

On the controllability of a free-boundary problem for the 1D heat equation

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Abstract

This paper deals with the local null control of a free-boundary problem for the classical 1D heat equation with distributed controls, locally supported in space. In the main result we prove that, if the final time T is fixed and the initial state is sufficiently small, there exist controls that drive the state exactly to rest at time $t = T$.

Keywords: null controllability, free-boundary problems, 1D heat equation, Carleman estimates.

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1 Introduction. The main result

Assume that $L_0 > 0$, $T > 0$, $0 < a < b < L_* < L_0$ and $y_0 \in L^\infty(0, L_0)$ are given. In this paper, we consider free-boundary problems of the following kind:

Find $L \in C^1([0, T])$ with $L(t) > 0$ for all t and a function $y = y(x, t)$ such that

$$L(0) = L_0, \tag{1}$$

$$\begin{cases} y_t - y_{xx} = v1_\omega, & (x, t) \in Q_L, \\ y(0, t) = 0, \quad y(L(t), t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), & x \in (0, L_0) \end{cases} \tag{2}$$

and, moreover,

$$y_x(L(t), t) = -L'(t), \quad t \in (0, T). \tag{3}$$

Here, we have used the notation

$$Q_L = \{ (x, t) : x \in (0, L(t)), \quad t \in (0, T) \}, \quad \omega = (a, b);$$

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as usual, 1_ω denotes the characteristic function of ω .

In (1)–(3), v is a control and (L, y) is an associated state. Recall that, for any $y_0 \in H_0^1(0, L_0)$ and $v \in L^2(\omega \times (0, T))$, there exists at least one *local in time* solution to (2)–(3). In other words, there exist $T_* \in (0, T]$ and a couple (L, y) with

$$\begin{cases} L \in C^1([0, T_*)), & L(t) \geq L_* \text{ in } [0, T_*), \\ y, y_x, y_{xx}, y_t \in L^2(Q_L^*), & \text{with } Q_L^* := \{(x, t) : x \in (0, L(t)), t \in (0, T_*)\}, \\ L(0) = L_0, & y(x, 0) = y_0(x) \text{ in } (0, L_0), \end{cases}$$

such that the PDE in (2) is satisfied in Q_L^* in the distributional sense and the boundary conditions and (3) are fulfilled in $(0, T_*)$. Furthermore, either $T_* = T$, or at least one of the following identities hold:

$$\lim_{t \rightarrow T_*^-} L(t) = b \quad \text{or} \quad \lim_{t \rightarrow T_*^-} |L'(t)| = +\infty.$$

This is established in [7]; see also [16].

Free-boundary problems similar to (1)–(3) are motivated by many different applications:

- Solidification processes and, in particular, the so called *Stefan problem*, see [11, 12].
- The analysis and computation of free surface flows, see [18, 22, 25].
- Fluid-solid interaction, see [3, 21, 24].
- Gas flow through porous media, see [1, 6, 23].
- Tumor growth and other problems from mathematical biology, see [13, 14], etc.

Note that, for a fixed control v , (1) and (2) are not enough to identify the state (L, y) . We need an additional information and this is furnished by (3). In many of these areas, this condition is completely natural. For instance, in tumor growth modelling, y can be viewed as a pressure (cells are pushed towards the low density regions) and (3) says that, on the moving tumor boundary, the growth speed is proportional to the pressure gradient. Of course, this is a version of the well known *Darcy's law* for porous media.

The main goal in this paper is to analyze the null controllability of (1)–(3). By definition, it will be said that (1)–(3) is *null-controllable* if, for any $y_0 \in H_0^1(0, L_0)$, there exist a control $v \in L^2(\omega \times (0, T))$, a function $L \in C^1([0, T])$ and an associated solution $y = y(x, t)$ satisfying (1)–(3) and

$$y(x, T) = 0, \quad x \in (0, L(T)). \quad (4)$$

Notice that, if L and (v, y) solve (1)–(4) and (for instance) $y(\cdot, T) \in H^2(0, L(T))$, then $L'(T) = 0$. Consequently, the null controllability of (1)–(3) is a useful property from the viewpoint of control theory: roughly speaking, if it is fulfilled, we only have to “work” during a finite time interval in order to get the desired behavior of the system.

The controllability of linear and nonlinear PDEs has been the objective of a lot of papers the last years. In the context of the linear and semilinear heat equation, the main contributions have been

obtained in [8, 20, 5, 15, 10, 4]. On the other hand, to our knowledge, for parabolic free-boundary problems, controllability questions have not been considered in depth; see however [2, 17].

In the sequel, C denotes a generic positive constant; C_0, C_1 , etc. are other positive (specific) constants; when it makes sense, the extension by zero in space of any function f is denoted by f^* .

The main result in this paper is the following:

Theorem 1.1 *Let us assume that $L_0 > 0$, $T > 0$ and $0 < a < b < L_* < L_0 < B$ are given. Then (1)–(3) is locally null-controllable. More precisely, there exists $\varepsilon > 0$ such that, if $y_0 \in H_0^1(0, L_0)$ and $\|y_0\|_{H_0^1(0, L_0)} \leq \varepsilon$, there exist L and (v, y) , with*

$$L \in C^1([0, T]), \quad L_* \leq L(t) \leq B, \quad v \in L^2(\omega \times (0, T)), \quad y^* \in C^0([0, T]; H_0^1(0, B)),$$

that satisfy (1)–(3) and (4).

The plan of the proof is the following:

- First, we prove that there exists $\varepsilon > 0$ such that, whenever $\|y_0\|_{H_0^1(0, L_0)} \leq \varepsilon$, for each $\beta > 0$ there exist uniformly bounded L_β and (v_β, y_β) satisfying (1)–(3) and

$$\|y_\beta(\cdot, T)\|_{L^2(0, L(T))} \leq \beta. \quad (5)$$

This is achieved by solving an appropriate fixed point equation in a closed convex set $\mathcal{M} \subset C^1([0, T])$:

$$L = \Lambda_\beta(L), \quad L \in \mathcal{M}.$$

- Then, we take limits as $\beta \rightarrow 0$. Thus, from the estimates deduced for L_β and (v_β, y_β) , we see that, at least for a subsequence, we have

$$\begin{aligned} L_\beta &\rightarrow L \text{ strongly in } C^1([0, T]), \\ v_\beta &\rightarrow v \text{ weakly in } L^2(\omega \times (0, T)), \\ y_\beta^* &\rightarrow y^* \text{ strongly in } C^0([0, T]; H_0^1(0, B)), \end{aligned}$$

where (v, y) is a control-state pair satisfying (1)–(3) and (4).

The rest of this paper is organized as follows.

In Section 2, we prove that, for any $L \in C^1([0, T])$ satisfying

$$L_* \leq L \leq B, \quad L(0) = L_0, \quad (6)$$

the linear system (2) is approximately and null-controllable. We also prove that the solutions to (2) satisfy an additional regularity property. Section 3 is devoted to prove Theorem 1.1. In Section 4, we present some additional comments and we give some indications on future work. Finally, Section 5 is an Appendix where we sketch the proof of a Carleman estimate.

2 Some controllability results for the classical heat equation in a non-cylindrical domain

In this section, we assume that $L_0 > 0$, $T > 0$ and $0 < a < b < L_* < L_0 < B$ are given.

We fix $y_0 \in H_0^1(0, L_0)$ and we assume that $L \in C^1([0, T])$ is a prescribed function satisfying (6); in particular, note that $L(t) > b$ for all $t \in [0, T]$.

Throughout this paper, we will use the notation

$$N_L := \|L'\|_\infty, \quad N_0 := \|y_0\|_{H_0^1(0, L_0)}. \quad (7)$$

2.1 The problems and the results

Let us consider the linear system

$$\begin{cases} y_t - y_{xx} = v1_\omega, & (x, t) \in Q_L, \\ y(0, t) = 0, \quad y(L(t), t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), & x \in (0, L_0). \end{cases} \quad (8)$$

For every $v \in L^2(\omega \times (0, T))$ and every $y_0 \in H_0^1(0, L_0)$, there exists exactly one solution to (8), with

$$y, y_x, y_{xx}, y_t \in L^2(Q_L)$$

and, consequently,

$$y^* \in C^0([0, T]; H_0^1(0, B)).$$

Indeed, an appropriate change of variable allows to rewrite (8) as a similar problem for a parabolic PDE of the form

$$z_\tau - z_{\xi\xi} + h(\xi, \tau)z_\xi = w$$

in a cylindrical domain, with a bounded coefficient h and a square-integrable right hand side w ; see the precise definitions of ξ and τ below, after (21).

The following controllability result is satisfied by (8):

Theorem 2.1 *For any $y_0 \in H_0^1(0, L_0)$ and any $\beta > 0$ there exist pairs (v_β, y_β) , with*

$$v_\beta \in L^2(\omega \times (0, T)), \quad y_\beta^* \in C^0([0, T]; H_0^1(0, B)),$$

satisfying (8) and

$$\|y_\beta(\cdot, T)\|_{L^2(0, L(T))} \leq \beta. \quad (9)$$

Furthermore, v_β can be found such that

$$\|v_\beta\|_{L^2(\omega \times (0, T))} \leq C_1 \|y_0\|_{H_0^1(0, L_0)}, \quad (10)$$

where C_1 depends on N_L , L_ , B , ω and T , but is independent of β .*

An immediate consequence is the null controllability of (8):

Corollary 2.1 For any $y_0 \in H_0^1(0, L_0)$, there exist pairs (v, y) , with

$$v \in L^2(\omega \times (0, T)), \quad y^* \in C^0([0, T]; H_0^1(0, B)),$$

satisfying (8) and (4). Furthermore, v can be found such that

$$\|v\|_{L^2(\omega \times (0, T))} \leq C_1 \|y_0\|_{H_0^1(0, L_0)}, \quad (11)$$

where C_1 only depends on N_L, L_*, B, ω and T .

The proof of Theorem 2.1 follows rather standard arguments. The main tool is a global Carleman estimate for the solution to the *adjoint system* of (8), that is given by

$$\begin{cases} -\varphi_t - \varphi_{xx} = g(x, t) & (x, t) \in Q_L, \\ \varphi(0, t) = 0, \quad \varphi(L(t), t) = 0, & t \in (0, T), \\ \varphi(x, T) = \varphi^T(x), & x \in (0, L(T)), \end{cases} \quad (12)$$

where $g \in L^2(Q_L)$ and $\varphi^T \in L^2(0, L(T))$.

This will be established in the next section.

2.2 A Carleman estimate

In the sequel, we will denote by Σ_L the lateral boundary of Q_L :

$$\Sigma_L = \{ (x, t) : x = 0 \text{ or } x = L(t), \quad 0 < t < T \}.$$

We begin with a simple result:

Lemma 2.1 Let ω_0 be a non-empty open set with $\overline{\omega_0} \subset \omega$. There exists a function $\alpha_0 \in C^1(\overline{Q_L})$ with $\alpha_{0,xx} \in C^0(\overline{Q_L})$ such that

$$\begin{cases} \alpha_0(x, t) = 0 & \forall (x, t) \in \Sigma_L, \\ |\alpha_{0,x}| > 0 & \text{in } \overline{Q_L} \setminus (\omega_0 \times (0, T)) \text{ and} \\ \alpha_0(x, t) = 1 - \frac{x-b}{L(t)-b} & \forall x \in (b, L(t)), \quad \forall t \in [0, T]. \end{cases}$$

For the proof, it suffices to take (for instance)

$$\alpha_0(x, t) := \begin{cases} \frac{x}{a}, & \text{if } 0 \leq x < a, \\ 1 + p \left(\frac{2(x-a)}{b-a}, \frac{b-a}{2a} \right), & \text{if } a \leq x < \frac{a+b}{2}, \\ 1 + p \left(\frac{2(b-x)}{b-a}, \frac{b-a}{2(L(t)-b)} \right), & \text{if } \frac{a+b}{2} \leq x < b, \\ \frac{L(t)-x}{L(t)-b}, & \text{if } b \leq x \leq L(t), \end{cases}$$

where

$$p(w, z) := zw + (10 - 6z)w^3 + (8z - 15)w^4 + (6 - 3z)w^5.$$

In particular, notice that $|\alpha_{0,x}| \geq \eta$ in $\overline{Q_L} \setminus (\omega_0 \times (0, T))$, where

$$\frac{1}{\eta} = \max \left\{ a, \max_{0 \leq t \leq T} (L(t) - b) \right\}.$$

Let us introduce the weights ξ and α and the functions α_1 and γ , with

$$\xi(x, t) := \frac{e^{\lambda \alpha_1(x, t)}}{\gamma(t)}, \quad \alpha(x, t) := \frac{e^{2\lambda \|\alpha_1\|_\infty} - e^{\lambda \alpha_1(x, t)}}{\gamma(t)}, \quad \alpha_1(x, t) := \alpha_0(x, t) + 1, \quad \gamma(t) := t^k (T - t)^k,$$

where $\lambda > 0$ and $k \geq 2$ are real numbers.

We will present now a Carleman estimate for the solution φ of the adjoint system (12). It is inspired by the ideas in Fursikov-Imanovilov [15] and is contained in the following theorem:

Theorem 2.2 *Let α_0 , α , γ and ξ be the functions defined above. There exist positive constants λ_0 , s_0 and C_0 , only depending on N_L , L_* , B , ω and T , such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$, any $g \in L^2(Q_L)$ and any $\varphi^T \in L^2(0, L(T))$, one has*

$$\begin{aligned} & \iint_{Q_L} e^{-2s\alpha} \left[(s\xi)^{-1} (|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2 (s\xi) |\varphi_x|^2 + \lambda^4 (s\xi)^3 |\varphi|^2 \right] dx dt \\ & + \int_0^T e^{-2s\alpha(L(t), t)} \lambda s \xi(L(t), t) |\varphi_x(L(t), t)|^2 dt + \int_0^T e^{-2s\alpha(0, t)} \lambda s \xi(0, t) |\varphi_x(0, t)|^2 dt \\ & \leq C_0 \left(\iint_{Q_L} e^{-2s\alpha} |g|^2 dx dt + \iint_{\omega \times (0, T)} e^{-2s\alpha} \lambda^4 (s\xi)^3 |\varphi|^2 dx dt \right), \end{aligned} \quad (13)$$

where φ is the corresponding solution to (12).

The proof is given in the Appendix (see Section 5).

2.3 An observability inequality and the controllability properties of (8)

The first consequence of Theorem 2.2 is an *observability inequality*. We will consider the homogeneous adjoint system

$$\begin{cases} -\varphi_t - \varphi_{xx} = 0, & (x, t) \in Q_L, \\ \varphi(0, t) = 0, \quad \varphi(L(t), t) = 0, & t \in (0, T), \\ \varphi(x, T) = \varphi^T(x), & x \in (0, L(T)), \end{cases} \quad (14)$$

where $\varphi^T \in L^2(0, L(T))$.

Proposition 2.1 *There exists $C_2 > 0$, only depending on N_L , L_* , B , ω and T , such that, for any $\varphi^T \in L^2(0, L(T))$, the associated solution to (14) satisfies*

$$\int_0^{L_0} |\varphi(x, 0)|^2 dx \leq C_2 \iint_{\omega \times (0, T)} |\varphi|^2 dx dt. \quad (15)$$

Proof: Let us take $\lambda = \lambda_0$ and $s = s_0$ in (13). Then

$$\iint_{Q_L} e^{-2s_0\alpha} \xi^3 |\varphi|^2 dx dt \leq C \iint_{\omega \times (0, T)} e^{-2s_0\alpha} \xi^3 |\varphi|^2 dx dt$$

and, consequently,

$$\int_{T/4}^{3T/4} \int_0^{L(t)} |\varphi|^2 dx dt \leq C \int_{T/4}^{3T/4} \int_0^{L(t)} e^{-2s_0\alpha\xi^3} |\varphi|^2 dx dt \leq C \iint_{\omega \times (0, T)} |\varphi|^2 dx dt. \quad (16)$$

On the other hand, by multiplying the PDE in (14) by φ and integrating in $(0, L(t))$, we get the identities

$$-\frac{1}{2} \frac{d}{dt} \left(\int_0^{L(t)} |\varphi|^2 dx \right) + \frac{1}{2} L'(t) |\varphi(L(t), t)|^2 + \int_0^{L(t)} |\varphi_x|^2 dx = 0 \quad \forall t \in (0, T).$$

Since $\varphi(L(t), t) \equiv 0$, we deduce that

$$\frac{d}{dt} \left(\int_0^{L(t)} |\varphi(x, t)|^2 dx \right) \geq 0$$

and, integrating in time, we deduce that

$$\int_0^{L(0)} |\varphi(x, 0)|^2 dx \leq \int_0^{L(t)} |\varphi(x, t)|^2 dx \quad \forall t \in (0, T)$$

and

$$\frac{T}{2} \int_0^{L(0)} |\varphi(x, 0)|^2 dx \leq \int_{T/4}^{3T/4} \int_0^{L(t)} |\varphi(x, t)|^2 dx dt. \quad (17)$$

From (16) and (17), we find (15) and the proof is done. \square

By duality, using classical arguments, it is not difficult to deduce from Proposition 2.1 the controllability result in Theorem 2.1; see [5]. Thus, let $y_0 \in H_0^1(0, L_0)$ and $\beta > 0$ be given and let us introduce the function $J_{L, \beta}$, with

$$\begin{cases} J_{L, \beta}(\varphi^T) := \frac{1}{2} \iint_{\omega \times (0, T)} |\varphi|^2 dx dt + \beta \|\varphi^T\|_{L^2(0, L(T))} + (\varphi(\cdot, 0), y_0)_{L^2(0, L_0)} \\ \forall \varphi^T \in L^2(0, L(T)), \end{cases} \quad (18)$$

where, for each $\varphi^T \in L^2(0, L(T))$, we have denoted by φ the unique solution to (14). Then, $J_{L, \beta}$ possesses a unique minimizer $\hat{\varphi}_\beta^T$ and the “best” control is given by

$$v_\beta = \hat{\varphi}_\beta|_{\omega \times (0, T)},$$

where $\hat{\varphi}_\beta$ is the solution to (14) associated to $\hat{\varphi}_\beta^T$.

Furthermore, from the inequality $J_{L, \beta}(\hat{\varphi}_\beta^T) \leq J_{L, \beta}(0) = 0$ and (15), we deduce that

$$\begin{aligned} & \frac{1}{2} \iint_{\omega \times (0, T)} |\hat{\varphi}_\beta|^2 dx dt + \beta \|\hat{\varphi}_\beta^T\|_{L^2(0, L(T))} \\ & \leq -(\hat{\varphi}_\beta(\cdot, 0), y_0)_{L^2(0, L_0)} \\ & \leq \frac{1}{4C_2} \|\hat{\varphi}_\beta(\cdot, 0)\|_{L^2(0, L_0)}^2 + C_2 \|y_0\|_{L^2(0, L_0)}^2 \\ & \leq \frac{1}{4} \iint_{\omega \times (0, T)} |\hat{\varphi}_\beta|^2 dx dt + C_2 \|y_0\|_{L^2(0, L_0)}^2 \end{aligned}$$

and, consequently,

$$\|v_\beta\|_{L^2(\omega \times (0,T))}^2 \leq \iint_{\omega \times (0,T)} |\hat{\varphi}_\beta|^2 dx dt + 2\beta \|\hat{\varphi}_\beta^T\|_{L^2(0,L(T))} \leq 4C_2 \|y_0\|_{L^2(0,L_0)}^2. \quad (19)$$

This way, we find the estimates (10) and (11), where the constant C_1 only depends on N_L , L_* , B , ω and T ; see for instance [15, 9] for more details.

2.4 A regularity property

Let (v, y) be a control-state pair furnished by Theorem 2.1 or Corollary 2.1 and let us introduce the set

$$R_L := Q_L \cap \{ (x, t) : x > b' \},$$

where $b < b' < L_*$. We will see in this section that, for some $\kappa \in (0, 1/2]$ only depending on N_L , L_* , B , ω , T and N_0 , one has

$$y \in C_{x,t}^{1+\kappa, \kappa/2}(\overline{R_L}), \quad (20)$$

where $C_{x,t}^{1+\kappa, \kappa/2}(\overline{R_L})$ is the space of functions $z \in C^0(\overline{R_L})$ that possess continuous partial derivatives with respect to x in $\overline{R_L}$ and satisfy

$$\sup_{(x,t), (x',t') \in \overline{R_L}} \left(\frac{|z(x,t) - z(x',t')|}{|x - x'| + |t - t'|^{\kappa/2}} + \frac{|z_x(x,t) - z_x(x',t')|}{|x - x'|^\kappa + |t - t'|^{\kappa/2}} \right) < +\infty.$$

Indeed, let $\delta > 0$ be sufficiently small to have $b < b' - \delta < b' + \delta < L_*$. In view of Theorems 10.1 and 11.1 in [19], pp. 204–211, there exists $\kappa \in (0, 1/2]$ depending on N_L , L_* , B , ω , T and N_0 such that

$$y \in C_{x,t}^{1+\kappa, \kappa/2}([b' - \delta, b' + \delta] \times [0, T]). \quad (21)$$

Let us now introduce the following change of variables in R_L :

$$\xi = \frac{1}{L(t) - b'} (b'(L(t) - x) + L_0(x - b')), \quad \tau = (L_0 - b')^2 \int_0^t \frac{1}{(L(s) - b')^2} ds$$

and let us set $z(\xi, \tau) := y(x, t)$. Then, it is not difficult to see that z is well defined in the rectangle $\overline{R_*}$, where

$$R_* := (b', L_0) \times (0, S_L), \quad S_L := (L_0 - b')^2 \int_0^T \frac{1}{(L(t) - b')^2} ds$$

and, moreover,

$$\begin{cases} z_\tau - z_{\xi\xi} + h(\xi, \tau)z_\xi = 0, & (\xi, \tau) \in R_*, \\ z(b', \tau) = y(b', t), \quad z(L_0, \tau) = 0, & \tau \in (0, S_L), \\ z(\xi, 0) = y_0(\xi), \quad \xi \in (0, L_0), \end{cases}$$

where $h \in C^0(\overline{R_*})$ and $\|h\|_\infty$ is bounded by a constant only depending on N_L , b' , L_0 and T . Consequently, taking again into account Theorems 10.1 and 11.1 in [19] and the facts that y satisfies (21) and $z(b', \tau) \equiv y(b', t)$, we deduce that

$$z \in C_{\xi, \tau}^{1+\kappa, \kappa/2}(\overline{R_*}),$$

whence we get (20).

Now, let the function V_L be given by

$$V_L(t) := y_x(L(t), t) \quad \forall t \in [0, T]. \quad (22)$$

Obviously, $V_L \in C^\kappa([0, T])$. Furthermore, from the estimates in the proofs of Theorems 10.1 and 11.1 in [19], we also have

$$\|V_L\|_{C^\kappa([0, T])} \leq C_3 \|y_0\|_\infty, \quad (23)$$

where C_3 only depends on N_L , L_* , B , ω , T and N_0 and is non-decreasing with respect to N_L and N_0 .

3 Proof of the main result

We will first see that there exists $\varepsilon > 0$ such that, if $N_0 = \|y_0\|_{H_0^1(0, L_0)} \leq \varepsilon$, for any small $\beta > 0$, there exist uniformly bounded $L_\beta \in C^1([0, T])$ and control-pairs (v_β, y_β) satisfying (1)–(3) and (9). To this purpose, we will use a fixed point argument.

Thus, let $\beta > 0$ and $R > 0$ be given, let us set

$$\mathcal{M} := \{ \ell \in C^1([0, T]) : L_* \leq \ell \leq B, \ell(0) = L_0, N_\ell := \|\ell'\|_\infty \leq R \}$$

and let us introduce the mapping $\Lambda_\beta : \mathcal{M} \mapsto C^1([0, T])$, defined as follows: for each $\ell \in \mathcal{M}$,

$$L = \Lambda_\beta(\ell) \quad \text{if and only if} \quad L(t) = L_0 - \int_0^t y_x(\ell(s), s) ds \quad \forall t \in [0, T], \quad (24)$$

where y is the solution to the linear system

$$\begin{cases} y_t - y_{xx} = v1_\omega, & (x, t) \in Q_\ell, \\ y(0, t) = 0, \quad y(\ell(t), t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), & x \in (0, L_0), \end{cases} \quad (25)$$

and v is the β -control of minimal norm in $L^2(\omega \times (0, T))$ associated to ℓ , i.e. the unique solution to the following extremal problem:

$$\begin{cases} \text{Minimize} & \iint_{\omega \times (0, T)} |v|^2 dx dt \\ \text{Subject to} & v \in L^2(\omega \times (0, T)), \quad (25), \quad (9). \end{cases} \quad (26)$$

Let us see that, if N_0 is small enough (depending on R but not on β), we can apply Schauder's Theorem to Λ_β in \mathcal{M} and deduce the existence of a fixed point.

Obviously, \mathcal{M} is a non-empty bounded, closed, convex subset of $C^1([0, T])$. In view of Theorem 2.1 and the strict convexity of the norm in $L^2(\omega \times (0, T))$, (26) possesses exactly one solution for each $\ell \in \mathcal{M}$ and the mapping Λ_β is well-defined.

- Let us see that Λ_β is continuous.

Thus, let us assume that $\ell_n \in \mathcal{M}$ for all $n \geq 1$ and $\ell_n \rightarrow \ell$ in $C^1([0, T])$. Let us denote by (v^n, y^n) the control-state pair associated to ℓ_n (v^n is the solution to (26) for $\ell = \ell_n$ and y^n is the corresponding state). Then, we know that

$$v^n = \hat{\varphi}_n|_{\omega \times (0, T)},$$

where $\hat{\varphi}_n$ is the solution to (14) for $L = \ell_n$ and $\varphi^T = \hat{\varphi}_n^T$, while $\hat{\varphi}_n^T$ is the unique minimizer of $J_{\ell_n, \beta}$ (see (18) for the definition of this functional).

From (19), we see that $\|\hat{\varphi}_n^T\|_{L^2(0, \ell(T))}$ is uniformly bounded. Consequently, at least for a subsequence (again indexed by n), one has:

$$\begin{aligned} \hat{\varphi}_n^T &\rightarrow \hat{\varphi}^T \text{ weakly in } L^2(0, \ell(T)), \\ \hat{\varphi}_n &\rightarrow \hat{\varphi} \text{ strongly in } L^2(\omega \times (0, T)), \\ \hat{\varphi}_n(\cdot, 0) &\rightarrow \hat{\varphi}(\cdot, 0) \text{ strongly in } L^2(0, L_0), \end{aligned}$$

where $\hat{\varphi}$ is the solution to (14) for $L = \ell$ and $\varphi^T = \hat{\varphi}^T$. In particular, this implies:

$$\liminf J_{\ell_n, \beta}(\hat{\varphi}_n^T) \geq J_{\ell, \beta}(\hat{\varphi}^T). \quad (27)$$

Let us check that $\hat{\varphi}^T$ is the unique minimizer of $J_{\ell, \beta}$. This will prove that *the whole sequence* $\{\hat{\varphi}_n^T\}$ converges to $\hat{\varphi}^T$.

Thus, let $\varphi^T \in L^2(0, \ell(T))$ be given and let the functions $\varphi_n^T \in L^2(0, \ell_n(T))$ be such that their extensions by zero converge strongly in $L^2(0, B)$ to the extension by zero of φ^T . Let us denote by φ_n (resp. φ) the solution to (14) with $L = \ell_n$ and $\varphi^T = \varphi_n^T$ (resp. $L = \ell$). Then, it is clear that $\varphi_n \rightarrow \varphi$ strongly in $L^2(\omega \times (0, T))$ and $\varphi_n(\cdot, 0) \rightarrow \varphi(\cdot, 0)$ strongly in $L^2(0, L_0)$ and, consequently,

$$J_{\ell_n, \beta}(\varphi_n^T) \rightarrow J_{\ell, \beta}(\varphi^T). \quad (28)$$

A direct consequence of (27) and (28) is that

$$J_{\ell, \beta}(\hat{\varphi}^T) \leq \liminf J_{\ell_n, \beta}(\hat{\varphi}_n^T) \leq \liminf J_{\ell_n, \beta}(\varphi_n^T) = J_{\ell, \beta}(\varphi^T)$$

and, since φ^T is arbitrary, $\hat{\varphi}^T$ minimizes $J_{\ell, \beta}$ in $L^2(0, \ell(T))$.

An immediate consequence is that the controls v^n converge strongly to the solution to (26). Also, the states y^n converges strongly (for instance in the L^2 sense) to the associated state y . Finally, from the estimates in Section 2.4, it becomes clear that $y_x^n(\ell_n(\cdot), \cdot) \rightarrow y_x(\ell(\cdot), \cdot)$ strongly in $C^0([0, T])$ and $\Lambda_\beta(\ell_n) \rightarrow \Lambda_\beta(\ell)$ strongly in $C^1([0, T])$.

Hence, Λ_β is continuous.

- The mapping Λ_β is also compact. In fact, all the $\Lambda_\beta(\ell)$ with $\ell \in \mathcal{M}$ belong to a fixed compact set of $C^1([0, T])$ (independent of β).

This can be seen as follows: in view of (24) and the estimates (23) (where V_L is given by (22) and C_3 only depends on N_L, L_*, B, ω, T and N_0), the $\Lambda_\beta(\ell)$ belong to a fixed bounded set of $C^{1+\kappa}([0, T])$ for some $\kappa \in (0, 1/2]$ that is independent of ℓ ; this, together with the compactness of the embedding $C^{1+\kappa}([0, T]) \hookrightarrow C^1([0, T])$, allows to conclude.

• Finally, let us see that, if N_0 is small enough (uniformly with respect to β), Λ_β maps \mathcal{M} into itself. In fact, this is clear from (23): by construction, $\Lambda_\beta(\ell) \in C^1([0, T])$ and $\Lambda_\beta(\ell)(0) = L_0$; we also have

$$|\Lambda_\beta(\ell)'(t)| = |y_x(\ell(t), t)| \leq C_3 \|y_0\|_\infty \leq C_4 N_0 \quad \text{and} \quad |\Lambda_\beta(\ell)(t) - L_0| \leq C_3 T \|y_0\|_\infty \leq C_4 T N_0$$

for all $t \in [0, T]$, where C_3 and C_4 depend on R, L_*, B, ω, T and N_0 and are non-decreasing with respect to N_0 . Therefore, if we choose ε sufficiently small and $N_0 \leq \varepsilon$, we will have

$$N_0 \leq \min \left(\frac{R}{C_4}, \frac{B - L_0}{C_4 T}, \frac{L_0 - L_*}{C_4 T} \right),$$

$$|\Lambda_\beta(\ell)'(t)| \leq R \quad \text{and} \quad L_* \leq \Lambda_\beta(\ell)(t) \leq B \quad \forall t \in [0, T],$$

whence the desired property $\Lambda_\beta(\ell) \in \mathcal{M}$ holds.

Let L_β be a fixed point of Λ_β for each $\beta > 0$. Then, it is clear that L_β satisfies, together with some v_β and y_β , (1)–(3) and (9). Moreover, L_β and v_β are uniformly bounded in $C^{1+\kappa}([0, T])$ and $L^2(\omega \times (0, T))$, respectively.

Consequently, our assertion is proved.

Now, at least for a subsequence, one has

$$L_\beta \rightarrow L \text{ strongly in } C^1([0, T]) \text{ and } v_\beta \rightarrow v \text{ weakly in } L^2(\omega \times (0, T))$$

as $\beta \rightarrow 0$. Obviously, L satisfies (1). Also, it is clear that the unique solution to (2) satisfies (3) and (4).

Hence, Theorem 1.1 is proved.

4 Some additional comments

In Theorem 1.1, we have established the null controllability of (1)–(3) with distributed controls, locally supported in space. It is unknown whether a similar global controllability result holds. Notice that the proof of such a result seems difficult: for general initial data in $H_0^1(0, L_0)$, even when $v \equiv 0$, the existence of a global in time solution to the free-boundary problem (1)–(3) fails, see [7].

To our knowledge, there are, at least, two other ways to prove Theorem 1.1 via a fixed point argument:

- We can work directly with $\beta = 0$ and introduce other controls for (25) (not necessarily of minimal norm in $L^2(\omega \times (0, T))$). A good candidate is the control furnished by the Fursikov-Imanuvilov method, described in [15]. In this case, it can be proved that the mapping Λ_0 is continuous.

- Alternatively, we can introduce a multi-valued mapping (assigning to each ℓ the whole family of null-controls) and, then, apply Kakutani's fixed point theorem.

On the other hand, it would be interesting to extend Theorem 1.1, at least, to the following situations:

1. Free-boundary problems for semilinear parabolic PDEs with (Lipschitz-continuous) nonlinear terms of the form

$$\begin{cases} y_t - y_{xx} + f(y, y_x) = v1_\omega, & (x, t) \in Q_L, \\ y(0, t) = 0, \quad y(L(t), t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), & x \in (0, L_0), \\ y_x(L(t), t) = -L'(t), & t \in (0, T), \end{cases}$$

2. Two-phase Stefan-like problems with the following structure:

$$\begin{cases} y_t - ay_{xx} = v1_\omega, & (x, t) \in Q_L, \\ z_t - bz_{xx} = 0, & (x, t) \in D_L, \\ y(0, t) = 0, \quad y(L(t), t) = z(L(t), t) = 0, \quad z(B, t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), & x \in (0, L_0), \\ z(x, 0) = z_0(x), & x \in (L_0, B), \\ (ay_x - bz_x)(L(t), t) = -L'(t), & t \in (0, T), \end{cases}$$

where D_L stands for the set

$$D_L = \{ (x, t) : x \in (L(t), B), \quad t \in (0, T) \}.$$

3. Radially symmetric and star-shaped systems, etc.

Some of these extensions will be considered in the next future.

Finally, let us mention that the exact controllability to the free trajectories of (1)–(3) is also a challenging and very interesting question.

5 Appendix: proof of Theorem 2.2

Let us introduce ψ , with $\varphi := e^{s\alpha}\psi$. Then

$$\begin{aligned} \varphi_t &= s\alpha_t e^{s\alpha}\psi + e^{s\alpha}\psi_t, & \varphi_x &= s\alpha_x e^{s\alpha}\psi + e^{s\alpha}\psi_x, \\ \varphi_{xx} &= e^{s\alpha}(\psi_{xx} + s^2\alpha_x^2\psi + 2s\alpha_x\psi_x + s\alpha_{xx}\psi). \end{aligned}$$

Taking into account that

$$\begin{aligned} \alpha_x &= -\xi_x = -\lambda\alpha_{0,x}\xi, & \alpha_{xx} &= -\lambda^2\alpha_{0,x}^2\xi - \lambda\alpha_{0,xx}\xi, \\ \alpha_t &= -k\frac{T-2t}{(T-t)^{k+1}t^{k+1}} \left(e^{2\lambda\|\alpha_1\|_\infty} - e^{\lambda\alpha_1(x,t)} \right) - \lambda\alpha_{0,t}\xi, \end{aligned}$$

we see that, for λ sufficiently large, one has:

$$|\alpha_x| \leq C\lambda\xi, \quad |\alpha_{xx}| \leq C\lambda^2\xi, \quad |\alpha_t| \leq C\xi^{1+1/k} + C\lambda\xi \leq C\lambda\xi^{3/2}$$

(recall that $k \geq 2$; also, note that, here and henceforth, λ and the constants C depend on N_L). We also have that $\psi(x, 0) \equiv 0$ in $(0, L(0))$ and $\psi(x, T) \equiv 0$ in $(0, L(T))$.

In the new variable ψ , the PDE in (12) reads

$$(\psi_t + 2s\alpha_x\psi_x) + (\psi_{xx} + s^2\alpha_x^2\psi) = -e^{-\alpha}g - s\alpha_{xx}\psi - s\alpha_t\psi.$$

Let us introduce the following notation:

$$\begin{cases} U\psi := \psi_t + 2s\alpha_x\psi_x, \\ V\psi := \psi_{xx} + s^2\alpha_x^2\psi. \end{cases}$$

Then one has:

$$U\psi + V\psi = -e^{-\alpha}g - s\alpha_{xx}\psi - s\alpha_t\psi.$$

whence

$$\begin{aligned} & \|U\psi\|_{L^2(Q_L)}^2 + \|V\psi\|_{L^2(Q_L)}^2 + 2(U\psi, V\psi)_{L^2(Q_L)} \\ &= \|e^{-\alpha}g + s\alpha_{xx}\psi + s\alpha_t\psi\|_{L^2(Q_L)}^2 \\ &\leq C \iint_{Q_L} e^{-2\alpha}|g|^2 dx dt + C \iint_{Q_L} \lambda^4 s^2 \xi^3 |\psi|^2 dx dt. \end{aligned} \tag{29}$$

After some computations, we see that

$$\begin{aligned} (U\psi, V\psi)_{L^2(Q_L)} &= s \iint_{Q_L} (-\alpha_{xx})|\psi_x|^2 dx dt \\ &\quad - \iint_{Q_L} (3s^3\alpha_x^2\alpha_{xx} + 2s^2\alpha_x\alpha_{xt})|\psi|^2 dx dt \\ &\quad + \int_0^T \left(s\alpha_x(L(t), t) - \frac{1}{2}L'(t) \right) |\psi_x(L(t), t)|^2 dt - \int_0^T s\alpha_x(0, t)|\psi_x(0, t)|^2 dt. \end{aligned}$$

Note that

$$-\alpha_x^2\alpha_{xx} = \lambda^4|\alpha_{0,x}|^4\xi^3 + \lambda^3|\alpha_{0,x}|^2\alpha_{0,xx}\xi^3 \geq C\lambda^4\xi^3 - C\lambda^3\xi^3 \quad \text{in } \overline{Q_L} \setminus (\omega_0 \times (0, T)).$$

On the other hand,

$$\alpha_x(L(t), t) \geq C > 0 \quad \text{and} \quad -\alpha_x(0, t) \geq C > 0 \quad \text{in } (0, T).$$

Consequently, we get the following for all sufficiently large s and λ :

$$\begin{aligned} (U\psi, V\psi)_{L^2(Q_L)} &\geq C \iint_{Q_L} ((s\xi)\lambda^2|\psi_x|^2 + (s\xi)^3\lambda^4|\psi|^2) dx dt \\ &\quad + Cs\lambda \int_0^T (\xi(L(t), t)|\psi_x(L(t), t)|^2 + \xi(0, t)|\psi_x(0, t)|^2) dx dt \\ &\quad - C \iint_{\omega_0 \times (0, T)} ((s\xi)\lambda^2|\psi_x|^2 + (s\xi)^3\lambda^4|\psi|^2) dx dt. \end{aligned} \tag{30}$$

From (29) and (30), we deduce that

$$\begin{aligned}
& \|U\psi\|_{L^2(Q_L)}^2 + \|V\psi\|_{L^2(Q_L)}^2 + \iint_{Q_L} ((s\xi)\lambda^2|\psi_x|^2 + (s\xi)^3\lambda^4|\psi|^2) dx dt \\
& \quad + s\lambda \iint_0^T (\xi(L(t), t)|\psi_x(L(t), t)|^2 + \xi(0, t)|\psi_x(0, t)|^2) dx dt \\
& \leq C \left(\iint_{Q_L} e^{-2\alpha}|g|^2 dx dt + \iint_{\omega_0 \times (0, T)} (s\xi)^3\lambda^4|\psi|^2 dx dt \right) \\
& \quad + C \iint_{\omega_0 \times (0, T)} (s\xi)\lambda^2|\psi_x|^2 dx dt.
\end{aligned} \tag{31}$$

Now, we can argue as in [15] to get (13) from (31); see also [9]. The steps are the following:

- First, we note that

$$\|V\psi\|_{L^2(Q_L)}^2 \geq C \iint_{Q_L} (s\xi)^{-1}|\psi_{xx}|^2 dx dt - C \iint_{Q_L} (s\xi)^3\lambda^4|\psi|^2 dx dt. \tag{32}$$

- Then, we see that

$$\|U\psi\|_{L^2(Q_L)}^2 \geq C \iint_{Q_L} (s\xi)^{-1}|\psi_t|^2 dx dt - C \iint_{Q_L} (s\xi)\lambda^2|\psi_x|^2 dx dt. \tag{33}$$

- Also, we prove that, for each $\varepsilon > 0$, we can find C_ε such that

$$\iint_{\omega_0 \times (0, T)} (s\xi)\lambda^2|\psi_x|^2 dx dt \leq \varepsilon \iint_{Q_L} (s\xi)^{-1}|\psi_{xx}|^2 dx dt + C_\varepsilon \iint_{\omega \times (0, T)} (s\xi)^3\lambda^4|\psi|^2 dx dt. \tag{34}$$

Putting (31)–(34) together, we deduce that

$$\begin{aligned}
& \iint_{Q_L} ((s\xi)^{-1}(|\psi_t|^2 + |\psi_{xx}|^2) + (s\xi)\lambda^2|\psi_x|^2 + (s\xi)^3\lambda^4|\psi|^2) dx dt \\
& \quad + s\lambda \int_0^T (\xi(L(t), t)|\psi_x(L(t), t)|^2 + \xi(0, t)|\psi_x(0, t)|^2) dt \\
& \leq C \left(\iint_{Q_L} e^{-2\alpha}|g|^2 dx dt + \iint_{\omega \times (0, T)} (s\xi)^3\lambda^4|\psi|^2 dx dt \right).
\end{aligned}$$

Finally, coming back to the original variable φ , it is very easy to obtain (13).

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