OPTIMAL DESIGN UNDER THE ONE-DIMENSIONAL WAVE EQUATION

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Abstract

An optimal design problem governed by the wave equation is examined in detail. Specifically, we seek the time-dependent optimal layout of two isotropic materials on a 1-d domain by minimizing a functional depending quadratically on the gradient of the state with coefficients that may depend on space, time and design. As it is typical in this kind of problems, they are ill-posed in the sense that there is not an optimal design. We therefore examine relaxation by using the representation of two-dimensional ($(x,t) \in \mathbb{R}^2$) divergence free vector fields as rotated gradients. By means of gradient Young measures, we transform the original optimal design problem into a non-convex vector variational problem, for which we can compute an explicit form of the "*constrained quasiconvexification* " of the cost density. Moreover, this quasiconvexification is recovered by first or second-order laminates which give us the optimal microstructure at every point. Finally, we analyze the relaxed problem and some numerical experiments are performed. The perspective is similar to the one developed in previous papers for linear elliptic state equations. The novelty here lies in the state equation (the wave equation), and our contribution consists in understanding the differences with respect to elliptic cases.

1 Introduction

Optimal design problems in conductivity and elasticity have been extensively studied from various perspectives. For the homogenization viewpoint, see [1]. For more simulation-oriented approaches, see [4, 9]. For treatments based on variational reformulations, see [23]. In many of these examples, the state equation is assumed to

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be isotropic. There has also been attempts to understand non-isotropic situations ([20] and references therein).

Suppose we choose two diagonal, non-isotropic, 2×2 matrices of the form

$$A_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_\beta = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$

and consider the state equation

$$\text{div } ([\chi(x)A_\alpha + (1-\chi(x))A_\beta]\nabla u) = 0 \quad \text{in} \quad \Omega$$

where $\Omega \subset \mathbb{R}^2$ is a bounded, regular, simply-connected domain. It is easy to see that we can also write

$$\text{div } \left(\begin{pmatrix} 1 & 0 \\ 0 & \chi(x)\alpha + (1-\chi(x))\beta \end{pmatrix} \nabla u \right) = 0 \quad \text{in} \quad \Omega,$$

and even more so

$$\text{div } (u_{x_1}, [\chi(x)\alpha + (1-\chi(x))\beta]u_{x_2}) = 0.$$

In the case where $\Omega = (0,T) \times (0,1)$, and we take both α and β negative, we see that we have a 1-d wave equation as state equation, and 2-d optimal designs can be interpreted as 1-d time-dependent optimal designs. For this reason, we change the notation and write $x(= x_2)$ for the spatial variable, $t(= x_1)$ for the time variable, and replace α, β by $-\alpha, -\beta$, respectively, so that we focus on such a wave equation. We also change accordingly the domain, and consider initial and boundary condition as is usual in hyperbolic problems. Yet notice that the non-isotropic elliptic example is contained in our analysis too.

Optimal control problems in the coefficients are rather well-known in the elliptic case. Homogenization has been the main tool to deal with these when cost functionals do not depend on derivatives of the state. As indicated earlier [1] is still, as far as we can tell, an up-to-date reference for the use of homogenization in optimal design problems. Our interest in understanding optimal design problems for cost functionals depending on gradients of the state led us to explore the use of gradient Young measures in this kind of problems ([23], [24]). See [27] for a pioneering situation that is quite instructive to better understand how different optimal design problems with quadratic cost functionals in the gradient can be. Other references that have treated similar situations from the perspective of homogenization are [11], [14]. Our perspective does not require a full understanding of the G-closure problem (as in homogenization) as the emphasis is n ot placed on the tensors that can be obtained by mixtures (the G-closure), but rather on the set of pairs of vectors that can be related through some tensor of the G-closure. Hence, although intimately connected to homogenization, our approach focuses directly on pairs of fields that can occur in relaxed state equations. In this way, we can treat cost functionals depending on the gradient of the state directly without further ingredients.

Having succeeded, to some degree, in understanding the elliptic situation in the conductivity setting (even producing numerical simulations as in [9] of optimal microstructures), a next natural step is to examine the same optimal design problems with quadratic cost functionals on the gradient of the state under a hyperbolic state equation, so as to better understand the differences introduced on the analysis and on the numerics because of this hyperbolic nature. One issue here is the phenomenon of concentration of cost (energy). It is well-understood that Young measures cannot capture concentration effects. We resolve easily this issue by demanding some extra regularity on initial data. Notice that if designs were not allowed to vary with time, the situation would be much simpler as it would not require "homogenization" or the formation of microstructure on this time variable. Many of the standard homogenization facts could be used and exploited. Indeed, one of our main goals is to grasp this dependence of designs on time.

Except for the works of Lurie ([15], [16], [17]), this sort of problems have not been addressed in the literature. In particular, he has been investigating over the years the analogue of the G-closure for dynamic designs. Being motivated by realistic applications, one main concerned in his work is to understand the relationship and interaction between the dynamic nature of the problem and the "dynamics" of microstructure. In a sense, even in a static situation, microstructure (laminates) is something dynamic as it is a never-ending refinement process. When this process interacts with real time, some funny situations may occur (including the formation of shocks). This is well-documented for instance in [17], where restrictions between the velocity of formation of microstructure and the dynamics of the state equation are explicitly given so that undesirable behavior is ruled out. We have avoided altogether this issue as we model dynamic microstructure through families of probability measures (Young measures) depending both on space and time so that there is only one dynamic process associated with time. Even so, it is interesting to stress that the laminates we get in our numerical simulations (Section 7) are of the kind that satisfy these requirements in the best way possible as they are oriented parallel to the time axis. This is the most favorable situation in Lurie's work. See also the comments after Conjecture 1. Another main difference of our work with that of Lurie is that we are interested from the beginning on a cost functional which is quadratic in the gradient of the state. Our methods allow to treat directly this sort of problems without understanding first the G-closure set. A main concern in the work of Lurie is to better understand the G-closure set corresponding to a dynamical situation, and, in particular, to discover the differences with respect to its elliptic counterpart.

Other works dealing with optimal control problems under the wave equation in greater dimensions can be found in [6], where the control is a time dependent coefficient, and under other constraints on modes where there is vibration. In this sense another work in which the authors examine time-harmonic solutions of the wave equation is [3], where they prove a relaxation result for this problem and very interesting results of existence of classical solutions for some particular cases. In the wave equation literature, we can find a huge family of optimal control problems where the design variable is not in the highest derivative term. When the control term acts on the first order derivative in time, the term is known as a ”damping” term. These problems are of a different nature physically as well as mathematically. Some relevant references in this topic are [5, 10, 12].

1.1 Problem Statement

We will thus study the following optimal design problem. We consider a design domain $\Omega = (0,1) \subset \mathbb{R}$, a positive time $T > 0$, and a maximum amount of one material at our disposal $V_\alpha \in (0,1)$. The optimal design problem consists in deciding, for each time $0 < t < T$, the best distribution in Ω of the two materials in order to minimize the time-dependent cost functional depending on the square of the gradient (with respect to both variables (t,x)) of the underlying state. More precisely, let us denote by (P) the problem that consists in minimizing

$$(P) \qquad I(\chi) = \int_0^T \int_\Omega \left[u_t^2(t,x) + a(t,x,\chi)u_x^2(t,x)\right] \, dx\, dt$$

where u is the unique solution of

$$\begin{aligned} u_{tt} - div([\alpha\chi + \beta(1-\chi)]u_x) = 0 \quad &\text{in} \quad (0,T)\times(0,1), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad &\text{in} \quad \Omega, \qquad (1) \\ u(t,0) = 0,\ u(t,1) = 0 \quad &\text{in} \quad [0,T], \qquad (2) \end{aligned}$$

and the functions a, u_0 and u_1 are known. We have put vanishing boundary conditions on both end-points and source term for simplicity. Functions of time $f(t)$ and $g(t)$ on both points could be considered as well. The function $\chi \in L^\infty([0,T] \times \Omega; \{0,1\})$ is the design variable, and it indicates where we place the α-material for each time t. Since χ is a binary variable $a(t,x,\chi) \in \{a(t,x,0), a(t,x,1)\}$, we can write

$$a(t,x,\chi) = \chi(t,x)a_\alpha(t,x) + (1-\chi(t,x))a_\beta(t,x),$$

where

$$a_\alpha(t,x) = a(t,x,1), a_\beta(t,x) = a(t,x,0).$$

In addition, we make the assumption $0 < \alpha < \beta$, and

$$a_\alpha(t,x) + \alpha \geq 0, \quad a_\beta(t,x) + \beta \geq 0.$$

The amount of the α-material is given, and therefore we have to enforce the volume constraint

$$\int_\Omega \chi(t,x)dx \leq V_\alpha|\Omega|, \quad \forall t \in [0,T].$$

The lack of classical solutions for this sort of problems is well understood (see. Theorem 11, [22]). In this sense we propose and analyze a relaxation of the problem.

Our approach is based on an equivalent variational reformulation of the original optimal design problem as a non-convex vector variational problem. As in other situations examined under this perspective [2, 23], we change a scalar problem with differential constraints by a vector variational problem with integral constraints (where the state equation is implicit in the new cost function). It is well-known that the non-existence of optimal solution for vector variational problem is intimately associated with the lack of quasiconvexity of the cost functional, and in this sense we propose to analyze the "*constrained quasiconvexification*" for this last problem by using gradient Young measures as generalized solutions of variational problems. We compute an explicit relaxation of the original optimal design problem in the form of a relaxed (quasiconvexified) variational problem.

It is elementary to check (this is done with some detail in Section 2), the equivalence of our dynamic optimal design problem with the following non-convex, vector variational problem

$$(VP) \qquad \min_{U} \ \hat{I}\ (U) = \int_0^T \int_\Omega W(t,x,\nabla U(t,x))dxdt$$

subject to

$$\begin{aligned}
&U = (U^{(1)}, U^{(2)}) \in H^1([0,T]\times\Omega)^2,\\
&U^{(1)}(0,x) = u_0(x),\ U_t^{(1)}(0,x) = u_1(x) \quad \text{in} \quad \Omega,\\
&U^{(1)}(t,0) = f(t),\ U^{(1)}(t,1) = g(t) \quad \text{in} \quad [0,T],\\
&\int_\Omega V(t,x,\nabla U(t,x))dx \le V_\alpha |\Omega| \quad \forall t \in [0,T].
\end{aligned}$$

The two integrands involved are

$$W(t,x,A) = \begin{cases} a_{11}^2 + a_\alpha(t,x)a_{12}^2, & \text{if} \quad A \in \Lambda_\alpha,\\ a_{11}^2 + a_\beta(t,x)a_{12}^2, & \text{if} \quad A \in \Lambda_\beta \setminus \Lambda_\alpha,\\ +\infty, & \text{else,} \end{cases}$$

$$V(t,x,A) = \begin{cases} 1, & \text{if } A \in \Lambda_\alpha,\\ 0, & \text{if } A \in \Lambda_\beta \setminus \Lambda_\alpha,\\ +\infty, & \text{else.} \end{cases}$$

Here

$$\Lambda_\gamma = \{A \in M^{2\times 2} : M_{-\gamma}A^{(1)} - RA^{(2)} = 0\}, \quad \gamma = \alpha, \beta, \tag{3}$$

where $A^{(i)}$ is the i-th row of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and

$$M_{-\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

1.2 Results Statement

To write down an explicit relaxation, put

$$h(t,x) = \beta a_\alpha(t,x) - \alpha a_\beta(t,x)$$

and for

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad s \in \mathbb{R},$$

set

$$\psi(F,s) = F_{12}F_{21} + \frac{\alpha}{s(\beta-\alpha)^2}(\beta F_{12} + F_{21})^2 + \frac{\beta}{(1-s)(\beta-\alpha)^2}(\alpha F_{12} + F_{21})^2.$$

Consider the variational problem,

$$(RP) \qquad \min_{U,s} \int_0^T \int_\Omega \varphi(t,x,\nabla U(t,x), s(t,x))\,dxdt$$

subject to

$$\begin{aligned} &U \in H^1([0,T]\times\Omega)^2, \quad tr(\nabla U(t,x)) = 0, \\ &U^{(1)}(0,x) = u_0(x), \; U_t^{(1)}(0,x) = u_1(x) \quad \text{in} \quad \Omega, \\ &U^{(1)}(t,1) = f(t), \; U^{(1)}(t,0) = g(t) \quad \text{in} \quad [0,T], \\ &0 \le s(t,x) \le 1, \int_\Omega s(t,x)\,dx \le V_\alpha|\Omega| \quad \forall t \in [0,T], \end{aligned}$$

where $\varphi(t,x,F,s)$ is explicitly given by the surprising expression

$$\begin{cases} \dfrac{h}{s\beta(\beta-\alpha)^2}(\beta^2|F_{12}|^2 + |F_{21}|^2 + 2\beta F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\beta}{\beta}F_{12}F_{21} \\ \hfill \text{if } h(x,t) \ge 0, \psi(F,s) \le 0, \\ \dfrac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2|F_{12}|^2 + |F_{21}|^2 + 2\alpha F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\alpha}{\alpha}F_{12}F_{21}, \\ \hfill \text{if } h(x,t) \le 0, \psi(F,s) \le 0, \\ -detF + \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(\big((1-s)\beta^2(\alpha+a_\alpha) + s\alpha^2(\beta+a_\beta)\big)|F_{12}|^2 \\ \quad +\big((1-s)(\alpha+a_\alpha) + s(\beta+a_\beta)\big)|F_{21}|^2 + 2\big((\alpha+a_\alpha)\beta - sh\big)F_{12}F_{21}\Big) \\ \hfill \text{if } \psi(F,s) \ge 0. \end{cases}$$

tr stands above for the trace of a matrix. All that matters is that this integrand φ is known in closed form.

Theorem 1 *Suppose that the initial data u_0 and u_1 have the regularity*

$$u_0 \in H^2(0,1) \cap H_0^1(0,1), \quad u_1 \in H_0^1(0,1).$$

Then the variational problem (RP) is a relaxation of the initial optimization problem (P) in the sense that

a) the infima of both problems coincide;

b) there are optimal solutions for the relaxed problem (RP);

c) these solutions codify (in the sense of the Young measures) the optimal microstructures of the original optimal design problem.

For the interpretation of Young measure solutions in this statement, we refer the reader to the already-mentioned contributions in the elliptic case. It is closely related to relaxation in vector, non-convex variational problems ([8]). These optimal Young measures carry the information of optimal microstructures, both on the local distribution of materials, and on the geometry of optimal microarrangements. See more on this interpretation in Section 7.

In addition, we can provide explicitly optimal microstructures.

Theorem 2 *Optimal, dynamic microarrangements of the two materials leading to the relaxed formulation are always laminates which can be given in a completely explicit form.*

The formulae for all of these laminates are given later at the end of Section 4, where we compute these optimal microstructures corresponding to first and second order laminates.

The main new contribution here is therefore to understand the character of the hyperbolic state law, and the differences it introduces with respect to the better known elliptic case. Some of these differences are related to the fact that the manifolds Λ_γ are two 2-dimensional subspaces whose intersection is another 1-dimensional manifold. Moreover there are rank-one connections within those manifolds. An interesting consequence is that the relaxed integrand is finite everywhere (except for the condition involving the trace) in contrast with the elliptic case where the relaxed integrand is finite only in a certain (quasi)convex subset. An important issue is that optimal Young measures gives us the necessary information about the behavior of minimizing sequences of the original optimal design problem.

A subsequent important step is to explore the relaxed problem (RP) in some particular cases, like the ones described in Section 5, with the objective of producing numerical simulations of optimal time-dependent structures [18]. For some particular situations in the (static) elliptic case, it has been shown that a simple relaxation consists in replacing the original discrete design variable $\chi \in L^\infty(\Omega, \{0, 1\})$ by its convex envelop $s \in L^\infty(\Omega, [0, 1])$. For the (dynamic) hyperbolic case with $a_\alpha = a_\beta = 1$, some numerical experiments (see Section 7) suggest that the above assertion is true. In this regard, we establish the following conjecture (examined briefly in Section 6).

Conjecture 1 *Suppose the coefficients* $a_\alpha = 1$, $a_\beta = 1$. *The optimization problem*

$$(\widetilde{RP}) \qquad \min_s \tilde{I}(s) = \int_0^T \int_\Omega u_t^2(t,x) + u_x^2(t,x)\, dx\, dt$$

where u *is the unique solution of*

$$\begin{aligned} u_{tt} - div\Big([\alpha s + \beta(1-s)]u_x\Big) &= 0 \quad in \quad (0,T)\times(0,1), \\ u(0,x) = u_0(x),\ u_t(0,x) &= u_1(x) \quad in \quad \Omega, \\ u(t,0) = f(t),\ u(t,1) &= g(t) \quad in \quad [0,T], \\ \int_\Omega s(t,x)dx &\le V_\alpha|\Omega|, \qquad \forall t \in [0,T], \\ 0 \le s(t,x) &\le 1. \end{aligned}$$

is equivalent to the original optimal design problem (P) *in the sense that*

a) the infima of both problems coincide, i.e., $\inf(\widetilde{RP}) = \inf(P)$*;*

b) the above optimal design problem $(\widetilde{RP})$ *admits optimal solutions;*

c) these solutions (in the sense of Young measures) show that optimal microstructures are first order laminates with normal $n = (0,1)$ *and volume fraction* s*.*

One can pass from (RP) to $(\widetilde{RP})$ simply by minimizing the general relaxed integrand $\varphi(t,x,F,s)$ on the auxiliary variable $F^{(2)}$, the second row of F, keeping all other variables fixed (and taking $a_\alpha = 1$, $a_\beta = 1$). This is an elementary calculus exercise (Section 6). This vector, the second row of F, was introduced as an auxiliary field to go from the original formulation (P) to its variational form (VP). After relaxation, in which this auxiliary vector plays an important role, we eliminate it by minimizing over it, so that we are back to a state law which is the result of this minimization process. More importantly, first-order laminates involved in this process (the passage from (RP) to $(\widetilde{RP})$) always correspond to normal direction $n = (0,1)$, i.e., the optimal laminates have to be arranged in the perpendicular direction to the space-axis with volume fraction $s(t,x)$. These are, in particular, laminates of the class considered by Lurie in the previously cited works. Our conjecture establishes that this procedure should capture the optimal relaxed state law.

Notice how this process cannot produce a general relaxation theorem (this is in fact our previous theorem for (RP)), as it is tailored and computed for the particular choice of the coefficients $a_\alpha = 1$, $a_\beta = 1$. It asserts that among the many relaxed wave-like equations that can be produced by mixing dynamically the two materials, the one providing optimal microstructures for the particular choice of the coefficients $a_\alpha = 1$, $a_\beta = 1$, is precisely the one obtained by replacing $\chi \in \{0,1\}$ by $s \in [0,1]$ (as in the situation in [27]). For other choices of the coefficients a_α and a_β, the optimal relaxed equation would eventually be different. One can see this phenomenon for the elliptic situation in [9]. The importance of having this more "economic" relaxation (compared Conjecture 1 with Theorem 1) is that simulations can be performed for these, while it is out of the question to use directly (RP). We have written this in the form of a conjecture because, even though the passage from (RP) to $(\widetilde{RP})$ is elementary, its formal rigorous proof requires a careful analysis. It has been shown to be correct in a number of situations in the elliptic case ([26]). For our situation here, showing the validity of the conjecture is in progress ([18]).

The paper is organized as follows. In Section 2, we describe in more detail the equivalent variational reformulation as well as a general relaxation result when integrands are not continuous and may take on infinite values abruptly. As there is nothing new here compared to other previous works in the elliptic case, our description is rather a remainder included here for the sake of completeness. Sections 3 and 4 are technical in nature but interesting, as we first compute a lower bound of the *constrained quasiconvexification* (Section 3), by using in a fundamental way the weak continuity of the determinant. Section 4 is concerned with the search for laminates furnishing the precise value of the lower bound in an attempt to show equality of the three convex hulls (poly-, quasi- and rank-one convex hulls), as it is standard in this kind of calculation. In Section 5, we show some particular examples of this relaxation for different and interesting choices of the coefficients a_α, a_β. Finally, in Section 6 we analyze the relaxed problem and propose a simpler relaxation, while in Section 7 we numerically solve it by using a gradient descent method.

2 Reformulation and relaxation

The lack of classical solution of the original optimal design problem is well-established. We propose to reformulate the problem as a vector variational problem to which we apply suitable tools to study its relaxation. We follow a similar approach to the one in [2, 23].

Under the hypothesis of simple-connectedness of Ω (an interval), there exists a potential $v \in H^1((0,T)\times\Omega)$ such that the state equation can be recast as

$$-div(u_t(t,x), -[\alpha\chi(t,x) + \beta(1-\chi(t,x))]u_x(t,x)) = 0 \quad \text{in } [0,T]\times\Omega$$

where the div operator is consider now with respect to the variables t and x. The state equation is equivalent to the pointwise constraint

$$\begin{pmatrix} u_t(t,x) \\ -[\alpha\chi(t,x) + \beta(1-\chi(t,x))]u_x(t,x) \end{pmatrix} = R\nabla v(t,x) \quad a.e.\ (t,x) \in [0,T]\times\Omega$$

where R is the counterclockwise $\pi/2$-rotation in the space-time plane. If we let $\Lambda_{-\gamma}$ be as in (3), this constraint reads

$$\begin{pmatrix} \nabla u(t,x) \\ \nabla v(t,x) \end{pmatrix} \in \Lambda_{-\alpha} \cup \Lambda_{-\beta} \quad a.e.\ (t,x) \in [0,T]\times\Omega. \tag{4}$$

It is clear that we can identify the design variable χ with the vector field $U = (u, v)$ complying with (4); and conversely, a pair $U = (u, v)$ which verifies (4) determines a characteristic function χ, so that we can consider the new design variable $U = (U^{(1)}, U^{(2)}) = (u, v)$, where $U : \mathbb{R}^2 \to \mathbb{R}^2$ and $\nabla U(t, x) \in \mathbb{R}^{2\times 2}$, under the main constraint (4).

Therefore, by using the above statement and the notation in the Introduction, it is easy to check that the original optimal design problem (P) is equivalent to the variational problem (VP).

We have so recast our optimal design problem as a typical variational problem. We see that it is a non-convex vector problem that we are going to analyze by seeking its relaxation. We use Young measures as a main tool in the computation of the suitable density for the relaxed problem. In this sense, we can rely on the following relaxation result [2] whose main idea has been a useful tool in other different places [2, 23, 25].

We note the initial condition (1) by I.C., the boundary condition (2) by B.C. and put

$$
\begin{aligned}
m = \inf \Big\{ & \int_\Omega \int_0^T W(t, x, \nabla U(t, x)) dt dx : \\
& U \in H^1((0, T) \times \Omega)^2, U^{(1)} \text{ satisfies the B.C. and the I. C.}, \\
& \qquad \int_\Omega V(t, x, \nabla U(t, x)) dx \le V_\alpha |\Omega|, \forall t \in [0, T] \Big\}.
\end{aligned}
$$

We know [2] that

$$
\begin{aligned}
m \ge \bar{m} = \inf \Big\{ & \int_\Omega \int_0^T CQW(t, x, \nabla U(t, x), s(t, x)) dt dx : \\
& U \in H^1([0, T] \times \Omega)^2, U^{(1)} \text{ satisfies the B.C. and the I. C.}, \\
& \qquad 0 \le s(t, x) \le 1, \int_\Omega s(t, x) dx \le V_\alpha |\Omega|, \forall t \in [0, T] \Big\},
\end{aligned}
$$

where $CQW(t, x, F, s)$ is defined by,

$$
CQW(t, x, F, s) = \inf \Big\{ \int_{M^{2\times 2}} W(t, x, A) d\nu(A) : \nu \in \mathcal{A}(F, s) \Big\}
$$

with

$$
\begin{aligned}
\mathcal{A}(F, s) = \Big\{ & \nu : \nu \text{ is a homogeneous } H^1\text{-Young measure}, \\
& F = \int_{M^{2\times 2}} A d\nu(A), \int_{M^{2\times 2}} V(t, x, A) d\nu(A) = s \Big\}. \qquad (5)
\end{aligned}
$$

Notice that the previous inequality will be an equality when W is a Carathéodory function with appropriate growth constrains. However, in our situation it is still possible to prove this equality despite the fact that W is not a Carathéodory function. Let us consider the following minimization problem

$$\tilde{m} = \inf \Big\{ \int_\Omega \int_0^T \int_{M^{2\times 2}} W(t,x,A)d\nu_{t,x}(A)dtdx : \nu \in \mathcal{B}(B.C., I.C., t_0) \Big\}$$

where

$$\begin{aligned}\mathcal{B}(B.C., I.C., V_\alpha) = \Big\{ &\nu : H^1\text{-Young meas.}, \; supp(\nu_{t,x}) \subset \Lambda_\alpha \cup \Lambda_\beta, \\ &\exists U \in H^1([0,T]\times\Omega)^2, U^{(1)} \text{ satisfies the I.C. and B.C.}, \\ &\nabla U(t,x) = \int_{M^{2\times 2}} A d\nu_{t,x}(A), \\ &\int_\Omega \int_{M^{2\times 2}} V(t,x,A)d\nu_{t,x}(A)dx \le V_\alpha |\Omega|, \forall t \in [0,T] \Big\}.\end{aligned}$$

We have the following result, whose proof is essentially identical to the one in ([2]).

Theorem 3 *([2]) Suppose that the initial data u_0 and u_1 have the regularity*

$$u_0 \in H^2(0,1) \cap H^1_0(0,1), \quad u_1 \in H^1_0(0,1).$$

Then the equalities

$$m = \bar{m} = \tilde{m}$$

hold. Moreover, for each measure $\nu \in \mathcal{B}(B.C., I.C., V_\alpha)$ such that $supp(\nu_{x,t}) \subset \Lambda_\alpha \cup \Lambda_\beta$ a.e. $(t,x) \in [0,T]\times\Omega$, there exists a sequence $\{\nabla U_k\}$ such that,

i) $U_k \in (H^1([0,T]\times\Omega))^2$, $U^{(1)}$ *satisfies the I.C. and B.C.*, $\{|\nabla U_k|^2\}$ *is equi-integrable,*

ii) $\nabla U_k(t,x) \in \Lambda_\alpha \cup \Lambda_\beta$, *a.e.* $(t,x) \in [0,T]\times\Omega \; \forall k$, $\int_\Omega V(t,x,\nabla U_k(t,x))dx \le V_\alpha|\Omega|$, $\forall t \in [0,T] \forall k$

iii) $\lim_{k\to\infty} \int_0^T \int_\Omega W(t,x,\nabla U_k(t,x))dxdt = \int_0^T \int_\Omega \int_{M^{2\times 2}} W(t,x,A)d\nu_{t,x}(A)dxdt$

The only remark worth noticing refers to the regularity of initial data. The proof of this theorem in the elliptic case in [2] relies in a fundamental way in the elliptic character of the manifolds Λ_γ to discard concentrations of the sequence $\{|\nabla U_k|^2\}$. For the hyperbolic case, this equiintegrability can also be shown, in a standard way, based on the regularity of solutions for the wave equation (with uniformly elliptic spatial part) coming from the regularity of initial conditions (see [13]).

3 The lower bound: polyconvexification.

We would like to compute explicitly the *constrained quasiconvexification* defined as

$$CQW(t,x,F,s) = \inf\Big\{ \int_{M^{2\times 2}} W(t,x,A)d\nu(A) : \nu \in \mathcal{A}(F,s)\Big\}$$

where $\mathcal{A}(F,s)$ is given in (5). Since the variable $(t,x) \in [0,T]\times\Omega$ can be consider as a parameter, we drop this dependence to simplify the notation. In this form, the constrained quasiconvexification can be expressed as

$$\begin{aligned}\inf_{\nu}\Big\{ \int_{M^{2\times 2}} W(A)d\nu(A) : F = \int_{M^{2\times 2}} Ad\nu(A),\\ \int_{M^{2\times 2}} V(A)d\nu(A) = s,\ \forall t\in[0,T]\Big\}\end{aligned} \tag{6}$$

with ν a homogeneous H^1-Young measure with $supp(\nu) \subset \Lambda_\alpha \cup \Lambda_\beta$.

For (F,s) (and (t,x)) fixed, we are going to compute the value of (6), i.e. $CQW(t,x,F,s)$. The main difficulty here is that we do not know explicitly the set of the admissible measures, which we note as $\mathcal{A}$. We propose the following strategy. Consider two classes of family of probability measures $\mathcal{A}_*, \mathcal{A}^*$ such that

$$\mathcal{A}_* \subset \mathcal{A} \subset \mathcal{A}^*.$$

We first calculate the minimum over the greater class of probability measures $\mathcal{A}^*$, and then we check that the optimal value is attained by at least one measure over the narrower class $\mathcal{A}_*$. This fact tells us that the optimal value so achieved is the same in $\mathcal{A}$, and hence we will have in fact computed the exact value $CQW(t,x,F,s)$.

Following [23], we choose $\mathcal{A}^*$ as the set of polyconvex measures, which are not necessarily gradient Young measures, and therefore obtain a lower bound (the (constrained) polyconvexification). The main property of these measures is that they commute with the determinant. This constraint can be imposed in a more-or-less manageable way. We also choose $\mathcal{A}_*$ as the class of *laminates* which is a subclass of the gradient Young measures. By working with this class, we would get an upper bound (the (constrained) rank-one convexification).

The *polyconvexification* $CPW(F,s)$ can be computed through the following optimization problem

$$\min_{\nu} \int_{M^{2\times 2}} W(A)d\nu(A)$$

where,

$$\begin{aligned}\nu \in \mathcal{A}(F,s) = \Big\{\nu : \nu \text{ is a homogeneous Young measure,}\\ \text{which commutes with det, } F = \int_{M^{2\times 2}} Ad\nu(A), \quad (7)\\ \int_{M^{2\times 2}} V(A)d\nu(A) = s\Big\}. \quad (8)\end{aligned}$$

From (8) we have the following decomposition

$$\nu = t\nu_\alpha + (1-t)\nu_\beta, \quad supp(\nu_\gamma) \subset \Lambda_\gamma,\ \gamma = \alpha, \beta,$$

and therefore, from (7)

$$F = s\int_{\Lambda_\alpha} A d\nu_\alpha(A) + (1-s)\int_{\Lambda_\beta} A d\nu_\beta(A). \tag{9}$$

If we put

$$F_\gamma = \int_{\Lambda_\gamma} A d\nu_\gamma(A), \quad \gamma = \alpha, \beta,$$

we have $F_\gamma \in \Lambda_\gamma$ for $\gamma = \alpha, \beta$, so from this property and (9), we have a non-compatible system on F_γ unless

$$F_{11} + F_{22} = 0, \text{ i.e. } tr(F) = 0.$$

Let us suppose henceforth that this compatibility condition holds. This condition lets us simplify the problem from 2×2 matrices to 3-d vectors, using the identification

$$F = \begin{pmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{z} & -\boldsymbol{x} \end{pmatrix} \longleftrightarrow (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}).$$

Therefore the manifolds Λ_γ can be rewritten as

$$\Lambda_\gamma = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^3 : \boldsymbol{z} + \gamma \boldsymbol{y} = 0\}.$$

In this way, the above system does not uniquely determined its solution. Indeed

$$F_\alpha = (\lambda, y_\alpha, -\alpha y_\alpha),\ F_\beta = (\frac{x - s\lambda}{1-s}, y_\beta, -\beta y_\beta)$$

where

$$y_\alpha = \frac{1}{s(\beta-\alpha)}(\beta \boldsymbol{y} + \boldsymbol{z}), \qquad y_\beta = \frac{-1}{(1-s)(\beta-\alpha)}(\alpha \boldsymbol{y} + \boldsymbol{z})$$

and $\lambda \in \mathbb{R}$. We can check that if $A = (a_1, a_2, a_3) \in \Lambda_\gamma$ with $\gamma = \alpha, \beta$, then

$$\det A = -a_1^2 + \gamma a_2^2,$$

and by using the important constraint about the commutativity with det, we know that

$$\begin{aligned}
\det F =& \int_{\mathbb{R}^3} \det A d\nu(A) \\
=& s\int_{\mathbb{R}^3} \det A d\nu_\alpha(A) + (1-s)\int_{\mathbb{R}^3} \det A d\nu_\beta(A) \\
=& -\int_{\mathbb{R}} a_1^2 d\nu^{(1)}(A) + s\alpha \int_{\mathbb{R}} a_2^2 d\nu_\alpha^{(2)}(A) + (1-s)\beta \int_{\mathbb{R}} a_2^2 d\nu_\beta^{(2)}(A)
\end{aligned}$$

where ν_γ^i designates the projection of ν_γ onto the i-th component.

On the other hand, we can write the cost functional in the form

$$\int_{\mathbb{R}^3} W(A)d\nu(A) = \int_{\mathbb{R}^3} a_1^2 d\nu(A) + sa_\alpha \int_{\mathbb{R}^3} a_2^2 d\nu_\alpha(A) + (1-s)a_\beta \int_{\mathbb{R}^3} a_2^2 d\nu_\beta(A)$$

so if we put

$$S_1 = \int_{\mathbb{R}^3} a_1^2 d\nu(A), \quad S_\gamma = \int_{\Lambda_\gamma} a_2^2 d\nu_\gamma(A), \text{ with } \gamma = \alpha, \beta,$$

and use Jensen's inequality, we have the constraints

$$S_1 \geq \boldsymbol{x}^2, \qquad S_\gamma \geq y_\gamma^2 \qquad \gamma = \alpha, \beta.$$

By using the notation just introduced, the above inequalities and the constraint on the determinant, the constrained polyconvexification is given by the following linear programming problem

$$\underset{(S_1, S_\gamma, x_\gamma)}{\text{minimize}} \;\; S_1 + sa_\alpha S_\alpha + (1-s)a_\beta S_\beta$$

subject to,

$$\begin{aligned} -\det F &= S_1 - s\alpha S_\alpha - (1-s)\beta S_\beta, \\ S_1 &\geq \boldsymbol{x}^2, S_\gamma \geq y_\gamma^2, \text{ with } \gamma = \alpha, \beta, \end{aligned}$$

We can eliminate S_1, by replacing its value from the equality constraint in the cost functional. By so doing, only the variables (S_α, S_β) occur, with inequality constraints (see Figure 1 for a geometrical interpretation of the progamming problem). It is easy to solve this problem. Under the conditions $a_\alpha \geq -\alpha$ and $a_\beta \geq -\beta$, the optimal value depends on the relative position of the oblique line and the P point. Namely, the optimal solution can be attained on P, P_1 or P_2.

We put the function

$$\psi(F,s) = \boldsymbol{y}\boldsymbol{z} + \frac{\alpha}{s(\beta-\alpha)^2}(\beta\boldsymbol{y} + \boldsymbol{z})^2 + \frac{\beta}{(1-s)(\beta-\alpha)^2}(\alpha\boldsymbol{y} + \boldsymbol{z})^2,$$

the optimal value is

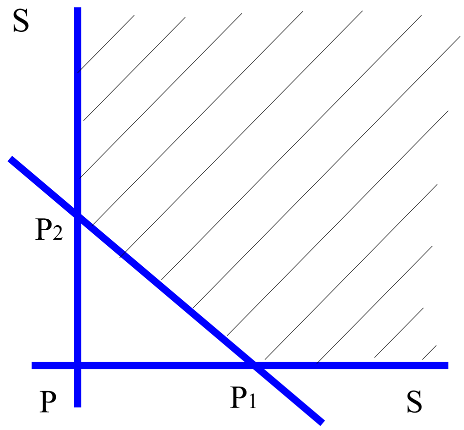

Figure 1: New mathematical programming problem.

$$\frac{h}{s\beta(\beta-\alpha)^2}(\beta^2\boldsymbol{y}^2+\boldsymbol{z}^2+2\beta\boldsymbol{yz})+\boldsymbol{x}^2-\frac{a_\beta}{\beta}\boldsymbol{yz}$$
$$\text{if } h(x,t)\geq 0, \psi(F,s)\leq 0,$$
$$\frac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2\boldsymbol{y}^2+\boldsymbol{z}^2+2\alpha\boldsymbol{yz})+\boldsymbol{x}^2-\frac{a_\alpha}{\alpha}yz,$$
$$\text{if } h(x,t)\leq 0, \psi(F,s)\leq 0,$$
$$-detF+\frac{1}{s(1-s)(\beta-\alpha)^2}\Big(\big((1-s)\beta^2(\alpha+a_\alpha)+s\alpha^2(\beta+a_\beta)\big)\boldsymbol{y}^2$$
$$+\big((1-s)(\alpha+a_\alpha)+s(\beta+a_\beta)\big)\boldsymbol{z}^2+2\big((\alpha+a_\alpha)\beta-s\beta\big)\boldsymbol{yz}\Big)$$
$$\text{if } \psi(F,s)\geq 0.$$

In addition, the optimal value is attained on

$$P_1 : S_\alpha = y_\alpha^2 \text{ and } S_1 = \boldsymbol{x}^2 \quad \text{if } h(x,t)\geq 0, \psi(F,s)\leq 0, \tag{10}$$

$$P_2 : S_\beta = y_\beta^2 \text{ and } S_1 = \boldsymbol{x}^2 \quad \text{if } h(x,t)\leq 0, \psi(F,s)\leq 0, \tag{11}$$

$$P : S_\alpha = y_\alpha^2 \text{ and } S_\beta = y_\beta^2 \quad \text{if } \psi(F,s)\geq 0. \tag{12}$$

Therefore we have an explicit computation of the constrained polyconvexification given by

$$CPW(F,s)=\begin{cases}\dfrac{h}{s\beta(\beta-\alpha)^2}(\beta^2y^2+z^2+2\beta\boldsymbol{yz})+\boldsymbol{x}^2-\dfrac{a_\beta}{\beta}\boldsymbol{yz} \\ \qquad\qquad\qquad \text{if } h(x,t)\geq 0,\psi(s,F)\leq 0, tr(F)=0, \\ \dfrac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2\boldsymbol{y}^2+\boldsymbol{z}^2+2\alpha\boldsymbol{yz})+\boldsymbol{x}^2-\dfrac{a_\alpha}{\alpha}\boldsymbol{yz}, \\ \qquad\qquad\qquad \text{if } h(x,t)\leq 0,\psi(s,F)\leq 0, tr(F)=0, \\ \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(\big((1-s)\beta^2(\alpha+a_\alpha)+s\alpha^2(\beta+a_\beta)\big)\boldsymbol{y}^2 \\ \qquad\qquad +\big((1-s)(\alpha+a_\alpha)+s(\beta+a_\beta)\big)\boldsymbol{z}^2 \\ \qquad\qquad +2\big((\alpha+a_\alpha)\beta-s\beta\big)\boldsymbol{yz}\Big)-\det F \\ \qquad\qquad\qquad \text{if } \psi(s,F)\geq 0, tr(F)=0, \\ +\infty \qquad\qquad \text{if } tr(F)\neq 0\end{cases}$$

We claim that in fact this is an exact value. This amounts to finding laminates which yield this same optimal value.

4 Optimal microstructures: laminates

We have the lower bound given by the polyconvexification, and we will show that this bound is in fact attained. To this end, we seek an optimal microstructure (a laminate) whose second moments recover the value of the bound.

We try to find $\nu=s\nu_\alpha+(1-s)\nu_\beta$, a laminate with $supp(\nu_\gamma)\subset\Lambda_\gamma$, $\gamma=\alpha,\beta$, $s\in(0,1)$, and first moment F. We have different optimal conditions depending of the sign of ψ and h, and we analyze different cases accordingly.

4.1 Case $\psi\geq 0$

We start with the case when $\psi(F,s)\geq 0$ holds. In this case the optimal conditions (12) tell us that

$$S_{\alpha,2}=y_\alpha^2,\quad S_{\beta,2}=y_\beta^2$$

and therefore, by the strict convexity of the square function, we can deduce that

$$\nu_\gamma^{(2)}=\delta_{y_\gamma},\gamma=\alpha,\beta.$$

Hence

$$F_\alpha=(\lambda,y_\alpha,-\alpha y_\alpha),\ F_\beta=\left(\frac{\boldsymbol{x}-s\lambda}{1-s},y_\beta,-\beta y_\beta\right), \tag{13}$$

with $\lambda\in\mathbb{R}$ arbitrary. This means that for every $\lambda\in\mathbb{R}$ we can decompose F as a convex combination of two matrices in $\Lambda_\alpha,\Lambda_\beta$ respectively, and satisfying the volume constraint, see Figure 2.

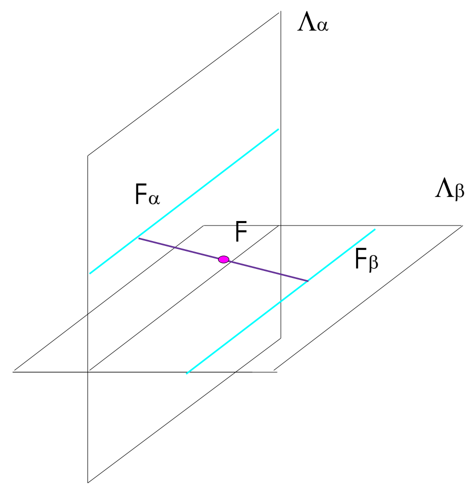

Figure 2: Infinite decompositions of F.

The next step is to check that there exist some $\lambda \in \mathbb{R}$ such that $rank(F_\alpha - F_\beta) = 1$. After some algebra, we can write

$$rank(F_\alpha - F_\beta) = 1 \Leftrightarrow C_{F,s}(\lambda) = 0$$

where

$$C_{F,s}(\lambda) = -detF - s(\lambda^2 - \alpha y_\alpha^2) - (1-s)((\frac{F_{11} - s\lambda}{1-s})^2 - \beta y_\beta^2)$$

is a second degree polynomial on λ. It is easy to check that the discriminant of $C_{F,s}$ is $\psi(F,s)$, and so that their roots are

$$\lambda_i = x + (-1)^i \sqrt{\frac{1-s}{s}\psi(F,s)} \quad i = 1,2.$$

Therefore for all pair (F,s) such that $\psi(F,s) \geq 0$, there exist two first order laminate

$$\nu = s\delta_{F_{\alpha,i}} + (1-s)\delta_{F_{\beta,i}} \qquad i = 1,2$$

where

$$F_{\alpha,i} = \begin{pmatrix} \lambda_i & F_{\alpha,12} \\ -\alpha F_{\alpha,12} & -\lambda_i \end{pmatrix}, \quad F_{\beta,i} = \begin{pmatrix} \frac{x - s\lambda_i}{1-s} & F_{\beta,12} \\ -\beta F_{\beta,12} & -\frac{x-s\lambda_i}{1-s} \end{pmatrix}$$

and they provide the optimal value of the polyconvexification.

Thanks to the spatial identification $F = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, we can observe the above computations from a geometric point of view (see Figure 3). For any matrix $F = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ the determinant is $\det F = -(\boldsymbol{x}^2 + \boldsymbol{y}\boldsymbol{z})$, this means that for any matrix F there exist a cone $\{\boldsymbol{x}^2 + \boldsymbol{y}\boldsymbol{z} = 0\}$ of rank one directions through this matrix. From

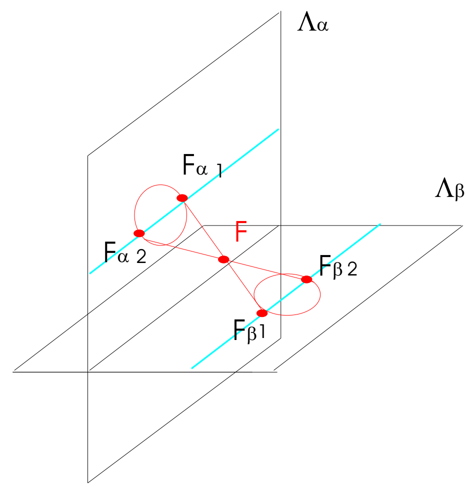

Figure 3: Two first order laminates

optimality conditions we obtain an explicit identification of F_γ $\gamma = \alpha, \beta$, up to the first component (13), which let us a degree of freedom in the search of the optimal decomposition. Geometrically, we notice that the intersection between the manifolds Λ_γ and the rank one cone are ellipses, whose intersection with the admissible F_γ are two points $F_{\gamma,i},\ \gamma = \alpha, \beta$ and $i = 1, 2$.

4.2 Case $\psi \leq 0$

We study now the other case, $\psi(F, s) < 0$. In this situation, we have two different optimal conditions depending of the sign of h. We treat the case $h \geq 0$. The other case is similar.

From the optimal condition for this case (10), we have

$$S_{\alpha,2} = y_\alpha^2, \quad S_1 = \boldsymbol{x}^2$$

and by using similar arguments as above, we can deduce

$$\nu_\alpha^{(2)} = \delta_{y_\alpha}, \quad \nu^{(1)} = \delta_x$$

where

1. $\nu_\alpha = \delta_{F_\alpha}$ with
$$F_\alpha = (\boldsymbol{x}, y_\alpha, -\alpha y_\alpha), \tag{14}$$

2. by using that F is the first moment of ν, there exists a unique decomposition
$$F = sF_\alpha + (1 - s)F_\beta$$

with $F_\gamma \in \Lambda_\gamma, \quad \gamma = \alpha, \beta$ where F_α is of the form just indicated, and

$$F_\beta = (\boldsymbol{x}, y_\beta, -\beta y_\beta). \tag{15}$$

Consider a pair (F, s) such that $\psi(F, s) < 0$. After an elementary manipulation, we get

$$\begin{gathered}\psi(F,s) \leq 0 \Longleftrightarrow \\ -(\beta-\alpha)^2 \boldsymbol{y}\boldsymbol{z} s^2 + \Big(\alpha\beta(\alpha-\beta)\boldsymbol{y}^2 + (\beta-\alpha)\boldsymbol{z}^2 + (\beta-\alpha)^2\boldsymbol{y}\boldsymbol{z}\Big)s \\ +\Big(\alpha\beta^2\boldsymbol{y}^2 + \alpha\boldsymbol{z}^2 + 2\alpha\beta\boldsymbol{y}\boldsymbol{z}\Big) \leq 0.\end{gathered}$$

Let $P_F(s)$ be this second degree polynomial in s for fixed F. The set where $\psi(F, s) \leq 0$ is the set where P_F has solutions in $[0, 1]$, and s lies between those two solutions. There exist real solutions if the discriminant is non-negative, and, in addition, it is easy to check that $P_F(0), P_F(1)$ are positive[1] if $F \notin \Lambda_\alpha \cup \Lambda_\beta$. Therefore there are positive solutions if P_F is decreasing at 0.

After some algebra the discriminant is

$$\begin{aligned}g(F) = \alpha^2\beta^2\boldsymbol{y}^4 + \boldsymbol{z}^4 + (\alpha^2 + 4\beta\alpha + \beta^2)\boldsymbol{y}\boldsymbol{z} \\ +2\alpha\beta\boldsymbol{y}^3\boldsymbol{z} + 2(\alpha+\beta)\boldsymbol{z}^3\boldsymbol{y} \geq 0,\end{aligned}$$

and the decreasing condition

$$h(F) = (\alpha+\beta)\boldsymbol{y}\boldsymbol{z} + \alpha\beta\boldsymbol{y}^2 + \boldsymbol{z}^2 \leq 0.$$

Therefore the set of pairs (F, s) where $\psi(F, s) \leq 0$ can be described as

$$\{(F,s) \in M^{2\times 2} \times \mathbb{R} : g(F) \geq 0,\ h(F) \leq 0,\ s \in (r_1, r_2)\}$$

where

$$r_i = \frac{1}{2} - \frac{1}{2(\beta-\alpha)\boldsymbol{y}\boldsymbol{z}}\Big(\alpha\beta\boldsymbol{y}^2 - \boldsymbol{z}^2 + (-1)^i\sqrt{g(F)}\Big) \quad i = 1, 2.$$

We thus have a characterization of the set $\psi(F, s) \leq 0$. We now look for rank-one connections between both manifolds.

We would like to write

$$F = rB_\alpha + (1-r)B_\beta$$

with $r \in (0, 1),\ B_\gamma \in \Lambda_\gamma,\ (B_\gamma)_1 = x,\ \gamma = \alpha, \beta,$ and $rank(B_\alpha - B_\beta) = 1$.

On the one hand,

$$\left.\begin{array}{l} B_\gamma \in \Lambda_\gamma \\ (B_\gamma)_1 = \boldsymbol{x} \end{array}\right\} \Rightarrow B_\gamma = (\boldsymbol{x}, y_\gamma, -\gamma y_\gamma) \quad \gamma = \alpha, \beta.$$

[1] $P_F(0) = \alpha|\beta\boldsymbol{y} + \boldsymbol{z}|^2, P_F(1) = \beta|\alpha\boldsymbol{y} + \boldsymbol{z}|^2$

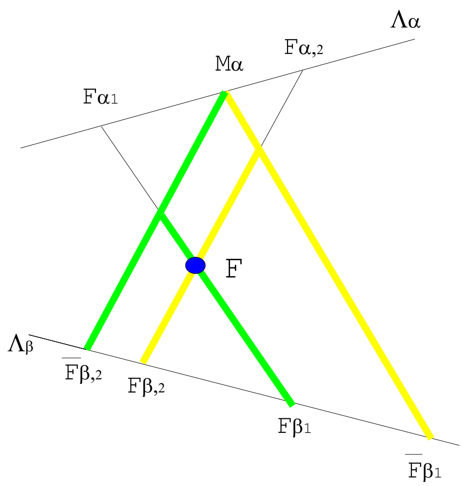

Figure 4: Second order laminates.

The constraint on the vanishing determinant can be rewritten, after some manipulation, as

$$P_F(r) = 0,$$

whose roots are r_i. We can therefore guarantee that there exist two rank-one directions between Λ_α and Λ_β with barycenter F.

We are now in a position to find an optimal second order laminate which recovers the lower bound given by the polyconvexification. We take $\nu_\alpha = \delta_{F\alpha}$ and ν_β as a convex combination of two Dirac masses supported in the β manifold (see Figure 4).

Put

$$F_{\beta,i} = (\boldsymbol{x}, y_{\beta,i}, -\beta y_{\beta,i})$$

with

$$y_{\beta,i} = \frac{-1}{(1-r_i)(\beta-\alpha)}(\alpha\boldsymbol{y} + \boldsymbol{z}), \quad i = 1, 2.$$

Since $r_1 \leq s \leq r_2$, it is clear that y_β is a convex combination of $F_{\beta,i}, i = 1, 2$.

If we consider $\bar{F}_{\beta,i} = F_\alpha + l_i(F_{\beta,i} - F_{\alpha,i})$ with l_i such that $\bar{F}_{\beta,i} \in \Lambda_\beta$, and take

$$l_i = \frac{r_i}{s}, \quad \rho_{i,j} = \frac{(1-r_j)(r_i - s)}{r_i - r_j}, \quad \tau_{i,j} = \frac{(r_j - s)(r_i - 1)}{r_j(1+r_i) + s(1-r_j)}, \tag{16}$$

we can define the second-order laminate with support on $\Lambda_\alpha \cup \Lambda_\beta$, barycenter F, and mass in Λ_α equal to s, by putting

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\beta,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\beta,j}} + (1-\rho_{i,j})\delta_{F_\alpha})$$

with $i, j \in \{1,2\}$, $i \neq j$ where,

$$\det(\bar{F}_{\beta,j} - F_\alpha) = 0$$

and

$$\det(F_{\beta,i} - (\rho_{i,j}\bar{F}_{\beta,j} + (1-\rho_{i,j})F_\alpha)) = 0.$$

Again, using the spatial identification we can interpret geometrically the above analytical computations. We lost the degree of freedom of the first component of the matrices F_α and F_β, since these matrices are explicitly determined by (14) and (15), and their first component is $\boldsymbol{x}$ in both cases. This fact lets us simplify the spatial situation to a 2-d case in the plane determined by the first component equal to $\boldsymbol{x}$. The intersection between the manifolds Λ_γ - the cone of rank one directions - and that reduces to two matrices in each manifold, which we noted by $F_{\gamma,i}$. From these matrices connected by rank one directions we can obtain a second order laminate with volume fraction s on Λ_α and $(1-s)$ on Λ_β. This construction is shown in the Figure 4, where the spatial situation is reduced to the plane of the first component equal to $\boldsymbol{x}$. A similar result holds for the other point where the optimal value is attained $(h(x,s) \leq 0)$.

We summarize all of these computations of optimal laminates leading to the relaxed integrand φ.

When $\psi(F,s) \geq 0$ there exist two optimal first-order laminates leading to the value of the relaxed integrand φ

$$\nu = s\delta_{F_{\alpha,i}} + (1-s)\delta_{F_{\beta,i}} \qquad i = 1,2 \tag{17}$$

where,

$$F_{\alpha,i} = \begin{pmatrix} \lambda_i & F_{\alpha,12} \\ -\alpha F_{\alpha,12} & -\lambda_i \end{pmatrix}, \quad F_{\beta,i} = \begin{pmatrix} \frac{F_{11}-s\lambda_i}{1-s} & F_{\beta,12} \\ -\beta F_{\beta,12} & -\frac{F_{11}-s\lambda_i}{1-s} \end{pmatrix}$$

with

$$\lambda_i = F_{11} + (-1)^i \sqrt{\frac{1-s}{s}\psi(F,s)} \quad i = 1,2,$$

$$F_{\alpha,12} = \frac{1}{s(\beta-\alpha)}(\beta F_{12} + F_{21}), \quad F_{\beta,12} = \frac{-1}{(1-s)(\beta-\alpha)}(\alpha F_{12} + F_{21}).$$

When $\psi(F,s) \leq 0$ and $h(x,t) \geq 0$, there exist two optimal second-order laminates

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\beta,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\beta,j}} + (1-\rho_{i,j})\delta_{F_\alpha}) \tag{18}$$

with $i, j \in \{1,2\}$, $i \neq j$ where the scalars are

$$\rho_{i,j} = \frac{(1-r_j)(r_i-s)}{r_i-r_j}, \quad \tau_{i,j} = \frac{(r_j-s)(r_i-1)}{r_j(1+r_i)+s(1-r_j)}$$

and the matrices are

$$F_\alpha = \begin{pmatrix} F_{11} & F_{\alpha,12} \\ -\alpha F_{\alpha,12} & -F_{11} \end{pmatrix}, \quad F_{\beta,i} = \begin{pmatrix} F_{11} & F_{\beta,12,i} \\ -\beta F_{\beta,12,i} & -F_{11} \end{pmatrix}$$

with

$$F_{\beta,12,i} = \frac{-1}{(1-r_i)(\beta-\alpha)}(\alpha F_{12} + F_{21}), \quad i = 1,2$$

$$r_i \frac{1}{2} - \frac{1}{2(\beta-\alpha)F_{12}F_{21}}\left(\alpha\beta|F_{12}|^2 - |F_{21}|^2 + (-1)^i\sqrt{g(F)}\right) \quad i = 1,2$$

$$\bar{F}_{\beta,i} = F_\alpha + l_i(F_{\beta,i} - F_{\alpha,i}), \quad l_i = \frac{r_i}{s}.$$

Similarly, when $\psi(F,s) \leq 0$ and $h(x,t) \leq 0$, the optimal microstructure is another second-order laminate given by

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\alpha,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\alpha,j}} + (1-\rho_{i,j})\delta_{F_\beta})$$

with $i,j \in \{1,2\}$, $i \neq j$ where the scalars are

$$\rho_{i,j} = \tfrac{r_j(r_i-s)}{r_i-r_j}, \quad \tau_{i,j} = \tfrac{(s-r_j)r_i}{r_i(r_j-1)+r_j(1-s)},$$

and the matrices involved are

$$F_\beta = \begin{pmatrix} F_{11} & F_{\beta,12} \\ -\beta F_{\beta,12} & -F_{11} \end{pmatrix}, \quad F_{\alpha,i} = \begin{pmatrix} F_{11} & F_{\alpha,12,i} \\ -\beta F_{\alpha,12,i} & -F_{11} \end{pmatrix}$$

with

$$F_{\alpha,12,i} = \frac{1}{r_i(\beta-\alpha)}(\beta F_{12} + F_{21}), \quad i = 1,2$$

$$\bar{F}_{\alpha,i} = F_\beta - l_i(F_{\beta,i} - F_{\alpha,i}), \quad l_i = \frac{1-r_i}{1-s}.$$

5 Some particular examples

In this section we would like to emphasize some particular examples where, by using Theorem 1, we can compute explicitly the relaxed cost functional.

Example 1 - An interesting and academic example is the corresponding to $a_\alpha(t,x) = \alpha$, $a_\beta(t,x) = \beta$ so that $h \equiv 0$, the cost functional can be written as

$$\int_0^T \int_\Omega \left[u_t^2(t,x) + (\alpha\chi + \beta(1-\chi))u_x^2(t,x)\right] dxdt,$$

and the *constrained quasiconvexification* is

$$\varphi(F,s)\begin{cases} F_{11}^2 - F_{12}F_{21} & \text{if } \psi(s,F)\leq 0,\\ -\det F + \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(2\alpha\beta(s\alpha+(1-s)\beta)\Big)|F_{12}|^2 & \\ \qquad +2\big((1-s)\alpha+s\beta\big)|F_{21}|^2+4\alpha\beta F_{12}F_{21}\Big) & \\ & \text{if } \psi(s,F)\geq 0. \end{cases}$$

Example 2 - Another interesting case occurs when we take $a(t,x,\chi)=1$, the most simple quadratic cost function but very interesting from the mathematical point of view. In this case the relaxed cost functional is

$$\int_0^T\int_\Omega \left[u_t^2(t,x)+u_x^2(t,x)\right]dxdt,$$

and therefore $a_\alpha(t,x)=a_\beta(t,x)=1$. Hence

$$h(t,x)=\beta-\alpha,$$

and the *constrained quasiconvexification* simplifies to $\varphi(F,s)=$

$$\begin{cases} \dfrac{1}{s\beta(\beta-\alpha)}(s\beta(\beta-\alpha)|F_{11}|^2+\beta^2|F_{12}|^2+|F_{21}|^2+(s\alpha+\beta(2-s))F_{12}F_{21}) & \\ & \text{if } \psi(s,F)\leq 0,\\ -\det F+\dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big((1-s)\beta^2(\alpha+1)+s\alpha^2(\beta+1)\Big)|F_{12}|^2 & \\ \qquad +\big((1-s)\alpha+s\beta+1\big)|F_{21}|^2+2\big(\beta(1-s)+\alpha(s+\beta)\big)F_{12}F_{21}\Big) & \\ & \text{if } \psi(s,F)\geq 0. \end{cases} \tag{19}$$

Example 3 - The last case lies in the border line for our computations to be valid. We take $a_\alpha(t,x)=-\alpha$ and $a_\beta(t,x)=-\beta$ so that h identically vanishes. The cost functional is

$$\int_0^T\int_\Omega \left[u_t^2(t,x)-(\alpha\chi+\beta(1-\chi))u_x^2(t,x)\right]dxdt,$$

and for this case the relaxed integrand surprisingly is -det (recall the restriction on the trace)

$$\varphi(F,s)=F_{11}^2+F_{12}F_{21}=-\det F.$$

Note that depending on the choice of the coefficients a_α, a_β we obtain different cost densities for the relaxed problem, yet this choice of the coefficients is independent of the state equation. It is interesting to remark that for all these examples the optimal laminates correspond to the ones computed in the last section (17) when $\psi\geq 0$ and (18) when $\psi\leq 0$, which are independents of a_α, a_β.

6 Analysis of (RP) in the quadratic case

In this section we would like to analyze the quadratic case which is Example 2 in the preceding section, and thus focus on (RP) where the cost density is given by (19).

From the previous sections we know that this problem admits optimal solutions, and moreover we know that such optimal solutions are first or second-order laminates depending on the sign of the function ψ. An interesting fact is that all functions involved are quadratic in the vector gradient variable and therefore regular, yet it is the presence of gradients and the pointwise constraint that make the problem difficult to examine.

One first attempt would lead us to look at optimality conditions introducing several multipliers to keep track of the restrictions. This makes the problem more difficult in the sense that we have to solve a system of partial differential equations. Instead we follow a similar strategy as in [9]. The pointwise constraint given by ψ depends only on the variables F_{12}, F_{21} , therefore we try to find the "optimal" relationship between these two variables. The next lemma is completely elementary.

Lemma 1 *For fixed s, the optimal solution of the quadratic, mathematical programming problem*

$$\text{Minimize in } F_{(21)} : \varphi(F, s)$$

occurs when

$$(\alpha s + \beta(1 - s))F_{12} + F_{21} = 0.$$

In addition, the associated optimal microstructures are first-order laminates with volume fraction s for the α-material and orientation of layers always vertical (along the time axis):

$$s(t, x)\delta_{F_\alpha} + (1 - s(t, x))\delta_{F_\beta}$$

with normal direction of lamination $n = (0, 1)$. Having in mind the trace condition $F_{11} + F_{22} = 0$ the optimal value of the cost function simplifies to

$$F_{11}^2 + F_{12}^2. \tag{20}$$

The idea is then to replace the complicated cost function φ by the expression (20) and then minimize under the constraints

$$(\alpha s + \beta(1 - s))F_{12}(t, x) + F_{21}(t, x) = 0, \qquad F_{11}(t, x) + F_{22}(t, x) = 0$$

i.e.

$$\begin{pmatrix} F_{11}(t, x) \\ -[\alpha s(t, x) + \beta(1 - s(t, x))]F_{12}(t, x) \end{pmatrix} = TF^{(2)}(x, t), a.e.(t, x) \in [0, T] \times \Omega,$$

which is equivalent to

$$div \begin{pmatrix} F_{11}(t,x) \\ -[\alpha s(t,x) + \beta(1 - s(t,x))]F_{12}(t,x) \end{pmatrix} = 0$$

Therefore we can write the minimization problem in terms of the original variable $U^{(1)} = u$ leading to the new relaxed problem (stated in Conjecture 1):

$$(\widetilde{RP}) \qquad \min_s \int_0^T \int_\Omega u_t^2(t,x) + u_x^2(t,x)\, dx\, dt$$

where u is the unique solution of

$$\begin{aligned} u_{tt} - div([\alpha s + \beta(1-s)]u_x) = 0 \quad &\text{in} \quad (0,T) \times (0,1), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad &\text{in} \quad \Omega, \\ u(t,0) = 0,\ u(t,1) = 0 \quad &\text{in} \quad [0,T], \end{aligned} \tag{21}$$

This new problem may be seen as the continuous version of the original design problem in which the function $\chi(x,t)$ is replaced by the continuous function $s(x,t)$. We cannot prove directly that the above problem admits optimal solutions, though we claim, by our conjecture, that it indeed does because of the particular form of the problem and not as a consequence of general results. A deeper and exhaustive analysis of this problem is still in progress (see [18]). Hopefully, the existence of solutions of these problems will be proved. We support numerically our conclusion in the next section. All we can say at this point is contained in the following assertion.

Lemma 2 *The equalities*

$$\inf(P) = \inf(\widetilde{RP}) = \min(RP)$$

hold.

Proof. It is easy to see that

$$\inf(P) \geq \inf(\widetilde{RP})$$

and

$$\inf(\widetilde{RP}) \geq \min(RP)$$

and using the relaxation Theorem 3

$$\inf(P) = \min(RP)$$

holds, therefore we have all equalities. ■

7 Numerical simulations

We address in this section the numerical resolution of the problem $(\widetilde{RP})$ in according with the Conjecture 1 for $a_\alpha(t,x) = 1$ and $a_\beta(t,x) = 1$. We first describe the algorithm of minimization and then present some numerical experiments.

7.1 Algorithm of minimization

We briefly present the resolution of the relaxed problem $(\widetilde{RP})$ using a gradient descent method. In this respect, we compute the first variation of the cost function.

For any $\eta \in \mathbb{R}^+$, $\eta << 1$, and any $s_1 \in L^\infty((0,T)\times\Omega)$, we associate with the perturbation $s^\eta = s + \eta s_1$ of s the derivative of $\tilde{I}$ with respect to s in the direction s_1 as follows :

$$\frac{\partial \widetilde{I}(s)}{\partial s}\cdot s_1 = \lim_{\eta\to 0}\frac{\widetilde{I}(s+\eta s_1) - \widetilde{I}(s)}{\eta}.$$

We obtain the following result.

Theorem 4 *If $(u_0,u_1) \in (H^2(\Omega)\cap H^1_0(\Omega))\times H^1_0(\Omega)$, then the derivative of $\widetilde{I}$ with respect to s in any direction s_1 exists and takes the form*

$$\frac{\partial \widetilde{I}(s)}{\partial s}\cdot s_1 = \int_0^T\int_\Omega s_1\Big((a_\alpha - a_\beta)u_x^2 + (\alpha-\beta)u_x p_x\Big)\,dxdt \tag{22}$$

where u is the solution of (21) and p is the solution in $C^1([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))$ of the adjoint problem

$$\begin{cases} div(p_t, -[s\alpha + (1-s)\beta]p_x) = div(u_t, a(t,x,s)u_x) & in \quad (0,T)\times\Omega,\\ p = 0 & on \quad (0,T)\times\partial\Omega,\\ p(T,x) = 0, \quad p_t(T,x) = u_t(T,x) & in \quad \Omega. \end{cases} \tag{23}$$

Sketch of the proof. Let us explain briefly how we obtain the expression (22). We introduce the lagrangian

$$\mathcal{L}(s,\phi,\psi) = \int_0^T\int_\Omega (\phi_t^2 + a(t,x,s)\phi_x^2)\,dxdt + \int_0^T\int_\Omega\Big[\phi_{tt} - div([\alpha s + \beta(1-s)]\phi_x)\Big]\psi\,dxdt$$

for any $s \in L^\infty((0,T)\times\Omega)$, $\phi \in C([0,T];H^2(\Omega)\cap H^1_0(\Omega))\cap C^1([0,T];H^1_0(\Omega))$ and $\psi \in C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))$ and then write formally that

$$\frac{d\mathcal{L}}{ds}\cdot s_1 = \frac{\partial}{\partial s}\mathcal{L}(s,\phi,\psi)\cdot s_1 + < \frac{\partial}{\partial\phi}\mathcal{L}(s,\phi,\psi), \frac{\partial\phi}{\partial s}\cdot s_1 > + < \frac{\partial}{\partial\psi}\mathcal{L}(s,\phi,\psi), \frac{\partial\psi}{\partial s}\cdot s_1 > .$$

The first term is

$$\frac{\partial}{\partial s}\mathcal{L}(s,\phi,\psi)\cdot s_1=\int_0^T\!\!\int_\Omega s_1\Big((a_\alpha-a_\beta)\phi_x^2+(\alpha-\beta)\phi_x\psi_x\Big)\,dxdt \tag{24}$$

for any s,ϕ,ψ whereas the third term is equal to zero if $\phi=u$ solution of (21). We then determine the solution p so that, for all $\phi\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];H_0^1(\Omega))$, we have

$$<\frac{\partial}{\partial\phi}\mathcal{L}(s,\phi,p),\frac{\partial\phi}{\partial s}\cdot s_1>=0,$$

which leads to the formulation of the adjoint problem (23). Next, writing that $\widetilde{I}(s)=\mathcal{L}(s,u,p)$, we obtain (22) from (24). ■

In order to take into account the volume constraint on s, we introduce the Lagrange multiplier function $\gamma\in L^\infty((0,T);\mathbb{R})$ and the functional

$$\widetilde{I}_\gamma(s)=\widetilde{I}(s)+\int_0^T\gamma(t)\int_\Omega s(t,x)dxdt.$$

Using Theorem 4, we obtain that the derivative of $\widetilde{I}_\gamma$ is

$$\frac{\partial\widetilde{I}_\gamma(s)}{\partial s}\cdot s_1=\int_0^T\!\!\int_\Omega s_1((a_\alpha-a_\beta)u_x^2+(\alpha-\beta)u_xp_x)\,dxdt+\int_0^T\gamma(t)\int_\Omega s_1dxdt$$

which permits to define the following descent direction :

$$s_1(x,t)=-((a_\alpha-a_\beta)u_x^2+(\alpha-\beta)u_xp_x+\gamma(t)),\quad\forall x\in\Omega,\forall t\in(0,T). \tag{25}$$

Consequently, for any function $\eta\in L^\infty(\Omega\times(0,T),\mathbb{R}^+)$ with $||\eta||_{L^\infty(\Omega\times(0,T))}$ small enough, we have $\widetilde{I}_\gamma(s+\eta s_1)\le\widetilde{I}_\gamma(s)$. The multiplier function γ is then determined in order that, for any function $\eta\in L^\infty(\Omega\times(0,T),\mathbb{R}^+)$, $||s+\eta s_1||_{L^1(\Omega)}=V_\alpha|\Omega|$ leading to

$$\gamma(t)=\frac{(\int_\Omega s(t,x)dx-V_\alpha|\Omega|)-\int_\Omega\eta(t,x)\Big((a_\alpha-a_\beta)u_x^2+(\alpha-\beta)u_xp_x\Big)\,dx}{\int_\Omega\eta(t,x)dx} \tag{26}$$

, for all $t\in(0,T)$. At last, the function η is chosen so that $s+\eta s_1\in[0,1]$, for all $x\in\Omega$ and $t\in(0,T)$. A simple and efficient choice consists in taking $\eta(t,x)=\varepsilon s(t,x)(1-s(t,x))$ for all $x\in\Omega$ and $t\in(0,T)$ with ε small real positive. Consequently, the descent algorithm to solve numerically the relaxed problem $(\widetilde{RP})$ may be structured as follows :

Let $\Omega\subset\mathbb{R}^N$, $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, $L\in(0,1)$, $T>0$, and $\varepsilon<1$, $\varepsilon_1<<1$ be given.

- Initialization of the density function $s^0 \in L^\infty(\Omega;]0,1[)$;
- For $k \geq 0$, iteration until convergence (i.e. $|\widetilde{I}(s^{k+1}) - \widetilde{I}(s^k)| \leq \varepsilon_1 |\widetilde{I}(s^0)||$) as follows :
  - Computation of the solution u_{s^k} of (21) and then the solution p_{s^k} of (23), both corresponding to $s = s^k$.
  - Computation of the descent direction s_1^k defined by (25) where the multiplier γ^k is defined by (26).
  - Update the density function in Ω:

$$s^{k+1} = s^k + \varepsilon s^k (1 - s^k) s_1^k$$

with $\varepsilon \in \mathbb{R}^+$ small enough in order to ensure the decrease of the cost function and $s^{k+1} \in L^\infty(\Omega \times (0,T), [0,1])$.

7.2 Numerical experiments in the quadratic case

In this section, we present some numerical simulations for $\Omega = (0,1)$ in the quadratic case, i.e, $(a_\alpha, a_\beta) = (1,1)$. On a numerical viewpoint, we highlight that the numerical resolution of the descent algorithm is *a priori* difficult because the descent direction (25) depends on the derivative of u and p, both solution of a wave equation with space and time coefficients only in $L^\infty((0,T) \times \Omega; \mathbb{R}_+^\star)$. To the knowledge of the authors, there does not exist any numerical analysis for this kind of equation. We use a C^0-finite element approximation for u and p with respect to x and a finite difference centered approximation with respect to t. Moreover, we add a vanishing viscosity and dispersive term of the type $\epsilon^2 \text{div}([s\alpha + (1-s)\beta] u_{xtt})$ with ϵ of order of h - the space discretization parameter. This term has the effect to regularize the descent term (25) and to lead to a convergent algorithm. At last, this provides an implicit and unconditionally stable scheme, consistent with (21) and (23), and of order two in time and space.

In the sequel, we treat the following two simple and smooth initial conditions on $\Omega = (0,1)$:

- **Case 1** : $u_0(x) = \sin(\pi x), \quad u_1(x) = 0 \quad ;$
- **Case 2** : $u_0(x) = \exp^{-80(x-0.5)^2}, \quad u_1(x) = 0,$

and we discuss the results with respect to the value of α, β and V_α. Results are obtained with $h = dt = 10^{-2}$, $\varepsilon_1 = 10^{-5}$, $s^0(t,x) = V_\alpha$ on $[0,T] \times \Omega$ and $\varepsilon = 10^{-2}$ (see the algorithm).

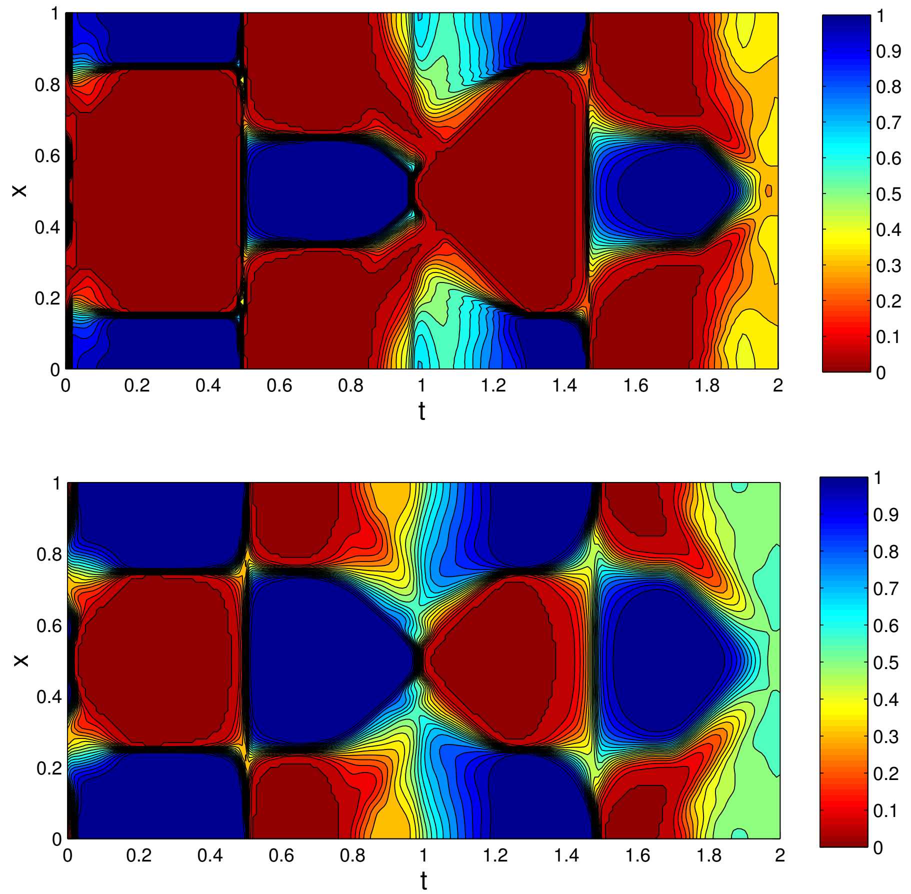

Figure 5: **Case 1** - $T = 2$, $(\alpha, \beta) = (1, 1.1)$ - Iso-values of the limit density - Top : $V_\alpha = 0.3$ - $\tilde{I}(s^{lim}) \approx 9.7451$ - Bottom: $V_\alpha = 0.5$ - $\tilde{I}(s^{lim}) \approx 9.5613$.

7.2.1 Case 1

We first consider the case 1, with $T = 2$ and $(\alpha, \beta) = (1, 1.1)$. Figure 5 depicts the iso-values of the optimal limit density s^{lim} (obtained at the convergence of the descent algorithm) for $V_\alpha = 0.3$ (top of the figure) and $V_\alpha = 0.5$ (bottom of the figure) respectively. For these values of α and β, we observe that the limit densities are characteristics functions taking either the value 0 or the value 1. As a consequence the relaxed problem $(\widetilde{RP})$ coincides with the original one (P) which is well-posed in the class of characteristics function. This validates Conjecture 1 in this case. Moreover, we observe that the limit densities are independent of the choice of the initialization s^0. This suggests that $\widetilde{I}$ admits a unique minimum.

Figure 6 represents the corresponding evolution of the energy $E(t) = 1/2 \int_\Omega (y_t^2 + [s\alpha + (1-s)\beta] y_x^2) dx$ with respect to t. Due to the time dependence of the coefficients of the state equation, the system is not conservative nor necessarily dissipative.

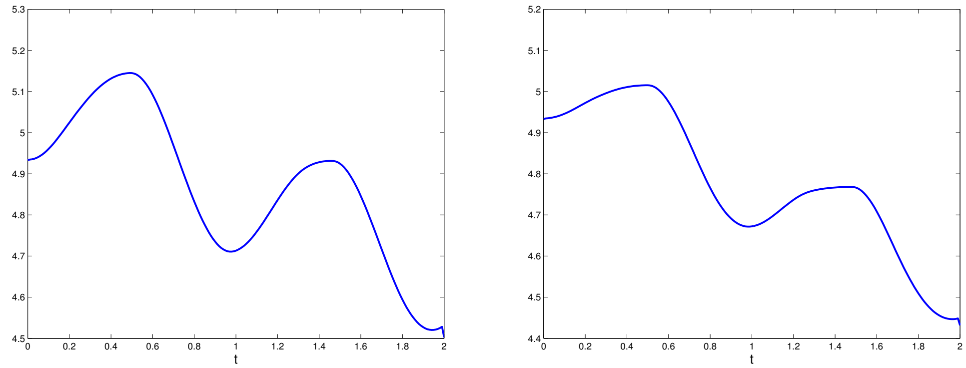

Figure 6: **Case 1** - $T = 2$, $(\alpha, \beta) = (1, 1.1)$ - $E(t)$ vs. t - Left : $V_\alpha = 0.3$ - $\widetilde{I}(s^{lim}) \approx 9.7451$ - Right : $V_\alpha = 0.5$ - $\widetilde{I}(s^{lim}) \approx 9.5613$.

Results are qualitatively different if we now consider a larger gap $\beta - \alpha$. Figures 7 and 8 represent the result obtained with $(\alpha, \beta) = (1, 6)$. We observe that the limit density are no more characteristic functions and take values in $(0, 1)$. This clearly indicates that the original problem may be no well-posed and justify the search of a relaxed formulation. We also observe that this property depends also on the value of V_α: for V_α or $1 - V_\alpha$ arbitrarily small, numerical simulation leads to bi-valued limit densities for all α and β.

We have also observed that as soon as the gap is large enough, the limit of the density depends on the initialization s^0 highlighting the existence of several infima for $\widetilde{I}$. We found that the choice s^0 constant on $(0, T) \times \Omega$ - which have the advantage to not favor any distribution between α and β - leads to the lowest value of $\widetilde{I}(s^{lim})$. Moreover, for this choice, the algorithm appears robust, stable and convergent with respect to the discretization parameters h and Δt. Under theses circumstances, we suspect that the infimum of $(\widetilde{RP})$ (see Lemma 2) is in fact a minimum.

Remark that the relaxation analysis and the results presented in the previous sections are unchanged if we consider the weaker volume constraint:

$$\int_0^T \int_\Omega s(t, x) dx dt \leq V_\alpha |\Omega| T. \tag{27}$$

Figure 9 depict the limit densities for $V_\alpha = 0.5$ for $(\alpha, \beta) = (1, 1.1)$ (Top) and $(\alpha, \beta) = (1, 6)$ (Bottom) respectively. Furthermore, as expected, these densities leads to a better distribution of materials: we obtain $\widetilde{I}(s^{lim}) \approx 9.2147$ and $\widetilde{I}(s^{lim}) \approx 3.4709$ respectively (to be compared with $\widetilde{I}(s^{lim}) \approx 9.5613$ and $\widetilde{I}(s^{lim}) \approx 6.1439$ for the initial volume constraint $\int_\Omega s(t, x) dx \leq V_\alpha |\Omega|$, for all t).

7.2.2 Case 2

We now present some results for the second case. Similarly to the first case, the optimal density takes value in $(0, 1)$ if and only if the gap $\beta - \alpha$ is large enough.

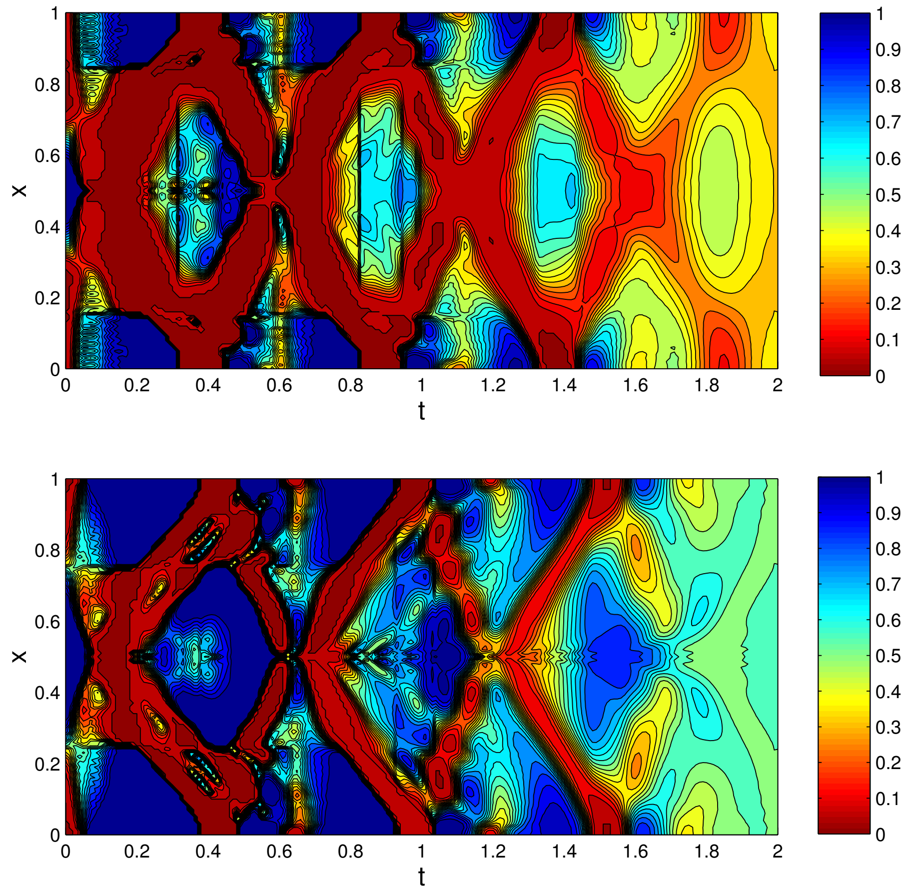

Figure 7: **Case 1** - $T = 2$, $(\alpha, \beta) = (1, 6)$ - Iso-values of the limit density - Top : $V_\alpha = 0.3$ - $\tilde{I}(s^{lim}) \approx 7.9567$ - Bottom: $V_\alpha = 0.5$ - $\tilde{I}(s^{lim}) \approx 6.1439$.

The pictures also clearly highlight that the optimal distribution is related to the propagation of the components of the solution on the cylinder $(0, T) \times (0, 1)$. For this case, we observe that the two volume constraint give similar results on the density and the optimal cost (see Figures 10 and 12). Furthermore, in the case $(\alpha, \beta) = (1, 10)$, we observe Figure 13 the strong damping mechanism of the optimal distribution and explain why, for t sufficiently large, the value of the cost function is less sensitive to the density s (i.e. for t large, the variations of s with respect to x and t are low).

7.2.3 Construction of a characteristic density associated to s^{lim}

In the case where the optimal density s^{lim} is not in $L^\infty((0, T) \times \Omega; \{0, 1\})$, one may associate with s^{lim} a characteristic function $s^{pen} \in L^\infty((0, T) \times \Omega; \{0, 1\})$ whose cost $\widetilde{I}(s^{pen})$ is arbitrarily near from $\widetilde{I}(s^{lim})$. Following [21], one may proceed as

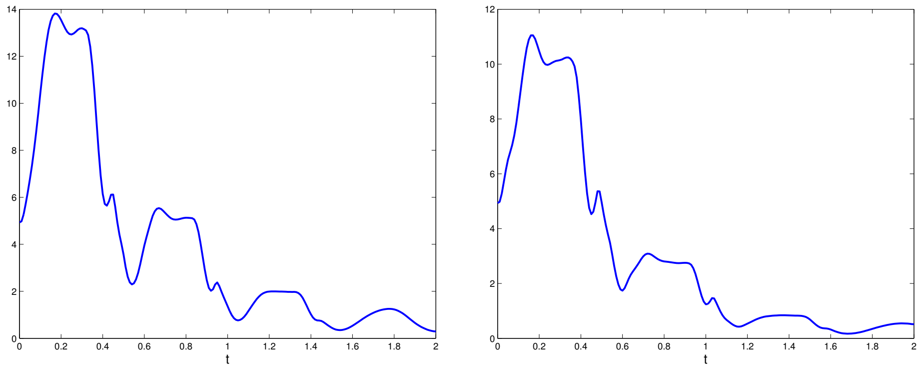

Figure 8: **Case 1** - $T=2$, $(\alpha,\beta)=(1,2)$ - $E(t)$ vs. t - Left : $V_\alpha = 0.3$ - $\widetilde{I}(s^{lim}) \approx 7.9567$ - Right : $V_\alpha = 0.5$ - $\widetilde{I}(s^{lim}) \approx 6.1439$.

follows: we first decompose the cylinder $(0,T)\times\Omega$ into $M\times N$ cells such that $(0,T)\times\Omega = \cup_{i=1,M}[t_i,t_{i+1}]\times\cup_{j=1,N}[x_j,x_{j+1}]$. Then, we associate with each cell the mean value $m_{i,j}\in[0,1]$ defined by

$$m_{i,j} = \frac{1}{(t_{i+1}-t_i)(x_{j+1}-x_j)}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}} s^{lim}(t,x)dx\,dt. \qquad (28)$$

At last, we define the function $s^{pen}_{M,N}$ in $L^\infty([0,T]\times\Omega)$ by

$$s^{pen}_{M,N}(t,x) = \sum_{i=1}^{M}\sum_{j=1}^{N}\chi_{[t_i,(1-\sqrt{m_{i,j}})t_i+\sqrt{m_{i,j}}t_{i+1}]\times[x_j,(1-\sqrt{m_{i,j}})x_j+\sqrt{m_{i,j}}x_{j+1}]}(t,x). \qquad (29)$$

We easily check that $||s^{pen}_{M,N}||_{L^1((0,T)\times\Omega)} = ||s^{lim}||_{L^1((0,T)\times\Omega)}$, for all $M,N>0$. Thus, the bi-valued function $s^{pen}_{M,N}$ takes advantage of the information codified in the density s^{lim}.

In order to illustrate this, we consider the case 1 with $T=1$, $(\alpha,\beta)=(1,2)$ and $V_\alpha=0.5$. Figure 14 depicts the corresponding optimal density s^{lim} and the associated function $s^{pen}_{M,N}$ for $M=N=30$. We obtain $\widetilde{I}(s^{pen}_{30,30})=5.62$ and $\widetilde{I}(s^{lim}) = 4.7584$ respectively. By letting M and N go to infinity, we expect to converge to the value $\widetilde{I}(s^{lim})$ and then construct a minimizing sequence of domains $\omega_{M,N}$ such that $\chi_{\omega_{\infty,\infty}}$ be the infimum for I (see Table 1). We refer to [19] for more example.

$M=N$	10	20	30	40	50
$\widetilde{I}(s^{pen}_{M,N})$	7.45	6.21	5.62	5.09	4.93

Table 1: Case 1 - $T=1$ - $(\alpha,\beta)=(1,3)$ - $V_\alpha=0.5$ - Values of the cost function $\widetilde{I}(s^{pen}_{M,N})$

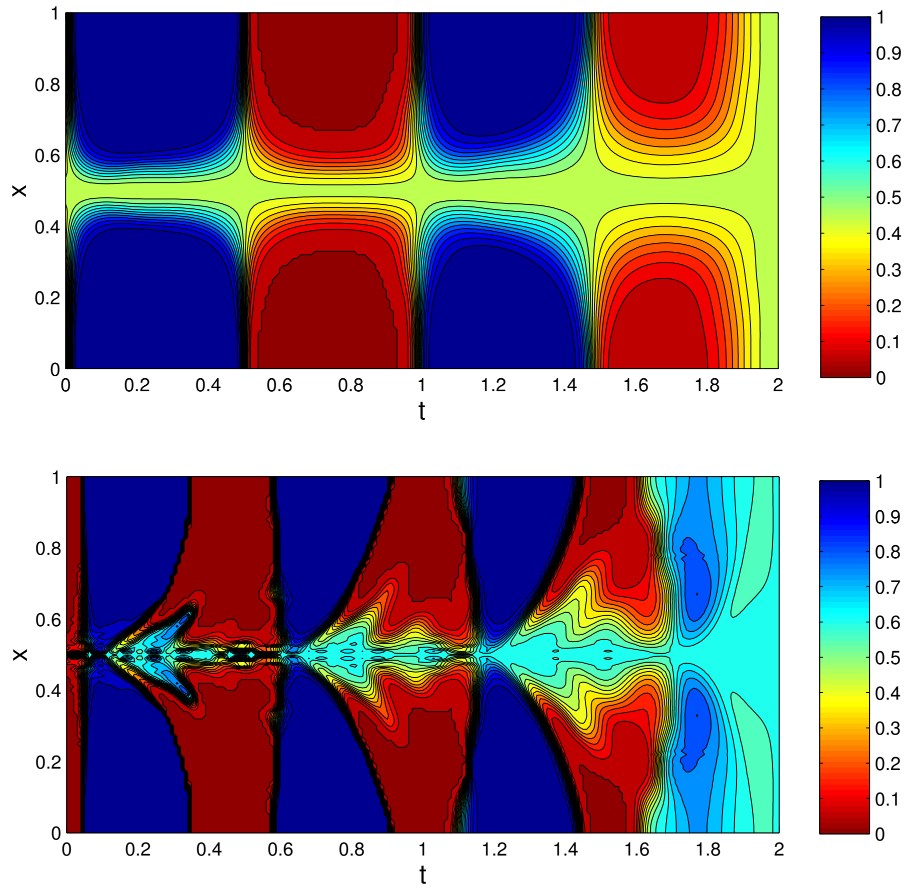

Figure 9: **Case 1** with the volume constraint (27) - $T = 2, V_\alpha = 0.5$ - Iso-value of the limit density - Top : $(\alpha, \beta) = (1, 1.1)$ $\widetilde{I}(s^{lim}) \approx 9.2147$ - Bottom : $(\alpha, \beta) = (1, 6)$ - $\widetilde{I}(s^{lim}) \approx 4.3109$.

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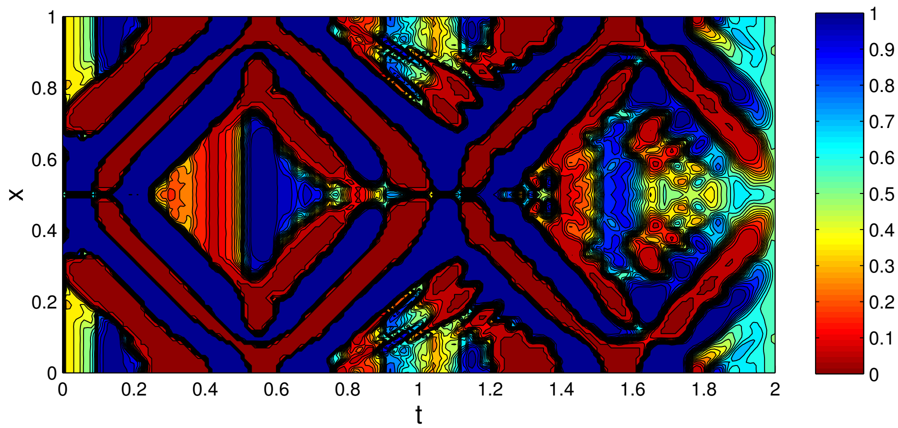

Figure 10: **Case 2** - $T = 2$, $(\alpha, \beta) = (1, 1.1)$ - Iso-value of the limit density - $V_\alpha = 0.5$ - $\widetilde{I}(s^{lim}) \approx 15.48$.

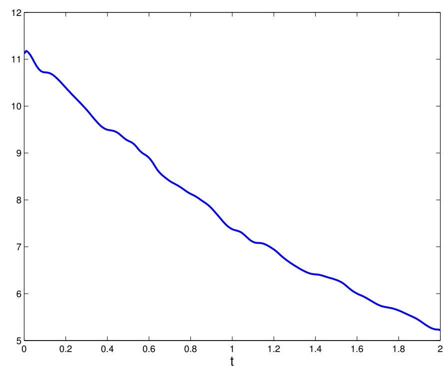

Figure 11: **Case 2** - $T = 2$, $(\alpha, \beta) = (1, 1.1)$ - $E(t)$ vs. t - $V_\alpha = 0.5$ - $\widetilde{I}(s^{lim}) \approx 15.48$.

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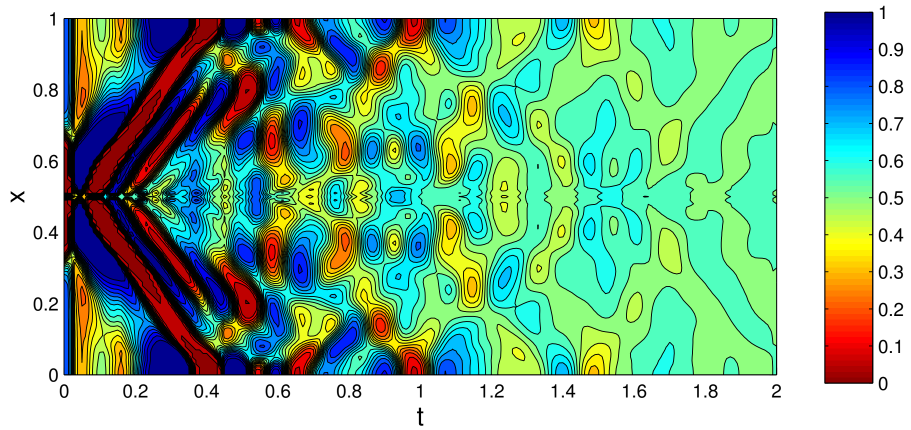

Figure 12: **Case 2** - $T = 2$, $(\alpha, \beta) = (1, 6)$ - Iso-value of the limit density - $V_\alpha = 0.5$ - $\widetilde{I}(s^{lim}) \approx 4.5414$.

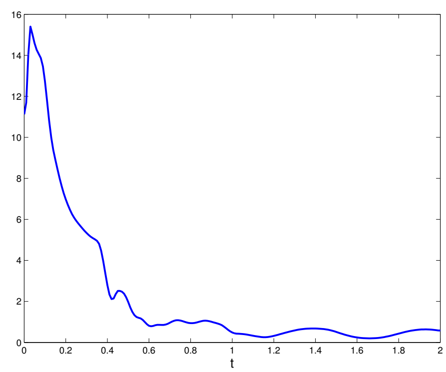

Figure 13: **Case 2**- $T = 2$, $(\alpha, \beta) = (1, 6)$ - $E(t)$ vs. t - $V_\alpha = 0.5$ - $\widetilde{I}(s^{lim}) \approx 4.5414$.

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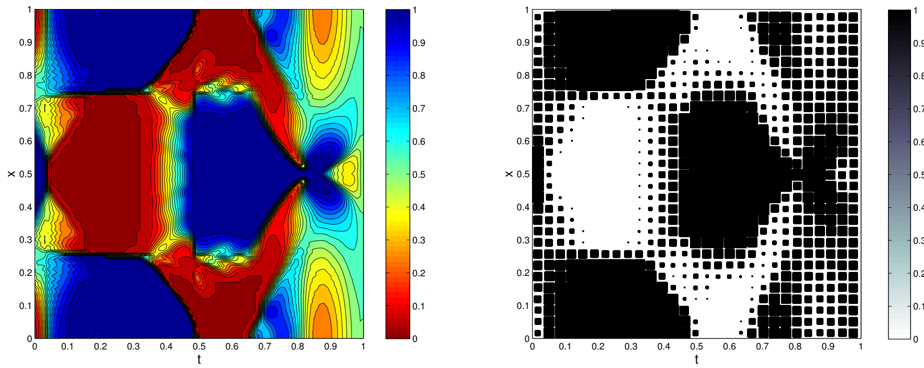

Figure 14: **Case 1**: $T = 1$, $(\alpha, \beta) = (1, 2)$ - $V_\alpha = 0.5$ $s^{pen}_{30,30}$ - $\widetilde{I}(s^{lim}) \approx 4.7584$- $\widetilde{I}(s^{pen}_{30,30}) \approx 5.62$.

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