ISOTROPIC SUBMANIFOLDS OF PSEUDO-RIEMANNIAN SPACES

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ABSTRACT

The family of all the submanifolds of a given Riemannian or pseudo-Riemannian manifold is large enough to classify them into some interesting subfamilies such as minimal (maximal), totally geodesic, Einstein, etc. Most of these have been extensively studied by many authors, but as far as we know, no paper has hitherto been published on the class of isotropic submanifolds. The purpose of this paper is therefore to gain a better understanding of this interesting class of submanifolds that arise naturally in mathematics and physics by studying their relationships with other closely distinguished families.

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1 Introduction

A submanifold N_s^n of a pseudo-Riemannian space M_ν^m is said to be *totally geodesic* if the geodesic lines of N_s^n are also geodesic lines in M_ν^m. A totally geodesic submanifold is characterized by the fact that for every normal vector of the corresponding second fundamental form h vanishes: $h(X,Y) = 0$ for any vectors fields X, Y. The existence of totally geodesic submanifolds in a general Riemannian manifold is exceptional. Conversely, the existence of many such totally-geodesic submanifolds characterize some special manifolds, e.g. symmetric spaces. The submanifold N_s^n is called *totally umbilical* if the second fundamental form h is proportional to the first, $h(X,Y) = g(X,Y)\mathbf{H}$, where $\mathbf{H} = (1/n)trace(h)$ is the mean curvature vector, and is *pseudo-umbilical* if $g(h(X,Y),\mathbf{H}) = \rho\, g(X,Y)$. Obviously every totally umbilical submanifold is pseudo-umbilical. An *Einstein* manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor is proportional to the metric.

A *minimal* submanifold is a submanifold with $\mathbf{H} \equiv 0$, therefore any totally geodesic submanifold is minimal. The analogous submanifolds in spacetimes are *spacelike maximal* submanifolds. They are all pseudo-umbilical, and have been used to prove positivity of

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mass, analyse the space of solutions of Einstein equations and in numerical integration schemes for Einstein equations ([3], p. 155). In the pseudo-Euclidean space $\mathbb{R}^m_\nu$, maximal submanifolds are the harmonic submanifolds.

In this article we are interested in a distinguised family of submanifolds of a pseudo-Riemannian manifold: the *isotropic* submanifolds. Isotropy is a notion that naturally appear in Particle Physics, Kinetic theory, Electromagnetics, Optics, Cosmology, Materials science, etc. In the sixties, the fascinating aspect of unexpected features in Einstein's theory came to development namely the existence of singularities in solutions. An outside observer does not perceive the singularity itself but an event horizon. Conversely light cannot escape the singularity, it becomes infinitely redshifted as the horizon is approached. Then, modulo some reservations it has been shown that horizons are isotropic hypersurfaces [16]. Hence theoretical physicists have widely used isotropic hypersurfaces with their degenerate metric induced by the embedding into the Lorentzian spacetime. The concept of isotropic submanifold of a Riemannian manifold was introduced by B. O' Neill [20], who studied the general properties of such a class of submanifold. A pseudo-Riemannian submanifold is isotropic if, roughly speaking, the geometry of the submanifold is the same regardless of direction. This notion remind us of the cosmological principle that no matter where we look in the Universe, we still see the same distribution of objetcs (see [8] p. 835 and Prop. 5.1). The isotropic submanifolds can be considered as a generalization of both, totally geodesic and totally umbilical submanifolds.

In a previous paper ([10]) we studied spacelike isotropic surfaces in a Lorentzian manifold and, in particular in the standard four-dimensional model spaces: Minkowski, De Sitter and anti De Sitter spaces. Now, the aim of the present paper is to place the notion of isotropic submanifold in a wider setting by studying its interrelationships with other distinguised families of submanifold such as minimal, pseudo-umbilical, Einstein, etc., into a pseudo-Riemannian manifold M^m_ν. For this end, we shall use the advantageous language of immersion theory.

Thus, we first introduce in Section 2 the necessary notation and recall some useful facts concerning an isometric immersion of a pseudo-Riemannian manifold N^n_s $(n \geq 2)$ into the pseudo-Riemannian manifold M^m_ν. We develop in Section 3 the notion of isotropic submanifold. In particular, every totally umbilical submanifold is obviously an isotropic submanifold. We give some characterizations of isotropic submanifolds and include an example showing that isotropy in a Riemannian or in a pseudo-Riemannian environment may have different consequences.

The class of constant discriminant isotropic immersions (for instance, into a sphere) seems to be too large to classify them. We first show that in such case an isotropic submanifold is also pseudo-umbilical. We give some examples of isometric embeddings between Riemannian or pseudo-Riemannian manifolds. As we remarked in Section 3, every totally umbilical submanifold is isotropic, and now we prove that the converse is false. Actually we can prove a slightly stronger result. In fact, we give an example of a submanifold with dimension $n \geq 3$ which is isotropic but not pseudo-umbilical. As a consequence, we shall sometimes assume the submanifold to be isotropic and pseudo-umbilical.

In Section 5 we consider the Ricci tensor of an isotropic pseudo-Riemannian submanifold of a pseudo-Riemanian space form of constant sectional curvature c. An useful formula is obtained for this tensor which allows us to obtain several applications. Thus, if the submanifold is also maximal and its dimension is $n \geq 3$, then it is an Einstein manifold (when $n = 2$ this result is false). We include an example that exhibits how different things may happen depending on the surronding space being Riemannian or pseudo-Riemannian.

In Section 6 we consider isotropic spacelike submanifolds N^n ($n \geq 3$) of a pseudo-Riemannian space of constant curvature c, $M^m_\nu(c)$. When the isotropy function λ is approppiately bounded above then strong topological conditions are obtained for N^n. In particular, if the background space is $\mathbb{R}^m_\nu$, the hyperbolic space $\mathbb{H}^m_\nu$ or the pseudo-sphere $\mathbb{S}^m_\nu$, aditional information is derived. Next we use Bochner's technique to show that when N^n is compact and admits a nontrivial conformal vector field, the isotropy function is bounded as desired. On the contrary, when the isotropy function is bounded below by the same function as before, then N^n is non-compact. We give an example to illustrate this result.

Finally, in Section 7 we specialize in the case when the index of the submanifold $s = 1$, that is, a Lorentzian submanifold N^n_1 of a pseudo-Riemannian space form $M^m_\nu(c)$. When N^n_1 is compact and admits a spatially conformal reference frame (or a projective vector field), then we prove that the isotropy function λ is bounded below by the same function as in Section 6. If in addition N^n_1 is a maximal submanifold, then λ is a constant. By changing the hypothesis on the existence of the spatially conformal reference frame by the existence of a timelike conformal vector field, we get the same bound for λ. We close this Section with some further consequences obtained when we assume that λ reaches the bound.

2 Preliminaries

Let M^m_ν be a m-dimensional pseudo-Riemannian manifold with metric tensor g of signature $(\nu, m-\nu)$. Let $\phi : N^n_s \to M^m_\nu$ be an isometric immersion of a connected pseudo-Riemannian manifold N^n_s of dimension $n \geq 2$ and signature $(s, n-s)$. For all local formulae and computations we may assume ϕ is an imbedding and thus we shall often identify $p \in N^n_s$ with $\phi(p) \in M^m_\nu$. The tangent space $T_pN^n_s$ is identified with the subspace $\phi_*(T_pN^n_s)$ of $T_pM^m_\nu$, and the normal space is denoted by $T^\perp_p N^n_s$. We will use letters X, Y, Z (resp. ξ, η, ζ) to denote vectors fields tangent (resp. normal) to N^n_s. Let $\tilde{\nabla}$ and ∇ be the Levi-Civita connections of M^m_ν and N^n_s, respectively. Then, the Gauss-Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (1)$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \quad (2)$$

where h denotes the second fundamental form of ϕ, A_ξ the shape operator, and D is the normal connection. The shape operator and the second fundamental form are related by $g(A_\xi X, Y) = g(h(X,Y), \xi)$.

A point $p \in N^n_s$ is *umbilic* [21] provided there exists a vector $\xi_p \in T^\perp_p N^n_s$ such that for all $u, v \in T_pN^n_s$ then $h(u,v) = g(u,v)\xi_p$. When every point of N^n_s is umbilic the immersion ϕ is called *totally umbilical.* In such case, it is well known that $h(X,Y) = g(X,Y)\mathbf{H}$ where $\mathbf{H} = (1/n)\mathrm{trace}_g\,(h)$ is the *mean curvature vector* of ϕ. We also recall that a point $p \in N^n_s$ is *flat* if $h = 0$ at p, and the immersion ϕ is said to be *totally geodesic* provided every point of N^n_s is flat.

The immersion ϕ is called *pseudo-umbilical* if its second fundamental form satisfies $g(h(X,Y),\mathbf{H}) = \rho g(X,Y)$ for some function ρ. Necessarily this function ρ is given by $\rho = g(\mathbf{H},\mathbf{H})$. Clearly, any totally umbilical immersion is pseudo-umbilical.

Denote by $\tilde{R}$ and R the curvature tensor of M^m_ν and N^n_s, respectively. The equation of Gauss is given by

$$(\tilde{R}(X,Y)Z)^\top = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, , \tag{3}$$

where $(\tilde{R}(X,Y)Z)^\top$ denotes the tangential component of $\tilde{R}(X,Y)Z$.

The notion of *discriminant* $\mathcal{D}$ of the immersion ϕ for a non-degenerate plane can be introduced in a similar way to the case of an immersion between Riemannian manifolds [20]. In fact, the *discriminant* $\mathcal{D}_p$ *of* ϕ *at* $p \in N^n_s$ is $\mathcal{D}_p(\Pi) = \mathcal{K}(\Pi) - \tilde{\mathcal{K}}(\Pi)$, where $\mathcal{K}(\Pi)$ and $\tilde{\mathcal{K}}(\Pi)$ denote the sectional curvature of N^n_s and M^m_ν, respectively, on a non-degenerate plane Π of $T_pN^n_s$. The immersion ϕ is said to be of *constant discriminant at* $p \in N^n_s$ if $\mathcal{D}_p(\Pi)$ is a constant for all non-degenerate planes Π of $T_pN^n_s$. Notice that if ϕ is of constant discriminant at each point of N^n_s, then the discriminant can be viewed as a function $\mathcal{D} : N^n_s \to \mathbb{R}$.

From the view-point of infinitesimal transformations, a vector field X is said to preserve a certain geometric quantity if the Lie derivative $\mathcal{L}_X$ of this quantity vanishes. On a pseudo-Riemannian manifold (M,g) a vector field X is called *isometric* or *Killing* if it preserves the metric in the sense of $\mathcal{L}_X g = 0$. The vector field X is called *conformal* if it preserves the conformal class of the metric in the sense that $\mathcal{L}_X g = 2\rho g$ holds for some function $\rho : M \longrightarrow \mathbb{R}$.

A *projective* vector field U is a smooth vector field whose flow preserves the geodesic structure of M without necessarily preserving the affine parameter of any geodesic. An *affine* vector field is a projective vector field preserving geodesics and preserving the affine parameter. It is known that U is projective if and only if there exists a 1-form ω on M such that $\mathcal{L}_U\nabla(X,Y) = \omega(X)Y + \omega(Y)X$ for all $X,Y \in \mathfrak{X}(M)$, and $\omega = (1/(n+1))d(\mathrm{div}(U))$ (see [22, Props. 5.27, 5.28] for instance). Note that U is affine if and only if $\omega = 0$.

Let $\mathbb{R}^m_\nu$ be the n-dimensional pseudo-Euclidean space with inner product of signature ν given by

$$\langle x,y\rangle = -\sum_{i=1}^{\nu} x_iy_i + \sum_{i=\nu+1}^{m} x_iy_i$$

where $x = (x_1,\ldots,x_m)$, $y = (y_1,\ldots,y_m)$. For a positive number c, the standard space form $\mathbb{S}^m_\nu(c)$ is the hypersurface $\mathbb{S}^m_\nu(c) = \{x \in \mathbb{R}^{m+1}_\nu : \langle x,x\rangle = 1/c\}$ with the induced metric of signature ν and constat curvature c and it is called a pseudo-sphere. For a negative number c, the standard space form $\mathbb{H}^m_\nu(c)$ is the hypersurface $\mathbb{H}^m_\nu(c) = \{x \in \mathbb{R}^{m+1}_{\nu+1} : \langle x,x\rangle = 1/c\}$ with the induced metric of signature ν and curvature $c < 0$. This is called the pseudo-hyperbolic space. We simply denote $\mathbb{S}^m_\nu(1)$ and $\mathbb{H}^m_\nu(-1)$ by $\mathbb{S}^m_\nu$ and $\mathbb{H}^m_\nu$, respectively. For short we shall write $M^m_\nu(c)$ to indicate $\mathbb{S}^m_\nu(c)$, $\mathbb{R}^m_\nu$ or $\mathbb{H}^m_\nu(c)$, according to $c > 0$, $c = 0$ or $c < 0$, respectively.

3 Isotropic immersion

We first recall that an isometric immersion $\phi : N^n_s \to M^m_\nu$ is called *isotropic at* $p \in N^n_s$ ([10, 19]) if

$$g(h(u,u), h(u,u)) = \lambda(p) \in \mathbb{R} \tag{4}$$

does not depends on the choice of the *unit* tangent vector $u \in T_pN^n_s$, and ϕ is said to be *isotropic* if ϕ is isotropic at each point of N^n_s. In such a case, the smooth function $\lambda : N^n_s \to \mathbb{R}$ defined by equation (4) is called the *isotropy function*, and the isometric immersion ϕ is said to be λ-*isotropic*. In particular, if λ is a constant function the immersion is called *constant isotropic*.

Remark 3.1 (a) It is clear that every totally umbilical immersion $\phi : N^n_s \to M^m_\nu$ is an isotropic immersion with $\lambda = g(\mathbf{H},\mathbf{H})$. Notice that the function $g(\mathbf{H},\mathbf{H})$ is constant when ϕ is totally umbilical and M^m_ν is of constant sectional curvature c [13, p. 11]. Thus, pseudo-spheres $\mathbb{S}^n_s(c)$ and pseudo-hyperbolic spaces $\mathbb{H}^n_s(c)$ into a pseudo-Euclidean space are constant isotropic submanifolds with $\lambda = g(\mathbf{H},\mathbf{H}) = c$. (b) Every isotropic immersion $\phi : N^n_s \to M^m_\nu$ with $0 < s < n$ and $s = \nu$, or $\nu - s = m - n$, is totally umbilical. In fact, in both cases $T^\perp_p N^n_s$ is a defined metric space. Thus, from isotropy condition (4) we obtain $h(x,x) = 0$ for any lightlike tangent vector x, and this means that our immersion is totally umbilical [15]. In particular, every isotropic immersion between Lorentzian manifolds is totally umbilical. (c) Any isotropic hypersurface $\phi : N^n_s \to M^{n+1}_\nu$ is totally umbilical. Notice that necessarily $s = \nu$ or $s = \nu - 1$. Thus, if the index s satisfies $0 < s < n$, the hypersurface is totally umbilical as before. It is easy to prove the defined cases $s = 0$ and $s = n$ (see [9]). (d) The composition of isotropic immersions is an isotropic immersion, and the isotropy function of the composition is the sum of the corresponding isotropy functions.

Now we provide a characterization of isotropic immersions [8].

Lemma 3.2 *Let $\phi : N^n_s \to M^m_\nu$ be an isometric immersion. Then, the following conditions are equivalent:*

1. *ϕ is isotropic.*
2. *There exists a smooth function $\lambda : N^n_s \to \mathbb{R}$ such that*

$$A_{h(X,X)}X = \lambda g(X,X)X$$

for all $X \in \mathfrak{X}(N^n_s)$.

3. *There exists a smooth function $\lambda : N^n_s \to \mathbb{R}$ such that*

$$A_{h(X,Y)}Z + A_{h(Y,Z)}X + A_{h(Z,X)}Y = \lambda\{g(X,Y)Z + g(Y,Z)X + g(Z,X)Y\}$$

for all $X, Y, Z \in \mathfrak{X}(N^n_s)$.

As a consequence of this lemma we have the following.

Theorem 3.3 *Let $\phi : N^n_s \to M^m_\nu$ be an isometric immersion. Then, the following conditions are equivalent.*

1. *ϕ is isotropic.*

2. *There exists a smooth funtion $\lambda : N_s^n \to \mathbb{R}$ such that*

$$\begin{aligned}3A_{h(X,Y)}Z &= \lambda\{g(X,Y)Z + g(Y,Z)X + g(X,Z)Y\} \\ &\quad +R(Z,X)Y - (\tilde{R}(Z,X)Y)^\top \\ &\quad +R(Z,Y)X - (\tilde{R}(Z,Y)X)^\top \end{aligned} \tag{5}$$

for any $X, Y, Z, W \in \mathfrak{X}(N_s^n)$.

Furthermore, the condition 2 holds with λ a constant function if and only if ϕ is constant isotropic.

Proof. It follows from Gauss equation (3) that

$$\begin{aligned}3A_{h(X,Y)}Z &= A_{h(X,Y)}Z + A_{h(Y,Z)}X + A_{h(Z,X)}Y \\ &\quad +R(Z,X)Y - (\tilde{R}(Z,X)Y)^\top \\ &\quad +R(Z,Y)X - (\tilde{R}(Z,Y)X)^\top.\end{aligned}$$

Now, by statement *3* of Lemma 3.2, we find equation (5). □

Remark 3.4 From Theorem 3.3 one can easily prove that every isotropic immersion $\phi : N_s^n \to M_\nu^m$ satisfies

$$n^2 g(\mathbf{H}, \mathbf{H}) = (n+2)n\lambda - 2g(h,h) \tag{6}$$

where $g(h,h)$ is defined by $g(h,h) = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j g(h(e_i,e_j), h(e_i,e_j))$ for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on N_s^n, $\varepsilon_i = g(e_i, e_i)$. As a consequence of equation (6), a λ-isotropic immersion between Riemannian manifolds *is totally-umbilical if and only if* $\lambda = g(\mathbf{H}, \mathbf{H})$. However, in the indefinite case the corresponding result is false. In fact, let $f_1, \ldots, f_\ell : \mathbb{R}_s^n \to \mathbb{R}$ be smooth functions and consider the isometric immersion $\phi : \mathbb{R}_s^n \to \mathbb{R}_{s+\ell}^{n+2\ell}$ defined by $\phi(x) = (f_1(x), \ldots, f_\ell(x), x, f_1(x), \ldots, f_\ell(x))$. It is easy to check that ϕ is 0-isotropic with $g(\mathbf{H}, \mathbf{H}) = 0$. Notice that ϕ is totally umbilical if and only if f_j is given by (see [8])

$$f_j(x_1, \ldots, x_n) = a_j \left(-\sum_{i=1}^{s} x_i^2 + \sum_{i=s+1}^{n} x_i^2 \right) + \sum_{i=1}^{n} b_j^i x_i + c_j,$$

with $a_j, b_j^1, \ldots, b_j^n, c_j \in \mathbb{R}$, $j = 1, \ldots \ell$.

4 The constant discriminant case

We need the following result for later use.

Lemma 4.1 *Let $\phi : N_s^n \to M_\nu^m$ be an isometric immersion. Then, ϕ is of constant discriminant at each point of N_s^n if and only if there exists a smooth function $\mathcal{D} : N_s^n \to \mathbb{R}$ such that*

$$A_{h(Y,Z)}X - A_{h(X,Z)}Y = \mathcal{D}\{g(Y,Z)X - g(X,Z)Y\} \tag{7}$$

for any $X, Y, Z \in \mathfrak{X}(N_s^n)$.

Proof. Suppose that ϕ is of constant discriminant $\mathcal{D}_p$ at $p \in N_s^n$. Let T be the 4-covariant tensor on $T_pN_s^n$ defined by

$$\begin{aligned} T(x,y,z,w) &= g(h(x,w),h(y,z)) - g(h(x,z),h(y,w)) \\ &\quad -\mathcal{D}_p \{g(x,w)g(z,y) - g(x,z)g(y,w)\}. \end{aligned} \tag{8}$$

Then, T is a curvaturelike function, i. e., T has the following symmetries:

(i) $T(x,y,z,w) + T(y,z,x,w) + T(z,x,y,w) = 0$,

(ii) $T(y,x,z,w) = -T(x,y,z,w)$,

(iii) $T(x,y,w,z) = -T(x,y,z,w)$,

(iv) $T(z,w,x,y) = T(x,y,z,w)$.

From Gauss equation (3) we have

$$\mathcal{D}_p = \frac{g(h(x,x),h(y,y)) - g(h(x,y),h(x,y))}{g(x,x)g(y,y) - g(x,y)^2},$$

for all $x, y \in T_pN_s^n$ such that $g(x,x)g(y,y) - g(x,y)^2 \neq 0$. Therefore $T(x,y,y,x) = 0$ and this means that $T = 0$ [21, p. 79]. This concludes the proof.

□

Theorem 4.2 *Let $\phi : N_s^n \to M_\nu^m$ be a λ-isotropic immersion with constant discriminant $\mathcal{D}$ at each point of N_s^n. Then, ϕ is pseudo-umbilical and*

$$3ng(\mathbf{H},\mathbf{H}) = (n+2)\lambda + 2(n-1)\mathcal{D}. \tag{9}$$

In particular, we have

(a) $\lambda = \mathcal{D}$ if and only if $\lambda = g(\mathbf{H},\mathbf{H})$.

(b) $\lambda = -\{2(n-1)/(n+2)\}\mathcal{D}$ if and only if $g(\mathbf{H},\mathbf{H}) = 0$.

Proof. From Theorem 5 and equation (7), we have

$$\begin{aligned} 3A_{h(X,Y)}Z &= (\lambda - \mathcal{D})\{g(Y,Z)X + g(X,Z)Y\} \\ &+ (\lambda + 2\mathcal{D})g(X,Y)Z. \end{aligned} \tag{10}$$

Consider an orthonormal reference frame $\{e_1, \ldots, e_n\}$ on N_s^n. From Eq. (10) we obtain

$$g(\mathbf{H}, h(e_i,e_j)) = \begin{cases} \varepsilon_i \rho & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

where $\rho = ((n+2)\lambda + (2n-2)\mathcal{D})/3n$, and then the immersion is pseudo-umbilical with $3ng(\mathbf{H},\mathbf{H}) = (n+2)\lambda + 2(n-1)\mathcal{D}$. The other statements are trivial.

□

As a immediate consequence we have the following.

Corollary 4.3 *Let $\phi : N_s^n \to M_\nu^m$ be a λ-isotropic immersion of a pseudo-Riemannian manifold N_s^n of constant sectional curvature k into a pseudo-Riemannian manifold M_ν^m of constant sectional curvature c. Then, ϕ is pseudo-umbilical and*

$$3ng(\mathbf{H}, \mathbf{H}) = (n + 2)\lambda + 2(n - 1)(k - c). \tag{11}$$

In particular, if $k = c$ then ϕ is 0-isotropic if and only if $g(\mathbf{H}, \mathbf{H}) = 0$.

Remark 4.4 It is well known that an isometric immersion between two Riemannian spaces of the same constant sectional curvature is totally geodesic. In this case, this result is irrelevant when $k = c$. However, in the following example we obtain an isotropic immersion of $\mathbb{S}^n$ in $\mathbb{S}_1^{n+2}$ with lightlike mean curvature vector, but the immersion is not totally geodesic.

Example 4.5 Let $f : \mathbb{S}^n \to \mathbb{R}$ be a smooth function. The isometric immersion $\psi : \mathbb{S}^n \to \mathbb{S}_1^{n+2}$ defined by

$$\psi(x) = (f(x), \iota(x), f(x))$$

is 0-isotropic, where $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ denotes the canonical inclusion. In fact, a straightforward computation shows that the second fundamental form h of ψ is given by ([14])

$$h(X, Y) = \big((\nabla^2 f)(X, Y) + fg(X, Y)\big)\,\eta,$$

where η denotes the lightlike vector $(1, 0, \ldots, 0, 1) \in \mathbb{R}_1^{n+3}$ and $\nabla^2 f$ is the Hessian of f in $\mathbb{S}^n$. Thus, the immersion ψ is 0-isotropic with mean curvature vector

$$\mathbf{H} = \frac{1}{n}\,(\Delta f + nf)\,\eta,$$

where $\Delta = \mathrm{trace}_g(\nabla^2)$ is the Laplacian operator. On the other hand, an eigenfunction of $-\Delta$ corresponding to the eigenvalue n is the restriction to $\mathbb{S}^n$ of a homogeneous polynomial of degree 1 ([4]), and therefore the immersion ψ is totally umbilical if and only if f is an affine function. Notice that $\mathbb{S}^n$ is compact, $\mathrm{Iso}(\mathbb{S}^n)$ is finite and the first Betti number of $\mathbb{S}^n$ is zero.

Corollary 4.6 *Let $\phi : N_s^2 \to M_\nu^m$ be a λ-isotropic immersion. Then, ϕ is pseudo-umbilical and $3g(\mathbf{H}, \mathbf{H}) = 2\lambda + \mathcal{D}$.*

Proof. This follows immediately from Theorem 4.2. Notice that every surface has constant discriminant.

□

Recall that we noted in Remark 3.1 (a) in Section 3 that every totally umbilical immersion $\phi : N_s^n \to M_\nu^m$ is an isotropic immersion. The converse is not true in general (Remark 3.4). Here we can prove a slightly sharper result: an isotropic immersion $\phi : N_s^n \to M_\nu^m$ with $n \geq 3$ does not need to be even pseudo-umbilical. In fact, we have the following counterexample.

Example 4.7 Consider the complex Euclidean space $\mathbb{C}^n$, $n \geq 3$ and let $\gamma : \mathbb{R} \to \mathbb{C}$ be the smooth complex curve defined by

$$\gamma(t) = \sqrt{\cosh(2t)}\,\exp\{i \arctan(\tanh 2t)\}.$$

It is easy to prove that $\phi : \mathbb{R}\times\mathbb{S}^{n-1} \to \mathbb{C}^n$, $\phi(t, x_1, \ldots, x_n) = \gamma(t)(x_1, \ldots, x_n)$ is an isometric immersion when $\mathbb{R}\times\mathbb{S}^{n-1}$ is endowed with the induced metric $g = \cosh 2t\ (dt^2 + g_0)$, where g_0 denotes the standard Euclidean induced metric of $\mathbb{S}^{n-1}$. Let $\{\tilde{e}_2, \ldots, \tilde{e}_n\}$ be a local orthonormal basis of $\mathbb{S}^{n-1}$ with respect to the metric g_0. Now we define

$$e_1 = \frac{1}{|\gamma|}(\partial_t, 0), \qquad e_i = \frac{1}{|\gamma|}(0, \tilde{e}_i), \qquad 2 \le i \le n,$$

and then we get a local orthonormal basis $\{e_i\}$ of ϕ and $\{e_i, Je_i\}$ a local orthonomal basis of $\mathbb{C}^n$. We apply Lemma 4.2 in reference [17] and we find that the second fundamental form h of ϕ is given by

$$\begin{cases} h(e_1, e_1) = aJe_1 & h(e_2, e_2) = \cdots = h(e_n, e_n) = -aJe_1, \\ h(e_1, e_j) = -aJe_j & h(ej, e_k) = 0, \qquad 2 \le j \ne k \le n, \end{cases}$$

where $a = -(\cosh 2t)^{-3/2}$. A straightforward computation gives that ϕ is an isotropic immersion with isotropy function $\lambda = a^2$ which satisfies

$$g(h(e_1, e_1), \mathbf{H}) = -g(h(e_j, e_j), \mathbf{H}) = a(2 - n)/n,\ 2 \le j \le n.$$

Thus, as $n \ge 3$ we have that ϕ is non-pseudo-umbilical.

5 The Ricci tensor of an isotropic submanifold

Recall that the Ricci tensor of a pseudo-Riemannian manifold N^n_s with metric tensor g is defined by $\mathrm{Ric}(X, Y) = \mathrm{trace}_g(Z \mapsto R(Z, X)Y)$. A pseudo-Riemannian manifold is an *Einstein manifold* when the Ricci tensor satisfies $\mathrm{Ric} = \mu g$ for some constant $\mu \in \mathbb{R}$. Pseudo-Riemannian manifolds N^n_s with constant sectional curvature c are simplest examples of Einstein manifolds, where $\mu = c(n-1)$. Conversely, for $n = 2$ or $n = 3$, every Einstein manifold has constant sectional curvature. However, there are 4-dimensional Einstein manifolds with non-constant sectional curvature. It is also well known that a connected manifold N^n_s with $n \ge 3$ and $\mathrm{Ric} = \mu g$ for some function $\mu : N^n_s \to \mathbb{R}$, is Einstein.

For a pseudo-Riemannian space form $M^m_\nu(c)$ with constant sectional curvature c we have the following result.

Theorem 5.1 *Let $\phi : N^n_s \to M^m_\nu(c)$ be a λ-isotropic immersion. Then,*

(a) The Ricci tensor of N^n_s is given by

$$\begin{aligned} \mathrm{Ric}(X, Y) &= \{c(n-1) - \lambda(n+2)/2\}g(X, Y) \\ &\quad +3ng(h(X, Y), \mathbf{H})/2. \end{aligned} \tag{12}$$

(b) If $n \ge 3$, ϕ is pseudo-umbilical if and only if N^n_s is an Einstein manifold.

Proof. (a) Using the Gauss equation (3) we have

$$\mathrm{Ric}(X, Y) = c(n-1)g(X, Y) + ng(h(X, Y), \mathbf{H}) - b(X, Y), \tag{13}$$

where $b(X,Y)$ is defined by $b(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(h(X,e_i), h(Y,e_i))$ for a local orthonormal frame $\{e_1, \cdots, e_n\}$ on N_s^n, $\varepsilon_i = g(e_i, e_i)$. Applying Theorem 3.3, a straightforward computation leads to

$$3b(X,Y) = \{c(n-1) + \lambda(n+2)\}g(X,Y) - \mathrm{Ric}(X,Y). \tag{14}$$

Now, equation (12) follows from (13) and (14).
(b) Assume ϕ is pseudo-umbilical. Then, equation (12) yields

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \{c(n-1) - \lambda(n+2)/2\}g(X,Y) \\ &\quad +3ng(h(X,Y), \mathbf{H})/2, \\ &= \{c(n-1) - \lambda(n+2)/2 + 3ng(\mathbf{H}, \mathbf{H})/2\}g(X,Y). \end{aligned} \tag{15}$$

As $n \geq 3$, then $\mu = c(n-1) - \lambda(n+2)/2 + 3ng(\mathbf{H}, \mathbf{H})/2$ is a constant function [5] and, consequently, N_s^n is a Einstein manifold as desired. The converse can be obtained by a similar way.

□

As a consequence of this result we have the following.

Corollary 5.2 *Let $\phi : N_s^n \to M_\nu^m(c)$ be an isotropic immersion of an Einstein manifold N_s^n. Then, ϕ is constant isotropic if and only if $g(\mathbf{H}, \mathbf{H})$ is a constant function.*

Corollary 5.3 *Let $\phi : N_s^n \to M_\nu^m(c)$ be an isotropic and pseudo-umbilical immersion with $n \geq 3$. Then, ϕ is constant isotropic if and only if $g(\mathbf{H}, \mathbf{H})$ is a constant function.*

Corollary 5.4 *Let $\phi : N_s^n \to M_\nu^m(c)$ be an isotropic immersion, $n \geq 3$ and $\mathbf{H} = 0$. Then, ϕ is constant isotropic and N_s^n is Einstein.*

Remark 5.5 The last result is not true for $n = 2$. In fact, every non-planar holomorphic curve with respect to some orthogonal complex structure on $\mathbb{R}^4$ is a minimal isotropic and non-constant isotropic surface (see [12]).

Corollary 5.6 *Let $\phi : N_s^3 \to M_\nu^m(c)$ be an isotropic immersion with $\mathbf{H} = 0$. Then, ϕ is constant isotropic and N_s^3 has constant sectional curvature.*

Now we show a nice application of Theorem 5.1.

Example 5.7 Let N^n a compact Riemannian manifold. Then N^n has a unique kernel of the heat equation $K : N^n \times N^n \times \mathbb{R}_0^+ \to \mathbb{R}$. Let d the distance function on N^n. Then N^n is called a *strongly harmonic manifold* if there exists a function $\Psi : \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ such that $K(p,q,t) = \Psi(d(p,q),t)$ for any $p, q \in N^n$ and $t \in \mathbb{R}_0^+$. Compact symmetric spaces of rank one are known examples of strongly harmonic manifolds. Let μ_k be the k-th nonzero eigenvalue of the Laplacian Δ. Denote by V_k be the eigenspace of Δ with eigenvalue μ_k. On V_k we define a inner product by $\langle\langle f, g\rangle\rangle = \int_{N^n} fg * 1$ for $f, g \in V_k$. The space V_k endowed with $\langle\langle \cdot, \cdot \rangle\rangle$ is a finite dimensional Euclidean space. Let $\varphi_k^1, \ldots, \varphi_k^m$ be a orthonormal basis of V_k. Then the mapping

$$\varphi_k : N^n \to \mathbb{R}^m; p \mapsto c_k(\varphi_k^1(p), \ldots, \varphi_k^m(p))$$

defines a constant isotropic immersion for some suitable constant c_k [12]. It is known [6] that $\varphi_k(N^n)$ is minimal in a certain Euclidean sphere of $\mathbb{R}^m$. Then, by applying Theorem 5.1, we conclude that *strongly harmonic manifolds are Einstein manifolds* for dimension $n \geq 3$.

Proposition 5.8 *Let $\phi : N_s^n \to M_\nu^m(c)$ be an isotropic immersion of an Einstein manifold N_s^n. Then, if N_s^n admits an umbilical point, ϕ is constant isotropic with isotropy constant $\lambda = g(\mathbf{H}, \mathbf{H})$.*

Proof. If ϕ is totally umbilical, from Remark 3.1 the result follows. Assume ϕ is non-totally umbilical and define $\mathcal{U} = \{p \in N_s^n : p \text{ is an umbilical point of } \phi\}$. Clearly, $\mathcal{U}$ and $N_s^n - \mathcal{U}$ are non-empty subsets of N_s^n. It follows from Gauss equation (3) that $\mathrm{Ric} = (n-1)(c+\lambda)g$ on $\mathcal{U}$. Consequently, as N_s^n is Einstein, ϕ is constant isotropic. On the other hand, by virtue of Eq. (15) we known that $\mathrm{Ric} = \{c(n-1) - \lambda(n+2)/2 + 3ng(\mathbf{H}, \mathbf{H})/2\}g$ on $N_s^n - \mathcal{U}$. Then, as N_s^n is Einstein, $(n-1)(c+\lambda) = c(n-1) - \lambda(n+2)/2 + 3ng(\mathbf{H}, \mathbf{H})/2$ and, consequently, $\lambda = g(\mathbf{H}, \mathbf{H})$ on N_s^n.

□

Remark 5.9 As a consequence of this result and Remark 3.4 we have that *an Einstein manifold isotropically immersed into a Riemannian space form, which admits a umbilical point, is totally umbilical.* But in the indefinite case the corresponding result it not true. In fact, consider the surface $S = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ and let $f : S \to \mathbb{R}$ be a smooth function such that $f > 0$ on

$$U = \{(x, y) \in S : 2 - \sqrt{1 - x^2} < y < \sqrt{4 - x^2}\}$$

and on

$$V = \{(x, y) \in S : -\sqrt{4 - x^2} < y < -2 + \sqrt{1 - x^2}\},$$

while $f = 0$ on $S - (U \cup V)$. Let $\phi : S \to \mathbb{R}_1^4$ by the isometric immersion given by

$$\phi(x, y) = \begin{cases} (f(x, y), x, y, f(x, y)) & \text{if } y \geq 0, \\ (f(x, y), x, y, -f(x, y)) & \text{if } y < 0. \end{cases}$$

Then ϕ is a non-totally umbilical 0-isotropic immersion with $g(\mathbf{H}, \mathbf{H}) = 0$ and admitting flat points.

Corollary 5.10 *Let $\phi : N_s^n(k) \to M_\nu^m(c)$ be an isotropic immersion of a pseudo-Riemannian manifold N_s^n with constant sectional curvature k and admitting an umbilical point. Then, ϕ is constant isotropic with isotropy constant $\lambda = g(\mathbf{H}, \mathbf{H}) = k - c$.*

Proof. This follows immediately from Proposition 5.8 and Corollary 4.3.

□

6 Spacelike isotropic submanifolds in pseudo-Riemannian space forms

Theorem 6.1 *Let $\phi : N^n \to M^m_\nu(c)$ be an isotropic and pseudo-umbilical immersion of a complete spacelike submanifold N^n with $n \geq 3$. Suppose that the isotropic function λ satisfies*

$$\lambda < \frac{2(n-1)}{n+2}c + \frac{3n}{n+2}g(\mathbf{H}, \mathbf{H}). \tag{16}$$

Then, N^n is compact, $\mathrm{Iso}(N^n)$ *is finite and the first Betti number of N^n is zero.*

Proof. From Theorem 5.1 we have $\mathrm{Ric} = \mu g$ with $\mu = c(n-1) - \lambda(n+2)/2 + 3ng(\mathbf{H}, \mathbf{H})/2$. Now, inequality (16) gives that μ is a constant positive function. The result follows from the classical Myers' theorem [21] and Corollary 5.5 in [22].

□

Remark 6.2 It is well-known [11] that there exist no compact maximal submanifolds in the pseudo-Euclidean space $\mathbb{R}^m_\nu$ and the pseudo-hyperbolic space $\mathbb{H}^m_\nu(c)$. However, there exist compact maximal submanifolds in pseudo-Riemannian spheres. So, by applying Corollary 5.4 and Theorem 6.1, we can state the following results.

Corollary 6.3 *Let N^n be a complete maximal isotropic submanifold into the pseudo-Euclidean space $\mathbb{R}^m_\nu$, $n \geq 3$. Then, N^n is constant isotropic and the isotropy constant satisfies $\lambda \geq 0$.*

Corollary 6.4 *Let N^n be a complete maximal isotropic submanifold into the pseudo-hyperbolic space $\mathbb{H}^m_\nu$, $n \geq 3$. Then, N^n is constant isotropic and the isotropy constant satisfies $\lambda \geq 2(1-n)/(n+2)$.*

For complete maximal isotropic submanifolds into pseudo-spheres we have the following.

Corollary 6.5 *Let N^n be a complete maximal isotropic submanifold in the pseudo-sphere $\mathbb{S}^m_\nu$ with $n \geq 3$. Then, either N^n is totally geodesic, or N^n is constant isotropic with isotropy constant $\lambda > 0$.*

Proof. As the dimension $n \geq 3$, from Corollary 5.4, N^n is constant isotropic. If $\nu = 0$ or $\nu = m - n$ the results is clear. Suppose that $0 < \nu < m - n$. If the isotropy constant $\lambda \leq 0$, Eq. (12) gives

$$\mathrm{Ric}(X, X) \;=\; \{(n-1) - \lambda(n+2)/2\}g(X, X) \geq (n-1)g(X, X).$$

Then, by Theorem 3.1 in [2] N^n is totally geodesic, and this concludes the proof.

□

Remark 6.6 Notice that the only complete maximal n-submanifolds in $\mathbb{S}^{n+\nu}_\nu$ are the totally geodesic ones [18]. However, when the normal bundle is not definite, there exists complete maximal constant isotropic submanifolds in pseudo-spheres which are non-totally geodesic. In fact, let G be the matrix $G = \mathrm{diag}[\varepsilon_1, \ldots, \varepsilon_{n+1}]$, $\varepsilon_1 = \ldots = \varepsilon_s = -1$ and

$\varepsilon_{s+1} = \ldots = \varepsilon_{n+1} = 1$. Let us denote by $\mathfrak{so}(n+1,s)$ the space of selfadjoint operators on $\mathbb{R}^{n+1}_s$ of trace 0, that is, $\mathfrak{so}(n+1,s) = \{P \in \mathfrak{gl}(n+1,\mathbb{R}) : P^t G = GP, \mathrm{trace}(P) = 0\}$, P^t standing for the transpose of P. Let us consider the map $\phi : \mathbb{S}^n_s(n/2(n+1)) \to \mathfrak{so}(n+1,s)$ defined by $\phi(x) = \sqrt{n/4(n+1)}(xx^t G - (2/n)I)$, where x is regarded as a 1-column matrix and I denote the unix matrix of degree $n+1$. It is easy to see that ϕ is an isometric immersion provided that $\mathfrak{so}(n+1,s)$ is endowed with the metric $g(P,Q) = \mathrm{trace}(PQ)$, so that $\mathfrak{so}(n+1,s) \cong \mathbb{R}^m_{s(n+1-s)}$ with $m = n(n+3)/2$. The image of ϕ is contained in $\mathbb{S}^{m-1}_{s(n+1-s)}$ as a constant isotropic submanifold with zero mean curvature vector [7]. Now, by Corollary 4.3, $\lambda = (n-1)/(n+1)$. Take $s = 0$ in the definition of ϕ and let $\iota : \mathbb{S}^{m-1} \to \mathbb{S}^m_1$ be the standard totally geodesic inclusion. Then, as the composition of isotropic immersions is also isotropic, it is easy to see that $\iota \circ \phi : \mathbb{S}^n(n/2(n+1)) \to \mathbb{S}^m_1$ is a complete maximal constant isotropic immersion with $\lambda = (n-1)/(n+1) > 0$.

Proposition 4.7 in [1] proved that there exists no pseudo-umbilical compact spacelike submanifolds with lightlike parallel mean curvature vector in a pseudo-hyperbolic space. This result combined with Corollary 5.3 and Theorem 5.1 gives the following.

Corollary 6.7 *Let N^n be a compact spacelike isotropic submanifold into the pseudo-hyperbolic space $\mathbb{H}^m_\nu$, $n \geq 3$. Suppose that N^n is pseudo-umbilical with lightlike parallel mean curvature vector. Then, N^n is constant isotropic and the isotropy constant satisfies $\lambda \geq 2(1-n)/(n+2)$.*

On the other hand, appliying the Bochner' technique we have

Theorem 6.8 *Let $\phi : N^n \to M^m_\nu(c)$ be an isotropic and pseudo-umbilical immersion of a compact spacelike manifold N^n $(n \geq 3)$ admiting a nontrivial conformal vector field X. Then, the isotropy function λ satisfies*

$$\lambda \leq \frac{2(n-1)}{n+2}c + \frac{3n}{n+2}g(\mathbf{H},\mathbf{H}). \tag{17}$$

If the equality holds at a point $p \in N^n$, then X is parallel, the first Betti number of N^n is not zero and Iso(N^n) *is finite. Moreover, if N^n is homogeneous, N^n is isometric to a flat n-dimensional Riemannian torus.*

Proof. From Theorem 5.1 N^n is Einstein and $\mathrm{Ric} = \mu g$ with $\mu = c(n-1) - \lambda(n+2)/2 + 3ng(\mathbf{H},\mathbf{H})/2$. By applying Bochner formula [22, Theorem 5.10],

$$\int_{N^n} \{\mu g(X,X) + \mathrm{trace}(\nabla X \circ \nabla X) - (\mathrm{div}(X))^2\} = 0. \tag{18}$$

Since X is conformal, $\mathrm{trace}(\nabla X \circ \nabla X) = -\|\nabla X\|^2 + 2\mathrm{div}(X)$. Thus, using the classical divergence theorem in Eq. (18) we get

$$\int_{N^n} \{\mu g(X,X) - \|\nabla X\|^2 - (\mathrm{div}(X))^2\} = 0. \tag{19}$$

But $\mu \geq 0$ because X is nontrivial, and inequality (17) is fulfilled. If the equality holds at a point $p \in N^n$, then $\mu = 0$ and from Eq. (19), X is parallel. In this case N^n is Ricci-flat and by Theorems 1.84 and 7.61 in [5] the result follows.

□

Corollary 6.9 *Let $\phi : N^n \to \mathbb{S}^m_\nu$ be a maximal isotropic immersion of a compact manifold N^n ($n \geq 3$) admitting a nontrivial conformal vector field X. Then, ϕ is constant isotropic with isotropy constant*

$$\lambda \leq \frac{2(n-1)}{n+2}. \tag{20}$$

If the equality holds, then X is parallel, the first Betti number of N^n is not zero and $\mathrm{Iso}(N^n)$ *is finite. Moreover, if N^n is homogeneous, N^n is isometric to a flat n-dimensional Riemannian torus.*

Proof. This follows immediately from Corollary 5.4 and Theorem 6.8.

□

When the spacelike submanifold N^n is homogeneous we have:

Corollary 6.10 *Let $\phi : N^n \to M^m_\nu(c)$ be an isotropic and pseudo-umbilical immersion of a homogeneous spacelike submanifold N^n with $n \geq 3$. Suppose that the isotropic function λ satisfies*

$$\lambda > \frac{2(n-1)}{n+2}c + \frac{3n}{n+2}g(\mathbf{H},\mathbf{H}). \tag{21}$$

Then, N^n is non-compact.

Proof. It is sufficient to observe that a Riemannian homogeneous space admits a nontrivial Killing field ([21, Corollary 9.38]). Applying Theorem 6.8 the result follows.

□

Next, we give an example satisfying the assumption of the last corollary.

Example 6.11 We define the isometric immersion $\psi : \mathbb{H}^n \to \mathbb{H}^{n+2}_1$ by $\psi(x) = (f(x), \iota(x), f(x))$, where $f : \mathbb{H}^n \to \mathbb{R}$ is a smooth function and $\iota : \mathbb{H}^n \to \mathbb{R}^{n+1}$ denotes the canonical inclusion. The second fundamental h and the mean curvature vector $\mathbf{H}$ of ϕ are given by

$$h(X,Y) = \big((\nabla^2 f)(X,Y) - fg(X,Y)\big)\,\eta, \qquad \mathbf{H} = \frac{1}{n}\,(\Delta f - nf)\,\eta,$$

where η denotes the lightlike vector $(1,0,\ldots,0,1) \in \mathbb{R}^{n+3}_2$, $\nabla^2 f$ is the Hessian of f in $\mathbb{H}^n$ and $\Delta f = \mathrm{trace}_g(\nabla^2 f)$ is the Laplacian. Then, ψ is 0-isotropic, pseudo-umbilical and satisfies (21).

7 Lorentzian isotropic submanifolds in pseudo-Riemannian space forms

Recall that a *reference frame* on a Lorentzian manifold N^n_1 is a vector field U on N^n_1 which satisfies $g(U,U) = -1$. In General Relativity, a reference frame in a spacetime is seen as a vector field that each of its integral curves is an observer (i.e. a particle of unit mass). There are remarkable families of reference frames of geometric interest. In fact, a reference frame U is said to be *spatially conformal* (resp. *spatially stationary* or *rigid*) if $\mathcal{L}_U g(X,Y) = 2\rho g(X,Y)$, where $\rho : N^n_1 \to \mathbb{R}$ (resp. $\mathcal{L}_U g(X,Y) = 0$) for all $X, Y \perp U$. A Lorentzian manifold which admits a timelike conformal (resp. Killing) vector field is called conformally stationary (resp. stationary) [23]. In General Relativity spatially conformal vector fields model sets of observers which see a constant metric in the spatial part for them. Spatially stationary reference frames model observers which see an expansion or compression along their proper time. The existence of conformal symmetries is a quite general and useful assumption to study Einstein equations.

Note that if X is a timelike conformal (resp. Killing) vector field on N_1^n, then the reference frame $U = (1/\sqrt{-g(X,X)})X$ is spatially conformal (resp. spatially stationary). However, there exist spatially conformal reference frames which cannot be obtained in that way. In fact, if $n = 2$ then every reference frame is indeed spatially conformal. But a time-orientable incomplete Lorentzian torus does not admit a timelike conformal vector field [23].

The study of spatially conformal reference frames on a Lorentzian manifold has been shown to be useful in solving several mathematical problems. In particular, in [23, 24] it has been proved that if N_1^n is an n-dimensional ($n \geq 3$) compact Lorentzian manifold is Einstein with $\mathrm{Ric} = \mu g$, $\mu \in \mathbb{R}$, and admits a spatially conformal reference frame (or a timelike projective vector field), then $\mu \leq 0$. Using this result, Theorem 5.1 allows us to state the following.

Theorem 7.1 *Let $\phi : N_1^n \to M_\nu^m(c)$ be an isotropic and pseudo-umbilical immersion of a compact Lorentz manifold N_1^n ($n \geq 3$). Then, if N_1^n admits a spatially conformal reference frame (or a timelike projective vector field), the isotropy function λ satisfies*

$$\lambda \geq \frac{2(n-1)}{n+2}c + \frac{3n}{n+2}g(\mathbf{H},\mathbf{H}). \tag{22}$$

Corollary 7.2 *Let $\phi : N_1^n \to M_\nu^m(c)$ be an isotropic immersion with $\mathbf{H} = 0$ of a compact Lorentz manifold N_1^n ($n \geq 3$). Then, if N_1^n admits a spatially conformal reference frame (or a timelike projective vector field), ϕ is constant isotropic with isotropy constant*

$$\lambda \geq \frac{2(n-1)}{n+2}c. \tag{23}$$

Proof. This follows immediately from Corollary 5.4 and Theorem 7.1.

□

Analogously, by using a uniqueness result about compact Ricci-flat Lorentzian manifolds admitting a timelike conformal vector field, Corollary 3.9 in [23], we have the following.

Theorem 7.3 *Let $\phi : N_1^n \to M_\nu^m(c)$ be an isotropic and pseudo-umbilical immersion of a compact Lorentz manifold N_1^n ($n \geq 3$) admitting a timelike conformal vector field X. Then, the isotropy function λ satisfies*

$$\lambda \geq \frac{2(n-1)}{n+2}c + \frac{3n}{n+2}g(\mathbf{H},\mathbf{H}). \tag{24}$$

If the equality holds at a point, then X is parallel, the first Betti number of N_1^n is not zero and the Levi-Civita connection of N_1^n is Riemannian. Moreover, if any of the following conditions

(1) N_1^n is homogeneous,

(2) N_1^n is flat (in particular if $n = 3$),

(3) $n = 4$,

holds, then N_1^n is isometric (up to a finite covering in the cases (2) and (3)) to a flat n-dimensional Lorentzian torus.

Remark 7.4 It is well known that in the Riemannian case a homogeneous Ricci-flat manifold is flat, but in the Lorentzian case the corresponding result is not true. Observe that in the last theorem the hypothesis of pseudo-umbilicity can be replaced by minimality.

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