisfying

$$D_n(x) - D_n(x,k) \ge (A - \varepsilon_i)k \qquad \text{for every } k \in [M_i, n]. \tag{7}$$

Moreover, since $\lim\limits_{n\to\infty} D_n(x)/n = A$, one gets J_i such that for every $k \ge J_i$,

$$(A - \varepsilon_i)k \le D_k(x) \le (A + \varepsilon_i)k. \tag{8}$$

Take $K_i := \max\{M_i, J_i, p_i\}$. Then there are infinitely many n such that (7) and (8) hold for every $k \in [K_i, n]$. We define the sequence (n_i) as follows. Take $n_1 > K_1 + K_2$ such that (7) and (8) hold for $k \in [K_1, n_1]$. For $i \ge 2$, pick $n_i > \max\{n_{i-1}, K_{i+1}\}$ such that $D_{n_i}(x) \ge \max\{D_{n_{i-1}}(x), An_{i-1}\}$ and (7), (8) hold for $k \in [K_i, n_i]$. Then $(K_i), (n_i)$ satisfy (i).
Let $i \ge 1$ and $k \in [K_i, n_i]$. By (8), $|D_k(x) - Ak| \le \varepsilon_i k \le Ak/2^i$. Furthermore,

$$\begin{aligned} D_k(x) + D_{n_i}(x,k) &\le D_k(x) + D_{n_i}(x) - (A - \varepsilon_i)k \quad \text{by (7)} \\ &\le D_{n_i}(x) + 2\varepsilon_i k \le D_{n_i}(x) + \frac{2\varepsilon_i D_k(x)}{A - \varepsilon_i} \quad \text{by (8)} \\ &\le D_{n_i}(x) + \min\{1/2^i, \alpha_i\} D_k(x) \quad \text{by (6)}. \end{aligned}$$

As $d(a_k(x)y, a_{n_i}(x)y) = d(a_k(x)y, a_k(x)a_{n_i-k}(T^k x)y) \le D_{n_i}(x,k)$, (5) holds. □

Recall that a geodesic space Y is *Busemann convex* if given any pair of geodesic paths $\gamma_1 : [0, l_1] \to Y$ and $\gamma_2 : [0, l_2] \to Y$ with $\gamma_1(0) = \gamma_2(0)$, one has $d(\gamma_1(tl_1), \gamma_2(tl_2)) \le t d(\gamma_1(l_1), \gamma_2(l_2))$ for every $t \in [0,1]$.

3.1 Proof of Theorem 1.1

Let $x \in E$ be such that $\lim_{n\to\infty} D_n(x)/n = A$. For every $i \in \mathbb{N}$ there exist $s_i > 0$ such that $\alpha_i = \inf_{r \ge s_i} \Psi\left(y, r, 1/2^i\right) > 0$ (by (C2)) and $p_i \in \mathbb{N}$ such that $D_n(x) \ge s_i$ for $n \ge p_i$. Apply Lemma 3.1 to obtain the sequences (K_i) and (n_i) satisfying properties (i)-(ii) thereof. For each $j \in \mathbb{N}$, we denote by γ_j the geodesic path that joins y and $a_{n_j}(x)y$ and for each $j, k \in \mathbb{N}$ we let

$$M_{j,k} = \min\{D_k(x), D_{n_j}(x)\}.$$

Claim 1: For all $j \in \mathbb{N}$ and all $k \in [K_j, n_j]$,

$$d(a_k(x)y, \gamma_j(M_{j,k}) \leq \frac{D_k(x)}{2^j}. \tag{9}$$

Proof of claim: If $D_k(x) > D_{n_j}(x)$, we apply (5) to get that

$$d(a_k(x)y, \gamma_j(D_{n_j}(x))) = d(a_k(x)y, a_{n_j}(x)y) \leq \frac{D_k(x)}{2^j}.$$

Assume now that $D_k(x) \leq D_{n_j}(x)$. Since $k \geq p_j$, we get that $D_k(x) \geq s_j$ and so $\alpha_j \leq \Psi(y, D_k(x), 1/2^j)$. Then (5) yields that

$$\begin{aligned} D_k(x) + d(a_k(x)y, a_{n_j}(x)y) &\leq D_{n_j}(x) + \alpha_j D_k(x) \\ &\leq D_{n_j}(x) + \Psi(y, D_k(x), 1/2^j) D_k(x). \end{aligned}$$

Apply now property (C) to obtain that $d(a_k(x)y, \gamma_j(D_k(x))) \leq D_k(x)/2^j$. ∎

For all $i \in \mathbb{N}$, we have that $n_i \in [K_{i+1}, n_{i+1}]$ and $D_{n_i}(x) \leq D_{n_{i+1}}(x)$, so we can apply (9) (with $j := i+1$ and $k := n_i$) to conclude that

$$d(\gamma_{i+1}(D_{n_i}(x)), \gamma_i(D_{n_i}(x))) = d(\gamma_{i+1}(D_{n_i}(x)), a_{n_i}(x)y) \leq \frac{D_{n_i}(x)}{2^{i+1}}. \tag{10}$$

Fix now $R > 0$ and define $I(R)$ as the smallest integer for which $D_{n_{I(R)}}(x) \geq R$. Let $i \geq I(R)$ be arbitrary. Since $(D_{n_i}(x))_i$ is nondecreasing, one has that $D_{n_i}(x) \geq R$. By Busemann convexity, we get that $d(\gamma_{i+1}(R), \gamma_i(R)) \leq (R/D_{n_i}(x)) d(\gamma_{i+1}(D_{n_i}(x)), \gamma_i(D_{n_i}(x)))$ and an application of (10) gives us

$$d(\gamma_{i+1}(R), \gamma_i(R)) \leq \frac{R}{2^{i+1}}. \tag{11}$$

Applying repeatedly (11), we get that for all $m \in \mathbb{N}$,

$$d(\gamma_{i+m}(R), \gamma_i(R)) \leq \sum_{j=1}^{m} \frac{1}{2^{i+j}} R \leq \frac{R}{2^i}.$$

Thus, $(\gamma_i(R))_{i \geq I(R)}$ is Cauchy, hence converges to $\gamma(R) := \lim_{i\to\infty} \gamma_i(R)$. Furthermore,

$$d(\gamma(R), \gamma_i(R)) \leq \frac{R}{2^i} \quad \text{for all } i \geq I(R). \tag{12}$$

It is easy to see that γ is a geodesic ray starting at y.

We shall prove in the sequel that $\lim_{k\to\infty} d(\gamma(Ak), a_k(x)y)/k = 0$. Let $k \in \mathbb{N}$. Then there exists i such that $k \in [n_{i-1}, n_i)$, which yields $k \in (K_i, n_i)$.

Claim 2: $|M_{i,k} - Ak| \leq Ak/2^i$.

Proof of claim: If $D_k(x) \leq D_{n_i}(x)$, then by Lemma (3.1).(ii), we get that $|M_{i,k} - Ak| = |D_k(x) - Ak| \leq Ak/2^i$. If $D_k(x) > D_{n_i}(x)$, since $|D_{n_i}(x) - An_i| \leq An_i/2^{i+1}$ (by Lemma (3.1).(ii))), we have that

$$(A - A/2^i)k \leq (A - A/2^{i+1}) n_i \leq D_{n_i}(x) < D_k(x) \leq (A + A/2^i)k.$$

On gets immediately that $|M_{i,k} - Ak| = |D_{n_i}(x) - Ak| \leq Ak/2^i$. ■

By (9), $d(a_k(x)y, \gamma_i(M_{i,k})) \leq D_k(x)/2^i \leq 2Ak/2^i$. Since $D_{n_{i+1}}(x) > Ak$ and $D_{n_i}(x) \geq M_{i,k}$, we have that $i+1 \geq I(Ak)$ and $i \geq I(M_{i,k})$, so we can apply (12) and (11) to get that $d(\gamma(Ak), \gamma_{i+1}(Ak)) \leq Ak/2^{i+1}$ and $d(\gamma_{i+1}(M_{i,k}), \gamma_i(M_{i,k})) \leq M_{i,k}/2^{i+1} \leq 2Ak/2^{i+1}$. Thus,

$$\begin{aligned} d(\gamma(Ak), a_k(x)y) &\leq d(\gamma(Ak), \gamma_{i+1}(Ak)) + d(\gamma_{i+1}(Ak), \gamma_{i+1}(M_{i,k})) \\ &\quad + d(\gamma_{i+1}(M_{i,k}), \gamma_i(M_{i,k})) + d(\gamma_i(M_{i,k}), a_k(x)y) \\ &\leq \frac{Ak}{2^{i+1}} + |Ak - M_{i,k}| + \frac{2Ak}{2^{i+1}} + \frac{2Ak}{2^i} \leq \frac{9Ak}{2^{i+1}}. \end{aligned}$$

This implies that $\lim_{k\to\infty} \frac{1}{k} d(\gamma(Ak), a_k(x)y) = 0$. By Busemann convexity it follows easily that the obtained geodesic ray is unique.

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