COMPACTNESS PROPERTIES OF COMMUTATORS OF BILINEAR FRACTIONAL INTEGRALS

ÁRPÁD BÉNYI, WENDOLÍN DAMIÁN, KABE MOEN, AND RODOLFO H. TORRES

ABSTRACT. Commutators of a large class of bilinear operators and multiplication by functions in a certain subspace of the space of functions of bounded mean oscillations are shown to be jointly compact. Under a similar commutation, fractional integral versions of the bilinear Hilbert transform yield separately compact operators.

1. INTRODUCTION

The smoothing effect of commutators of linear operators is nowadays a well known and very useful fact. For the purposes of this paper, "smoothing" will mean the improvement of boundedness to the stronger condition of compactness. A pilar for such considerations in linear setting is the work of Uchiyama [23], where he showed that linear commutators of Calderón-Zygmund operators and pointwise multiplication with a symbol belonging to an appropriate subspace of the John-Nirenberg space *BMO* are compact. Thus, indeed, these commutators behave better than just being bounded, a result earlier proved by Coifman, Rochberg and Weiss [10]. Once compactness is established, one can derive a Fredholm alternative for equations with appropriate coefficients in all L^p spaces with $1 < p < \infty$, as in the work of Iwaniec and Sbordone [15]. Similarly, the theory of compensated compactness of Coifman, Lions, Meyer and Semmes [9] or the integrability theory of Jacobians, see, for example, the work of Iwaniec [14], owe a lot to the smoothing effect of commutators.

Bilinear commutators are naturally appearing operators in harmonic analysis, which leads to the equally relevant question about their smoothing behavior. For a bilinear operator T, and b an appropriately smooth function, we will consider the following *bilinear commutators*:

$$[T,b]_1(f,g) = T(bf,g) - bT(f,g),$$
$$[T,b]_2(f,g) = T(f,bg) - bT(f,g).$$

The notion of compactness in bilinear setting goes back to Calderón foundational article [4]. Using the terminology in the work of Bényi and Torres [1], we will be considering here the joint compactness (or simply compactness) and separate compactness of such bilinear operators.

Given three normed spaces X, Y, Z, a bilinear operator $T : X \times Y \to Z$ is said to be *(jointly) compact* if the set $\{T(x, y) : \|x\|, \|y\| \leq 1\}$ is precompact in Z. Writing $B_{1,X}$ for the closed unit ball in X, the definition of compactness specifically requires that if $\{(x_n, y_n)\} \subseteq B_{1,X} \times B_{1,Y}$, then the sequence $\{T(x_n, y_n)\}$ has a convergent subsequence in Z. Clearly, any compact bilinear operator T is continuous.

We say that $T : X \times Y \to Z$ is *compact in the first variable* if $T_y = T(\cdot, y) : X \to Z$ is compact for all $y \in Y$. T is called *compact in the second variable* if $T_x = T(x, \cdot) : Y \to Z$ is compact for all $x \in Y$. Finally, T is called *separately compact* if T is compact both in the first and second variable. While, in general, it is only true that separate compactness implies separate continuity, if we further consider one of the spaces X or Y to be Banach, the boundedness of T follows from separate compactness as well. For more on these notions of compactness and their basic properties, we refer the interested reader to [1].

Throughout this paper, the relevant space for the multiplicative symbols in our commutators will be a subspace of BMO, which we denote by CMO^1. Recall that BMO consists of all locally integrable functions b with $\|b\|_{BMO} < \infty$, where $\|b\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |b - b_Q| \, dx$, with the supremum taken over all cubes $Q \in \mathbb{R}^n$ and $b_Q = \frac{1}{|Q|} \int_Q b \, dx$ denoting the average of b on Q. We define CMO to be the closure of $C_c^\infty(\mathbb{R}^n)$ in the BMO norm.

Consider for the moment T to be a bilinear Calderón-Zygmund operator as defined by Grafakos and Torres [13]. For simplicity, we further assume that the kernels K and ∇K satisfy the appropriate decay conditions in such theory. If $b, b_1, b_2 \in BMO$, the bilinear commutators can be (formally) expressed in the form

$$[T, b]_1(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z)(b(y) - b(x)) f(y) g(z) \, dy dz,$$

$$[T, b]_2(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z)(b(z) - b(x)) f(y) g(z) \, dy dz.$$

The main result proved in [1] confirms that the smoothing effect of commutators of such operators with CMO symbols is present in the bilinear setting as well; thus also extending Uchiyama's result mentioned before to bilinear commutators. In fact, one has the following.

Theorem A. *Let T be a bilinear Calderón-Zygmund operator. If $b \in CMO$, $1/p + 1/q = 1/r$, $1 < p, q < \infty$ and $1 \leq r < \infty$, then, for $i = 1, 2$, $[T, b]_i : L^p \times L^q \to L^r$ is compact.*

The proof of Theorem A, as well as the main results proved in this work and other compactness results in the literature, make use of a known characterization

---

[1]The notion *CMO* for this space is not uniformly used throughout the literature. See [1] for remarks and references about this notation.

of precompactness in Lebesgue spaces, known as the Fréchet-Kolmogorov-Riesz theorem; see, for example, Yosida's book [25].

Theorem B. *Let $1 \leq r < \infty$. A subset $\mathcal{K} \subseteq L^r$ is compact if and only if the following three conditions are satisfied:*

(a) *$\mathcal{K}$ is bounded in L^r;*
(b) *$\lim_{A \to \infty} \int_{|x|>A} |f(x)|^r \, dx = 0$ uniformly for $f \in \mathcal{K}$;*
(c) *$\lim_{t \to 0} \|f(\cdot + t) - f\|_{L^r} = 0$ uniformly for $f \in \mathcal{K}$.*

The goal of this paper is two fold. First, it aims to extend Theorem A to a larger class of bilinear operators, denoted by $\{T_\alpha\}_{\alpha>0}$, that has as limiting case for $\alpha = 0$ the bilinear Calderón-Zygmund operators. This is obtained in Theorem 2.1 below. Second, it investigates a more singular version of these operators, whose limiting case is the bilinear Hilbert transform, and shows that the smoothing phenomenon, albeit weaker, is still present under commutation. This is achieved in Theorem 3.2.

2. COMPACTNESS FOR COMMUTATORS OF THE CLASS $\{T_\alpha\}$

We begin by defining the larger class of bilinear operators $\{T_\alpha\}$, with α in some appropriate open interval contained in $\mathbb{R}$.

Fix $0 < \alpha < 2n$ and let $K_\alpha(x,y,z)$ be a kernel on $\mathbb{R}^{3n}$ defined away from $x = y = z$ that satisfies

$$(1) \qquad |K_\alpha(x,y,z)| \lesssim \frac{1}{(|x-y| + |x-z|)^{2n-\alpha}}$$

and

$$(2) \qquad |K_\alpha(x,y,z) - K_\alpha(x+h,y,z)| \lesssim \frac{|h|}{(|x-y| + |x-z|)^{2n-\alpha+1}},$$

with the analogous estimates in the y and z variables. We consider the bilinear operator T_α

$$(3) \qquad T_\alpha(f,g)(x) = \int_{\mathbb{R}^{2n}} K_\alpha(x,y,z) f(y) g(z) \, dy \, dz$$

defined a priori for, say, f, g bounded and with compact support. It is easy to see that they extend with the same integral definition (3) to bounded operators from $L^p \times L^q \to L^r$ provided $0 < \alpha < 2n, 1 < p, q < \infty, \alpha/n < 1/p + 1/q$, and $1/r = 1/p + 1/q - \alpha/n$. Clearly, the analog kernels for $\alpha = 0$ correspond to a *bilinear Calderón-Zygmund kernel*, see again [13].

The typical example of the above operators is, of course, the *bilinear Riesz potential operator I_α*, given by the kernel

$$K_\alpha(x,y,z) = \frac{1}{(|x-y| + |x-z|)^{2n-\alpha}}.$$

The first relevant observation about the family $\{T_\alpha\}$ is that, with respect to bound-edness, its commutators behave similarly as in the "end-point" case $\alpha = 0$.

Theorem C. *Let* $0 < \alpha < 2n$, $1 < p, q < \infty$, $r \geq 1$, $\frac{\alpha}{n} < \frac{1}{p} + \frac{1}{q}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$ *and* $b \in BMO$. *The following estimates hold:*

$$\|[T_\alpha, b]_1(f, g)\|_{L^r} \lesssim \|b\|_{BMO} \|f\|_{L^p} \|g\|_{L^q},$$
$$\|[T_\alpha, b]_2(f, g)\|_{L^r} \lesssim \|b\|_{BMO} \|f\|_{L^p} \|g\|_{L^q}.$$

As usual, the notation $x \lesssim y$ indicates that $x \leq Cy$ with a positive constant C independent of x and y. For a proof of the above boundedness properties, see the papers by Chen and Xue [7] and Lian and Wu [18]. In the linear case the corresponding result goes back to the work of Chanillo [5]. The results for the multi-linear Calderón-Zygmund case used in Theorem A were addressed by Pérez and Torres [21], Tang [22], and Lerner, Ombrosi, Pérez, Torres and Trujillo-González [17].

Our real interest, however, lies in the possibility of improving boundedness to compactness. In the linear case, the compactness of the commutators of fractional integrals and multiplication by appropriate functions has already received some attention in several contexts. See, for example, the work of Chen, Ding and Wang [8] where the compactness in the usual Lebesgue measure case is traced back to Wang [24]. See also Betancor and Fariña's work [3] for the setting of non-doubling measures; the boundedness in this case was obtained by Chen and Sawyer [6].

We note that Theorem B intrinsically assumes that $r \geq 1$. The boundedness result in Theorem C of the operators $[T_\alpha, b]_1$, $[T_\alpha, b]_2$ when $r > 1$, and $1/p + 1/q < 1$, can be alternately obtained as follows. The kernel bound (1), implies that

$$|T_\alpha(f, g)(x)| \lesssim \int_{\mathbb{R}^{2n}} \frac{|f(y)||g(z)|}{(|x - y| + |x - z|)^{2n - \alpha}} \, dy \, dz = I_\alpha(|f|, |g|)(x).$$

As shown by Moen [19], the operator I_α satisfies appropriate weighted estimates. Therefore, so does T_α, and we can use the "Cauchy integral trick". An exposition of this "trick" can be found in the proof of Theorem 3.1, which deals with the more singular versions BI_α of the operators T_α. Our first main result is an extension of Theorem A that encompasses the commutators of the family $\{T_\alpha\}_{0 < \alpha < 2n}$.

Theorem 2.1. *Let* $0 < \alpha < 2n$, $1 < p, q < \infty$, $1 \leq r < \infty$, $\frac{\alpha}{n} < \frac{1}{p} + \frac{1}{q}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$, *and let* $b \in CMO$. *If* T_α *is the bilinear operator defined by* (3) *whose kernel* K_α *satisfies* (1) *and* (2), *then* $[T_\alpha, b]_1, [T_\alpha, b]_2 : L^p \times L^q \to L^r$ *are compact.*

Proof. We will work with $[T_\alpha, b]_1$; by symmetry, the proof for $[T_\alpha, b]_2$ is the same. By the form of the norm estimates in Theorem C, density, and the results about limits of compact bilinear operators in the operator norm proved in [1], we may assume that $b \in C_c^\infty$. Denote by $B_{1,p}$ and $B_{1,q}$ the unit balls in L^p and L^q, respectively and let $\mathcal{K} = [T_\alpha, b]_1(B_{1,p}, B_{1,q})$. Since $[T_\alpha, b]_1$ is a bounded operator, see Theorem C, it is clear that $\mathcal{K}$ is a bounded set in L^r, thus fulfilling condition (a) in Theorem B. We now aim to show that condition (b) in Theorem B holds.

We introduce the following two indices:

$$\alpha_p = \alpha(1/p + 1/q)^{-1} 1/p \text{ and } \alpha_q = \alpha(1/p + 1/q)^{-1} 1/q.$$

Clearly, $\alpha_p + \alpha_q = \alpha$. Since $1/p + 1/q - \alpha/n > 0$, there exist $s_p > p > 1$ and $s_q > q > 1$ such that

$$1/s_p = 1/p - \alpha_p/n \ \text{ and } \ 1/s_q = 1/q - \alpha_q/n.$$

Now, since $p, q > 1$, we see that $n > \max(\alpha_p, \alpha_q)$. In particular, this yields

$$(|x-y| + |x-z|)^{2n-\alpha} = (|x-y| + |x-z|)^{(n-\alpha_p)+(n-\alpha_q)} \geq |x-y|^{n-\alpha_p} |x-z|^{n-\alpha_q}.$$

Pick now $R > 1$ large enough so that $R > 2\max\{|x| : x \in \operatorname{supp} b\}$. Using (1) we see that, for $|x| > R$, we have

$$|[T_\alpha, b]_1(f,g)(x)| \lesssim \|b\|_\infty \int_{\mathbb{R}^n} \int_{y \in \operatorname{supp} b} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-\alpha}} \, dy dz$$

$$\leq \|b\|_\infty \int_{y \in \operatorname{supp} b} \int_{\mathbb{R}^n} \frac{|f(y)||g(z)|}{|x-y|^{n-\alpha_p}|x-z|^{n-\alpha_q}} \, dz dy$$

$$\lesssim \frac{\|b\|_\infty}{|x|^{n-\alpha_p}} \int_{y \in \operatorname{supp} b} |f(y)| \int_{\mathbb{R}^n} \frac{|g(z)|}{|x-z|^{n-\alpha_q}} \, dz dy$$

$$\lesssim \frac{\|b\|_\infty I_{\alpha_q}(|g|)(x) \|f\|_{L^p}}{|x|^{n-\alpha_p}} |\operatorname{supp} b|^{1/p'}.$$

Here, we abused a bit the notation and wrote I_α also for the linear Riesz potential, $I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(x)}{|x-y|^{n-\alpha}} \, dy$. Next, we observe that, since $s_p(n - \alpha_p) = np > n$, the function $|x|^{s_p(\alpha_p - n)}$ is integrable at infinity. Therefore, for a given $\varepsilon > 0$, we will be able to select an $R = R(\varepsilon)$ (but independent of f and g) such that

$$\left(\int_{|x|>R} |x|^{s_p(\alpha_p - n)} \, dx \right)^{1/s_p} < \varepsilon.$$

Notice now that the indices $s_p, s_q > 1$ satisfy $1/r = 1/s_p + 1/s_q$. Therefore, we can raise the previous point-wise estimate on $|[T_\alpha, b]_1(f,g)(x)|$ to the power r, integrate over $|x| > R$, and use the Hölder inequality and the $L^q \to L^{s_q}$ boundedness of I_{α_q} to get

$$\left(\int_{|x|>R} |[T_\alpha, b]_1(f,g)(x)|^r \, dx \right)^{1/r} \lesssim \varepsilon \|f\|_{L^p} \|I_{\alpha_q}(|g|)\|_{L^{s_q}} \lesssim \varepsilon \|f\|_{L^p} \|g\|_{L^q};$$

this, in turn, proves that condition (b) in Theorem B is satisfied.

Next, we will use the smoothness of b and that of the kernel K_α to show that condition (c) in Theorem B holds; specifically, we want to show that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |[T_\alpha, b]_1(f,g)(x+t) - [T_\alpha, b]_1(f,g)(x)|^r \, dx = 0.$$

We use the following splitting from [1]:

$$[T_\alpha, b]_1(f,g)(x+t) - [T_\alpha, b]_1(f,g)(x) = A(x) + B(x) + C(x) + D(x),$$

where, for $\delta > 0$ to be chosen later, we have

$$A(x) = \iint_{|x-y|+|x-z|>\delta} (b(x+t) - b(x)) K_\alpha(x,y,z) f(y) g(z) \, dy dz$$

$$B(x) = \iint_{|x-y|+|x-z|>\delta} (b(x+t) - b(y)) (K_\alpha(x+t,y,z) - K_\alpha(x,y,z)) f(y) g(z) \, dy dz$$

$$C(x) = \iint_{|x-y|+|x-z|\leq\delta} (b(y) - b(x)) K_\alpha(x,y,z) f(y) g(z) \, dy dz$$

$$D(x) = \iint_{|x-y|+|x-z|\leq\delta} (b(x+t) - b(y)) K_\alpha(x+t,y,z) f(y) g(z) \, dy dz$$

The term A is easy to handle with the mean value theorem; we have

$$|A(x)| \lesssim |t| \|\nabla b\|_\infty I_\alpha(|f|,|g|)(x).$$

Consequently, we obtain

(4) $$\|A\|_{L^r} \lesssim |t| \|\nabla b\|_{L^\infty} \|f\|_{L^p} \|g\|_{L^q}.$$

We now consider the terms B, C and D.[2] We start with B.

$$|B(x)| \leq \iint_{|x-y|+|x-z|>\delta} (b(x+t) - b(y)) (K_\alpha(x+t,y,z) - K_\alpha(x,y,z)) f(y) g(z) \, dy dz$$

$$\leq 2\|b\|_\infty \iint_{|x-y|+|x-z|>\delta} |K_\alpha(x+t,y,z) - K_\alpha(x,y,z)| \, |f(y)| \, |g(z)| \, dy dz$$

$$\lesssim |t| \|b\|_\infty \iint_{|x-y|+|x-z|>\delta} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-\alpha+1}} \, dy dz$$

$$\lesssim |t| \|b\|_\infty \iint_{\max(|x-y|,|x-z|)>\frac{\delta}{2}} \frac{|f(y)||g(z)|}{\max(|x-y|,|x-z|)^{2n-\alpha+1}} \, dy dz$$

$$= |t| \|b\|_\infty \sum_{k=0}^\infty \iint_{2^{k-1}\delta<\max(|x-y|,|x-z|)\leq 2^k\delta} \frac{|f(y)||g(z)|}{\max(|x-y|,|x-z|)^{2n-\alpha+1}} \, dy dz$$

$$\leq |t| \|b\|_\infty \sum_{k=0}^\infty \frac{1}{(2^k\delta)^{2n-\alpha+1}} \iint_{\max(|x-y|,|x-z|)\leq 2^k\delta} |f(y)||g(z)| \, dy dz.$$

Note now that

$$\{(y,z) \in \mathbb{R}^{2n} : \max(|x-y|,|x-z|) \leq 2^k\delta\} \subset B_{2^{k+1}\delta}(x) \times B_{2^{k+1}\delta}(x),$$

where $B_r(x)$ denotes the ball of radius r centered at x. Therefore, we can further estimate

$$|B(x)| \lesssim \frac{|t| \|b\|_\infty}{\delta} \sum_{k=0}^\infty \frac{|B_{2^k\delta}(x)|^{\frac{\alpha}{n}}}{2^k} \frac{1}{|B_{2^k\delta}(x)|} \int_{B_{2^k\delta}(x)} |f(y)| \, dy \frac{1}{|B_{2^k\delta}(x)|} \int_{B_{2^k\delta}(x)} |g(z)| \, dz$$

$$\lesssim |t| \|b\|_\infty \frac{1}{\delta} \mathcal{M}_\alpha(f,g)(x) \Big(\sum_{k=0}^\infty 2^{-k} \Big) = \frac{2|t| \|b\|_\infty}{\delta} \mathcal{M}_\alpha(f,g)(x),$$

---

[2]We actually obtain estimates for these terms that slightly improve on the corresponding ones for $\alpha = 0$ in [1].

where

$$\mathcal{M}_\alpha(f,g)(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \left(\fint_Q |f(y)| \, dy \right) \left(\fint_Q |g(z)| \, dz \right).$$

Since the operator $\mathcal{M}_\alpha(f,g)$ is pointwise smaller than $I_\alpha(|f|,|g|)$, we get $\mathcal{M}_\alpha : L^p \times L^q \to L^r$. In turn, this yields

(5)
$$\|B\|_{L^r} \lesssim \frac{|t| \|b\|_\infty}{\delta} \|f\|_{L^p} \|g\|_{L^q}.$$

Let us now estimate the C term.

$$|C(x)| \le \iint_{|x-y|+|x-z| \le \delta} |b(y) - b(x)| \, |K_\alpha(x,y,z)| \, |f(y)| \, |g(z)| \, dydz$$

$$\lesssim \|\nabla b\|_\infty \iint_{|x-y|+|x-z| \le \delta} \frac{|x-y|}{(|x-y|+|x-z|)^{2n-\alpha}} |f(y)||g(z)| \, dydz$$

$$\le \|\nabla b\|_\infty \iint_{|x-y|+|x-z| \le \delta} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-\alpha-1}} \, dydz$$

$$\lesssim \|\nabla b\|_\infty \sum_{k=0}^\infty \iint_{2^{-k-1}\delta < \max(|x-y|,|x-z|) \le 2^{-k}\delta} \frac{|f(y)||g(z)|}{\max(|x-y|,|x-z|)^{2n-\alpha-1}} \, dydz$$

$$\lesssim \|\nabla b\|_\infty \sum_{k=0}^\infty \frac{2^{-k}\delta}{(2^{-k}\delta)^{2n-\alpha}} \iint_{\max(|x-y|,|x-z|) \le 2^{-k}\delta} |f(y)||g(z)| \, dydz$$

$$\lesssim \delta \|\nabla b\|_\infty \mathcal{M}_\alpha(f,g)(x).$$

From here, we get

(6)
$$\|C\|_{L^r} \lesssim \delta \|\nabla b\|_\infty \|f\|_{L^p} \|g\|_{L^q}.$$

For the last term D we have an identical estimate to the C term, except that x is now replaced by $x+t$. We have

$$|D(x)| \le \iint_{|x-y|+|x-z| \le \delta} |b(x+t) - b(y)| \, |K_\alpha(x+t,y,z)f(y)g(z)| \, dydz$$

$$\lesssim \|\nabla b\|_\infty \iint_{|x+t-y|+|x+t-z| \le \delta+2|t|} \frac{|x+t-y||f(y)||g(z)|}{(|x+t-y|+|x+t-z|)^{2n-\alpha}} \, dydz$$

$$\le \|\nabla b\|_\infty \iint_{|x+t-y|+|x+t-z| \le \delta+2|t|} \frac{|f(y)||g(z)|}{(|x+t-y|+|x+t-z|)^{2n-\alpha}} \, dydz$$

$$\lesssim (\delta+|t|) \|\nabla b\|_\infty \mathcal{M}_\alpha(f,g)(x+t).$$

Thus, as above, we get

(7)
$$\|D\|_{L^r} \lesssim (\delta+|t|) \|\nabla b\|_\infty \|f\|_{L^p} \|g\|_{L^q}.$$

Let $1 > \varepsilon > 0$ be given. For each $0 < |t| < \varepsilon^2$ we now select $\delta = |t|/\varepsilon$. Estimates (4), (5), (6) and (7) then prove

$$\|[T_\alpha,b]_1(f,g)(\cdot+t) - [T_\alpha,b]_1(f,g)(\cdot)\|_{L^r} \lesssim \varepsilon \|f\|_{L^p} \|g\|_{L^q},$$

that is, condition (c) in Theorem B holds. $\qquad\square$

Remark. *Iterated commutators* can be considered as well. For example, one can look at operators of the form

$$[T, b_1, b_2](f, g) = [[T, b_1]_1, b_2]_2(f, g) = [T, b_1]_1(f, b_2 g) - b_2[T, b_1]_1(f, g).$$

For bilinear Calderón-Zygmund operators, the boundedness of such operators was studied in [22], see also the work by Pérez, Pradolini, Torres and Trujillo-González [20], while for bilinear fractional integrals they were addressed in [18]. As pointed out in [1], the compactness of the iterated commutators is actually easier to prove. The interested reader may adapt the arguments in [1] to our current situation $\{T_\alpha\}$.

3. SEPARATE COMPACTNESS FOR COMMUTATORS OF THE CLASS $\{BI_\alpha\}$

We will now examine a more singular family of bilinear fractional integral operators,

$$BI_\alpha(f, g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} \, dy.$$

These operators were first introduced by Grafakos in [11], and later studied by Grafakos and Kalton [12] and Kenig and Stein [16]. We can view them as fractional versions of the bilinear Hilbert transform

$$BHT(f, g)(x) = p.v. \int_{\mathbb{R}} \frac{f(x-y)g(x+y)}{y} \, dy.$$

For $i = 1, 2$ and $b \in BMO$, we define the commutators $[BI_\alpha, b]_i$ similarly to those of the operators T_α. First, we prove that the commutators $[BI_\alpha, b]_i$, $i = 1, 2$, are bounded. Our proof makes use of what we call the "Cauchy integral trick".

Theorem 3.1. *Let* $0 < \alpha < n$, $1 < p, q, r < \infty$, $\frac{1}{p} + \frac{1}{q} < 1$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$, *and* $b \in BMO$. *Then, for* $i = 1, 2$, *we have*

$$\|[BI_\alpha, b]_i(f, g)\|_{L^r} \leq C\|b\|_{BMO}\|f\|_{L^p}\|g\|_{L^q}.$$

Proof. We will work with the commutator in the first variable; the proof for the second variable is identical. We define $s > 1$ by $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. As observed by Bernicot, Maldonado, Moen and Naibo [2], see also [19], if $1 < s < r$ satisfy $\frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n}$, then BI_α is bounded on appropriate product weighted Lebesgue spaces; we have

(8) $$BI_\alpha : L^p(w_1^p) \times L^q(w_2^q) \to L^r(w_1^r w_2^r)$$

where $w_1, w_2 \in A_{s,r}$, that is, for $i = 1, 2$,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w_i^r \, dx \right) \left(\frac{1}{|Q|} \int_Q w_i^{-s'} \, dx \right)^{\frac{r}{s'}} < \infty.$$

Without loss of generality, we may assume $f, g \in C_c^\infty(\mathbb{R}^n)$ and b is real valued. For $z \in \mathbb{C}$, consider the holomorphic function (in z)

$$T_z(f, g; \alpha) = e^{zb} BI_\alpha(e^{-zb} f, g),$$

and notice that by the Cauchy integral formula, for $\varepsilon > 0$,

$$[BI_\alpha, b]_1(f, g) = -\frac{d}{dz} T_z(f, g; \alpha) \Big|_{z=0} = -\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{T_z(f, g; \alpha)}{z^2} \, dz.$$

Since $r > 1$, we can use Minkowski's integral inequality to obtain

$$\|[BI_\alpha, b]_1(f,g)\|_{L^r} \leq \frac{1}{2\pi\varepsilon^2} \int_{|z|=\varepsilon} \|T_z(f,g;\alpha)\|_{L^r} |dz|$$

and

$$\|T_z(f,g;\alpha)\|_{L^r}^r = \int_{\mathbb{R}^n} \left(|BI_\alpha(e^{-zb}f,g)|e^{(\mathrm{Re}\,z)b}\right)^r dx.$$

For $\varepsilon > 0$, $\varepsilon \lesssim \|b\|_{BMO}^{-1}$, and $|t| \leq \varepsilon$, by John-Nirenberg's inequality, we have $e^{tb} \in A_{s,r}$. Therefore, by (8) with $w_1 = e^b$ and $w_2 = 1$, we have

$$\|T_z(f,g;\alpha)\|_{L^r} = \left(\int_{\mathbb{R}^n} \left(|BI_\alpha(e^{-zb}f,g)|e^{(\mathrm{Re}\,z)b}\right)^r dx\right)^{1/r}$$

$$\leq C\left(\int_{\mathbb{R}^n}(|e^{-zb}f|e^{(\mathrm{Re}\,z)b})^p dx\right)^{1/p}\left(\int_{\mathbb{R}^n}|g|^q dx\right)^{1/q}$$

$$= C\|f\|_{L^p}\|g\|_{L^q}.$$

The desired result follows from here. $\qquad\square$

Theorem 3.2. *Let* $0 < \alpha < n$, $1 < p,q,r < \infty$, $\frac{1}{p} + \frac{1}{q} < 1$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$, *and* $b \in CMO$. *Then,* $[BI_\alpha, b]_1, [BI_\alpha, b]_2 : L^p \times L^q \to L^r$ *are separately compact.*

Proof. We will work again with the commutator in the first variable. By a change of variables, this commutator can be rewritten as

$$[BI_\alpha, b]_1(f,g)(x) = \int_{\mathbb{R}^n} \frac{b(y) - b(x)}{|x-y|^{n-\alpha}} f(y)g(2x-y)\,dy.$$

We may assume that $b \in C_c^\infty(\mathbb{R}^n)$ and aim to prove that the conditions (a), (b) and (c) of Theorem B hold for the family of functions $[BI_\alpha, b]_1(f,g)$, where $g \in L^q$ is fixed and $f \in B_{1,p}$.

By Theorem 3.1, we already know that condition (a) is satisfied. Thus, we concentrate on proving (b) and (c).

The estimates that yield (b) are reminiscent of the ones used in the proof of Theorem 2.1. Assume $R > 1$ is large enough so that $|x| \geq R$ implies $x \notin \mathrm{supp}\, b$. Then

$$|[BI_\alpha, b]_1(f,g)(x)| \leq \|b\|_\infty \int_{\mathrm{supp}\,b} |x-y|^{\alpha-n}|f(y)g(2x-y)|\,dy$$

$$\lesssim \|b\|_\infty |x|^{\alpha-n} \int_{\mathrm{supp}\,b} |f(y)g(2x-y)|\,dy$$

$$\leq \|b\|_\infty |x|^{\alpha-n}\left(\int_{\mathrm{supp}\,b}|f(y)|^{q'}\,dy\right)^{1/q'}\|g\|_{L^q}$$

Let us write $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1 = \frac{1}{q} + \frac{1}{q'}$; so $q' < p$. As such, we can further estimate

$$\left(\int_{\mathrm{supp}\,b}|f(y)|^{q'}\,dy\right)^{1/q'} \leq |\mathrm{supp}\,b|^{\frac{1}{q'}-\frac{1}{p}}\|f\|_{L^p} = |\mathrm{supp}\,b|^{\frac{1}{s'}}\|f\|_{L^p}.$$

Now, we raise to the power r and integrate with respect to x over the set $|x| > R$. Notice that, since $s > 1$, we have $\frac{1}{r} = \frac{1}{s} - \frac{\alpha}{n} < \frac{n-\alpha}{n} \Leftrightarrow r(n-\alpha) > n$. This allows us, for a given $\varepsilon > 0$, to control

$$\int_{|x|>R} |[BI_\alpha, b]_1(f,g)(x)|^r \, dx < \varepsilon^r$$

by taking $R = R(\varepsilon) > 0$ sufficiently large; which shows that, indeed, (b) is satisfied.

We are left to show the continuity condition (c), that is,

$$\lim_{t \to 0} \|[BI_\alpha, b]_1(f,g)(\cdot + t) - [BI_\alpha, b]_1(f,g)\|_{L^r} = 0,$$

uniformly for $\|f\|_{L^p} \le 1$ and $g \in L^q$ fixed. First, we lump our fixed function g into a general kernel

$$[BI_\alpha, b]_1(f,g)(x) = \int_{\mathbb{R}^n} \frac{b(y) - b(x)}{|x-y|^{n-\alpha}} f(y) g(2x - y) \, dy$$

$$= \int_{\mathbb{R}^n} (b(y) - b(x)) K_g(x,y) f(y) \, dy$$

where

$$K_g(x,y) = \frac{g(2x-y)}{|x-y|^{n-\alpha}}.$$

Second, we split the commutator $[BI_\alpha, b]_1$ by following the decomposition used for $[T_\alpha, b]_1$. Namely, we write

$$[BI_\alpha, b]_1(f,g)(x+t) - [BI_\alpha, b]_1(f,g)(x) = A(x) + B(x) + C(x) + D(x),$$

where

$$A(x) = \int_{|x-y|>\delta} (b(x+t) - b(x)) K_g(x,y) f(y) \, dy,$$

$$B(x) = \int_{|x-y|>\delta} (b(x+t) - b(y))(K_g(x+t,y) - K_g(x,y)) f(y) \, dy,$$

$$C(x) = \int_{|x-y|\le\delta} (b(y) - b(x)) K_g(x,y) f(y) \, dy,$$

$$D(x) = \int_{|x-y|\le\delta} (b(x+t) - b(y)) K_g(x+t,y) f(y) \, dy.$$

We will now estimate each term in this decomposition. For A, the estimate is immediate. We clearly have $|A(x)| \le |t| \|\nabla b\|_\infty BI_\alpha(|f|, |g|)(x)$. Since BI_α is $L^p \times L^q \to L^r$ bounded, we get $\|A\|_{L^r} \lesssim |t| \|f\|_{L^p} \|g\|_{L^q}$.

The estimate for the B term is the most delicate. To facilitate the ease of reading, we postpone it until the end of the proof.

We estimate C as follows:

$$|C(x)| \leq \int_{|x-y|\leq\delta} |b(y) - b(x)| \frac{|g(2x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy$$

$$\leq \|\nabla b\|_\infty \int_{|x-y|\leq\delta} |x-y| \frac{|g(2x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy$$

$$\leq \delta \|\nabla b\|_\infty BM_\alpha(f,g)(x),$$

where BM_α is the associated bilinear fractional maximal operator,

$$BM_\alpha(f,g)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |f(x-y)g(x+y)| \, dy.$$

The estimate for $D(x)$ is similar; we now have

$$|D(x)| \leq (\delta + |t|)\|\nabla b\|_\infty BM_\alpha(f,g)(x+t).$$

Again, since $BM_\alpha(f,g) \lesssim BI_\alpha(|f|,|g|)$, we have $BM_\alpha : L^p \times L^q \to L^r$. Thus, similarly to A, we get $\|C\|_{L^r} \lesssim \delta \|f\|_{L^p} \|g\|_{L^q}$ and $\|D\|_{L^r} \lesssim (\delta + |t|)\|f\|_{L^p} \|g\|_{L^q}$.

Finally, we turn our attention to B.

$$|B(x)| \leq \int_{|x-y|>\delta} |b(x+t) - b(y)| \left| \frac{g(2x+2t-y)}{|x+t-y|^{n-\alpha}} - \frac{g(2x-y)}{|x-y|^{n-\alpha}} \right| |f(y)| \, dy$$

$$\leq 2\|b\|_\infty \int_{|x-y|>\delta} \left| \frac{g(2x+2t-y)}{|x+t-y|^{n-\alpha}} - \frac{g(2x-y)}{|x-y|^{n-\alpha}} \right| |f(y)| \, dy$$

$$\lesssim \int_{|x-y|>\delta} \left| \frac{1}{|x+t-y|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}} \right| |g(2x+2t-y)f(y)| \, dy$$

$$+ \int_{|x-y|>\delta} \frac{|g(2x+2t-y) - g(2x-y)||f(y)|}{|x-y|^{n-\alpha}} \, dy$$

$$= E(x) + F(x)$$

To estimate E, we note that

$$\left| \frac{1}{|x+t-y|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}} \right| \lesssim \frac{|t|}{|x-y|^{n-\alpha+1}},$$

which implies

$$E(x) \lesssim |t| \int_{|x-y|>\delta} \frac{|g(2x+2t-y)f(y)|}{|x-y|^{n-\alpha+1}} \, dy$$

$$\lesssim \frac{|t|}{\delta} BM_\alpha(f, \tau_{2t}g)(x).$$

Here, τ_a is the shift operator $\tau_a g(x) = g(x+a)$. It follows from the boundedness of BM_α that

$$\|E\|_{L^r} \lesssim \frac{|t|}{\delta} \|f\|_{L^p} \|g\|_{L^q}.$$

For $F(x)$ we have

$$F(x) \lesssim BM_\alpha(f, \tau_{2t}g - g)(x),$$

so

$$\|F\|_{L^r} \lesssim \|f\|_{L^p} \|\tau_{2t}g - g\|_{L^q}.$$

Since $g \in L^q$, for a given $\varepsilon > 0$ we can find $\gamma = \gamma(\varepsilon, g) > 0$ such that $|t| < \gamma$ implies

$$\|\tau_{2t}g - g\|_{L^q} < \varepsilon.$$

Finally, by choosing $|t| < \varepsilon^2$ and $\delta = |t|/\varepsilon$ we get that

$$\|[BI_\alpha, b]_1(f, g)(\cdot + t) - [BI_\alpha, b]_1(f, g)\|_{L^r} \lesssim \varepsilon.$$

This shows that (c) holds, thus finishing our proof for the compactness in the first variable.

We now show that $[BI_\alpha, b]_1$ is compact in the second variable, that is, $[BI_\alpha, b]_1(f, \cdot):$ $L^q \to L^r$ is compact for a fixed $f \in L^p$. Conditions (a) and (b) of Theorem B follow from similar calculations to those performed above. Thus we will check condition (c) of Theorem B. For $f \in L^p$ fixed and $g \in B_{1,q}$ we write

$$[BI_\alpha, b]_1(f, g)(x + t) - [BI_\alpha, b]_1(f, g)(x)$$

$$= \int_{\mathbb{R}^n} (b(2x + 2t - y) - b(x + t)) \frac{f(2x + 2t - y)g(y)}{|x + t - y|^{n-\alpha}} \, dy$$

$$- \int_{\mathbb{R}^n} (b(2x - y) - b(x)) \frac{f(2x - y)g(y)}{|x - y|^{n-\alpha}} \, dy$$

$$= \int_{\mathbb{R}^n} (b(2x + 2t - y) - b(x + t)) K_f(x + t, y)g(y) \, dy$$

$$- \int_{\mathbb{R}^n} (b(2x - y) - b(x)) K_f(x, y)g(y) \, dy$$

where this time we combine f with the kernel:

$$K_f(x, y) = \frac{f(2x - y)}{|x - y|^{n-\alpha}}.$$

Before proceeding further, we make one reduction. Notice that

$$[b, BI_\alpha]_1(f, g)(x + t) = \int_{\mathbb{R}^n} (b(2x + 2t - y) - b(x + t)) K_f(x + t, y)g(y) \, dy$$

(9) $$= \int_{\mathbb{R}^n} (b(2x + 2t - y) - b(2x - y)) K_f(x + t, y)g(y) \, dy$$

$$+ \int_{\mathbb{R}^n} b(2x - y) - b(x + t)) K_f(x + t, y)g(y) \, dy$$

The first term in the sum (9) is bounded by

$$2\|\nabla b\|_\infty |t| BI_\alpha(|f|, |g|)(x + t)$$

and the L^r norm of this quantity will go to zero uniformly for $g \in B_{1,q}$ as $t \to 0$. Thus it remains to estimate

$$\int_{\mathbb{R}^n} b(2x - y) - b(x + t)) K_f(x + t, y)g(y) \, dy$$

$$- \int_{\mathbb{R}^n} (b(2x - y) - b(x)) K_f(x, y)g(y) \, dy = G(x) + H(x) + I(x) + J(x)$$

where

$$G(x) = \int_{|x-y|>\delta} (b(x) - b(x+t))K_f(x,y)g(y)\,dy,$$

$$H(x) = \int_{|x-y|>\delta} (b(2x-y) - b(x+t))(K_f(x+t,y) - K_f(x,y))g(y)\,dy,$$

$$I(x) = \int_{|x-y|\leq\delta} (b(x) - b(2x-y))K_f(x,y)g(y)\,dy,$$

$$J(x) = \int_{|x-y|\leq\delta} (b(2x-y) - b(x+t))K_f(x+t,y)g(y)\,dy.$$

The estimates for $G, H, I,$ and J are handled similarly to the corresponding estimates for $A, B, C,$ and D above, again, with H being the most complicated. For example the estimates for $G, I,$ and J are as follows:

$$|G(x)| \leq |t| \|\nabla b\|_\infty BI_\alpha(f,g)(x),$$

$$|I(x)| \leq \delta \|\nabla b\|_\infty BM_\alpha(f,g)(x),$$

and

$$|J(x)| \leq (\delta + |t|)\|\nabla b\|_\infty BM_\alpha(f,g)(x+t).$$

Finally, for H we have

$$|H(x)| \lesssim \|b\|_\infty \left(\frac{|t|}{\delta} BM_\alpha(\tau_{2t}f,g)(x) + \frac{1}{\delta}BM_\alpha(\tau_{2t}f - f,g)(x) \right).$$

These estimates show that $[BI_\alpha, b]_1(f,g)$ is compact in the second variable as well, thus showing that it is separately compact. □

A close inspection of the proof of Theorem 3.2 shows that we barely miss proving joint compactness. Indeed, the only non-uniform estimate concerns the very last terms, which we denote by F and H, where we use the fact that we can make the quantity $\|\tau_{2t}g - g\|_{L^q}$ (or $\|\tau_{2t}f - f\|_{L^p}$) small by taking t sufficiently small and, crucially, dependent on g (or f). Thus, our method of proof only yields separate compactness. Compared to the nicely behaved operators T_α, we have in effect a weaker smoothing property of the commutators of the more singular bilinear fractional integrals, BI_α.

The remarks above motivate the following question.

Question 3.3. For $b \in CMO$, are the commutators $[BI_\alpha, b]_i, i = 1, 2$, jointly compact?

The techniques used in this section can be applied to commutators of the BHT, if, *a priori*, we know that its commutators are bounded. Specifically, if we assume that $[BHT, b]_1 : L^p \times L^q \to L^r$, then $[BHT, b]_1$ is separately compact for $b \in CMO$; a similar result holds for $[BHT, b]_2$. This leads to the following natural question about the bilinear Hilbert transform.

Question 3.4. For $b \in BMO$, are the commutators $[BHT, b]_i, i = 1, 2$, bounded from $L^p \times L^q \to L^r$?

REFERENCES

[1] Á. Bényi and R.H. Torres, *Compact bilinear operators and commutators*, Proc. Amer. Math. Soc., to appear.

[2] F. Bernicot, D. Maldonado, K. Moen, and V. Naibo, *Bilinear Sobolev-Poincaré inequalities and Leibniz-type rules*, J. Geom. Anal. to appear.

[3] J.J. Betancor and J.C. Fariña, *A note on compactness of commutators for fractional integrals associated with nondoubling measures*, Z. Anal. Anwend. **26** (2007), 331–339.

[4] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.

[5] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), 7–16.

[6] W. Chen and E. Sawyer, *A note on commutators of fractional integrals with RBMO(μ) functions,* Illinois J. Math. **46** (2002), 1287–1298.

[7] X. Chen and Q. Xue, *Weighted estimates for a class of multilinear fractional type operators*. J. Math. Anal. Appl. **362** (2010), 355–373.

[8] Y. Chen, Y. Ding, and X. Wang, *Compactness of commutators of Riesz potential on Morrey spaces*, Potential Anal. **30** (2009), 301–313.

[9] R. R. Coifman, P. L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces,* J. Math. Pures Appl. **72** (1993), 247–286.

[10] R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.

[11] L. Grafakos, *On multilinear fractional integrals*, Studia Math. **102** (1992), 49–56.

[12] L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. **319** (2001), 151–180.

[13] L. Grafakos and R.H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), 124–164.

[14] T. Iwaniec, *Nonlinear commutators and Jacobians,* J. Fourier Anal. Appl. **3** (2007), 775–796.

[15] T. Iwaniec and C. Sbordone, *Riesz transform and elliptic PDEs with VMO coefficients,* J. Anal. Math. **74** (1998), 183–212.

[16] C. Kenig and E. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. **6** (1999) 1–15.

[17] A. Lerner, S. Ombrosi, C. Pérez, R.H. Torres, and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220** (2009), 1222–1264.

[18] J. Lian and H. Wu, *A class of commutators for multilinear fractional integrals in nonhomogeneous spaces*, J. Inequal. Appl., vol. 2008, Article ID 373050, 17 pages.

[19] K. Moen, *New weighted estimates for bilinear fractional integrals*, Trans. Amer. Math. Soc., to appear.

[20] C. Pérez, G. Pradolini, R.H. Torres, and R. Trujillo-González, *End-points estimates for iterated commutators of multilinear singular integrals*, Bull. Lond. Math. Soc., to appear.

[21] C. Pérez and R.H. Torres, *Sharp maximal function estimates for multilinear singular integrals*, Contemp. Math. **320** (2003), 323–331.

[22] L. Tang, *Weighted estimates for vector-valued commutators of multilinear operators*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 897–922.

[23] A. Uchiyama, *On the compactness of operators of Hankel type*, Tôhoku Math. J. **30** (1978), no. 1, 163–171.

[24] S. Wang, *The compactness of the commutator of fractional integral operator* (in Chinese), Chin. Ann. Math. **8(A)** (1987), 475–482.

[25] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin (1995).

DEPARTMENT OF MATHEMATICS, WESTERN WASHINGTON UNIVERSITY, 516 HIGH STREET, BELLINGHAM, WA 98225, USA

E-mail address: arpad.benyi@wwu.edu

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SEVILLA, 41080 SEVILLA, SPAIN

E-mail address: wdamian@us.es

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, TUSCALOOSA, AL 35487, USA

E-mail address: kabe.moen@ua.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KS 66045, USA

E-mail address: torres@math.ku.edu