Large Galois images for Jacobian varieties of genus 3 curves

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Abstract

Given a prime number $\ell \geq 5$, we construct an infinite family of three-dimensional abelian varieties over $\mathbb{Q}$ such that, for any $A/\mathbb{Q}$ in the family, the Galois representation $\overline{\rho}_{A,\ell}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_6(\mathbb{F}_\ell)$ attached to the ℓ-torsion of A is surjective. Any such variety A will be the Jacobian of a genus 3 curve over $\mathbb{Q}$ whose respective reductions at two auxiliary primes we prescribe to provide us with generators of $\mathrm{Sp}_6(\mathbb{F}_\ell)$.

Introduction

Let ℓ be a prime number. This paper is concerned with realisations of the general symplectic group $\mathrm{GSp}_6(\mathbb{F}_\ell)$ as a Galois group over $\mathbb{Q}$, arising from the Galois action on the ℓ-torsion points of three-dimensional abelian varieties defined over $\mathbb{Q}$.

More precisely, let $g \geq 1$ be an integer. One can exploit the theory of abelian varieties defined over $\mathbb{Q}$ as follows. If A is an abelian variety of dimension g defined over $\mathbb{Q}$, let $A[\ell] = A(\bar{\mathbb{Q}})[\ell]$ denote the ℓ-torsion subgroup of the $\bar{\mathbb{Q}}$-points of A. The natural action of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $A[\ell]$ gives rise to a continuous Galois representation $\overline{\rho}_{A,\ell}$ taking values in $\mathrm{GL}(A[\ell]) \simeq \mathrm{GL}_{2g}(\mathbb{F}_\ell)$. If the abelian variety A is moreover principally polarised, the image of $\overline{\rho}_{A,\ell}$ lies inside the general symplectic group $\mathrm{GSp}(A[\ell])$ of $A[\ell]$ with respect to the symplectic pairing induced by the Weil pairing and the polarisation of A; thus, we have a representation

$$\overline{\rho}_{A,\ell} \,:\, G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}(A[\ell]) \simeq \mathrm{GSp}_{2g}(\mathbb{F}_\ell),$$

providing a realisation of $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$ as a Galois group over $\mathbb{Q}$ if $\overline{\rho}_{A,\ell}$ is surjective.

For any g, it is possible to give parametric families of genus g curves over $\mathbb{Q}$ whose Jacobian varieties have big monodromy modulo all primes $\ell > 2$ (cf. [7], Theorem 1.2). For any fixed odd prime ℓ, the Hilbert Irreducibility Theorem ensures the existence of infinitely many rational values of the parameters such that the corresponding specialisations give rise to Jacobian varieties with surjective ℓ-torsion representation. In this way one obtains many distinct realisations of $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$ as a Galois group over $\mathbb{Q}$.

We can use Galois representations attached to the torsion points of abelian varieties defined over $\mathbb{Q}$ to address the Inverse Galois Problem and its variations involving ramification conditions. For example, the Tame Inverse Galois Problem, proposed by B. Birch, asks if, given a finite group G, there exists a tamely ramified Galois extension $K/\mathbb{Q}$ with Galois group isomorphic to G. S. Arias-de-Reyna and N. Vila solved the Tame Inverse Galois problem for $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$ when $g = 1, 2$ and $\ell \geq 5$ is any prime number, by constructing a family of genus g curves C such that the Galois representation $\overline{\rho}_{\mathrm{Jac}(C),\ell}$ attached to the Jacobian variety $\mathrm{Jac}(C)$ is surjective and tamely ramified for every curve in the family (cf. [3], [4]). For both $g = 1$ and $g = 2$, the strategy entails determining a set of local conditions at auxiliary primes, (that is to say, prescribing a finite list of congruences that the defining equation of C should satisfy) which ensure the surjectivity of $\overline{\rho}_{\mathrm{Jac}(C),\ell}$, and a careful study of the ramification at ℓ in particularly favourable situations.

In fact, the strategy of ensuring surjectivity of the Galois representation attached to the ℓ-torsion of an abelian variety by prescribing local conditions at auxiliary primes works in great generality. Given a g-dimensional principally polarised abelian variety A over $\mathbb{Q}$, such that the Galois representation $\overline{\rho}_{A,\ell}$ is surjective, it is always possible to find some auxiliary primes p and q

depending on ℓ such that any abelian variety B defined over $\mathbb{Q}$ which is "close enough" to A with respect to the primes p and q (in a sense that can be made precise in terms of p-adic resp. q-adic neighbourhoods in moduli spaces of principally polarised g-dimensional abelian varieties with full level structure) also has a surjective ℓ-torsion Galois representation $\overline{\rho}_{B,\ell}$. This is a consequence of Kisin's results on local constancy in p-families of Galois representations; the reader can find a detailed explanation of this application of Kisin's result in [2], Section 4.2.

In this paper we focus on the case $g = 3$. Our aim is to find auxiliary primes p and q (depending on ℓ), and explicit congruence conditions on polynomials defining genus 3 curves, which ensure that any curve C, defined by an equation over $\mathbb{Z}$ satisfying these congruences, will have the property that the image of $\overline{\rho}_{\mathrm{Jac}(C),\ell}$ coincides with $\mathrm{GSp}_6(\mathbb{F}_\ell)$. In this way we obtain many distinct realisations of $\mathrm{GSp}_6(\mathbb{F}_\ell)$ as a Galois group $\mathrm{Gal}(K/\mathbb{Q})$ for some number field K.

To state our main result, we introduce the following notation: we will say that a polynomial $f(x, y)$ in two variables is of *3-hyperelliptic type* if it is of the form $f(x, y) = y^2 - g(x)$, where $g(x)$ is a polynomial of degree 7 or 8 and of *quartic type* if the total degree of $f(x, y)$ is 4.

Theorem 0.1. Let $\ell \geq 5$ be a prime number. There exists an odd prime number $q \neq \ell$ such that, for all odd prime numbers $p \notin \{q, \ell\}$, there exist polynomials $f_p \in \mathbb{Z}_p[x, y]$ and $f_q \in \mathbb{F}_q[x, y]$ of the same type (3-hyperelliptic or quartic), such that for any polynomial $f \in \mathbb{Z}[x, y]$ of the same type as f_p and f_q and satisfying

$$f(x, y) \equiv \overline{f}_q(x, y) \pmod{q} \quad \text{and} \quad f(x, y) \equiv f_p(x, y) \pmod{p^3},$$

the image of the Galois representation $\overline{\rho}_{\mathrm{Jac}(C),\ell}$ attached to the ℓ-torsion points of the Jacobian of the projective genus 3 curve C defined over $\mathbb{Q}$ by the equation $f(x, y) = 0$ is $\mathrm{GSp}_6(\mathbb{F}_\ell)$.

Moreover, when $\ell \geq 13$, the statement holds for all odd prime numbers $q \neq \ell$, such that $q > 1.82\ell^2$ and q is not a square modulo 7.

In Section 4 we state and prove a refinement of this Theorem (cf. Theorem 4.1). In fact, we have a very explicit control of the polynomial $f_p(x, y)$. In general we can say little about $f_q(x, y)$, but for any fixed $\ell \geq 13$ and any fixed $q \geq 1.82\ell^2$ we can find suitable polynomials $f_q(x, y)$ by an exhaustive search as follows: there exist only finitely many polynomials over $\mathbb{F}_q[x, y]$ of hyperelliptic or quartic type with non-zero discriminant. For each of these, we can compute the characteristic polynomial of the action of the Frobenius endomorphism on the Jacobian of the curve defined by $f_q(x, y) = 0$ by counting the $\mathbb{F}_{q^r}$-points of this curve, for $r = 1, 2, 3$, and check whether this polynomial is an ordinary q-Weil polynomial with non-zero trace, non-zero middle coefficient and which is irreducible modulo ℓ. Proposition 3.6 ensures that the search will terminate.

Note that the above result constitutes an explicit version of Proposition 4.6 of [2] in the case of principally polarised 3-dimensional abelian varieties. We can explicitly give the size of the neighbourhoods where surjectivity of $\overline{\rho}_{A,\ell}$ is preserved; in other words, we can give the powers of the auxiliary primes p and q such that any other curve defined by congruence conditions modulo these powers gives rise to a Jacobian variety with surjective ℓ-torsion representation.

The proof of Theorem 0.1 is based on two main pillars: the classification of subgroups of $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$ containing a non-trivial transvection, and the fact that one can force the image of $\overline{\rho}_{A,\ell}$ to contain a non-trivial transvection by imposing a specific type of ramification at an auxiliary prime. This strategy goes back to Le Duff [13] in the case of Jacobians of genus 2 hyperelliptic curves, and has been extended to the general case by Hall in [8], where he obtains a surjectivity result for $\overline{\rho}_{A,\ell}$ for almost all primes ℓ.

We already followed this strategy in [1] to formulate an explicit surjectivity result for g-dimensional abelian varieties (see Theorem 3.10 of loc. cit.): let A be a principally polarised g-dimensional abelian variety defined over $\mathbb{Q}$, such that the reduction of the Néron model of A at some prime p is semistable with toric rank 1, and the Frobenius endomorphism at some prime q of good reduction for A acts irreducibly and with trace $a \neq 0$ on the reduction of the Néron model of A at q. We proved that for each prime number $\ell \nmid 6pqa$, coprime with the order of the component

group of the Néron model of A at p, and such that the characteristic polynomial of the Frobenius endomorphism at q is irreducible mod ℓ, then the representation $\overline{\rho}_{A,\ell}$ is surjective.

Section 1 collects some notations and tools that we will use in the rest of the paper. In Section 2 we address the condition of semistable reduction of toric rank 1 at a prime p; we obtain a congruence condition modulo p^3 (cf. Theorem 2.2).

Let $A = \mathrm{Jac}(C)$ be an abelian variety. We give conditions ensuring that the reduction of the Néron model at a prime q is an absolutely simple abelian variety over $\mathbb{F}_q$ such that the characteristic polynomial of the Frobenius endomorphism at q is irreducible and has non-zero trace (cf. Theorem 3.1). We make use of Honda-Tate Theory in the ordinary case, which relates so-called ordinary Weil polynomials to isogeny classes of ordinary abelian varieties defined over finite fields of characteristic q. First, we need to prove the existence of a suitable prime q and a suitable ordinary Weil polynomial; this is the content of Proposition 3.6, whose proof is postponed to Section 5. This polynomial then provides us with an abelian variety A_q defined over $\mathbb{F}_q$; any abelian variety A such that the reduction of the Néron model of A at q coincides with A_q will satisfy the desired condition at q. At this point we use the fact that each principally polarised 3-dimensional abelian variety over $\mathbb{F}_q$ is the Jacobian of a genus 3 curve, which can be defined over $\mathbb{F}_q$ up to a quadratic twist. We elaborate on this argument in Section 3.

Once we have established congruence conditions at auxiliary primes p and q, we need to check that any curve C over $\mathbb{Z}$ whose defining equation satisfies these conditions will provide us with a Galois representation $\overline{\rho}_{\mathrm{Jac}(C),\ell}$ whose image is $\mathrm{GSp}_6(\mathbb{F}_\ell)$. This is carried out in Section 4.

Acknowledgements We want to thank David Zywina for sending us his preprint [25]. We are indebted to Jean-Pierre Serre and Sinnou David for helpful discussions and pointing us to the references in Remark 3.8. We are grateful for the hospitality of the Institut Henri Poincaré during a short visit. S. Arias-de-Reyna and N. Vila are partially supported by the project MTM2012-33830 of the Ministerio de Economía y Competitividad of Spain, C. Armana by a BQR 2013 Grant from Université de Franche-Comté and M. Rebolledo by the ANR Project Régulateurs ANR-12-BS01-0002. L. Thomas thanks the Laboratoire de Mathématiques de Besançon for its support.

1 Geometric preliminaries

1.1 Hyperelliptic curves and curves of genus 3

In this article, we will say that a *curve over a field* K is an algebraic variety over K whose irreducible components are of dimension 1. (In particular, a curve can be singular.)

A smooth geometrically connected projective curve C of genus $g \geq 1$ over a field K is *hyperelliptic* if there exists a degree 2 finite separable morphism from $C_{\overline{K}} = C \times_K \overline{K}$ to $\mathbb{P}^1_{\overline{K}}$. If K is algebraically closed or a finite field, then such a curve C has a *hyperelliptic equation* defined over K. That is to say, the function field of C is $K(x)[y]$ under the relation $y^2 + h(x)y = f(x)$ with $f, h \in K[x]$, $\deg(f) \in \{2g+1, 2g+2\}$, and $\deg(h) \leq g$; moreover, if $\mathrm{char}(K) \neq 2$, we can take $h = 0$. Indeed, in that case, the conic Q defined as the quotient of C by the group generated by the hyperelliptic involution has a K-rational point, hence is isomorphic to $\mathbb{P}^1_K$ (see e.g. [14, Section 1.3] for more details).[1] The curve C is the union of the two affine open schemes $U = \mathrm{Spec}\left(K[x,y]/(y^2 + h(x)y - f(x))\right)$ and $V = \mathrm{Spec}\left(K[t,w]/(w^2 + t^{g+1}h(1/t)y - t^{2g+2}f(1/t))\right)$ glued along $\mathrm{Spec}(K[x,y,1/x]/(y^2 + h(x)y - f(x)))$ via the identifications $x = 1/t, y = t^{-g-1}w$.

If $\mathrm{char}(K) \neq 2$, then any separable polynomial $f \in K[x]$ of degree $2g+1$ or $2g+2$ gives rise to a hyperelliptic curve C of genus g defined over K by glueing the open affine schemes U and V (with $h = 0$) as above. We will say that C is *given by the hyperelliptic equation* $y^2 = f(x)$. We will also say, as in the introduction, that a polynomial in two variables is of *g-hyperelliptic type* if it is of the form $y^2 - f(x)$ with f a polynomial of degree $2g+1$ or $2g+2$.

In this article, we are especially interested in curves of genus 3. If C is a non-hyperelliptic curve of genus 3 defined over a field K, then its canonical embedding $C \hookrightarrow \mathbb{P}^2_K$ identifies C with a

[1] When K is not algebraically closed nor a finite field, the situation can be more complicated (see [14, Section 4.1]).

smooth plane quartic curve defined over K. This means that the curve C has a model over K given by $\mathrm{Proj}(K[X,Y,Z]/F(X,Y,Z))$ where F is a degree 4 homogeneous polynomial with coefficients in K. Conversely, any smooth plane quartic curve is the image by a canonical embedding of a non-hyperelliptic curve of genus 3. If this curve is $\mathrm{Proj}(K[X,Y,Z]/F(X,Y,Z))$ where F is the homogenisation of a degree 4 polynomial $f \in K[x,y]$, we will say that C is the *quartic plane curve defined by the affine equation* $f(x,y)=0$. We will say, as in the introduction, that a polynomial in two variables is of *quartic type* if its total degree is 4.

1.2 Semistable curves and their generalised Jacobians

Now, we briefly recall the basic notions we need about semistable and stable curves, give the definition of the intersection graph of a curve and explain the link between this graph and the structure of their generalised Jacobian. The classical references we use are essentially [15] and [5]. For a nice overview which contains other references, the reader could also consult [18].

A curve C over a field k is said to be *semistable* if the curve $C_{\overline{k}} = C \times_k \overline{k}$ is reduced and has at most ordinary double points as singularities. It is said to be *stable* if moreover $C_{\overline{k}}$ is connected, projective of arithmetic genus ≥ 2, and if any irreducible component of $C_{\overline{k}}$ isomorphic to $\mathbb{P}^1_{\overline{k}}$ intersects the other irreducible components in at least three points. A proper flat morphism of schemes $\mathcal{C} \to S$ is said to be *semistable* (resp. *stable*) if it has semistable (resp. stable) geometric fibres.

Now let R be discrete valuation ring with fraction field K and residue field k. Let C be a smooth projective geometrically connected curve over K. A *model* of C over R is a normal scheme $\mathcal{C}/R$ such that $\mathcal{C} \times_R K \cong C$. We say that C has *semistable reduction* (resp. *stable reduction*) if C has a model $\mathcal{C}$ over R which is a semistable (resp. stable) scheme over R. If such a stable model exists, it is unique up to isomorphism and we call it *the stable model of* C *over* R (cf. [15, Definition 10/3.27 and Theorem 10/3.34]). If the curve C has genus $g \geq 1$, then it admits a minimal regular model $\mathcal{C}_{min}$ over R, unique up to unique isomorphism. Moreover, $\mathcal{C}_{min}$ is semistable if and only if C has semistable reduction, and if $g \geq 2$, this is equivalent to C having stable reduction (see [15, Theorem 10/3.34], or [18, Theorem 3.1.1] when R is strictly henselian).

Suppose now that C is a smooth projective geometrically connected curve of genus $g \geq 2$ over K and that it has semistable reduction. Let us denote by $\mathcal{C}$ its stable model over R and by $\mathcal{C}_{min}$ its minimal regular model over R. By [5, Corollary 9.7/2], which follows from Theorem 9.5.4 and Theorem 9.7.1 [*loc. cit.*], the Jacobian variety $J = \mathrm{Jac}(C)$ of C admits a Néron model $\mathcal{J}$ over R and the canonical morphism $\mathrm{Pic}^0_{\mathcal{C}/R} \to \mathcal{J}^0$ is an isomorphism. Note that since $\mathcal{C}_{min}$ is also semistable, we also have $\mathrm{Pic}^0_{\mathcal{C}_{min}/R} \cong \mathcal{J}^0$.

Moreover, the abelian variety J has semistable reduction, that is to say $\mathcal{J}^0_k \cong \mathrm{Pic}^0_{\mathcal{C}_k/k}$ is canonically an extension of an abelian variety by a torus T. As we will see, the structure of the algebraic group $\mathcal{J}^0_k$ (by which we mean the toric rank and the order of the component group of its geometric special fibre) is related to the intersection graphs of $\mathcal{C}_{\overline{k}}$ and $\mathcal{C}_{min,\overline{k}}$:

We define the *intersection graph* (or *dual graph* or simply *graph*) $\Gamma(X)$ of a curve X over $\overline{k}$ to be the graph whose vertices are the irreducible components of X, and where two irreducible components X_i and X_j are connected by as many edges as there are irreducible components in the intersection $X_i \cap X_j$. In particular, if the curve X is semistable, two components X_i and X_j are connected by one edge if there is a singular point lying on both X_i and X_j. Here $X_i = X_j$ is allowed. The *(intersection) graph without loops*, denoted by $\Gamma'(X)$, is the graph obtained by removing from $\Gamma(X)$ the edges corresponding to $X_i = X_j$.

Next, we paraphrase Example 9.2.8 of [5], which gives the toric rank in terms of the cohomology of the graph $\Gamma(\mathcal{C}_{\overline{k}})$:

Proposition 1.1 ([5], Ex. 9.2/8)**.** The Néron model $\mathcal{J}$ of the Jacobian of the curve C has semistable reduction. More precisely, let $X_1, \ldots, X_r$ be the irreducible components of $\mathcal{C}_k$, and let $\widetilde{X}_1, \ldots, \widetilde{X}_r$ be their respective normalisations. Then the canonical extension associated to $\mathrm{Pic}^0_{\mathcal{C}_k/k}$

is given by the exact sequence

$$1 \longrightarrow T \hookrightarrow \mathrm{Pic}^0_{\mathcal{C}_k/k} \xrightarrow{\pi^*} \prod_{i=1}^{r} \mathrm{Pic}^0_{\widetilde{X}_i/k} \longrightarrow 1$$

where the morphism π^* is induced by the morphisms $\pi_i : \widetilde{X}_i \longrightarrow X_i$. The rank of the torus T is equal to the rank of the cohomology group $H^1(\Gamma(\mathcal{C}_{\overline{k}}), \mathbb{Z})$.

We will use the preceding result in Sections 2 and 3. Note that the toric rank does not change if we replace $\mathcal{C}$ by $\mathcal{C}_{min}$.

The graph of $\mathcal{C}_{min,\overline{k}}$ also determines the order of the component group of the geometric special fibre $\mathcal{J}_{\overline{k}}$. Indeed, the scheme $\mathcal{C}_{min} \times R^{sh}$, where R^{sh} is the strict henselisation of R, fits the hypotheses of Proposition 9.6.10 of [5] which gives the order of the component group in terms of the graph of $\mathcal{C}_{min,\overline{k}}$; we reproduce it here for the reader's convenience.

Proposition 1.2 ([5], Prop. 9.6/10)**.** Let X be a proper and flat curve over a strictly henselian discrete valuation ring R with algebraically closed residue field $\overline{k}$. Suppose that X is regular and has geometrically irreducible generic fibre as well as a geometrically reduced special fibre $X_{\overline{k}}$. Assume that $X_{\overline{k}}$ consists of the irreducible components $X_1, \ldots, X_r$ and that the local intersection numbers of the X_i are 0 or 1 (the latter is the case if different components intersect at ordinary double points). Furthermore, assume that the intersection graph without loops $\Gamma'(X_{\overline{k}})$ consists of l arcs of edges $\lambda_1, \ldots, \lambda_l$, starting at X_1 and ending at X_r, each arc λ_i consisting of m_i edges. Then the component group $\mathcal{J}(R^{sh})/\mathcal{J}^0(R^{sh})$ has order $\sigma_{l-1}(m_1, \ldots, m_l) = \sum_{i=1}^{l} \prod_{j \neq i} m_j$.

We will use this result in the proof of Theorem 2.2 item 2.

2 Local conditions at p

Let $p > 2$ be a prime number.

Notation 2.1. Let $f(x, y) \in \mathbb{Z}_p[x, y]$ be a polynomial with $f(0, 0) = 0$ or $v_p(f(0, 0)) > 2$, and of one of the following types:

(H) $f(x, y) = y^2 - g(x)$, where $g(x) \in \mathbb{Z}_p[x]$ is of degree 7 or 8 and such that $g(x) \equiv x(x - p)m(x) \bmod p^2\mathbb{Z}_p[x]$, with $m \bmod p$ having simple non-zero roots in $\mathbb{F}_p$;

(Q) $f(x, y)$ is of total degree 4 and such that $f(x, y) \equiv px + x^2 - y^2 + x^4 + y^4 \bmod p^2\mathbb{Z}_p[x, y]$.

Theorem 2.2. Let $f(x, y) \in \mathbb{Z}_p[x, y]$ be a polynomial of type (H) or (Q).

1. The curve C defined by $f(x, y) = 0$ as explained in Section 1.1 is a smooth projective and geometrically connected curve of genus 3 over $\mathbb{Q}_p$ with stable reduction. Moreover, the stable reduction is geometrically integral with exactly one (ordinary double) singularity.

2. The Jacobian variety $\mathrm{Jac}(C)$ of the curve C has a Néron model $\mathcal{J}$ over $\mathbb{Z}_p$ which has semi-abelian reduction of toric rank 1. The component group of the geometric special fibre of $\mathcal{J}$ over $\overline{\mathbb{F}}_p$ has order 2.

Proof. 1. In Section 1, we gave a description of what we called the *projective curve defined by* f either as the union of two affine open schemes in Case (H), i.e. when C is the hyperelliptic curve of hyperelliptic equation f, or as $\mathrm{Proj}(\mathbb{Q}_p[X, Y, Z]/F(X, Y, Z))$ where F is the homogenisation of f in Case (Q). With these descriptions, smoothness over $\mathbb{Q}_p$ follows from the Jacobian criterion. This implies that C is a projective curve of genus 3. Consider the scheme $\mathcal{C}$ over $\mathbb{Z}_p$ defined, for each case of Notation 2.1 respectively, as follows:

(H) the union of the two affine subschemes

$$U = \mathrm{Spec}(\mathbb{Z}_p[x,y]/(y^2 - g(x))) \text{ and } V = \mathrm{Spec}(\mathbb{Z}_p[t,w]/(w^2 - g(1/t)t^8))$$

glued along $\mathrm{Spec}(\mathbb{Z}_p[x,y,1/x]/(y^2-g(x))$ via $x = 1/t, y = t^{-4}w$;

(Q) the scheme $\mathrm{Proj}(\mathbb{Z}_p[X,Y,Z]/(F))$, where F is the homogenisation of f.

The polynomials defining the affine schemes U and V and the quartic polynomial F are all irreducible over $\overline{\mathbb{Q}_p}$, hence over $\mathbb{Z}_p$. So the curve C is geometrically integral (hence geometrically irreducible and geometrically connected) and the model $\mathcal{C}$ is integral as a scheme over $\mathbb{Z}_p$. It follows in particular that $\mathcal{C}$ is flat over $\mathbb{Z}_p$ (cf. [15] Corollary 4.3.10).

We will now show that $\mathcal{C}_{\mathbb{F}_p}$ is semistable (i.e. reduced with only ordinary double points as singularities). Combined with flatness, this will imply that the scheme $\mathcal{C}$ is semistable over $\mathbb{Z}_p$. Since C has genus greater than 2, and $C = \mathcal{C}_{\mathbb{Q}_p}$ is smooth and geometrically connected, this is then equivalent to saying that C has stable reduction at p, as required (cf. [18] Theorem 3.1.1).

In Case (H), $\mathcal{C}_{\overline{\mathbb{F}}_p}$ is the union of the two affine subschemes $U' = \mathrm{Spec}(\overline{\mathbb{F}}_p[x,y]/(y^2 - x^2\bar{m}(x)))$ and $V' = \mathrm{Spec}(\overline{\mathbb{F}}_p[t,w]/(w^2 - \bar{m}(1/t)t^6))$, glued along $\mathrm{Spec}(\overline{\mathbb{F}}_p[x,y,1/x]/(y^2-\bar{g}(x))$ via $x = 1/t$ and $y = t^{-4}w$ (cf. [15] Example 10.3.5). In Case (Q), the geometric special fibre is $\mathrm{Proj}(\overline{\mathbb{F}}_p[X,Y,Z]/(F \bmod p))$. In both cases, the defining polynomials are irreducible over $\overline{\mathbb{F}}_p$. Hence, $\mathcal{C}_{\overline{\mathbb{F}}_p}$ is integral, i.e. reduced and irreducible.

Now we prove that $\mathcal{C}_{\overline{\mathbb{F}}_p}$ has only one ordinary double point as singularity. For Case (H), see e.g. [15] Examples 10.3.4, 10.3.5 and 10.3.29. For Case (Q), we proceed analogously to Liu's examples cited above: first consider the open affine subscheme of $\mathcal{C}_{\overline{\mathbb{F}}_p}$ defined by $U = \mathrm{Spec}(\overline{\mathbb{F}}_p[x,y]/f(x,y))$ for $f(x,y) = x^2 - y^2 + x^4 + y^4 \pmod p$. Since $\mathcal{C}_{\overline{\mathbb{F}}_p} \backslash U$ is smooth, it suffices to prove that U has only ordinary double singularities. Let $u \in U$. The Jacobian criterion shows that U is smooth at $u \neq (0,0)$. So suppose that $u = (0,0)$ and consider $\bar{f}(x,y) = x^2(1+x^2) - y^2(1-y^2) \in \overline{\mathbb{F}}_p[x,y]$. Since $2 \in \overline{\mathbb{F}}_p^\times$, there exist $h(x) = 1 + xc(x) \in \overline{\mathbb{F}}_p[[x]]$ and $g(y) = 1 + yd(y) \in \overline{\mathbb{F}}_p[[y]]$ such that $1 + x^2 = h(x)^2$ and $1 - y^2 = g(y)^2$, by ([15] Exercise 1.3.9). Then we have

$$\widehat{\mathcal{O}}_{U,u} \cong \overline{\mathbb{F}}_p[[x,y]]/(xh(x) + yg(y))(xh(x) - yg(y)) \cong \overline{\mathbb{F}}_p[[t,w]]/(tw).$$

It follows that $\mathcal{C}_{\overline{\mathbb{F}}_p}$ has only one singularity (at $[0:0:1]$) which is an ordinary double singularity (cf. [15] Proposition 7.5.15). We thus showed that $\mathcal{C}$ is the stable model of C over $\mathbb{Z}_p$ and that its special fibre is geometrically integral and has only one ordinary double singularity.

2. Let $\mathcal{C}$ be the stable model of C over $\mathbb{Z}_p$ as before, and let $\mathcal{C}_{min}$ be the minimal regular model of C. By [5] Corollary 9.7.2, the Jacobian variety $\mathrm{Jac}(C)$ has a Néron model $\mathcal{J}$ over $\mathbb{Z}_p$ and the canonical morphism $\mathrm{Pic}^0_{\mathcal{C}/\mathbb{Z}_p} \to \mathcal{J}^0$ is an isomorphism. In particular, $\mathcal{J}$ has semi-abelian reduction and $\mathcal{J}^0_{\mathbb{F}_p} \cong \mathrm{Pic}^0_{\mathcal{C}_{\mathbb{F}_p}/\mathbb{F}_p}$. Since $\mathcal{C}_{min}$ is also semistable (cf. [15] Theorem 10.3.34), we also have $\mathrm{Pic}^0_{\mathcal{C}_{min}/S} \cong \mathcal{J}^0$.

   By [5] Example 9.2.8, as recalled in Section 1, the toric rank of $\mathcal{J}^0_{\overline{\mathbb{F}}_p}$ is equal to the rank of the cohomology group of the dual graph of $\mathcal{C}_{\overline{\mathbb{F}}_p}$. Since $\mathcal{C}_{\overline{\mathbb{F}}_p}$ is irreducible and has only one ordinary double point, the dual graph consists of one vertex and one loop, so the rank of $\mathcal{J}^0_{\overline{\mathbb{F}}_p}$ is 1.

   To determine the order of the component group of the geometric special fibre $\mathcal{J}_{\overline{\mathbb{F}}_p}$, we apply [5] Proposition 9.6.10 to the minimal regular model $\mathcal{C}_{min} \times \mathbb{Z}_p^{sh}$, where $\mathbb{Z}_p^{sh}$ is the strict henselisation of $\mathbb{Z}_p$. This is still regular and semistable over $\mathbb{Z}_p^{sh}$ (cf. [15] Prop 10.3.15 a) or [18] Theorem 3.1). Let e denote the thickness of the ordinary double point of $\mathcal{C}_{\overline{\mathbb{F}}_p}$ (as defined in [15] Corollary 10.3.22 and Definition 10.3.23). Then by [15] Corollary 10.3.25, the geometric special fibre $\mathcal{C}_{min,\overline{\mathbb{F}}_p}$ of $\mathcal{C}_{min} \times \mathbb{Z}_p^{sh}$ consists of a chain of $e-1$ projective lines over $\mathbb{F}_p$ and one

component of genus 2 (where the latter corresponds to the irreducible component $\mathcal{C}_{\overline{\mathbb{F}}_p}$), which meet transversally at rational points. It follows from [5] Proposition 9.6.10 that the order of the component group $\mathcal{J}(\mathbb{Z}_p^{sh})/\mathcal{J}^0(\mathbb{Z}_p^{sh})$ of the geometric special fibre is equal to the thickness e.

We will now show that in both cases (H) and (Q), the thickness e is equal to 2, which will concludes the proof of Theorem 2.2. For this, in several places, we will use (as in part 1. of the proof) the well-known fact that every formal power series in $\mathbb{Z}_p[[x]]$ (resp. $\mathbb{Z}_p[[y]]$, $\mathbb{Z}_p[[x,y]]$) with constant term 1 (or more generally a unit square in $\mathbb{Z}_p$) is a square in $\mathbb{Z}_p[[x]]$ (resp. $\mathbb{Z}_p[[y]]$, $\mathbb{Z}_p[[x,y]]$) of some invertible formal power series.

Let U denote the affine subscheme $\mathrm{Spec}(\mathbb{Z}_p[x,y]/(f(x,y)))$ which contains the ordinary double point $P=[0:0:1]$. Firstly, we claim that, possibly after a finite extension of scalars $R/\mathbb{Z}_p$ which splits the singularity, in both cases we may write in $R[[x,y]]$:

$$\pm f(x,y) = x^2a(x)^2 - y^2b(y)^2 + p\alpha x + p^2yg(x,y) + p^r\beta \tag{1}$$

where $a(x)\in R[[x]]^\times, b(y)\in R[[y]]^\times, g(x,y)\in\mathbb{Z}_p[x,y], \alpha\in\mathbb{Z}_p^\times, \beta\in\mathbb{Z}_p$. Moreover, from the assumptions on f, it follows that either $\beta=0$, or $\beta\in\mathbb{Z}_p^\times$ and $r=v_p(f(0,0))>2$.

We prove the claim case by case:

(H) We have $f(x,y)=y^2-g(x)=y^2-x(x-p)m(x)+p^2h(x)$ for some $h(x)\in\mathbb{Z}_p[x]$. Since $h(x)=h(0)+xs(x)$ for some $s(x)\in\mathbb{Z}_p[x]$ and $m(x)+ps(x)=m(0)+ps(0)+xt(x)$ for some $t(x)\in\mathbb{Z}_p[x]$, we obtain

$$\begin{aligned} f(x,y) &= y^2 - x^2m(x) + px(m(x)+ps(x)) + p^2h(0) \\ &= y^2 - x^2(m(x)-pt(x)) + px(m(0)+ps(0)) + p^2h(0). \end{aligned}$$

Since $m(0)\not\equiv 0 \pmod p$, we have $m(0)-pt(0)\in\mathbb{Z}_p^\times$, hence if we extend the scalars to some finite extension R over $\mathbb{Z}_p$, in which $m(0)-pt(0)$ is a square, we get that $(m(x)-pt(x))$ is a square of some $a(x)$ in $R[[x]]^\times$. Then $-f(x,y)$ has the expected form. Note that $R/\mathbb{Z}_p$ is unramified because $p\neq 2$ and $m(0)\not\equiv 0 \pmod p$, so we still denote the ideal of R above $p\in\mathbb{Z}_p$ by p.

(Q) We have $f(x,y)=x^4+y^4+x^2-y^2+px+p^2h(x,y)$ for some $h(x,y)\in\mathbb{Z}_p[x,y]$. We may write $h(x,y)=\delta+x\gamma+x^2s(x)+yt(x,y)$ for some $\gamma,\delta\in\mathbb{Z}_p$, $s(x)\in\mathbb{Z}_p[x]$ and $t(x,y)\in\mathbb{Z}_p[x,y]$. We obtain

$$\begin{aligned} f(x,y) &= x^2(1+x^2) - y^2(1-y^2) + px + p^2(\delta + x\gamma + x^2s(x) + yt(x,y)) \\ &= x^2(1+x^2+p^2s(x)) - y^2(1-y^2) + px(1+p\gamma) + p^2yt(x,y) + p^2\delta. \end{aligned}$$

Since $1+x^2+p^2s(x)$ and $1-y^2$ have constant terms which are squares in $\mathbb{Z}_p^\times$, the formal power series are squares in $\mathbb{Z}_p[[x]]$, resp. $\mathbb{Z}_p[[y]]$. So $f(x,y)$ again has the desired form.

Next, we show that $e=2$ for $f(x,y)$ of the form (1) over $R[[x,y]]$. We have

$$\pm f(x,y) = \left(xa(x) + p\frac{\alpha}{2a(x)}\right)^2 - \left(yb(y) - p^2\frac{g(x,y)}{2b(y)}\right)^2 + p^2c(x,y),$$

where $c(x,y)=p^{r-2}\beta - \frac{\alpha^2}{4a(x)^2} + p^2\frac{g(x,y)^2}{4b(y)^2}$. Since either $\beta=0$ or $r>2$ and $\frac{\alpha^2}{4a(0)^2}\not\equiv 0 \pmod p$, the constant term γ of the formal power series c belongs to $R^\times$. It follows that $\gamma^{-1}c(x,y)$ is the square of some formal power series $d(x,y)\in R[[x,y]]^\times$. Defining the variables

$$u = \frac{xa(x)}{d(x,y)} + p\frac{\alpha}{2a(x)d(x,y)} - \frac{yb(y)}{d(x,y)} + p^2\frac{g(x,y)}{2b(y)d(x,y)}$$

and

$$v = \frac{xa(x)}{d(x,y)} + p\frac{\alpha}{2a(x)d(x,y)} + \frac{yb(y)}{d(x,y)} - p^2\frac{g(x,y)}{2b(y)d(x,y)},$$

we get $\widehat{O}_{U\times R,P}\cong R[[u,v]]/(uv\pm p^2\gamma)$. Since $\gamma\in R^\times$, it follows that $e=2$.

□

3 Local conditions at q

This section is devoted to the proof of the following key result. In the statement, the two conditions on the characteristic polynomial, namely non-zero trace and irreducibility modulo ℓ, are the ones appearing in Theorem 2.10 of [1] which is used to prove the main Theorem 0.1.

Theorem 3.1. Let $\ell \geq 13$ be a prime number. For every prime number $q > 1.82\ell^2$ which is not a square modulo 7, there exists a smooth geometrically connected curve C_q of genus 3 over $\mathbb{F}_q$ whose Jacobian variety $\mathrm{Jac}(C_q)$ is a 3-dimensional ordinary absolutely simple abelian variety such that the characteristic polynomial of its Frobenius endomorphism is irreducible modulo ℓ and has non-zero trace.

Moreover, for $\ell \in \{3, 5, 7, 11\}$, there exists a prime number $q > 1.82\ell^2$ which is not a square modulo 7, such that the same statement holds.

For any integer $g \geq 1$, a g-dimensional abelian variety over a finite field k of characteristic q is said to be *ordinary* if it has exactly q^g points over $\overline{k}$ of order dividing q.

The proof of Theorem 3.1 relies on Honda-Tate theory which relates ordinary abelian varieties to ordinary Weil polynomials:

Definition 3.2. A *Weil q-polynomial*, or simply a *Weil polynomial*, is a monic polynomial $P_q(X) \in \mathbb{Z}[X]$ of even degree $2g$ whose complex roots are all *Weil q-numbers*, i.e., algebraic integers with absolute value $\sqrt{q}$ under all of their complex embeddings. Moreover, a Weil q-polynomial is said to be *ordinary* if its middle coefficient is prime to q.

In particular, for $g = 3$, every Weil q-polynomial of degree 6 is of the form

$$P_q(X) = X^6 + aX^5 + bX^4 + cX^3 + qbX^2 + q^2aX + q^3 \tag{2}$$

for some integers a, b and c (see Proposition 3.4 of [10]). Such a Weil polynomial is ordinary if, moreover, $c \not\equiv 0 \pmod{q}$.

Conversely, not every polynomial of this form is a Weil polynomial. However, we will prove in Proposition 5.1 that for $q > 1.82\ell^2$, every polynomial as in (2) with $|a|, |b|, |c| < \ell$ is a Weil q-polynomial.

As an important example, the characteristic polynomial of the Frobenius endomorphism of an abelian variety over $\mathbb{F}_q$ is a Weil q-polynomial, by the Riemann hypothesis as proven by Deligne.

A variant of the Honda-Tate Theorem (Theorem 3.3 of [10]) states that the map which sends an ordinary abelian variety over $\mathbb{F}_q$ to the characteristic polynomial of its Frobenius endomorphism induces a bijection between the set of isogeny classes of ordinary abelian varieties of dimension $g \geq 1$ over $\mathbb{F}_q$ and the set of ordinary Weil q-polynomials of degree $2g$. Moreover, under this bijection, isogeny classes of simple ordinary abelian varieties correspond to irreducible ordinary Weil q-polynomials.

Hence, the proof of Theorem 3.1 consists in proving the existence of an irreducible ordinary Weil q-polynomial of degree 6 which gives rise to an isogeny class of simple ordinary abelian varieties of dimension 3. By Howe ([10], Theorem 1.2), such an isogeny class contains a principally polarised abelian variety A over $\mathbb{F}_q$, which we show to be the Jacobian variety of some smooth curve C_q defined over $\mathbb{F}_q$. If this abelian variety A is moreover absolutely simple, the curve is geometrically irreducible. Thus, it is a natural question whether the Weil q-polynomial determines if the abelian varieties in the isogeny class are absolutely simple.

In [11], Howe and Zhu give a sufficient condition for an abelian variety over a finite field to be absolutely simple; for ordinary varieties, this condition is also necessary. Let A be a simple abelian variety over a finite field, π its Frobenius endomorphism and m_A the minimal polynomial of π. Since A is simple, the subalgebra $\mathbb{Q}(\pi)$ of $\mathrm{End}(A) \otimes \mathbb{Q}$ is a field; it contains a filtration of subfields $\mathbb{Q}(\pi^d)$ for $d > 1$. If moreover A is ordinary, then the fields $\mathrm{End}(A) \otimes \mathbb{Q} = \mathbb{Q}(\pi)$ and $\mathbb{Q}(\pi^d)$

($d > 1$) are all CM-fields, i.e., totally imaginary quadratic extensions of a totally real field. A slight reformulation of Howe and Zhu's criterion is the following (see Proposition 3 and Lemma 5 of [11]):

Proposition 3.3 (Howe-Zhu's criterion for absolute simplicity)**.** Let A be a simple abelian variety over a finite field k. If $\mathbb{Q}(\pi^d) = \mathbb{Q}(\pi)$ for all integers $d > 0$, then A is absolutely simple. If A is ordinary, then the converse is also true, and if $\mathbb{Q}(\pi^d) \neq \mathbb{Q}(\pi)$ for some $d > 0$, then A splits over the degree d extension of k. Moreover, if $\mathbb{Q}(\pi^d)$ is a proper subfield of $\mathbb{Q}(\pi)$ such that $\mathbb{Q}(\pi^r) = \mathbb{Q}(\pi)$ for all $r < d$, then either $m_A \in \mathbb{Z}[X^d]$, or there exists a primitive d-th root of unity ζ_d such that $\mathbb{Q}(\pi) = \mathbb{Q}(\pi^d, \zeta_d)$.

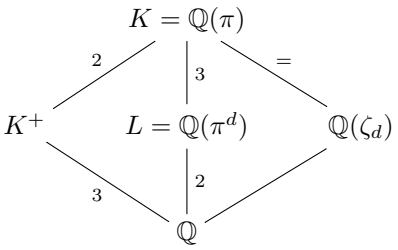
From this criterion, Howe and Zhu show how the characteristic polynomial of the Frobenius endomorphism of a simple ordinary abelian surface over a finite field can be used to easily determine the splitting behaviour of the surface over the algebraic closure [11, Theorem 6]. Elaborating on their condition, we prove the following proposition for dimension 3:

Proposition 3.4. Let A be an ordinary simple abelian variety of dimension 3 over a finite field k. Suppose that A is not absolutely simple. Then either the characteristic polynomial of the Frobenius endomorphism π of A has zero trace, or the smallest extension of k over which A splits has degree $d \in \{7, 14\}$. Moreover, we have $\mathbb{Q}(\pi) = \mathbb{Q}(\zeta_7)$ and $\mathbb{Q}(\pi^d) = \mathbb{Q}(\sqrt{-7})$, for some primitive 7-th root of unity $\zeta_7 \in \overline{\mathbb{Q}}$.

As a corollary, we find the following criterion, which we shall use to obtain an absolutely simple ordinary abelian variety over $\mathbb{F}_q$ in the proof of Theorem 3.1:

Corollary 3.5. Let q be a prime number and $P_q(X) = X^6 + aX^5 + bX^4 + cX^3 + qbX^2 + q^2aX + q^3 \in \mathbb{Z}[X]$ be an ordinary irreducible Weil q-polynomial. If q is not a square modulo 7, then the simple ordinary abelian varieties of dimension 3 over $\mathbb{F}_q$ in the isogeny class defined by P_q are absolutely simple.

Proof of Proposition 3.4. Let A be an ordinary simple but not absolutely simple abelian variety of dimension 3 over k. Let d be the smallest integer such that $\mathbb{Q}(\pi^d) \neq \mathbb{Q}(\pi)$. The field $K = \mathbb{Q}(\pi)$ is a CM-field of degree 6 over $\mathbb{Q}$, hence its proper CM-subfield $L = \mathbb{Q}(\pi^d)$ has to be a quadratic imaginary field.



Since A is simple, the characteristic polynomial of π is m_A. Suppose that m_A has non-zero trace, then $m_A \notin \mathbb{Z}[X^d]$. Hence, by Proposition 3.3, there exists ζ_d such that $\mathbb{Q}(\pi) = \mathbb{Q}(\pi^d, \zeta_d)$. It follows that $\phi(d) = 3$ or 6, where ϕ denotes the Euler totient function. However, $\phi(d) = 3$ has no solution, so we must have $\phi(d) = 6$ (i.e. $d \in \{7, 9, 14, 18\}$), $K = \mathbb{Q}(\zeta_d)$ and $K^+ = \mathbb{Q}(\zeta_d + \bar{\zeta}_d)$). Note that $\mathbb{Q}(\zeta_7) = \mathbb{Q}(\zeta_{14})$ and $\mathbb{Q}(\zeta_9) = \mathbb{Q}(\zeta_{18})$, and that they contain only one quadratic imaginary field; namely, $\mathbb{Q}(\sqrt{-7})$ for $d = 7$ (resp. 14), and $\mathbb{Q}(\sqrt{-3})$ for $d = 9$ (resp. $d = 18$) (see [23]). Let σ be a generator of the (cyclic) group $\mathrm{Gal}(K/L)$ of order 3. In their proof of ([11], Lemma 5), Howe and Zhu show that we can choose ζ_d such that $\pi^\sigma = \zeta_d \pi$. On the other hand, $\zeta_d^\sigma = \zeta_d^k$ for some integer k (which can be chosen to lie in $[0, d-1]$). Since σ is of order 3, we have $\pi = \pi^{\sigma^3} = \zeta_d^{(k^2+k+1)}\pi$, which gives $k^2 + k + 1 \equiv 0 \pmod{d}$. This rules out the case $d = 9$ (and 18), because -3 is neither a square modulo 9 nor a square modulo 18. So $d = 7$ or 14, $K = \mathbb{Q}(\zeta_7)$, $K^+ = \mathbb{Q}(\zeta_7 + \bar{\zeta}_7)$ and $\mathbb{Q}(\pi^d) = \mathbb{Q}(\sqrt{-7})$. □

Proof of Corollary 3.5. Let A be an ordinary simple abelian variety over $\mathbb{F}_q$ in the isogeny class defined by P_q, according to Honda-Tate theory.

Suppose that A is not absolutely simple. By Proposition 3.4, the smallest integer d such that $\mathbb{Q}(\pi^d) \neq \mathbb{Q}(\pi)$ is $d = 7$ or 14. Moreover, we have $\mathbb{Q}(\pi) = \mathbb{Q}(\zeta_7)$ and $\mathbb{Q}(\pi^d) = \mathbb{Q}(\sqrt{-7})$. The characteristic polynomial of π^d is of the form

$$X^6 + \alpha X^5 + \beta X^4 + \gamma X^3 + \beta q^d X^2 + \alpha q^{2d} X + q^{3d} \in \mathbb{Z}[X],$$

so it is the cube of a quadratic polynomial of discriminant -7 if and only if

$$\alpha^2 - 36q^d + 63 = 0, \quad \alpha^2 - 3\beta + 9q^d = 0 \quad \text{and} \quad \alpha^3 - 27\gamma + 54\alpha q^d = 0, \tag{3}$$

that is,

$$\alpha^2 = 9(4q^d - 7), \quad \beta = 3(5q^d - 7) \quad \text{and} \quad 3\gamma = \alpha(10q^d - 7). \tag{4}$$

From $\alpha^2 = 9(4q^d - 7)$ we find that 3 divides α and that $4q^d - 7$ is a square. Hence, -7 is a square modulo q, that is to say, q is a square modulo 7, which contradicts the hypothesis. The corollary follows. □

Finally, the existence of an ordinary irreducible Weil polynomial that satisfies the conditions of Corollary 3.5 is provided by the following proposition, whose proof relies on counting arguments and is postponed to Section 5:

Proposition 3.6. For any prime number $\ell \geq 13$ and any prime number $q > 1.82\ell^2$, there exists an ordinary Weil q-polynomial $P_q(X) = X^6 + aX^5 + bX^4 + cX^3 + qbX^2 + q^2aX + q^3$, with $a \not\equiv 0 \pmod{\ell}$, which is irreducible modulo ℓ. For $\ell \in \{3, 5, 7, 11\}$, there exists some prime number $q > 1.82\ell^2$ and an ordinary Weil q-polynomial as above. Moreover, for all $\ell > 3$, the coefficients a, b, c can be chosen to lie in $\mathbb{Z} \cap [-(\ell-1)/2, (\ell-1)/2]$.

Remark 3.7. Computations suggest that for $\ell \in \{5, 7, 11\}$ and *any* prime number $q > 1.82\ell^2$, there still exist integers a, b, c such that Proposition 3.6 holds. For $\ell = 3$, this is no longer true: our computations indicate that if q is such that $\left(\frac{q}{\ell}\right) = -1$,, then there are no suitable a, b, c, while if q is such that $\left(\frac{q}{\ell}\right) = 1$, then there are 4 suitable triples (a, b, c).

We now have all the ingredients to prove Theorem 3.1.

Proof of Theorem 3.1. Let ℓ and q be two distinct prime numbers as in Proposition 3.6 and such that q is not a square modulo 7, and let P_q be an ordinary Weil q-polynomial provided by this proposition. Since the polynomial P_q is irreducible modulo ℓ, it is a fortiori irreducible over $\mathbb{Z}$. It is also ordinary and of degree 6. Hence, by Honda-Tate theory, it defines an isogeny class $\mathcal{A}$ of ordinary simple abelian varieties of dimension 3 over $\mathbb{F}_q$. By Corollary 3.5, since q is not a square modulo 7, the abelian varieties in $\mathcal{A}$ are actually absolutely simple. Moreover, according to Howe ([10], Theorem 1.2), $\mathcal{A}$ contains a principally polarised abelian variety (A, λ).

Now, by Oort-Ueno theory ([16],Theorem 4), there exists a so-called good curve C defined over $\overline{\mathbb{F}}_q$ such that (A, λ) is $\overline{\mathbb{F}}_q$-isomorphic to $(\mathrm{Jac}(C), \mu_0)$, where μ_0 denotes the canonical polarisation on $\mathrm{Jac}(C)$.

A curve over $\overline{\mathbb{F}}_q$ is a *good curve* if it is either irreducible and non-singular or a non-irreducible stable curve whose generalised Jacobian variety is an abelian variety. In particular, the curve C is stable, and so semi-stable. Since the generalised Jacobian variety $\mathrm{Jac}(C) \cong \mathrm{Pic}^0_C$ is an abelian variety, the torus appearing in the short exact sequence of Proposition 1.1 is trivial. Hence, there is an isomorphism $\mathrm{Jac}(C) \cong \prod_{i=1}^r \mathrm{Pic}^0_{\widetilde{X}_i}$, where $\widetilde{X}_1, \ldots, \widetilde{X}_r$ denote the normalisations of the irreducible component of C over $\overline{\mathbb{F}}_q$. Since $\mathrm{Jac}(C)$ is absolutely simple, we conclude that $r = 1$, i.e., the curve C is geometrically irreducible.

We can therefore apply Theorem 9 of the appendix by Serre in [12] (see also the reformulation in Theorem 1.1 of [17]) and conclude that the curve C descends to $\mathbb{F}_q$. Indeed, there exists a smooth and geometrically irreducible curve C_q defined over $\mathbb{F}_q$ which is isomorphic to C over $\overline{\mathbb{F}}_q$. Moreover, either (A, λ) or a quadratic twist of (A, λ) is isomorphic to $(\mathrm{Jac}(C_q), \mu)$ over $\mathbb{F}_q$, where μ denotes the canonical polarisation of $\mathrm{Jac}(C_q)$. The characteristic polynomial of $\mathrm{Jac}(C_q)$ is $P_q(X)$ or $P_q(-X)$, since the twist may replace the Frobenius endomorphism with its negative.

Note that the polynomial $P_q(-X)$ is still an ordinary Weil polynomial which is irreducible modulo ℓ with non-zero trace, so that $\mathrm{Jac}(C_q)$ is still ordinary and absolutely simple. This proves Theorem 3.1. □

Remark 3.8. In the descent argument above, the existence of a non-trivial quadratic twist may occur in the non-hyperelliptic case only. This obstruction was first stated precisely by Serre, in a Harvard course [22]; it was derived from a precise reformulation of Torelli's theorem that Serre attributes to Weil [24]. In parallel, Sekiguchi proved the descent of the curve in [19] and [20] as a generalisation of the results of Oort and Ueno, but without pointing out such an obstruction; he solved the issue entirely with Sekino in [21].

Remark 3.9. David Zywina communicated to us that he has recently and independently developed a method for studying the image of Galois representations $\overline{\rho}_{\mathrm{Jac}(C),\ell}$ attached to the Jacobians of genus 3 plane quartic curves C, for a large class of such curves (see [25]).

A different proof of Theorem 3.1 could be given using his results. For example, Zywina applies his method to the plane quartic curve C given by the equation

$$x^3y - x^2y^2 + x^2z^2 + xy^3 - xyz^2 - xz^3 - y^4 + y^3z - y^2z^2 - yz^3 = 0$$

and proves that, for every prime ℓ, the image of $\overline{\rho}_{\mathrm{Jac}(C),\ell}$ coincides with $\mathrm{GSp}_6(\mathbb{F}_\ell)$. One could choose the polynomial $P_q(X)$ to be the characteristic polynomial of the Frobenius endomorphism of this curve C at infinitely many primes q (depending on the prime ℓ).

4 Proof of the main theorem

The goal of this section is to prove Theorem 0.1, by collecting together the results from Sections 2 and 3. We keep the notation introduced in Subsection 1.1; in particular, we will consider genus 3 curves defined by polynomials which are of 3-hyperelliptic or quartic type. We will prove the following refinement of Theorem 0.1:

Theorem 4.1. Let $\ell \geq 13$ be a prime number. For each prime number $q > 1.82\ell^2$ which is not a square modulo 7, there exists $f_q(x,y) \in \mathbb{F}_q[x,y]$ of 3-hyperelliptic or quartic type, such that if $f(x,y) \in \mathbb{Z}[x,y]$ is a lift of $f_q(x,y)$, of the same type, satisfying the following two conditions for some prime number $p \notin \{2, q, \ell\}$:

1. $f(0,0) = 0$ or $v_p(f(0,0)) > 2$;
2. $f(x,y)$ is congruent modulo p^2 to:

$$\begin{cases} y^2 - x(x-p)m(x) & \text{if } f_q(x,y) \text{ is of hyperelliptic type} \\ x^4 + y^4 + x^2 - y^2 + px & \text{if } f_q(x,y) \text{ is of quartic type} \end{cases}$$

for some $m(x) \in \mathbb{Z}_p[x]$ of degree 5 or 6 with simple non-zero roots modulo p;

then the projective curve C defined over $\mathbb{Q}$ by the equation $f(x,y) = 0$ is a smooth projective geometrically irreducible genus 3 curve, such that the image of the Galois representation attached to the ℓ-torsion of $\mathrm{Jac}(C)$ coincides with $\mathrm{GSp}_6(\mathbb{F}_\ell)$.

Moreover, if $\ell \in \{5, 7, 11\}$, the statement is true, replacing "For each prime q" by "There exists an odd prime q".

Remark 4.2. Let $\ell \geq 5$ be a prime number. Note that it is easy to construct infinitely many polynomials $f(x,y) = 0$ satisfying the conclusion of Theorem 4.1: choose a polynomial $f_p(x,y)$ satisfying the conditions in Notation 2.1. Then it suffices to choose each coefficient of $f(x,y)$ as a lift of the corresponding coefficient of $f_q(x,y)$ to an element of $\mathbb{Z}$, which is congruent mod p^3 to the corresponding coefficient of $f_p(x,y)$. This also proves that Theorem 0.1 follows from Theorem 4.1.

For the convenience of the reader, we recall the contents of Theorem 3.10 from [1]: Let A be a principally polarised n-dimensional abelian variety defined over $\mathbb{Q}$. Assume that A has semistable reduction of toric rank 1 at some prime number p. Denote by Φ_p the group of connected components of the Néron model of A at p. Let q be a prime of good reduction of A and $P_q(X) = X^{2n} + aX^{2n-1} + \cdots + q^n \in \mathbb{Z}[X]$ the characteristic polynomial of the Frobenius endomorphism acting on the reduction of A at q. Then for all primes ℓ which do not divide $6pq|\Phi_p|a$ and are such that the reduction of $P_q(X)$ mod ℓ is irreducible in $\mathbb{F}_\ell$, the image of $\overline{\rho}_{A,\ell}$ coincides with $\mathrm{GSp}_{2n}(\mathbb{F}_\ell)$.

Proof of Theorem 4.1. Fix a prime $\ell \geq 5$. Let q and C_q be a prime, respectively a genus 3 curve over $\mathbb{F}_q$, provided by Theorem 3.1. The curve C_q is either a plane quartic or a hyperelliptic curve. More precisely, it is defined by an equation $f_q(x, y) = 0$, where $f_q(x, y) \in \mathbb{F}_q[x, y]$ is a quartic type polynomial in the first case and a 3-hyperelliptic type polynomial otherwise (cf. Subsection 1.1). Note that if $f(x, y) \in \mathbb{Z}[x, y]$ is a quartic (resp. 3-hyperelliptic type) polynomial which reduces to $f_q(x, y)$ modulo q, then it defines a smooth projective genus 3 curve over $\mathbb{Q}$ which is geometrically irreducible.

Let now $p \notin \{2, q, \ell\}$ be a prime. Assume that $f(x, y) \in \mathbb{Z}[x, y]$ is a polynomial of the same type as $f_q(x, y)$ which is congruent to $f_q(x, y)$ modulo q and also satisfies the two conditions of the statement of Theorem 4.1 for this p. We claim that the curve C defined over $\mathbb{Q}$ by the equation $f(x, y) = 0$ satisfies all the conditions of the explicit surjectivity result of ([1], Theorem 3.10). Namely, Theorem 2.2 implies that C is a smooth projective geometrically connected curve of genus 3 with stable reduction. Moreover, the Jacobian $\mathrm{Jac}(C)$ is a principally polarised 3-dimensional abelian variety over $\mathbb{Q}$, and its Néron model has semistable reduction at p with toric rank equal to 1. Furthermore, the component group Φ_p of the Néron model of $\mathrm{Jac}(C)$ at p has order 2. Finally, by the choice of q and C_q provided by Theorem 3.1, q is a prime of good reduction of $\mathrm{Jac}(C)$ such that the Frobenius endomorphism of the special fibre at q has Weil polynomial $P_q(t) = t^6 + at^5 + bt^4 + ct^3 + qbt^2 + q^2at + q^3$, which is irreducible modulo ℓ. Since the prime ℓ does not divide $6pq|\Phi_p|a$, we conclude that the image of the Galois representation $\overline{\rho}_{\mathrm{Jac}(C),\ell}$ attached to the ℓ-torsion of $\mathrm{Jac}(C)$ coincides with $\mathrm{GSp}_6(\mathbb{F}_\ell)$. □

5 Counting irreducible Weil polynomials of degree 6 - Proof of Proposition 3.6

In this section, we will prove Proposition 3.6 stated in Section 3. This proof is based on Proposition 5.1 as well as Lemmas 5.3 and 5.4 below. Let ℓ and q be distinct prime numbers. Proposition 5.1 ensures that for $q \gg \ell^2$, every polynomial $P_q(X) = X^6 + aX^5 + bX^4 + cX^3 + qbX^2 + q^2aX + q^3 \in \mathbb{Z}[X]$ with coefficients in $]-\ell, \ell[$ is a Weil polynomial. Lemmas 5.3 and 5.4, then allow us to show that the number of such polynomials which are irreducible modulo ℓ is strictly positive.

Proposition 5.1. Let ℓ and q be two prime numbers.

1. Suppose that $q > 1.67\ell^2$. Then every polynomial
$$X^4 + uX^3 + vX^2 + uqX + q^2 \in \mathbb{Z}[X]$$
with integers u, v of absolute value $< \ell$ is a Weil q-polynomial.

2. Suppose that $q > 1.82\ell^2$. Then every polynomial
$$P_q(X) = X^6 + aX^5 + bX^4 + cX^3 + qbX^2 + q^2aX + q^3 \in \mathbb{Z}[X], \tag{5}$$
with integers a, b, c of absolute value $< \ell$, is a Weil q-polynomial.

Remark 5.2. The power in ℓ is optimal, but the constants 1.67 and 1.82 are not.

Let us consider now the polynomials of the form $P_q(X) = X^6 + aX^5 + bX^4 + cX^3 + qbX^2 + q^2aX + q^3 \in \mathbb{Z}[X]$ with a, b, c in $[-(\ell-1)/2, (\ell-1)/2]$, $a, c \neq 0$ and whose discriminant Δ_{P_q} is not a square modulo ℓ. Let D_6^{*-} be the number of such polynomials and R_6 the number of such polynomials which are reducible modulo ℓ. Denoting by $\left(\frac{\cdot}{\ell}\right)$ the Legendre symbol, we have:

Lemma 5.3. Let $\ell > 3$, then $D_6^{*-} \geq \frac{1}{2}(\ell-1)^2\left(\ell - 1 - \left(\frac{q}{\ell}\right)\right) + \frac{1}{2}(\ell-1)\left(\frac{q}{\ell}\right)\left(1 - \left(\frac{-1}{\ell}\right)\right) - \ell(\ell-1)$.

Lemma 5.4. Let $\ell > 3$, then $R_6 \leq \frac{3}{8}\ell^3 - \frac{5}{8}\ell^2\left(\frac{q}{\ell}\right) - \ell^2 + \frac{3}{2}\ell\left(\frac{q}{\ell}\right) + \frac{5}{8}\ell - \frac{3}{8}\left(\frac{q}{\ell}\right) - \frac{1}{2}$.

We postpone the proofs of Proposition 5.1 as well as Lemmas 5.3 and 5.4 to the following subsections but use now those statements to prove Proposition 3.6. Before that, let us recall a result of Stickelberger, as proven by Carlitz in [6], which will also be useful for proving Lemmas 5.3 and 5.4: For any monic polynomial P of degree n with coefficients in $\mathbb{Z}$, and any odd prime number ℓ not dividing its discriminant Δ_P, the number s of irreducible factors of P modulo ℓ satisfies

$$\left(\frac{\Delta_P}{\ell}\right) = (-1)^{n-s}.$$

Proof of Proposition 3.6. Let $\ell > 3$ be a prime number. It Stickelberger's result that if P_q as in (5) is irreducible modulo ℓ, then $\left(\frac{\Delta_{P_q}}{\ell}\right) = -1$. Hence by Proposition 5.1, when $q > 1.82\ell^2$, we find that $(D_6^{*-} - R_6)$ is exactly the number of degree 6 ordinary Weil polynomials which have non-zero trace modulo ℓ and are irreducible modulo ℓ.

By Lemmas 5.3 and 5.4, we have

$$D_6^{*-} - R_6 \geq \frac{1}{8}\ell^3 + \frac{1}{8}\ell^2\left(\frac{q}{\ell}\right) - \frac{1}{2}\ell\left(\frac{-q}{\ell}\right) - \frac{3}{2}\ell^2 + \frac{1}{2}\left(\frac{-q}{\ell}\right) + \frac{15}{8}\ell - \frac{5}{8}\left(\frac{q}{\ell}\right),$$

which is strictly positive for all q, provided that $\ell \geq 13$.

For $\ell = 5, 7$ or 11, direct computations of $(D_6^{*-} - R_6)$ using SAGE show that it is strictly positive at least for any prime number $q \leq 300$. In particular, $q = 47$ for $\ell = 5$, $q = 97$ for $\ell = 7$, $q = 223$ for $\ell = 11$ will answer to the conditions of Proposition 3.6 (and will be moreover not squares modulo 7). For $\ell = 3$, computations suggest that all prime numbers q which are not squares modulo ℓ will fit the conditions, for instance $q = 19$ will do (see Remark 3.7). The proposition follows. □

5.1 Proof of Proposition 5.1

Recall that ℓ and q are two prime numbers.

We first consider degree 4 polynomials. One can prove that a polynomial $X^4 + uX^3 + vX^2 + uqX + q^2 \in \mathbb{Z}[X]$ is a q-Weil polynomial if and only if the integers u, v satisfy the following inequalities:

1. $|u| \leq 4\sqrt{q}$,
2. $2|u|\sqrt{q} - 2q \leq v \leq \frac{u^2}{4} + 2q$.

Now let $q > 1.67\ell^2$ and $Q(X) = X^4 + uX^3 + vX^2 + uqX + q^2 \in \mathbb{Z}[X]$ with $|u| < \ell, |v| < \ell$. Then $q \geq \frac{1}{16}\ell^2$ and, since $\ell \geq 2$, we have $q \geq \frac{1}{4}\ell^2 \geq \frac{1}{2}\ell$ so Condition 1 and the right hand side inequality in Condition 2 are satisfied. Finally, $q \geq \left(1 + \frac{1}{2\sqrt{3}}\right)^2 \ell^2$ so $\sqrt{q} \geq \left(1 + \frac{1}{2\sqrt{q}}\right)\ell$ and the left hand side inequality in Condition 2 is satisfied. This proves that Q is a Weil polynomial and the first part of the proposition.

Now we turn to degree 6 polynomials. The proof is similar to the degree 4 case. According to Haloui [9, Theorem 1.1], a degree 6 polynomial of the form (5) is a Weil polynomial if its coefficients satisfy the following conditions:

1. $|a| < 6\sqrt{q}$,
2. $4\sqrt{q}|a| - 9q < b \leq \frac{a^2}{3} + 3q$,
3. $-\frac{2a^3}{27} + \frac{ab}{3} + qa - \frac{2}{27}(a^2 - 3b^2 + 9q)^{\frac{3}{2}} \leq c \leq -\frac{2a^3}{27} + \frac{ab}{3} + qa + \frac{2}{27}(a^2 - 3b^2 + 9q)^{\frac{3}{2}}$,
4. $-2qa - 2\sqrt{q}b - 2q\sqrt{q} < c < -2qa + 2\sqrt{q}b + 2q\sqrt{q}$.

Now let $q > 1.82\ell^2$ and P_q a polynomial of the form (5) with $|a|, |b|, |c| < \ell$. Then we note:

(i) We have $q > \frac{1}{36}\ell^2$, so $\ell < 6\sqrt{q}$ and Condition 1 is satisfied.

(ii) The right hand side inequality of Condition 2 is satisfied since $\ell \leq 3q$. Moreover we have $q > (1 + \sqrt{17/8})\ell^2 \geq 4\ell^2(1 + \sqrt{1 + 9/4\ell})^2/81$. Hence $9q - 4\ell\sqrt{q} - \ell > 0$ and the left hand inequality of Condition 2 is satisfied.

(iii) A sufficient condition to have both inequalities in Condition 3 is

$$2\ell^3 + 9\ell^2 + 27q\ell - 2(-3\ell^2 + 9q)^{3/2} + 27\ell \leq 0.$$

A computation shows that this inequality is equivalent to $A \leq B$, with

$$A = \ell^6\left(\frac{28}{729} + \frac{1}{81\ell} + \frac{7}{108\ell^2} + \frac{1}{6\ell^3} + \frac{1}{4\ell^4}\right) \text{ and } B = q^3\left(1 - \frac{5}{4}\frac{\ell^2}{q} + \frac{\ell^4}{q^2}\left(\frac{8}{27} - \frac{1}{6\ell} - \frac{1}{2\ell^2}\right)\right).$$

Since $\ell \geq 2$, we have $A \leq \frac{4537}{46656}\ell^6$ and $B \geq q^3\left(1 - \frac{5}{4}\frac{\ell^2}{q} + \frac{19}{216}\frac{\ell^4}{q^2}\right)$. Furthermore, since the polynomial

$$\frac{4537}{46656}X^3 - \frac{19}{216}X^2 + \frac{5}{4}X - 1$$

has only one real root with approximate value 0.805, we find that $A \leq B$, because $q \geq 1.243\ell^2$.

(iv) Since $q > 1.82\ell^2$ and $\ell \geq 2$, we have $\ell\left(\frac{1}{2q} + \frac{1}{\sqrt{q}} + 1\right) \leq \ell\left(\frac{1}{22} + \frac{1}{\sqrt{11}} + 1\right) < \sqrt{q}$. Hence, $-2q\ell - 2\sqrt{q}\ell + 2q\sqrt{q} - \ell > 0$ and Condition 4 is satisfied.

This proves that P_q is a Weil polynomial and the second part of the proposition. □

5.2 Proofs of Lemmas 5.3 and 5.4

In this section, $\ell > 2$, $q \neq \ell$ are prime numbers and we, somewhat abusively, denote with the same letter an integer in $[-(\ell-1)/2, (\ell-1)/2]$ and its image in $\mathbb{F}_\ell$. We will use the following elementary lemma.

Lemma 5.5. Let $D \in \mathbb{F}_\ell^*$ and $\varepsilon \in \{-1, 1\}$. We have

$$\sharp\left\{x \in \mathbb{F}_\ell; \left(\frac{x^2 - D}{\ell}\right) = \varepsilon\right\} = \frac{1}{2}\left(\ell - 1 - \varepsilon - \left(\frac{D}{\ell}\right)\right);$$

and

$$\sharp\left\{(x, y) \in \mathbb{F}_\ell^2; \left(\frac{x^2 - Dy^2}{\ell}\right) = \varepsilon\right\} = \frac{1}{2}(\ell - 1)\left(\ell - \left(\frac{D}{\ell}\right)\right).$$

5.2.1 Estimates on the number of degree 4 Weil polynomials modulo ℓ

Proposition 5.6. 1. For $\varepsilon \in \{-1,1\}$, we denote by D_4^ε the number of degree 4 polynomials of the form $X^4+uX^3+vX^2+uqX+q^2 \in \mathbb{F}_\ell[X]$ with discriminant Δ such that $\left(\frac{\Delta}{\ell}\right)=\varepsilon$. Then

$$D_4^- = \frac{1}{2}(\ell-1)\left(\ell-\left(\frac{q}{\ell}\right)\right) \quad \text{and} \quad D_4^+ = \frac{1}{2}(\ell-3)\left(\ell-\left(\frac{q}{\ell}\right)\right)+1.$$

2. The number N_4 of degree 4 Weil polynomials with coefficients in $[-(\ell-1)/2,(\ell-1)/2]$ which are irreducible modulo ℓ satisfies

$$N_4 \leq \frac{1}{4}(\ell+1)(\ell-1). \tag{6}$$

3. The number T_4 of degree 4 Weil polynomials with coefficients in $[-(\ell-1)/2,(\ell-1)/2]$ with exactly two irreducible factors modulo ℓ satisfies

$$T_4 \leq \frac{1}{4}(\ell-3)\left(\ell-\left(\frac{q}{\ell}\right)\right)+\frac{1}{8}(\ell-1)(\ell+1). \tag{7}$$

Moreover, if $q > 1.67\ell^2$, Inequalities (6) and (7) are equalities.

Proof. 1. The polynomial $Q(X)=X^4+uX^3+vX^2+uqX+q^2$ has discriminant

$$\Delta = q^2\kappa^2\delta \quad \text{where} \quad \kappa=-u^2-8q+4v \quad \text{and} \quad \delta=(v+2q)^2-4qu^2.$$

So, since $q \in \mathbb{F}_\ell^*$, we have $\left(\frac{\Delta}{\ell}\right)=\left(\frac{\kappa}{\ell}\right)^2\left(\frac{\delta}{\ell}\right)$. Moreover, notice that if $\kappa=0$ then $\delta=(v-6q)^2$. Hence the set $\mathcal{D}_4^\varepsilon$ of $(u,v)\in\mathbb{F}_\ell^2$ such that $\left(\frac{\Delta}{\ell}\right)=\varepsilon$ is equal to

$$\begin{aligned}\mathcal{D}_4^\varepsilon &= \left\{(u,v)\in\mathbb{F}_\ell^2; \left(\frac{\delta}{\ell}\right)=\varepsilon\right\} \setminus \left\{(u,v)\in\mathbb{F}_\ell^2; \left(\frac{\delta}{\ell}\right)=\varepsilon \text{ and } \kappa=0\right\} \\ &= \left\{(u,v)\in\mathbb{F}_\ell^2; \left(\frac{\delta}{\ell}\right)=\varepsilon\right\} \setminus \left\{(u,v)\in\mathbb{F}_\ell^2; \left(\frac{v-6q}{\ell}\right)^2=\varepsilon \text{ and } u^2=4(v-2q)\right\}.\end{aligned}$$

It follows that

$$D_4^- = \sharp\mathcal{D}_4^- = \sharp\left\{(u,v)\in\mathbb{F}_\ell^2; \left(\frac{\delta}{\ell}\right)=-1\right\}$$

and

$$D_4^+ = \sharp\mathcal{D}_4^+ = \sharp\left\{(u,v)\in\mathbb{F}_\ell^2; \left(\frac{\delta}{\ell}\right)=1\right\} - \sharp\left\{(u,v)\in\mathbb{F}_\ell^2; v\neq 6q \text{ and } u^2=4(v-2q)\right\}.$$

Since $(u,v)\mapsto(v+2q,2u)$ is a bijection (because $\ell\neq 2$), by Lemma 5.5 we have

$$\sharp\left\{(u,v)\in\mathbb{F}_\ell^2; \left(\frac{\delta}{\ell}\right)=\varepsilon\right\} = \sharp\left\{(x,y)\in\mathbb{F}_\ell^2; \left(\frac{x^2-qy^2}{\ell}\right)=\varepsilon\right\} = \frac{(\ell-1)}{2}\left(\ell-\left(\frac{q}{\ell}\right)\right)$$

for any $\varepsilon\in\{\pm1\}$. This gives the result for D_4^-. Moreover

$$\begin{aligned}\sharp\left\{(u,v);\ v\neq 6q \text{ and } u^2=4(v-2q)\right\} &= \sharp\left\{(u,v);\ u^2=4(v-2q)\right\} - \sharp\{u\in\mathbb{F}_\ell; u^2=16q\} \\ &= \ell-1-\left(\frac{q}{\ell}\right).\end{aligned}$$

This gives the result for D_4^+.

2. By Stickelberger's result recalled at the beginning of the section, if a monic polynomial of degree 4 in $\mathbb{Z}[X]$ is irreducible modulo ℓ then it has non-square discriminant modulo ℓ. Conversely, if a monic degree 4 polynomial in $\mathbb{Z}[X]$ has non-square discriminant modulo ℓ, then it has one or three distinct irreducible factors in $\mathbb{F}_\ell[X]$. If the reduction of a degree 4 Weil polynomial with non-square discriminant modulo ℓ has three distinct irreducible factors in $\mathbb{F}_\ell[X]$, then it has the form
$$(X-\alpha')(X-q/\alpha')(X^2-B'X+q)$$
with $X^2-B'X+q$ irreducible in $\mathbb{F}_\ell[X]$ and $\alpha' \neq q/\alpha'$ in $\mathbb{F}_\ell^*$. By Lemma 5.5, there are
$$\frac{1}{4}\left(\ell-2-\left(\frac{q}{\ell}\right)\right)\left(\ell-\left(\frac{q}{\ell}\right)\right)$$
such polynomials with three irreducible factors. It follows that
$$N_4 \leq D_4^- - \frac{1}{4}\left(\ell-2-\left(\frac{q}{\ell}\right)\right)\left(\ell-\left(\frac{q}{\ell}\right)\right) \leq \frac{1}{4}(\ell-1)(\ell+1).$$

3. Similarly, a degree 4 Weil polynomial Q in $\mathbb{Z}[X]$ has exactly two distinct irreducible factors modulo ℓ if and only if $\left(\frac{\Delta_Q}{\ell}\right)=1$ and $Q \pmod{\ell}$ does not have four distinct roots in $\mathbb{F}_\ell$. By Lemma 5.5, there are
$$\frac{1}{8}\left(\ell-\left(\frac{q}{\ell}\right)-2\right)\left(\ell-\left(\frac{q}{\ell}\right)-4\right)$$
Weil polynomials with coefficients in $[-(\ell-1)/2,(\ell-1)/2]$ whose reduction modulo ℓ has four distinct roots in $\mathbb{F}_\ell$. It follows that
$$\begin{aligned} T_4 &\leq D_4^+ - \frac{1}{8}\left(\ell-\left(\frac{q}{\ell}\right)-2\right)\left(\ell-\left(\frac{q}{\ell}\right)-4\right) \\ &\leq \frac{1}{4}(\ell-3)\left(\ell-\left(\frac{q}{\ell}\right)\right)+\frac{1}{8}(\ell-1)(\ell+1). \end{aligned}$$

When $q > 1.67\ell^2$, these upper bounds for N_4 and T_4 are equalities, since in this case, by Proposition 5.1, every polynomial of the form $X^4+uX^3+vX^2+uqX+q^2$ with $|u|,|v|<\ell$ is a Weil polynomial. □

5.2.2 Proof of Lemma 5.4

Let P_q be a degree 6 Weil polynomial with coefficients in $[-(\ell-1)/2,(\ell-1)/2]$ and non-square discriminant modulo ℓ. We may drop the conditions $a \neq 0, c \neq 0$ to simplify computations for finding an upper bound for R_6. By Stickelberger's result, P_q has 1, 3 or 5 distinct irreducible factors in $\mathbb{F}_\ell[X]$. Note that a root α of P_q in $\bar{\mathbb{F}}_\ell$ is in $\mathbb{F}_\ell$ if and only q/α is also in $\mathbb{F}_\ell$. So a degree 6 Weil polynomial P_q with non-square discriminant modulo ℓ is reducible modulo ℓ if and only if its factorisation in $\mathbb{F}_\ell[X]$ is of one of the following types:

1. $P_q(X) \equiv (X-\alpha)(X-\frac{q}{\alpha})(X-\beta)(X-\frac{q}{\beta})(X^2-CX+q)$, with C^2-4q non-square modulo ℓ and $\alpha \neq q/\alpha$, $\beta \neq q/\beta$ and $\{\alpha, q/\alpha\} \neq \{\beta, q/\beta\}$; equivalently $P_q \equiv (X^2-AX+q)(X^2-BX+q)(X^2-CX+q)$where the first two quadratic polynomials are distinct and both reducible and the third one is irreducible;

2. $P_q(X) \equiv (X-\alpha)(X-\frac{q}{\alpha})Q$, where $\alpha \neq q/\alpha$ and the irreducible factor Q is the reduction of a degree 4 Weil polynomial;

3. P_q is the product of three distinct irreducible quadratic polynomials, i.e., $P_q(X) \equiv (X^2-CX+q)Q$ where X^2-CX+q is irreducible and Q is the reduction of a degree 4 Weil polynomial which has two distinct irreducible factors, both of which are distinct from X^2-CX+q.

We will count the number of polynomials of each type.

Type 1. By Lemma 5.5, there are $\frac{1}{2}\left(\ell-\left(\frac{q}{\ell}\right)\right)$ irreducible quadratic polynomials X^2-CX+q. Also by Lemma 5.5, there are $\frac{1}{2}\left(\ell-2-\left(\frac{q}{\ell}\right)\right)$ choices for reducible X^2-AX+q without a double root and then there are $\frac{1}{2}\left(\ell-2-\left(\frac{q}{\ell}\right)\right)-1$ choices for reducible X^2-BX+q without a double root and distinct from X^2-AX+q. It follows that there are $\frac{1}{16}\left(\ell-\left(\frac{q}{\ell}\right)\right)\left(\ell-\left(\frac{q}{\ell}\right)-2\right)\left(\ell-\left(\frac{q}{\ell}\right)-4\right)$ such polynomials.

Type 2. By Proposition 5.6 and Lemma 5.5, the number of polynomials with decomposition of this type is

$$\frac{1}{2}\left(\ell-\left(\frac{q}{\ell}\right)-2\right)N_4 \le \frac{1}{8}(\ell+1)(\ell-1)\left(\ell-\left(\frac{q}{\ell}\right)-2\right).$$

Type 3. Proposition 5.6 and Lemma 5.5 imply that there are

$$\le \frac{1}{2}\left(\ell-\left(\frac{q}{\ell}\right)\right)T_4 \le \frac{1}{8}\left(\ell-\left(\frac{q}{\ell}\right)\right)^2(\ell-3)+\frac{1}{16}(\ell-1)(\ell+1)\left(\ell-\left(\frac{q}{\ell}\right)\right)$$

polynomials of this type. [2]

Summing these three upper bounds yields the lemma. □

5.2.3 Proof of Lemma 5.3

The discriminant of P_q is $\Delta_{P_q}=q^6\Gamma^2\delta$, where

$$\Gamma = 8qa^4+9q^2a^2-42qa^2b+a^2b^2-4a^3c+108q^3-108q^2b+36qb^2-4b^3+54qac+18abc-27c^2$$

and $\delta=(c+2aq)^2-4q(b+q)^2$. Hence, we have

$$\begin{aligned} D_6^{*-} &= \sharp\left\{(a,b,c); a,c\neq 0, \Gamma\not\equiv 0 \bmod \ell \text{ and } \left(\frac{\delta}{\ell}\right)=-1\right\} \\ &= \sharp\left\{(a,b,c); a,c\neq 0, \left(\frac{\delta}{\ell}\right)=-1\right\} - \sharp\left\{(a,b,c); a,c\neq 0, \Gamma\equiv 0 \bmod \ell \text{ and } \left(\frac{\delta}{\ell}\right)=-1\right\} \\ &\ge M-W, \end{aligned}$$

where $M=\sharp\left\{(a,b,c); a,c\neq 0, \left(\frac{\delta}{\ell}\right)=-1\right\}$ and $W=\sharp\{(a,b,c); a\neq 0, \Gamma\equiv 0 \bmod \ell\}$.

Computation of M Since $\ell>2$ and $q\in\mathbb{F}_\ell^*$, for any fixed $c\in\mathbb{F}_\ell^\times$, the map $(a,b)\mapsto(c+2aq, b+q)$ is a bijection from $\mathbb{F}_\ell^*\times\mathbb{F}_\ell$ to $\mathbb{F}_\ell\backslash\{c\}\times\mathbb{F}_\ell$. From this and Lemma 5.5 we deduce that

$$\begin{aligned} M &= \sum_{c\in\mathbb{F}_\ell^*}\sharp\left\{(x,y)\in\mathbb{F}_\ell^2; x\neq c, \left(\frac{x^2-4qy^2}{\ell}\right)=-1\right\} \\ &= \sum_{c\in\mathbb{F}_\ell^*}\sharp\left\{(x,y)\in\mathbb{F}_\ell^2; \left(\frac{x^2-4qy^2}{\ell}\right)=-1\right\} - \sum_{c\in\mathbb{F}_\ell^*}\sharp\left\{y\in\mathbb{F}_\ell; \left(\frac{c^2-4qy^2}{\ell}\right)=-1\right\} \\ &= \frac{1}{2}(\ell-1)^2\left(\ell-\left(\frac{q}{\ell}\right)\right)-\sum_{c\in\mathbb{F}_\ell^*}M_c', \end{aligned}$$

where

$$M_c'=\sharp\left\{y\in\mathbb{F}_\ell; \left(\frac{c^2-4qy^2}{\ell}\right)=-1\right\}=\sharp\left\{y\in\mathbb{F}_\ell; \left(\frac{y^2-(c^2/4q)}{\ell}\right)=-\left(\frac{-q}{\ell}\right)\right\}.$$

By Lemma 5.5, if $\left(\frac{-q}{\ell}\right)=-1$, then

$$M_c'=\sharp\left\{y\in\mathbb{F}_\ell; \left(\frac{y^2-(c^2/4q)}{\ell}\right)=1\right\}=\frac{1}{2}\left(\ell-2-\left(\frac{q}{\ell}\right)\right)$$

[2] The first inequality is due to the fact that we do not take into account that X^2-CX+q has to be distinct from the factors of Q.

and if $\left(\frac{-q}{\ell}\right) = 1$, then

$$M'_c = \sharp\left\{y \in \mathbb{F}_\ell; \left(\frac{y^2 - (c^2/4q)}{\ell}\right) = -1\right\} = \frac{1}{2}\left(\ell - \left(\frac{q}{\ell}\right)\right).$$

This can be rewritten, for all q and ℓ, as $M'_c = \frac{1}{2}\left(\ell - 1 - \left(\frac{q}{\ell}\right) + \left(\frac{-q}{\ell}\right)\right)$. We obtain

$$M = \frac{1}{2}(\ell-1)^2\left(\ell - 1 - \left(\frac{q}{\ell}\right)\right) + \frac{1}{2}(\ell-1)\left(\frac{q}{\ell}\right)\left(1 - \left(\frac{-1}{\ell}\right)\right).$$

Computation of $W = \sharp\left\{(a,b,c) \in \mathbb{F}_\ell^3; a \neq 0, \Gamma = 0\right\}$. Note that Γ is a degree 2 polynomial in c over $\mathbb{F}_\ell[a,b]$:

$$\Gamma = -27c^2 + G_1 c + G_0, \quad \text{where} \quad G_1(a,b) = -2a(2a^2 - 27q - 9b)$$

and $G_0(a,b) = 8qa^4 + 9q^2a^2 - 42qa^2b + a^2b^2 + 108q^3 - 108q^2b + 36qb^2 - 4b^3$. The discriminant of Γ as a polynomial in c is $\gamma = 16(a^2 + 9q - 3b)^3$. So $\Gamma \equiv 0 \bmod \ell$ if and only if

$$\left(\left(\frac{\gamma}{\ell}\right) = 1 \text{ and } c = \frac{-1}{54}(-G_1 \pm \sqrt{\gamma})\right) \text{ or } \left(\gamma = 0 \text{ and } c = \frac{1}{54}G_1\right),$$

where $\sqrt{\gamma}$ denotes a square root of γ in $\mathbb{F}_\ell$. It follows that

$$\begin{aligned} W &= 2 \cdot \sharp\left\{(a,b) \in \mathbb{F}_\ell^2; a \neq 0, \left(\frac{\gamma}{\ell}\right) = 1\right\} + \sharp\left\{(a,b) \in \mathbb{F}_\ell^2; a \neq 0, \gamma = 0\right\} \\ &= 2 \cdot \sharp\left\{(a,b) \in \mathbb{F}_\ell^2; a \neq 0, \left(\frac{a^2 - 3(b-3q)}{\ell}\right) = 1\right\} + \sharp\left\{(a,b) \in \mathbb{F}_\ell^2; a \neq 0, a^2 = 3(b-3q)\right\}. \end{aligned}$$

Since $\ell > 3$, the map $b \mapsto 3(b-3q)$ is a bijection on $\mathbb{F}_\ell$. So we have

$$\begin{aligned} W &= 2 \cdot \sharp\left\{(x,y) \in \mathbb{F}_\ell^2; x \neq 0, \left(\frac{x^2 - y}{\ell}\right) = 1\right\} + \sharp\left\{(x,y) \in \mathbb{F}_\ell^2; x \neq 0, x^2 = y\right\} \\ &= 2 \cdot \sum_{y \in \mathbb{F}_\ell} \sharp\left\{x \in \mathbb{F}_\ell; \left(\frac{x^2 - y}{\ell}\right) = 1\right\} - 2 \cdot \sharp\left\{y \in \mathbb{F}_\ell; \left(\frac{-y}{\ell}\right) = 1\right\} + \sum_{y \in \mathbb{F}_\ell^*} \sharp\{x \in \mathbb{F}_\ell^*; x^2 = y\} \\ &= \sum_{y \in \mathbb{F}_\ell^*} \left(\ell - 2 - \left(\frac{y}{\ell}\right)\right) + 2(\ell-1) - (\ell-1) + (\ell-1) \end{aligned}$$

using Lemma 5.5 (the second term is the contribution of $y = 0$). This yields $W = \ell(\ell-1)$ and computing $M - W$ concludes the proof. □

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