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A GEOMETRIC INVERSE PROBLEM FOR THE BOUSSINESQ SYSTEM

A. Doubova, E. Fernández-Cara and M. González-Burgos

Universidad de Sevilla
Dpto. E.D.A.N., Aptdo. 1160, 41080, Sevilla, SPAIN

J.H. Ortega

Universidad del Bío-Bío
Dpto. de Ciencias Básicas, Casilla 447, Fernando May, Chillán, CHILE

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Abstract. In this work we present some results for the inverse problem of the identification of a single rigid body immersed in a fluid governed by the stationary Boussinesq equations. First, we establish a uniqueness result. Then, we show the way the observation depends on perturbations of the rigid body and we deduce some consequences. Finally, we present a new method for the partial identification of the body assuming that it can be deformed only through fields that, in some sense, are finite dimensional. In the proofs, we use various techniques, related to Carleman estimates, differentiation with respect to domains, data assimilation and controllability of PDEs.

1. Introduction and main results.

Let $\Omega \subset \mathbb{R}^N$ be a simply connected bounded open set ($N = 2$ or $N = 3$) whose boundary $\partial\Omega$ is of class $W^{2,\infty}$. Let γ be a nonempty open subset of $\partial\Omega$ and let us denote by 1_γ the characteristic function of γ.

Let D^* be a fixed nonempty open set such that $D^* \subset\subset \Omega$. We will consider the following family of subsets of Ω:

$$\begin{aligned}\mathcal{D} = \{\, D \subset \Omega : {} & D \text{ is a simply connected nonempty open set,} \\ & \partial D \text{ is of class } W^{2,\infty},\ D \subset\subset D^* \,\}.\end{aligned}$$

In this paper we will deal with the following inverse problem:

Given (φ, ψ) and (α, β) in appropriate spaces, find a set $D \in \mathcal{D}$ such that a solution (u, p, θ) of the Boussinesq system

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = \theta g, \quad \nabla\cdot u = 0 & \textit{in } \Omega\setminus\overline{D}, \\ -\kappa\Delta\theta + u\cdot\nabla\theta = 0 & \textit{in } \Omega\setminus\overline{D}, \\ u = \varphi, \quad \theta = \psi & \textit{on } \partial\Omega, \\ u = 0, \quad \theta = 0 & \textit{on } \partial D, \end{cases} \tag{1}$$

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satisfies the additional conditions

$$\sigma(u,p)\cdot n := (-p\,\mathrm{Id.} + 2\nu\, e(u))\cdot n = \alpha, \quad \kappa\frac{\partial\theta}{\partial n} = \beta \quad on \quad \gamma. \tag{2}$$

In (1), u, p and θ are respectively a velocity field, a pressure distribution and a temperature θ. The constant vector g is the gravitational force and $\nu > 0$ and $\kappa > 0$ are given constants, respectively representing the kinematic viscosity and thermal conductivity of the fluid. In (2), Id. is the identity matrix and $e(u)$ is the linear strain tensor, given by

$$e(u) = \frac{1}{2}(\nabla u + {}^t\nabla u).$$

All along this paper we will assume that, among other things, $(\varphi,\psi) \neq (0,0)$.

The interpretation of problem (1)–(2) is the following. We assume that a stationary Newtonian viscous fluid sensible to temperature effects fills an unknown domain $\Omega \setminus \overline{D}$ at rest. The velocity φ and the temperature ψ on the outer boundary $\partial\Omega$ are given and we are able to measure the *normal stresses* $\sigma(u,p)\cdot n$ and also the *normal heat flux* $\kappa\frac{\partial\theta}{\partial n}$ on $\gamma \subset \partial\Omega$. Then the question is whether we can determine D from Ω, φ, ψ and these measurements.

A related problem concerning a Navier-Stokes fluid was considered in [12]. A similar problem has been analyzed in [18]. There, instead of (1), one has

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega\setminus\overline{D}, \\ u = \varphi & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3}$$

and the role of the additional information (2) is replaced by

$$\frac{\partial u}{\partial n} = \alpha \quad \text{on} \quad \gamma. \tag{4}$$

Other problems of this kind have been analized by several authors; see for instance [1], [2], [3], [4], [5], [6], [7], [10], [11], [14], [19] and [20].

In what concerns the associated direct problem, i.e. the determination of (u,p,θ) (and then α and β) from Ω, D, φ and ψ, we have the following standard result:

Theorem 1. *Assume that $D \in \mathcal{D}$ and $(\varphi,\psi) \in H^{1/2}(\partial\Omega)^N \times H^{1/2}(\partial\Omega)$ satisfies*

$$\int_{\partial\Omega} \varphi\cdot n\, d\Gamma = 0. \tag{5}$$

(a) *For any $\nu > 0$ and any $\kappa > 0$,* (1) *possesses at least one solution (u,p,θ) that belongs to $H^1(\Omega\setminus\overline{D})^N \times L^2(\Omega\setminus\overline{D}) \times H^1(\Omega\setminus\overline{D})$ and verifies*

$$\begin{cases} \|u\|_{H^1} \le \dfrac{C}{\nu}\left(\|\psi\|_{H^{1/2}} + \dfrac{1}{\kappa}\|\varphi\|_{H^{1/2}}\|\psi\|_{H^{1/2}} + \nu\|\varphi\|_{H^{1/2}} + \|\varphi\|^2_{H^{1/2}}\right), \\ \|\theta\|_{H^1} \le C\left(\|\psi\|_{H^{1/2}} + \dfrac{1}{\kappa}\|\varphi\|_{H^{1/2}}\|\psi\|_{H^{1/2}} + \nu\|\varphi\|_{H^{1/2}} + \|\varphi\|^2_{H^{1/2}}\right), \end{cases} \tag{6}$$

where C only depends on Ω and D^* and $\sigma(u,p)\cdot n \in H^{-1/2}(\partial\Omega)^N$, $\frac{\partial\theta}{\partial n} \in H^{-1/2}(\partial\Omega)$.
(b) *There exists a positive constant $K_0 = K_0(\Omega, D^*)$ such that, if (φ,ψ) satisfies*

$$\begin{cases} K(\nu,\kappa,\varphi,\psi) := \|\psi\|_{H^{1/2}} + \dfrac{1}{\kappa}\|\varphi\|_{H^{1/2}}\|\psi\|_{H^{1/2}} + \nu\|\varphi\|_{H^{1/2}} + \|\varphi\|^2_{H^{1/2}} \\ \le K_0(\Omega, D^*)\dfrac{\nu^2\kappa}{\nu+\kappa}, \end{cases} \tag{7}$$

then the solution of (1) *is unique (p is unique up to a constant).*

(c) *If in addition* $r \in [2, +\infty)$ *and* $(\varphi, \psi) \in W^{2-1/r,r}(\partial\Omega)^N \times W^{2-1/r,r}(\partial\Omega)$*, the previous solutions of* (1) *satisfy* $(u, p, \theta) \in W^{2,r}(\Omega\setminus\overline{D})^N \times W^{1,r}(\Omega\setminus\overline{D}) \times W^{2,r}(\Omega\setminus\overline{D})$ *and* $\sigma(u, p) \cdot n \in W^{1-1/r,r}(\partial\Omega)^N$ *and* $\frac{\partial\theta}{\partial n} \in W^{1-1/r,r}(\partial\Omega)$.

For completeness, we will present the proof of this result in Section 6.

Remark 1. In the sequel, we will always assume that $(\varphi, \psi) \in H^{1/2}(\partial\Omega)^N \times H^{1/2}(\partial\Omega)$, (5) is satisfied, and (7) holds. Accordingly, for each $D \in \mathcal{D}$, we can speak of the unique solution $(u, p, \theta) \in H^1(\Omega\setminus\overline{D})^N \times L^2(\Omega\setminus\overline{D}) \times H^2(\Omega\setminus\overline{D})$ of (1).

On the other hand, it is clear from (6) that, if (7) is fulfilled, the weak solution to (1) satisfies

$$\|\nabla u\|_{L^2} + \frac{1}{\kappa}\|\nabla\theta\|_{L^2} \le C(\Omega, D^*)\frac{\nu+\kappa}{\nu\kappa}\,K(\nu, \kappa, \varphi, \psi) \le C(\Omega, D^*)K_0(\Omega, D^*)\nu. \quad (8)$$

In the sequel, we will have to consider several linear systems of the forms

$$\begin{cases} -\nu\Delta\xi + (u\cdot\nabla)\xi + (\xi\cdot\nabla)u + \nabla\chi = \rho g, \quad \nabla\cdot\xi = 0 & \text{in } \Omega\setminus\overline{D}, \\ -\kappa\Delta\rho + u\cdot\nabla\rho + \xi\cdot\nabla\theta = 0 & \text{in } \Omega\setminus\overline{D}, \\ \xi = \varphi, \quad \rho = \psi & \text{on } \partial\Omega, \\ \xi = 0, \quad \rho = 0 & \text{on } \partial D \end{cases} \quad (9)$$

and

$$\begin{cases} -\nu\Delta\xi - (\nabla\xi)^t u - (u\cdot\nabla)\xi + \nabla\chi = -\rho\nabla\theta, \quad \nabla\cdot\xi = 0 & \text{in } \Omega\setminus\overline{D}, \\ -\kappa\Delta\rho + - u\cdot\nabla\rho = \xi\cdot g & \text{in } \Omega\setminus\overline{D}, \\ \xi = \varphi, \quad \rho = \psi & \text{on } \partial\Omega, \\ \xi = 0, \quad \rho = 0 & \text{on } \partial D, \end{cases} \quad (10)$$

where $u \in H^1(\Omega\setminus\overline{D})^N$, $\nabla\cdot u = 0$ in $\Omega\setminus\overline{D}$ and $\theta \in H^1(\Omega\setminus\overline{D})$. Under these assumptions, there exists a constant $K_1(\Omega, D^*)$ such that, whenever

$$\|\nabla u\|_{L^2} + \frac{1}{\kappa}\|\nabla\theta\|_{L^2} \le K_1(\Omega, D^*)\nu, \quad (11)$$

the systems (9) and (10) possess exactly one weak solution. In view of (8), there exists a new constant $K_2(\Omega, D^*) \le K_0(\Omega, D^*)$ such that, if we have

$$K(\nu, \kappa, \varphi, \psi) \le K_2(\Omega, D^*)\frac{\nu^2\kappa}{\nu+\kappa} \quad (12)$$

and u and θ solve (together with some p) the nonlinear system (1), then the existence and uniqueness of weak solution is ensured for (9) and (10).

In the context of the inverse problem (1)–(2), the first property we will analyze is *uniqueness*. Thus, let D^0 and D^1 be two sets in $\mathcal{D}$ and let us consider the systems

$$\begin{cases} -\nu\Delta u^i + (u^i\cdot\nabla)u^i + \nabla p^i = \theta^i g, \quad \nabla\cdot u^i = 0 & \text{in } \Omega\setminus\overline{D^i}, \\ -\kappa\Delta\theta^i + u^i\cdot\nabla\theta^i = 0 & \text{in } \Omega\setminus\overline{D^i}, \\ u^i = \varphi, \quad \theta^i = \psi & \text{on } \partial\Omega, \\ u^i = 0, \quad \theta^i = 0 & \text{on } \partial D^i, \end{cases} \quad (13)$$

for $i = 0$ and $i = 1$. We have the following uniqueness result:

Theorem 2. *Assume that* $(\varphi, \psi) \in H^{1/2}(\partial\Omega)^N \times H^{1/2}(\partial\Omega)$, $(\varphi, \psi) \neq (0, 0)$ *and satisfies* (5) *and* (7)*. Let* D^0 *and* D^1 *be two sets in* $\mathcal{D}$*, let* (u^i, p^i, θ^i) *be the unique solution of* (13) *and let us set* $\alpha^i = \sigma(u^i, p^i)\cdot n$ *and* $\beta^i = \kappa\frac{\partial\theta^i}{\partial n}$ *for* $i = 0, 1$*. Then, if*

$$\alpha^0 = \alpha^1 \quad \text{and} \quad \beta^0 = \beta^1 \quad \text{on} \quad \gamma, \quad (14)$$

one has $D^0 = D^1$.

For the proof of this result, we will an argument already used, for instance, in [6] and [13]. To this end, we need an appropriate unique continuation property, which will be proved in Section 2. We will also be concerned by the way $\sigma(u,p)\cdot n$ and $\kappa\frac{\partial\theta}{\partial n}$ depend on (small) perturbations of D and some related consequences (see Theorem 3 and the remarks after it). In order to represent the deformations of a set $D\in\mathcal{D}$, let us introduce

$$\mathcal{W}=\{\, m\in W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^N)\,:\,\|m\|_{W^{2,\infty}}\le\varepsilon,\quad m=0\ \text{ in }\ \Omega\setminus D^*\,\},$$

where $\varepsilon>0$ is small enough. For each $m\in\mathcal{W}$, we define a new domain $D+m$ by

$$D+m=\{\, z\in\mathbb{R}^N\,:\, z=x+m(x),\, x\in D\,\}.$$

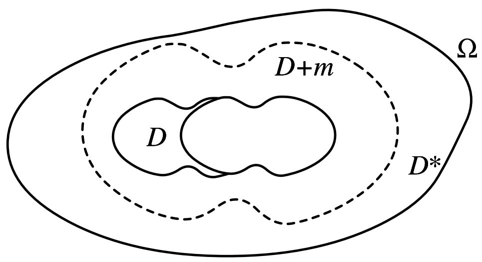

FIGURE 1. Deformations of D

It is then known that, if ε is small enough, for any $D\in\mathcal{D}$ and any $m\in\mathcal{W}$, one has again $D+m\in\mathcal{D}$; see for instance [24].

For each $m\in\mathcal{W}$, let us consider the "perturbed" Boussinesq system

$$\begin{cases} -\nu\Delta v+(v\cdot\nabla)v+\nabla q=\eta\, g, \quad \nabla\cdot v=0 & \text{in } \Omega\setminus\overline{(D+m)},\\ -\kappa\Delta\eta+v\cdot\nabla\eta=0 & \text{in } \Omega\setminus\overline{(D+m)},\\ v=\varphi,\quad \eta=\psi & \text{on } \partial\Omega,\\ v=0,\quad \eta=0 & \text{on } \partial(D+m). \end{cases} \tag{15}$$

In view of Theorem 1 and our assumptions on (φ,ψ), for each $m\in\mathcal{W}$ the Boussinesq system (15) possesses exactly one solution (v,q,η) that belongs to $H^2(\Omega\setminus\overline{(D+m)})^N\times H^1(\Omega\setminus\overline{(D+m)})\times H^2(\Omega\setminus\overline{(D+m)})$ and satisfies $\sigma(v,q)\cdot n\in H^{1/2}(\partial\Omega)^N$ and $\frac{\partial\theta}{\partial n}\in H^{1/2}(\partial\Omega)$. Let us denote by (u,p,θ) the solution of (1), i.e. the solution to (15) for $m=0$.

Our second aim in this paper is to deduce identities of the form,

$$\begin{cases} \sigma(v,q)\cdot n-\sigma(u,p)\cdot n=L_1m+o(m) & \text{on }\ \gamma,\\ \kappa\dfrac{\partial\eta}{\partial n}-\kappa\dfrac{\partial\theta}{\partial n}=L_2m+o(m) & \text{on }\ \gamma, \end{cases}$$

where L_1 and L_2 are linear operators and

$$\frac{o(m)}{\|m\|_{W^{2,\infty}}}\to 0\quad\text{as}\quad \|m\|_{W^{2,\infty}}\to 0. \tag{16}$$

Theorem 3. *Assume that $D\in\mathcal{D}$, $m\in\mathcal{W}$ and $(\varphi,\psi)\in H^{3/2}(\partial\Omega)^N\times H^{3/2}(\partial\Omega)$ satisfies* (5) *and* (12) *and let (v,q,η) and (u,p,θ) be the solutions of* (15) *and* (1)*, respectively. Then,*

(a) *We have*

$$\sigma(v,q)\cdot n - \sigma(u,p)\cdot n = \sigma(u',p')\cdot n + o(m) \quad \text{on} \ \ \gamma, \tag{17}$$

$$\kappa\frac{\partial\eta}{\partial n} - \kappa\frac{\partial\theta}{\partial n} = \kappa\frac{\partial\theta'}{\partial n} + o(m) \quad \text{on} \ \ \gamma, \tag{18}$$

where $o(m)$ *satisfies* (16) *and, for each* $m \in \mathcal{W}$, (u', p', θ') *is the solution of the associated linear problem*

$$\begin{cases} -\nu\Delta u' + (u'\cdot\nabla)u + (u\cdot\nabla)u' + \nabla p' = \theta' g, \quad \nabla\cdot u' = 0 & \text{in } \Omega\setminus\overline{D}, \\ -\kappa\Delta\theta' + u'\cdot\nabla\theta + u\cdot\nabla\theta' = 0 & \text{in } \Omega\setminus\overline{D}, \\ u' + (m\cdot\nabla)u \in H_0^1(\Omega\setminus\overline{D})^N, \\ \theta' + (m\cdot\nabla)\theta \in H_0^1(\Omega\setminus\overline{D}). \end{cases} \tag{19}$$

(b) *Assume that* $\xi \in C^2(\partial\Omega)$, $\operatorname{supp}\xi \subset\subset \gamma$ *and* $\xi \equiv 1$ *on* $\tilde\gamma$, *a relative open set of* $\partial\Omega$ *such that* $\tilde\gamma \subset\subset \gamma$. *Then, for any* $(\bar y, \bar z) \in C^2(\overline{\gamma})^N \times C^2(\overline{\gamma})$ *satisfying*

$$\int_\gamma \bar y\xi\cdot n\, d\Gamma = 0, \tag{20}$$

we have

$$\begin{cases} \displaystyle\int_\gamma (\sigma(v,q)\cdot n - \sigma(u,p)\cdot n)\cdot\bar y\xi\, d\Gamma + \kappa\int_\gamma\left(\frac{\partial\eta}{\partial n} - \frac{\partial\theta}{\partial n}\right)\bar z\xi\, d\Gamma \\ \displaystyle\quad = -\nu\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\cdot\frac{\partial y}{\partial n}\, d\Gamma - \int_{\partial D}(m\cdot n)\frac{\partial\theta}{\partial n}\kappa\frac{\partial z}{\partial n}\, d\Gamma + \ o(m). \end{cases} \tag{21}$$

Here, (y, π, z) *is the solution of the adjoint system*

$$\begin{cases} -\nu\Delta y - (\nabla y)^t u - (u\cdot\nabla)y + \nabla\pi = -z\nabla\theta, \quad \nabla\cdot y = 0 & \text{in } \Omega\setminus\overline{D}, \\ -\kappa\Delta z - u\cdot\nabla z = y\cdot g & \text{in } \Omega\setminus\overline{D}, \\ y = \bar y\xi, \quad z = \bar z\xi & \text{on } \partial\Omega, \\ y = 0, \quad z = 0 & \text{on } \partial D. \end{cases} \tag{22}$$

For the proof, our main tool will be the domain variation techniques introduced by F. Murat and J. Simon in [21] and [22]; see also [24] and [9]. We will present the proof of this result in Section 4.

Remark 2. Notice that, in view of (17)–(19), for each $m \in \mathcal{W}$ we can compute the local derivatives (u', p', θ') and then the differences $\sigma(v,q)\cdot\ n - \sigma(u,p)\cdot n$ and $\kappa\frac{\partial\eta}{\partial n} - \kappa\frac{\partial\theta}{\partial n}$ on γ up to second-order perturbations. On the other hand, we see from (21) that the same quantity can be easily computed using (y, π, z), which is independent of m.

Corollary 1. *Let the assumptions of Theorem 3 be satisfied and assume that* $\partial D \in W^{3,\infty}$ *and the perturbation* m *is of the form* $m = \lambda n + m^\perp$ *on* ∂D, *where* $\lambda \in \mathbb{R}$ *and* $m^\perp\cdot n = 0$. *Then, if* $(\bar y, \bar z)$ *satisfies* (20) *and*

$$K := \int_{\partial D}\left(\nu\frac{\partial u}{\partial n}\cdot\frac{\partial y}{\partial n} + \kappa\frac{\partial\theta}{\partial n}\frac{\partial z}{\partial n}\right) d\Gamma \neq 0,$$

we have:

$$\lambda = -\frac{1}{K}\left(\int_\gamma (\sigma(v,q)\cdot n - \sigma(u,p)\cdot n)\cdot\bar y\xi\, d\Gamma + \kappa\int_\gamma\left(\frac{\partial\eta}{\partial n} - \frac{\partial\theta}{\partial n}\right)\bar z\xi\, d\Gamma\right) + \ o(m).$$

Remark 3. Assume that $\partial\Omega$, φ and ψ are regular enough and we have already computed a first regular approximation $\tilde{D}$ to the solution of our inverse problem. Then, the associated solution $(\tilde{u}, \tilde{p}, \tilde{\theta})$ and consequently

$$\tilde{\alpha}|_\gamma \equiv \sigma(\tilde{u}, \tilde{p}) \cdot n|_\gamma \quad \text{and} \quad \tilde{\beta}|_\gamma = \kappa \frac{\partial \tilde{\theta}}{\partial n}|_\gamma$$

are known. Assume that we intend to compute a new (and possibly better) approximation of the form $\tilde{D} + m$ where $m = \lambda n + m^\perp$ on $\partial\tilde{D}$, $\lambda \in \mathbb{R}$ and $m^\perp \cdot n = 0$. From (21), for each $\bar{y}$ and $\bar{z}$ as in Corollary 1, we can write

$$\int_\gamma \left((\sigma(v,q) \cdot n - \tilde{\alpha}) \cdot \bar{y}\xi + \left(\kappa \frac{\partial \eta}{\partial n} - \tilde{\beta} \right) \bar{z}\xi \right) d\Gamma = -\lambda \tilde{K} + o(\lambda),$$

where

$$\tilde{K} := \int_{\partial\tilde{D}} \left(\nu \frac{\partial \tilde{u}}{\partial n} \cdot \frac{\partial y}{\partial n} + \kappa \frac{\partial \tilde{\theta}}{\partial n} \frac{\partial z}{\partial n} \right) d\Gamma$$

and (y, π, z) is the solution of (22). So, the "good" strategy is to choose λ, if possible, according to the formula:

$$\lambda = -\frac{1}{\tilde{K}} \left(\int_\gamma \left((\alpha - \tilde{\alpha}) \cdot \bar{y}\xi + (\beta - \tilde{\beta})\bar{z}\xi \right) d\Gamma \right).$$

Indeed, this is a way to ensure that, at least at first order, the projections of $\sigma(v,q) \cdot n|_\gamma$ and $\alpha|_\gamma$ in the direction of $\bar{y}$ and the projections of $\kappa \frac{\partial \eta}{\partial n}|_\gamma$ and $\beta|_\gamma$ in the direction of $\bar{z}$ coincide.

Remark 4. More generally, starting from an already computed candidate $\tilde{D}$ to the solution of problem (1)–(2), let us try to determine a better candidate of the form $\tilde{D} + m$, where $m \cdot n|_{\partial\tilde{D}} \in M$ and M is a finite dimensional space. Let $\{f_1, \dots, f_d\}$ be a basis of M. Then we can write

$$m \cdot n|_{\partial\tilde{D}} = \sum_{i=1}^d a_i f_i$$

for some a_i to be determined. Let us introduce d linearly independent functions $(\bar{y}^i, \bar{z}^i) \in C^2(\overline{\gamma})^N \times C^2(\overline{\gamma})$ satisfying (20). Using again (21), we see now that

$$\int_\gamma \left((\sigma(v,q) \cdot n - \tilde{\alpha}) \cdot \bar{y}_j\xi + \left(\kappa \frac{\partial \eta}{\partial n} - \tilde{\beta} \right) \bar{z}_j\xi \right) d\Gamma = -\sum_{i=1}^d \tilde{K}_{ij} a_i + o(m),$$

where

$$\tilde{K}_{ij} := \int_{\partial\tilde{D}} f_i \left(\nu \frac{\partial \tilde{u}}{\partial n} \cdot \frac{\partial y^j}{\partial n} + \kappa \frac{\partial \tilde{\theta}}{\partial n} \frac{\partial z^j}{\partial n} \right) d\Gamma \quad \forall\, i,j = 1, \dots, d$$

and, for each j, (y_j, π_j, z_j) is the solution of (22) corresponding to the data $\bar{y}_j$ and $\bar{z}_j$. Consequently, a strategy to compute the coefficients a_i is to solve (if possible) the system of equations

$$\sum_{i=1}^d \tilde{K}_{ij} a_i = -\langle \alpha - \tilde{\alpha}, \bar{y}_j\, 1_\gamma \rangle - \langle \beta - \tilde{\beta}, \bar{z}_j\, 1_\gamma \rangle, \quad 1 \le j \le d.$$

These ideas are being considered in a work in progress which will appear in the next future.

Our third aim in this paper is the (partial) identification of D. More precisely, we will analyze the following question. Let us assume that $D \in \mathcal{D}$ is known and thus we can solve the direct problem (1) and compute (α, β) from (2). Let us also assume that we know the observation (α^m, β^m) corresponding to a *modified* domain $D + m$, with $m \in \mathcal{W}$. Then we want to know whether we are able to compute $m \cdot n|_{\partial D}$ from D, (α, β) and (α^m, β^m). Our third main result is the following:

Theorem 4. *Assume that $(\varphi, \psi) \in H^{3/2}(\partial\Omega)^N \times H^{3/2}(\partial\Omega)$, $(\varphi, \psi) \not\equiv 0$ and satisfies (5) and (7) and the corresponding solution of (1) satisfies*

$$\left|\frac{\partial u}{\partial n}\right|^2 + \left|\frac{\partial \theta}{\partial n}\right|^2 \neq 0 \quad \text{on} \quad \partial D. \tag{23}$$

Also, assume that $m \in \mathcal{W}$ and $(m \cdot n)|_{\partial D}$ belongs to a finite dimensional space $M \subset W^{1,\infty}(\partial D)$. Then $(m \cdot n)|_{\partial D}$ can be computed explicitly, up to second-order terms, from Ω, D, M, α, β, α^m and β^m.

More precisely, there exists a computable function

$$H_{\Omega,D,M} : H^{1/2}(\gamma)^N \times H^{1/2}(\gamma) \mapsto M$$

such that

$$(m \cdot n)|_{\partial D} = H_{\Omega,D,M}(\alpha^m - \alpha, \beta^m - \beta) + o(m)$$

for all $m \in \mathcal{W}$ with $(m \cdot n)|_{\partial D} \in M$.

Remark 5. The assumption (23) is reasonable. Indeed, in view of the unique continuation results we will prove below (see corollary 2), the set of points of ∂D where (23) is not satisfied has no interior point. On the other hand, (23) is satisfied whenever, for instance, we have $(\varphi, \psi) \in W^{2-1/r,r}(\partial\Omega)^N \times W^{2-1/r,r}(\partial\Omega)$ for some $r > N$, $\psi \geq 0$, $\psi \not\equiv 0$ and the other assumptions on (φ, ψ) are fulfilled. This is a consequence of Hopf's maximum principle applied to the equation satisfied by θ; see for example [17]. In fact, in this case we obtain $\frac{\partial \theta}{\partial n} < 0$ on ∂D, which trivially implies (23).

To our knowledge, it is unknown whether (23) is implied by the other assumptions on (φ, ψ) imposed in Theorem 4.

For the proof of this theorem, we will use (again) domain variation techniques and also some recent results on data assimilation introduced by J.-P. Puel in [23]. For clarity, we will first present the argument in the case of the similar but simpler problem (3)–(4), which involves only the Laplace equation; see Section 5.

The rest of this paper is organized as follows. In Section 2, we will prove a unique continuation property needed for the proof of Theorem 2. Theorems 2, 3 and 4 are respectively proved in sections 3, 4 and 5. Finally, Section 6 deals with the proofs of Theorem 1 as well as other technical results.

2. **A unique continuation property.**

In this section, we will present a unique continuation property which will be used in the proof of Theorem 2. Let $G \subset \mathbb{R}^N$ be a bounded connected open set ($N = 2$ or $N = 3$) whose boundary ∂G is of class $W^{1,\infty}$. In the sequel, C denotes a generic positive constant.

We will prove the following result:

Theorem 5. *Let $\omega \subset G$ be a nonempty open set. Assume that $a \in L^\infty(G)^N$, $b \in L^\infty(G)^N$, $d \in L^\infty(G)$ and $\nabla \cdot a = \nabla \cdot b = 0$ in G. Then any solution $(v, q, \eta) \in H^1(G)^N \times L^2(G) \times H^1(G)$ of the linear system*

$$\begin{cases} -\nu\Delta v + (a\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = \eta g, & \nabla\cdot v = 0 \quad \text{in } G, \\ -\kappa\Delta\eta + a\cdot\nabla\eta + v\cdot\nabla d = 0 & \text{in } G, \end{cases} \tag{24}$$

satisfying

$$v = 0 \quad \text{and} \quad \eta = 0 \quad \text{in} \quad \omega$$

is zero everywhere, i.e. satisfies $v \equiv 0$ in G, $q \equiv$ Const. in G and $\eta \equiv 0$ in G.

The proof of this theorem is based on the ideas and results in [15]. It will be composed of four steps. First, we recall an appropriate local Carleman inequality (see Section 2.1). Then, in Section 2.2, using this Carleman inequality, we will prove the result of Theorem 5 but in a ball and for potentials a and b with sufficiently small L^∞ norms. Next, in Section 2.3, we will show the result in small balls. Finally, in Section 2.4, we will conclude the proof.

2.1. **Carleman inequality.** In [15], the following result can be found:

Proposition 1. *Let $U \subset \mathbb{R}^N$ be an open set, $K \subset U$ a nonempty compact set, $a_{jk} \in C^\infty(\mathbb{R}^N)$ for $1 \le j \le s$, $1 \le k \le N$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Let us set*

$$L_1 f = \sum_{j=1}^{s}\sum_{k=1}^{N} a_{jk}\partial_k f_j \quad \forall f = (f_1, \ldots, f_s) \in L^2(U)^s$$

and

$$a_0(x,\xi) = \sum_{j=1}^{N}\left(\xi_j^2 - (\partial_j\varphi(x))^2\right), \quad b_0(x,\xi) = 2\sum_{j=1}^{N}\xi_j\,\partial_j\varphi(x) \quad \forall (x,\xi) \in U \times \mathbb{R}^N$$

and let us assume that φ satisfies the following property:

$$\begin{cases} \nabla\varphi \text{ does not vanish in } U\text{; furthermore,} \\ \exists C_0 > 0 \text{ such that } \partial_\xi a_0(x,\xi)\cdot\partial_x b_0(x,\xi) - \partial_x a_0(x,\xi)\cdot\partial_\xi b_0(x,\xi) \ge C_0 \\ \text{for all } (x,\xi) \in U \times \mathbb{R}^N \text{ such that } a_0(x,\xi) = b_0(x,\xi) = 0. \end{cases} \tag{25}$$

Then, there exist constants $C_1 > 0$ and $h_1 > 0$ such that, for any couple $(y, F) \in H_0^1(U) \times L^2(U)^s$ satisfying $\mathrm{supp}\,(y) \cup \mathrm{supp}\,(F) \subset K$ *and $\Delta y - L_1 F \in L^2(U)$ and any $h \in (0, h_1)$, one has:*

$$\int_K e^{2\varphi/h}(|y|^2 + h^2|\nabla y|^2)\,dx \le C_1 \int_K e^{2\varphi/h}(h|F|^2 + h^3|\Delta y - L_1 F|^2)\,dx. \tag{26}$$

2.2. **A unique continuation property for small coefficients.** In this paragraph, we will deduce the result in Theorem 5 but for potentials with sufficiently small norm.

Let $B(0;r)$ be an open ball of radius $r > 0$ centered at the origin. We consider system (24) in $B(0;2)$, i.e.

$$\begin{cases} -\nu\Delta v + (a\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = \eta g, & \nabla\cdot v = 0 \quad \text{in } B(0;2), \\ -\kappa\Delta\eta + a\cdot\nabla\eta + v\cdot\nabla d = 0 & \text{in } B(0;2). \end{cases} \tag{27}$$

We have the following result:

Lemma 1. *Assume that* $a \in L^\infty(B(0;2))^N$, $b \in L^\infty(B(0;2))^N$, $d \in L^\infty(B(0;2))$ *and* $\nabla \cdot a = \nabla \cdot b = 0$ *in* $B(0;2)$. *Then there exists* $\epsilon > 0$ *such that, if*

$$\|a\|_\infty \le \epsilon \quad \text{and} \quad \|b\|_\infty \le \epsilon,$$

any solution $(v, q, \eta) \in H^1(B(0;2))^N \times L^2(B(0;2)) \times H^1(B(0;2))$ *of* (27) *satisfying* $v = 0$ *and* $\eta = 0$ *in* $B(0;1)$ *is zero everywhere.*

Proof. Let $(v, q, \eta) \in H^1(B(0;2))^N \times L^2(B(0;2)) \times H^1(B(0;2))$ be a solution of (27) satisfying $v = 0$ and $\eta = 0$ in $B(0;1)$. Since $q \equiv$ Const. in $B(0;1)$, it is not restrictive to assume that it also vanishes in $B(0;1)$.

• Step 1: Let us first see that Proposition 1 can be applied in this context for some appropriate choices of U, K, L_1 and φ. Let us choose $\varepsilon > 0$ and let us set

$$K = \{\, x \in \mathbb{R}^N \ : \ \frac{3}{4} \le |x| \le 2 - \varepsilon \,\} \quad \text{and} \quad U = \{x \in \mathbb{R}^N \ : \ \frac{1}{2} < |x| < 2\}.$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ be such that

$$\varphi(x) = e^{-\delta |x|^2} \quad \forall x \in \overline{B}(0;2), \tag{28}$$

where $\delta > 4$. Notice that

$$\partial_j \varphi(x) = -2\delta \varphi(x) x_j\,, \tag{29}$$

$$\partial_j \partial_k \varphi(x) = -2\delta \varphi(x) \delta_{jk} + 4\delta^2 \varphi(x) x_j x_k\,, \tag{30}$$

where δ_{jk} is the usual Kronecker's symbol.

Assume that $a_0(x,\xi) = b_0(x,\xi) = 0$, i.e.

$$\sum_{j=1}^N \xi_j^2 = \sum_{j=1}^N (\partial_j \varphi(x))^2 \quad \text{and} \quad \sum_{j=1}^N \xi_j\, \partial_j \varphi(x) = 0.$$

Then, in view of (29) and (30) we have

$$\begin{aligned} &\partial_\xi a_0(x,\xi) \cdot \partial_x b_0(x,\xi) - \partial_x a_0(x,\xi) \cdot \partial_\xi b_0(x,\xi) \\ &\qquad = 64\delta^3 \varphi(x)^3 \sum_{j=1}^N x_j^2 \left[\delta \sum_{k=1}^N x_k^2 - 1 \right] = 64\delta^3 \varphi(x)^3 |x|^2 [\delta |x|^2 - 1] \\ &\qquad \ge 16\delta^3 e^{-3\delta/4} \left(\tfrac{\delta}{4} - 1 \right). \end{aligned}$$

Consequently, (25) is satisfied by this function φ in this open set U.

• Step 2: We introduce a function $\zeta \in \mathcal{D}(\overset{\circ}{K})$ such that $\zeta = 1$ in $1 - \varepsilon \le |x| \le 2 - 2\varepsilon$. We put

$$\tilde{v} = \zeta v, \quad \tilde{q} = \zeta q, \quad \tilde{\eta} = \zeta \eta, \tag{31}$$

where $(v, q, \eta) \in H^1(B(0;2))^N \times L^2(B(0;2)) \times H^1(B(0;2))$ is a solution of (27). It is then clear that $(\tilde{v}, \tilde{q}, \tilde{\eta}) \in H_0^1(\overset{\circ}{K})^N \times L^2(\overset{\circ}{K}) \times H_0^1(\overset{\circ}{K})$.

From (27) we can readily see

$$-\nu \Delta \tilde{v} + \nabla \tilde{q} + \nabla \cdot (\tilde{v} b) = \tilde{\eta} g - (a \cdot \nabla)\, \tilde{v} + H_1 \quad \text{in} \quad \overset{\circ}{K}, \tag{32}$$

where $H_1 \in L^2(\overset{\circ}{K})$ is given by

$$H_1 = b(v \cdot \nabla)\zeta - 2\nu \nabla \zeta \cdot \nabla v - \nu v \Delta \zeta + (a \cdot \nabla \zeta) v + q \nabla \zeta. \tag{33}$$

This is true because $\nabla \cdot v = 0$ in U.

Taking the divergence in the first equation of (27), we see that

$$\Delta q = -\nabla \cdot ((a \cdot \nabla) v + (v \cdot \nabla) b) + \nabla \eta \cdot g = -\nabla \cdot ((a \cdot \nabla) v + (\nabla v)\, b) + \nabla \eta \cdot g. \tag{34}$$

Here, we have used that $\partial_i(v_j\partial_j b_i) = \partial_j(\partial_i v_j b_i)$, which is a consequence of the identities $\nabla\cdot v = \nabla\cdot b = 0$. Then, using (34) we deduce that $\tilde q$ satisfies

$$\Delta\tilde q + \nabla\cdot((a\cdot\nabla)\tilde v) + \nabla\cdot((\nabla\tilde v)b) = \nabla\tilde\eta\cdot g + H_2 \quad \text{in } \overset{\circ}{K}, \tag{35}$$

where $H_2 \in L^2(\overset{\circ}{K})$ is given as follows:

$$\begin{aligned} H_2 &= (a\cdot\nabla)v\cdot\nabla\zeta + \nabla\cdot((a\cdot\nabla\zeta)v) + \nabla\zeta\cdot((b\cdot\nabla)v) \\ &+ \nabla\cdot((b\cdot\nabla\zeta)v) - \eta\nabla\zeta\cdot g + 2\nabla\zeta\cdot\nabla q + q\Delta\zeta. \end{aligned} \tag{36}$$

Using now the second equation of (27), we get

$$-\kappa\Delta\tilde\eta + \nabla\cdot(d\tilde v) = -a\cdot\nabla\tilde\eta + H_3\,, \tag{37}$$

where $H_3 \in L^2(\overset{\circ}{K})^N$ is given by

$$H_3 = d\,v\cdot\nabla\zeta - 2\kappa\nabla\zeta\cdot\nabla\eta - \kappa\eta\Delta\zeta + (a\cdot\nabla\zeta)\eta. \tag{38}$$

Here, we have used that $\tilde v\cdot\nabla d = \nabla\cdot(d\,\tilde v) - d(\nabla\cdot\tilde v) = \nabla\cdot(d\,\tilde v) - d\,v\cdot\nabla\zeta$, which is again implied by the fact that $\nabla\cdot v = 0$.

• Step 3: We will now apply Proposition 1 several times.

(a) More precisely, let us first take $s = N+1$,

$$L_1 f = -\frac{1}{\nu}\partial_k f_0 - \frac{1}{\nu}\sum_{j=1}^{N}\partial_j f_j \quad \forall f = (f_0, f_1, \dots, f_N) \in L^2(U)^{N+1},$$

$y = \tilde v_k$ and $F = (\tilde q, \tilde v_1 b_k, \dots, \tilde v_N b_k)$. Thanks to (32), we have $(y, F) \in H_0^1(U)\times L^2(U)$, $\Delta y - L_1 F \in L^2(U)$ and $\operatorname{supp}(y)\cup\operatorname{supp}(F)\subset K$. Therefore, we can apply Proposition 1 and deduce that there exist $C > 0$ and $h_1 > 0$ such that, for any $h \in (0, h_1)$, the following holds:

$$\begin{aligned} &\int_K e^{2\varphi/h}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx \le Ch\int_K e^{2\varphi/h}(|\tilde q|^2 + |b\tilde v|^2)\,dx \\ &\quad + Ch^3\int_K e^{2\varphi/h}|\tilde\eta|^2\,dx + Ch^3\int_K e^{2\varphi/h}|(a\cdot\nabla)\tilde v|^2\,dx + Ch^3\int_K e^{2\varphi/h}|H_1|^2\,dx, \end{aligned}$$

($H_1 \in L^2(\overset{\circ}{K})$ is given by (33)). Then, we also have

$$\left\{ \begin{aligned} &\int_K e^{2\varphi/h}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx \le Ch\int_K e^{2\varphi/h}|\tilde q|^2\,dx + Ch^3\int_K e^{2\varphi/h}|\tilde\eta|^2\,dx \\ &\qquad\qquad + Ch^3\int_K e^{2\varphi/h}|H_1|^2\,dx \end{aligned} \right. \tag{39}$$

for $0 < h < h_2 := \min(h_1, C(\|a\|_\infty^{-2} + \|b\|_\infty^{-2})$. Notice that H_1 is independent of h and has the same support than $\nabla\zeta$. This will be used below.

In the sequel, our goal is to get suitable estimates for the first two terms in the right hand side of (39). We will be able to do this, using again Proposition 1.

(b) At this point, we will apply again Proposition 1. This time, we take $s = N$,

$$L_1 f = -\nabla\cdot f \quad \forall f = (f_1, \dots, f_N) \in L^2(U)^N,$$

$y = \tilde q$ and $F = (a\cdot\nabla)\tilde v + (\nabla\tilde v)b$. In view of (31) and (35), we have $(y, F) \in H_0^1(U)\times L^2(U)^N$, $\Delta y - L_1 F \in L^2(U)$ and $\operatorname{supp}(y)\cup\operatorname{supp}(F)\subset K$. Thus, we deduce from Proposition 1 that there there exist $C > 0$ and $h_3 > 0$ such that

$$\left\{ \begin{array}{l} \displaystyle\int_K e^{2\varphi/h}(|\tilde{q}|^2 + h^2|\nabla\tilde{q}|^2)\,dx \le Ch\int_K e^{2\varphi/h}(|(a\cdot\nabla)\tilde{v}|^2 + |(\nabla\tilde{v})b|^2)\,dx \\ \displaystyle\qquad +Ch^3\int_K e^{2\varphi/h}|\nabla\tilde{\eta}|^2\,dx + Ch^3\int_K e^{2\varphi/h}|H_2|^2\,dx \end{array}\right. \tag{40}$$

for any $h \in (0, h_3)$, where $H_2 \in L^2(\overset{\circ}{K})$ is given by (36). Notice that, again, H_2 has the same support than $\nabla\zeta$.

(c) In order to apply proposition 1 to $\tilde{\eta}$, let us now take $s = N$,

$$L_1 f = -\frac{1}{\kappa}\nabla\cdot f \quad \forall f = (f_1, \dots, f_N) \in L^2(U)^N,$$

$y = \tilde{\eta}$ and $F = d\tilde{v}$. Then, as a consequence of (37), we have again $(y, F) \in H_0^1(U) \times L^2(U)^N$, $\Delta y - L_1 F \in L^2(U)$ and $\mathrm{supp}\,(y, F) \subset K$ and there must exist $C > 0$ and $h_4 > 0$ such that

$$\left\{ \begin{array}{l} \displaystyle\int_K e^{2\varphi/h}(|\tilde{\eta}|^2 + h^2|\nabla\tilde{\eta}|^2)\,dx \le Ch\int_K e^{2\varphi/h}|d\tilde{v}|^2\,dx \\ \displaystyle +Ch^3\int_K e^{2\varphi/h}|a\cdot\nabla\tilde{\eta}|^2\,dx + Ch^3\int_K e^{2\varphi/h}|H_3|^2\,dx \end{array}\right.$$

for any $h \in (0, h_4)$, where $H_3 \in L^2(\overset{\circ}{K})$ is given by (38) (again, H_3 is supported by the same set than $\nabla\zeta$).

From this last inequality we deduce that, for any $h \in (0, h_5)$, where $h_5 = \min(h_4, C\|a\|_\infty^{-2})$, the following holds:

$$\int_K e^{2\varphi/h}(|\tilde{\eta}|^2 + h^2|\nabla\tilde{\eta}|^2)\,dx \le Ch\int_K e^{2\varphi/h}|d\tilde{v}|^2\,dx + Ch^3\int_K e^{2\varphi/h}|H_3|^2\,dx. \tag{41}$$

Now, we use (41) in (40) and we get that there exist positive constants R, C and $h_6 = \min(h_3, h_5)$ such that

$$\left\{ \begin{array}{l} \displaystyle\quad\int_K e^{2\varphi/h}(|\tilde{q}|^2 + h^2|\nabla\tilde{q}|^2)\,dx \le Rh(\|a\|_\infty^2 + \|b\|_\infty^2)\int_K e^{2\varphi/h}|\nabla\tilde{v}|^2\,dx \\ \displaystyle +Ch^2\int_K e^{2\varphi/h}|d\tilde{v}|^2\,dx + Ch^3\int_K e^{2\varphi/h}|H_2|^2\,dx + Ch^4\int_K e^{2\varphi/h}|H_3|^2\,dx \end{array}\right. \tag{42}$$

for any $h \in (0, h_6)$, where H_2 and H_3 are respectively given by (36) and (38).

(d) Replacing (41) and (42) in the right hand side of (39), we obtain that

$$\begin{aligned} \int_K e^{2\varphi/h}(|\tilde{v}|^2 + h^2|\nabla\tilde{v}|^2)\,dx &\le CRh^2(\|a\|_\infty^2 + \|b\|_\infty^2)\int_K e^{2\varphi/h}|\nabla\tilde{v}|^2\,dx \\ &\quad +C(h^3 + h^4)\int_K e^{2\varphi/h}|d\tilde{v}|^2\,dx \\ &\quad +C\int_K e^{2\varphi/h}\left(h^4|H_2|^2 + (h^5 + h^6)|H_3|^2 + h^3|H_1|^2\right)\,dx \end{aligned}$$

for any $h \in (0, h_7)$, where $h_7 = \min(h_2, h_6)$. Then, for some $R_0 > 0$ we also have

$$\left\{ \begin{array}{l} \displaystyle\int_K e^{2\varphi/h}(|\tilde{v}|^2 + h^2|\nabla\tilde{v}|^2)\,dx \le R_0h^2(\|a\|_\infty^2 + \|b\|_\infty^2)\int_K e^{2\varphi/h}|\nabla\tilde{v}|^2\,dx \\ \displaystyle\qquad + \int_K e^{2\varphi/h}(h^3|H_1|^2 + h^4|H_2|^2 + h^5|H_3|^2)\,dx \end{array}\right. \tag{43}$$

for any $h \in (0, h_8)$, where H_1, H_2 and H_3 are respectively given by (33), (36) and (38). Notice that h_8 can be chosen as follows:

$$h_8 = C \min(1, \|a\|_\infty^{-2}, \|b\|_\infty^{-2}, \|d\|_\infty^{-2/3}).$$

Let us assume that

$$\|a\|_\infty \le \epsilon \quad \text{and} \quad \|b\|_\infty \le \epsilon \quad \text{where} \quad \epsilon := \frac{1}{2\sqrt{R_0}}\,.$$

Then, we deduce from (43) that

$$\int_K e^{2\varphi/h}(|\tilde v|^2 + h^2|\nabla \tilde v|^2)\,dx \le C\int_K e^{2\varphi/h}(h^3|H_1|^2 + h^4|H_2|^2 + h^5|H_3|^2)\,dx \tag{44}$$

for any $h \in (0, h_8)$. Coming back to (41), we also find that

$$\int_K e^{2\varphi/h}(|\tilde\eta|^2 + h^2|\nabla\tilde\eta|^2)\,dx \le C\|d\|_\infty^2\int_K e^{2\varphi/h}(h^4|H_1|^2 + h^5|H_2|^2 + h^3|H_3|^2)\,dx \tag{45}$$

for any $h \in (0, h_8)$. On the other hand, from (40) and (44), the following holds:

$$\left\{\begin{array}{c} \displaystyle\int_K e^{2\varphi/h}(|\tilde q|^2 + h^2|\nabla\tilde q|^2)\,dx \\ \displaystyle\le C(1+\|d\|_\infty^2)\int_K e^{2\varphi/h}(h^2|H_1|^2 + h^3|H_2|^2 + h^4|H_3|^2)\,dx \end{array}\right. \tag{46}$$

for all $h \in (0, h_8)$.

• Step 4: In order to achieve the proof, we will now argue as in [15]. Recall that H_1, H_2 and H_3 are respectively given by (33), (36) and (38) and have the same support than $\nabla\zeta$. Therefore, H_1, H_2 and H_3 vanish outside the ring $2-2\varepsilon \le |x| \le 2$.

We see from (28) that φ is a radially decreasing positive function in U, then

$$\begin{array}{l} \displaystyle\int_K e^{2\varphi/h}(h^3|H_1|^2 + h^4|H_2|^2 + h^5|H_3|^2)\,dx \\ \displaystyle\qquad \le e^{\frac{2\varphi(2-2\varepsilon)}{h}}\int_K (h^3|H_1|^2 + h^4|H_2|^2 + h^5|H_3|^2)\,dx. \end{array} \tag{47}$$

On the other hand, we also have

$$\begin{array}{rcl} \displaystyle\int_K e^{2\varphi/h}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx & \ge & \displaystyle\int_{1\le|x|\le 2-3\varepsilon} e^{2\varphi/h}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx \\ & \ge & \displaystyle e^{\frac{2\varphi(2-3\varepsilon)}{h}}\int_{1\le|x|\le 2-3\varepsilon}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx \end{array} \tag{48}$$

Combining (44), (47) and (48), the following is found:

$$\int_{1\le|x|\le 2-3\varepsilon}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx \le Ch^3 e^{\frac{2}{h}(\varphi(2-2\varepsilon)-\varphi(2-3\varepsilon))}\int_U |H_4|^2\,dx, \tag{49}$$

where $H_4 \in L^2(\overset{\circ}{K})$ is independent of h. Using that $\varphi(2-3\varepsilon) - \varphi(2-2\varepsilon) > 0$, and passing to the limit in (49) as $h \to 0$, we get

$$\tilde v = 0 \quad \text{in} \quad 1 \le |x| \le 2 - 3\varepsilon.$$

As (31) shows, we have $\tilde v = \zeta v$ and, since $\zeta = 1$ in $1 \le |x| \le 2 - 3\varepsilon$, we finally deduce that $v = 0$ in $B(0; 2-3\varepsilon)$.

In a similar way, starting from (45) and (46), it can be proved that $q = 0$ and $\eta = 0$ in $B(0; 2-3\varepsilon)$. Since $\varepsilon > 0$ is arbitrarily small, we finally deduce that v, q and η vanish identically. This ends the proof of Lemma 1. □

2.3. **A unique continuation property in small balls.** In this paragraph, we will prove a result like Lemma 1 for not necessarily small coefficients but in a small ball. More precisely, we have:

Lemma 2. *Let G be an open connected such that $x_0 \in G$. Assume that $a \in L^\infty(G)^N$, $b \in L^\infty(G)^N$, $d \in L^\infty(G)$ and $\nabla \cdot a = \nabla \cdot b = 0$ in G. There exists $r_0 > 0$ such that, if $0 < r < r_0$, any solution $(v, q, \eta) \in H^1(G)^N \times L^2(G) \times H^1(G)$ of (24) satisfying $v = 0$ and $\eta = 0$ in $B(x_0; r)$ vanishes in $B(x_0; 2r)$. Furthermore, r_0 can be chosen as follows:*

$$r_0 = \min\left(\frac{\epsilon}{\|a\|_\infty}, \frac{\epsilon}{\|b\|_\infty}, \frac{\rho}{2}\right), \tag{50}$$

where ϵ is the constant furnished by Lemma 1 and ρ is such that $\overline{B}(x_0; \rho) \subset G$.

Proof. Let us assume that $(v, q, \eta) \in H^1(G)^N \times L^2(G) \times H^1(G)$ solves (24) and

$$v = 0 \text{ and } \eta = 0 \text{ in } B(x_0; r). \tag{51}$$

For any $x \in G$, we set $x' = \lambda(x - x_0)$ and $G' = \lambda(G - x_0)$, where $\lambda > 0$ will be fixed later on. Let us introduce the following notation:

$$v'(x') = v(x), \quad q'(x') = \frac{1}{\lambda} q(x), \quad \eta'(x') = \frac{1}{\lambda^2} \eta(x),$$
$$a'(x') = a(x), \quad b'(x') = b(x), \quad d'(x') = d(x).$$

Using (24) and (51), we have:

$$\begin{cases} -\nu\Delta' v' + (\lambda^{-1} a' \cdot \nabla')v' + (v' \cdot \nabla')(\lambda^{-1} b') + \nabla' q' = \eta' g, \quad \nabla' \cdot v' = 0 & \text{in } G', \\ -\kappa\Delta'\eta' + \lambda^{-1} a' \cdot \nabla'\eta' + v' \cdot \nabla'(\lambda^{-3} d') = 0 & \text{in } G'. \end{cases}$$

and $v' = 0$ and $\eta' = 0$ in $B(0; \lambda r)$. Now, if

$$\|\frac{a'}{\lambda}\|_\infty \le \epsilon, \quad \|\frac{b'}{\lambda}\|_\infty \le \epsilon \text{ and } \lambda r \ge 1, \tag{52}$$

we can apply Lemma 1 to (v', q', η') and deduce that

$$v' = 0 \text{ and } \eta' = 0 \text{ in } B(0; 2). \tag{53}$$

Let us take r_0 as in (50), where ρ is such that $\overline{B}(x_0; \rho) \subset G$ and let us assume that $0 < r < r_0$. Let us take $\lambda = 1/r$. Since $\lambda > \frac{1}{r_0} \ge \frac{\|a\|_\infty}{\epsilon}$, $\lambda > \frac{1}{r_0} \ge \frac{\|b\|_\infty}{\epsilon}$ and $\lambda r = 1$, we have (52), (53) and then $v \equiv 0$, $\eta \equiv 0$ in $B(x_0; 2r)$ and $q \equiv$ Const. in $B(x_0; 2r)$. This proves the lemma. □

2.4. **Proof of Theorem 5.** In this section we will achieve the proof of Theorem 5. Let (v, q, η) be a solution of (24) satisfying $v = 0$ in ω and $\eta = 0$ in ω. Let us assume that $\overline{B}(x_0; \rho_0) \subset \omega$, and let x_1 be another point in G. There exists $\widetilde{\gamma} \in C^\infty([0, 1])$ with $\widetilde{\gamma}(0) = x_0$, $\widetilde{\gamma}(1) = x_1$ and such that $\widetilde{\gamma}(t) \in G$ for all $t \in [0, 1]$. Let $\bar{U} \subset\subset G$ be a bounded open neighborhood of $\widetilde{\gamma}([0, 1])$. There exists $\rho_1 \in (0, \rho_0]$ such that $\overline{B}(x; \rho_1) \subset U$ for all $x \in \widetilde{\gamma}([0, 1])$. Let us set

$$r_0 = \min\left(\frac{\epsilon}{\|a\|_\infty}, \frac{\epsilon}{\|b\|_\infty}, \frac{\rho_1}{2}\right).$$

In view of Lemma 2, for $r \in (0, r_0)$ and any $x \in \widetilde{\gamma}([0, 1])$, the equalities $v = 0$ and $\eta = 0$ in $B(x; r)$ imply $v = 0$ and $\eta = 0$ in $B(x; 2r)$.

We fix now r with $0 < r < r_0$. It is then clear that

$$\sup\{\, t \in [0, 1] : u = 0 \text{ in } B(\gamma(\tau); r) \ \ \forall \tau \le t \,\} = 1.$$

Hence, $v = 0$ and $\eta = 0$ in $B(x_1; r)$. This ends the proof.

As a consequence, we obtain the following result:

Corollary 2. *Let* $\Gamma \subset \partial G$ *be a nonempty open set. Assume that* $a \in L^\infty(G)^N$, $b \in L^\infty(G)^N$, $d \in L^\infty(G)$*and* $\nabla \cdot a = \nabla \cdot b = 0$ *in* G*. Then any solution* $(v, q, \eta) \in H^1(G)^N \times L^2(G) \times H^1(G)$ *of* (24) *that satisfies*

$$v = 0 \quad and \quad \eta = 0 \quad on \quad \Gamma,$$

$$\sigma(v, q) \cdot n = 0 \quad and \quad \kappa \frac{\partial \eta}{\partial n} = 0 \quad on \quad \Gamma$$

is zero everywhere.

Proof. Let us fix a point $x_0 \in \Gamma$ and a number $r > 0$ such that

$$\overline{B}(x_0; r) \cap \partial G \subset \Gamma.$$

Let us set

$$G' = G \cup B(x_0; r).$$

Then we can define the triplet $(\tilde{v}, \tilde{q}, \tilde{\eta}) \in H^1(G') \times L^2(G') \times H^1(G')$ by extending by zero (v, q, η) to the whole set G', i.e. by setting

$$(\tilde{v}, \tilde{q}, \tilde{\eta})(x) = \begin{cases} (v, q, \eta), & \text{in } G, \\ (0, 0, 0), & \text{in } B(x_0; r) \cap G^c. \end{cases}$$

In this way, we obtain a solution $(\tilde{v}, \tilde{q}, \tilde{\eta})$ of (24) in G' which vanishes in $B(x_0; r) \cap G^c \subset G'$. By applying Theorem 5, we deduce that $\tilde{v} = 0$ in G', $\tilde{\eta} = 0$ in G' and $\tilde{q} \equiv$ Const. in G'. In particular, v and η vanish in G. This ends the proof. □

3. **Proof of Theorem 2.** Let D^0 and D^1 be two different open sets in $\mathcal{D}$, let (u^i, p^i) be the solution of the system

$$\begin{cases} -\nu \Delta u^i + (u^i \cdot \nabla) u^i + \nabla p^i = \theta^i g, \quad \nabla \cdot u^i = 0 & \text{in } \Omega \setminus \overline{D^i}, \\ -\kappa \Delta \theta^i + u^i \cdot \nabla \theta^i = 0 & \text{in } \Omega \setminus \overline{D^i}, \\ u^i = \varphi, \quad \theta^i = \psi & \text{on } \partial\Omega, \\ u^i = 0, \quad \theta^i = 0 & \text{on } \partial D^i \end{cases} \tag{54}$$

and let us set $\alpha^i = \sigma(u^i, p^i) \cdot n$, $\beta^i = \kappa \frac{\partial \theta^i}{\partial n}$ for $i = 0, 1$.

Assume that (14) holds. Let us consider the open sets $D^0 \cup D^1$ and $\mathcal{O}^0 = \Omega \setminus \overline{D^0 \cup D^1}$. Let $\mathcal{O}$ be the unique connected component of $\mathcal{O}^0$ such that $\partial\mathcal{O} = \partial\Omega$ (recall that D^0 and D^1 are subset of D^*) and let us introduce

$$w = u^0 - u^1, \quad \chi = \theta^0 - \theta^1 \quad \text{and} \quad \pi = p^0 - p^1 \quad \text{in} \quad \mathcal{O}.$$

Then $(w, \pi, \chi) \in H^1(\mathcal{O})^N \times L^2(\mathcal{O}) \times H^1(\mathcal{O})$ and verifies

$$\begin{cases} -\nu \Delta w + (u^0 \cdot \nabla) w + (w \cdot \nabla) u^1 + \nabla \pi = \chi g, \quad \nabla \cdot v = 0 & \text{in } \mathcal{O}, \\ -\kappa \Delta \chi + u^0 \cdot \nabla \chi + w \cdot \nabla \theta^1 = 0 & \text{in } \mathcal{O}, \\ w = 0, \quad \chi = 0 & \text{on } \partial\Omega, \\ \sigma(w, \pi) \cdot n = 0, \quad \kappa \dfrac{\partial \chi}{\partial n} = 0 & \text{on } \gamma. \end{cases}$$

We now apply the unique continuation result in Corollary 2 (observe that, as a consequence of the regularity hypotheses on φ and ψ, $u^0, u^1 \in L^\infty(\Omega)^N$ and $\nabla \cdot u^0 = \nabla \cdot u^1 \equiv 0$ in Ω) and we deduce that $w = 0$ and $\chi = 0$ in $\mathcal{O}$, that is to say,

$$u^0 = u^1 \text{ in } \mathcal{O} \quad \text{and} \quad \theta^0 = \theta^1 \text{ in } \mathcal{O}. \tag{55}$$

For instance, let us assume that $D^1 \setminus \overline{D^0}$ is nonempty and let us introduce the open set $D^2 = D^1 \cup ((\Omega \setminus \overline{D^0}) \cap (\Omega \setminus \overline{\mathcal{O}}))$. By hypothesis, $D^2 \setminus \overline{D}^0$ is nonempty. Moreover,

$\partial(D^2 \setminus \overline{D^0}) = \Gamma^0 \cup \Gamma^1$, where $\Gamma^0 = \partial(D^2 \setminus \overline{D^0}) \cap \partial D^0$ and $\Gamma^1 = \partial(D^2 \setminus \overline{D^0}) \cap \partial D^1$ (see Figure 2).

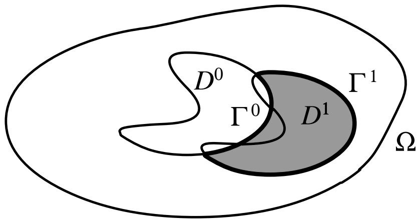

FIGURE 2. The dash set is $D^2 \setminus \overline{D^1}$

In view of (54) and (55), (u^0, p^0, θ^0) satisfies

$$\begin{cases} -\nu\Delta u^0 + (u^0 \cdot \nabla)u^0 + \nabla p^0 = \theta^0 g, \quad \nabla \cdot u^0 = 0 & \text{in } D^2 \setminus \overline{D^0}, \\ -\kappa\Delta\theta^0 + u^0 \cdot \nabla\theta^0 = 0 & \text{in } D^2 \setminus \overline{D^0}, \\ u^0 = u^1 = 0, \quad \theta^0 = \theta^1 = 0 & \text{on } \Gamma^1, \\ u^0 = 0, \quad \theta^0 = 0 & \text{on } \Gamma^0. \end{cases}$$

Of course, the uniqueness of the null solution implies $u^0 = 0$ in $D^2 \setminus \overline{D^0}$ and $\theta^0 = 0$ in $D^2 \setminus \overline{D^0}$. Consequently, from Theorem 5 we deduce that $u^0 \equiv 0$ in $\Omega \setminus \overline{D^0}$ and $\theta^0 \equiv 0$ in $\Omega \setminus \overline{D^0}$, which is impossible because $u^0 = \varphi$ on $\partial\Omega$ and $\theta^0 = \psi$ on $\partial\Omega$ where (φ, ψ) is not identically zero. This implies that $D^1 \setminus \overline{D^0}$ is the empty set.

We can prove in the same way that the set $D^0 \setminus \overline{D^1}$ is empty. Therefore, $D^0 = D^1$. This completes the proof.

4. **Proof of Theorem 3.** In order to prove the equality (17), we will apply the domain variation techniques introduced in [21, 22] and [24] and particularized in [9] to Navier-Stokes systems. Notice that the main difficulty to see that the mapping $m \mapsto (v, q, \eta)$ (where (v, q, η) the solution of (15)) is *differentiable* relies on the fact that (v, q, η) is a function defined for $x \in \Omega \setminus (\overline{D + m})$, a domain that depends on m.

The right way to proceed is as follows:

• First, we introduce a suitable change of variables, we rewrite the equations satisfied by (v, q, η) in a fixed domain $\Omega \setminus \overline{D}$ and we prove the existence of the derivative of the transported variable $(v, q, \eta) \circ (\mathrm{Id.} + m)$. This leads to the definition of the *total derivative* of (v, q, η) in the direction m:

$$(\dot{u}, \dot{p}, \dot{\theta})(m) := \lim_{t \to 0} \frac{(v^t, q^t, \eta^t) \circ (\mathrm{Id.} + tm) - (u, p, \theta)}{t},$$

where (v^t, q^t, η^t) denote the solution of (15) with m replaced by tm.

• Then, we prove the existence of the *local derivative* (u', p', θ') in the direction m, which is defined as follows: For any open set $\omega \subset\subset \Omega \setminus \overline{D}$, we put

$$(u', p', \theta')(m)|_\omega := \lim_{t \to 0} \frac{(v^t, q^t, \eta^t)|_\omega - (u, p, \theta)|_\omega}{t}.$$

Notice that this defines $(u', p', \theta')(m)$ in each open set $\omega \subset\subset \Omega \setminus \overline{D}$ and, consequently, in the whole domain $\Omega \setminus \overline{D}$.

Following the arguments of [9], we can prove the following result:

Lemma 3. *Assume* $(\varphi,\psi)\in H^{3/2}(\partial\Omega)^N\times H^{3/2}(\partial\Omega)$ *satisfies* (5) *and* (12). *Then*
(a) *The mapping* $m\mapsto(v,q,\eta)\circ(\mathrm{Id.}+m)$, *which is defined in* $\mathcal{W}$ *and takes values in* $H^1(\Omega\setminus\overline{D})^N\times L^2(\Omega\setminus\overline{D})\times H^1(\Omega\setminus\overline{D})$, *is differentiable at* 0, *with (total) derivative denoted by* $(\dot u,\dot p,\dot\theta)(m)$. *That is to say, there exists a linear continuous mapping* $m\mapsto(\dot u(m),\dot p(m),\dot\theta(m))$ *such that*

$$(v,q,\eta)\circ(\mathrm{Id.}+m)-(u,p,\theta)=(\dot u,\dot p,\dot\theta)(m)+o(m),\tag{56}$$

where $o(m)$ *satisfies* (16).
(b) *For each* $\omega\subset\subset\Omega\setminus\overline{D}$, *the mapping* $m\mapsto(v,q,\eta)|_\omega$, *which is defined in* $\mathcal{W}$ *and takes values in* $H^1(\omega)^N\times L^2(\omega)\times H^1(\omega)$, *is differentiable at* 0. *In other words,* $m\mapsto(v,q,\eta)$ *is locally differentiable. The local derivative at* 0 *in the direction* m *is denoted by* $(u',p',\theta')(m)$.
(c) *Furthermore,* $(u',p',\theta')(m)$ *is the unique solution of the linear system* (19) *and*

$$(\dot u,\dot p,\dot\theta)(m)=(u',p',\theta')(m)+(m\cdot\nabla)(u,p,\theta).\tag{57}$$

In view of (56) and (57), taking into account that $m=0$ in a neighborhood of $\partial\Omega$, we find that

$$\begin{aligned}&\sigma(v,q)\cdot n-\sigma(u,p)\cdot n=\sigma(u',p')\cdot n+o(m) &&\text{on } \gamma,\\ &\kappa\frac{\partial\eta}{\partial n}-\kappa\frac{\partial\theta}{\partial n}=\kappa\frac{\partial\theta'}{\partial n}+o(m) &&\text{on } \gamma.\end{aligned}$$

This proves (17) and (18).

Since $(\varphi,\psi)\in H^{3/2}(\Omega)^N\times H^{3/2}(\Omega)$, the solution (u,p,θ) of (1) satisfies $(u,p,\theta)\in H^2(\Omega\setminus\overline{D})^N\times H^1(\Omega\setminus\overline{D})\times H^2(\Omega\setminus\overline{D})$, so we have

$$\begin{aligned}&u'=0\ \text{ on }\ \partial\Omega\ \text{ and }\ u'=-(m\cdot n)\frac{\partial u}{\partial n}\ \text{ on }\ \partial D,\\ &\theta'=0\ \text{ on }\ \partial\Omega\ \text{ and }\ \theta'=-(m\cdot n)\frac{\partial \theta}{\partial n}\ \text{ on }\ \partial D.\end{aligned}$$

Let $\bar y\in C^2(\overline{\gamma})^N$ satisfy (20), with $\xi\in C^2(\partial\Omega)$, $\operatorname{supp}\xi\subset\subset\gamma$ and $\xi\equiv1$ on $\tilde\gamma$, a relative open set of $\partial\Omega$ such that $\tilde\gamma\subset\subset\gamma$. Let $\bar z\in C^2(\overline{\gamma})$ be given and let (y,π,z) be the associated solution of (22). We will justify that (21) holds. Multiplying the first equation of (19) by y and integrating by parts, we get

$$\left\{\begin{aligned}&\int_{\Omega\setminus\overline{D}}\sigma(u',p')\nabla y\,dx+\int_{\Omega\setminus\overline{D}}((u\cdot\nabla)u'+(u'\cdot\nabla)u)\cdot y\,dx\\ &\qquad=\int_{\partial\Omega}(\sigma(u',p')\cdot n)\cdot\bar y\xi\,d\Gamma+\int_{\Omega\setminus\overline{D}}\theta' g\cdot y\,dx.\end{aligned}\right.\tag{58}$$

Here, the first term of the left hand side can also be written in the form

$$\begin{aligned}\int_{\Omega\setminus\overline{D}}\sigma(u',p')\nabla y\,dx&=-\int_{\Omega\setminus\overline{D}}\nu\Delta y\cdot u'\,dx+\int_{\partial D}(\sigma(y,\pi)\cdot n)\cdot u'\,d\Gamma\\ &=-\int_{\Omega\setminus\overline{D}}\nu\Delta y\cdot u'\,dx-\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\cdot(\sigma(y,\pi)\cdot n)\,d\Gamma.\end{aligned}$$

The second term of the left hand side of (58) satisfies

$$\begin{aligned}&\int_{\Omega\setminus\overline{D}}((u\cdot\nabla)u'+(u'\cdot\nabla)u)\cdot y\,dx\equiv\int_{\Omega\setminus\overline{D}}\left(u_i\partial_iu'_jy_j+u'_i\partial_iu_jy_j\right)\,dx\\ &=\int_{\Omega\setminus\overline{D}}\left(-u_i\partial_iy_ju'_j+u'_i\partial_iu_jy_j\right)\,dx=\int_{\Omega\setminus\overline{D}}\left(-(u\cdot\nabla)y+(\nabla u)^ty\right)\cdot u'\,dx.\end{aligned}$$

Therefore, we obtain from (58) that

$$\begin{aligned}&\int_{\partial\Omega}(\sigma(u',p')\cdot n)\cdot\bar{y}\xi\,d\Gamma+\int_{\Omega\setminus\overline{D}}\theta' g\cdot y\,dx\\ &=-\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\cdot(\sigma(y,\pi)\cdot n)\,d\Gamma-\int_{\Omega\setminus\overline{D}}z\nabla\theta\cdot u'\,dx.\end{aligned}$$

On the boundary ∂D, since u and y vanish, we have $\frac{\partial u}{\partial n}\cdot n=\nabla\cdot u=0$ on ∂D and

$$\sigma(y,\pi)\cdot n=2\nu e(u)\cdot n-pn=\nu\frac{\partial y}{\partial n}+\nu(\nabla\cdot y)n-pn\quad\text{on }\partial D.$$

Consequently,

$$\frac{\partial u}{\partial n}\cdot(\sigma(y,\pi)\cdot n)=\nu\frac{\partial u}{\partial n}\cdot\frac{\partial y}{\partial n}\quad\text{on }\ \partial D$$

and

$$\left\{\begin{aligned}&\int_{\partial\Omega}(\sigma(u',p')\cdot n)\cdot\bar{y}\xi\,d\Gamma+\int_{\Omega\setminus\overline{D}}\theta' g\cdot y\,dx\\ &=-\nu\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\cdot\frac{\partial y}{\partial n}\,d\Gamma-\int_{\Omega\setminus\overline{D}}z\nabla\theta\cdot u'\,dx.\end{aligned}\right.\tag{59}$$

On the other hand, multiplying the third equation in (19) by z and integrating by parts, we have

$$\int_{\partial\Omega}\frac{\partial\theta'}{\partial n}\bar{z}\xi\,d\Gamma=\int_{\Omega\setminus\overline{D}}\kappa\nabla\theta'\cdot\nabla z\,dx+\int_{\Omega\setminus\overline{D}}(u\cdot\nabla\theta'+u'\cdot\nabla\theta)\,z\,dx.\tag{60}$$

The first term in the right hand side of this equality is as follows:

$$\begin{aligned}\int_{\Omega\setminus\overline{D}}\kappa\nabla\theta'\cdot\nabla z\,dx&=-\int_{\Omega\setminus\overline{D}}\kappa\theta'\Delta z\,dx+\int_{\partial D}\kappa\theta'\frac{\partial z}{\partial n}\,d\Gamma\\ &=-\int_{\Omega\setminus\overline{D}}\kappa\theta'\Delta z\,dx-\int_{\partial D}(m\cdot n)\frac{\partial\theta}{\partial n}\,\kappa\frac{\partial z}{\partial n}\,d\Gamma.\end{aligned}\tag{61}$$

The second term in the right hand side of (60) reads

$$\begin{aligned}\int_{\Omega\setminus\overline{D}}(u_i\partial_i\theta'+u_i'\,\partial_i\theta)\,z\,dx&=-\int_{\Omega\setminus\overline{D}}u_i\partial_i z\,\theta'\,dx+\int_{\Omega\setminus\overline{D}}u_i'\,\partial_i\theta z\,dx\\ &\equiv-\int_{\Omega\setminus\overline{D}}(u\cdot\nabla z)\,\theta'+\int_{\Omega\setminus\overline{D}}z\,\nabla\theta\cdot u'\,dx.\end{aligned}\tag{62}$$

From (60), (61) and (62), we deduce that

$$\begin{aligned}&\int_{\partial\Omega}\frac{\partial\theta'}{\partial n}\bar{z}\xi\,d\Gamma-\int_{\Omega\setminus\overline{D}}\theta' g\cdot y\,dx\\ &\qquad=-\int_{\partial D}(m\cdot n)\frac{\partial\theta}{\partial n}\,\kappa\frac{\partial z}{\partial n}\,d\Gamma+\int_{\Omega\setminus\overline{D}}z\nabla\theta\cdot u'\,dx.\end{aligned}\tag{63}$$

Finally, adding (59) and (63) we obtain that

$$\left\{\begin{aligned}&\int_{\gamma}(\sigma(u',p')\cdot n)\cdot\bar{y}\xi\,d\Gamma+\int_{\gamma}\frac{\partial\theta'}{\partial n}\bar{z}\xi\,d\Gamma\\ &\qquad=-\nu\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\cdot\frac{\partial y}{\partial n}\,d\Gamma-\int_{\partial D}(m\cdot n)\frac{\partial\theta}{\partial n}\,\kappa\frac{\partial z}{\partial n}\,d\Gamma.\end{aligned}\right.\tag{64}$$

Now, using (17) and (18) in (64) we get (21). This ends the proof of Theorem 3.

5. **Proof of Theorem 4.** To clarify the situation, we start presenting a sketch of the proof of Theorem 4 in the much more simple case of the Laplace equation.

5.1. **A simple case: the Laplace equation.** Given φ in an appropriate space, $D \in \mathcal{D}$ and $m \in \mathcal{W}$, we consider the following problems:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{D}, \\ u = \varphi & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial D \end{cases} \tag{65}$$

and

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \setminus \overline{D+m}, \\ v = \varphi & \text{on } \partial\Omega, \\ v = 0 & \text{on } \partial(D+m). \end{cases} \tag{66}$$

We now set $\alpha = \frac{\partial u}{\partial n}|_\gamma$ and $\alpha^m = \frac{\partial v}{\partial n}|_\gamma$. Then we have the following result:

Theorem 6. *Assume that $\varphi \in H^{3/2}(\partial\Omega)$ does not vanish identically. Also, assume that $m \in \mathcal{W}$ and $(m \cdot n)|_{\partial D}$ belongs to a finite dimensional space $M \subset W^{1,\infty}(\partial D)$. Then $(m \cdot n)|_{\partial D}$ can be computed explicitly, up to second-order terms, from Ω, D, M, α and α^m. More precisely, there exists a computable function $G_{\Omega,D,M}$ such that*

$$(m \cdot n)|_{\partial D} = G_{\Omega,D,M}(\alpha^m - \alpha) + o(m) \tag{67}$$

for all small m with $(m \cdot n)|_{\partial D} \in M$.

Sketch of the proof: We will proceed in three steps:

• Step 1: *Domain variations techniques.* Using domain variation techniques, we can write that

$$\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} = \frac{\partial u'}{\partial n} + o(m) \quad \text{on} \quad \gamma,$$

where $o(m)$ satisfies (16) and u' is the solution of

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega \setminus \overline{D}, \\ u' = 0 & \text{on } \partial\Omega, \\ u' = -(m \cdot n)\dfrac{\partial u}{\partial n} & \text{on } \partial D. \end{cases} \tag{68}$$

Therefore, we have

$$\frac{\partial u'}{\partial n} = \alpha^m - \alpha + o(m) \quad \text{on} \quad \gamma$$

and the proof is reduced to compute $(m \cdot n)|_{\partial D}$ from $\frac{\partial u'}{\partial n}|_\gamma$ up to second-order perturbations.

• Step 2: *A (non standard) data assimilation approach.* At this point, our approach is inspired by the techniques introduced by J.-P. Puel in [23]. Thus, assume that $m \in \mathcal{W}$ and $(m \cdot n)|_{\partial D} \in M$, where $M \subset W^{1,\infty}(\partial D)$ is a finite dimensional space. Then $\frac{\partial u'}{\partial n}|_{\partial D}$ belongs to a suitable finite dimensional space $E \subset H^{-1/2}(\partial D)$.

Notice that $\frac{\partial u}{\partial n} \in C^0(\partial D)$. For simplicity, let us assume that $\frac{\partial u}{\partial n} \neq 0$ on ∂D. Then, to determine $(m \cdot n)|_{\partial D}$, it suffices to compute the integrals

$$\int_{\partial D} (m \cdot n)\frac{\partial u}{\partial n}\, h\, d\Gamma, \quad h \in M.$$

Let $P_E : L^2(\partial D) \mapsto E$ be the usual orthogonal projector and let us assume that, for each $h \in M$, we can solve the following control problem: Find $w \in L^2(\gamma)$ such that the solution θ_h of

$$\begin{cases} -\Delta \theta_h = 0 & \text{in } \Omega \setminus \overline{D}, \\ \theta_h = w 1_\gamma & \text{on } \partial\Omega, \\ \dfrac{\partial \theta_h}{\partial n} = h & \text{on } \partial D \end{cases} \tag{69}$$

satisfies

$$P_E(\theta_h|_{\partial D}) = 0. \tag{70}$$

Then, using (68) and making some integrations by parts, we find:

$$\begin{aligned} -\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\,h\,d\Gamma &= \int_{\partial D\cup\partial\Omega} u'\,\frac{\partial \theta_h}{\partial n}\,d\Gamma \\ &= \int_{\partial D}\frac{\partial u'}{\partial n}\,P_E(\theta_h|_{\partial D})\,d\Gamma + \int_{\partial\Omega}\frac{\partial u'}{\partial n}\,\theta_h\,d\Gamma = \int_\gamma \frac{\partial u'}{\partial n}\,w\,d\Gamma. \end{aligned}$$

That is to say, we get the equalities

$$-\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\,h\,d\Gamma = \int_\gamma \frac{\partial u'}{\partial n}\,w\,d\Gamma,$$

where $\frac{\partial u'}{\partial n}$ is known (up to second-order terms; this is the consequence of step 1) and w can be computed solving (69)–(70).

This shows that $(m\cdot n)|_{\partial D}$ can be computed explicitly by solving as many control problems of the previous kind as dim M.

• Step 3: *Resolution of the exact finite dimensional control problem* (69)–(70). Finally, it can be seen that, for any $h \in L^2(\partial D)$, there exist w and θ_h such that (69)–(70) hold. Indeed, it is sufficient to apply a classical unique continuation property for the Laplace equation in combination with the arguments in [25]; see the proof of Lemma 4 for more details. This ends the proof of Theorem 6. □

5.2. **The general case: a Boussinesq system.** We will now follow the steps of the previous proof in order to deduce Theorem 4. We have $(\varphi,\psi) \in W^{2-1/r,r}(\partial\Omega)^N \times W^{2-1/r,r}(\partial\Omega)^N$ for some $r > N$ and thus $(u,p,\theta) \in W^{2,r}(\Omega\setminus\overline{D})^N \times W^{1,r}(\Omega\setminus\overline{D}) \times W^{2,r}(\Omega\setminus\overline{D})$. Let us assume that $m \in \mathcal{W}$, with $(m\cdot n)|_{\partial D} \in M$.

Let us also assume that $D \in \mathcal{D}$ is known, we have computed (α,β) from (2) solving the direct problem (1) and, also, that we know the observation (α^m,β^m) corresponding to the modified domain $D+m$, that is to say,

$$\alpha^m = \sigma(v,q)\cdot n|_\gamma \quad\text{and}\quad \beta^m = \kappa\frac{\partial \eta}{\partial n}|_\gamma\,,$$

where (v,q,η) is the solution of (15). We recall that our goal is to compute explicitly $(m\cdot n)|_{\partial D}$ from (α,β) and (α^m,β^m).

• Step 1: *Domain variations.* Thanks to Theorem 3, using domain variation techniques, we get the following identities:

$$\begin{aligned} &\alpha^m - \alpha \equiv \sigma(v,q)\cdot n - \sigma(u,p)\cdot n = \sigma(u',p')\cdot n + o(m) \quad\text{on}\quad \gamma, \\ &\beta^m - \beta \equiv \kappa\frac{\partial\eta}{\partial n} - \kappa\frac{\partial\theta}{\partial n} = \kappa\frac{\partial\theta'}{\partial n} + o(m) \quad\text{on}\quad \gamma, \end{aligned}$$

where $(u',p',\theta') \in H^1(\Omega\setminus\overline{D})^N \times L^2(\Omega\setminus\overline{D}) \times H^1(\Omega\setminus\overline{D})$ is the unique solution to

$$\begin{cases} -\nu\Delta u' + (u'\cdot\nabla)u + (u\cdot\nabla)u' + \nabla p' = \theta' g, \quad \nabla\cdot u' = 0 & \text{in } \Omega\setminus\overline{D}, \\ -\kappa\Delta\theta' + u'\cdot\nabla\theta + u\cdot\nabla\theta' = 0 & \text{in } \Omega\setminus\overline{D}, \\ u' = 0, \quad \theta' = 0 & \text{on } \partial\Omega, \\ u' = -(m\cdot n)\dfrac{\partial u}{\partial n}, \quad \theta' = -(m\cdot n)\dfrac{\partial\theta}{\partial n} & \text{on } \partial D \end{cases} \tag{71}$$

and $o(m)$ satisfies (16).

Therefore, the proof of Theorem 4 is reduced to compute $(m\cdot n)|_{\partial D}$ from $\sigma(u',p')\cdot n|_\gamma$ and $\kappa\frac{\partial\theta'}{\partial n}|_\gamma$ up to second-order terms. This will be done in the next step.

Notice that, up to now, we have not used the fact that $(m\cdot n)|_{\partial D}$ belongs to a finite dimensional space.

• STEP 2: *A (non standard) data assimilation approach.* Let us now assume that $(m\cdot n)|_{\partial D}\in M\subset W^{1,\infty}(\partial D)$, with $\dim M<+\infty$. Then, in view of (71), we have

$$(\sigma(u',p')\cdot n|_{\partial D}\,,\kappa\frac{\partial\theta'}{\partial n}|_{\partial D})\in E,$$

where $E\subset H^{-1/2}(\partial D)^N\times H^{-1/2}(\partial D)$ is another finite dimensional space.

As before, we use an argument inspired by the techniques in [23]. We will use the fact that the quantities

$$\int_{\partial D}(m\cdot n)\left(\left|\frac{\partial u}{\partial n}\right|^2+\left|\frac{\partial\theta}{\partial n}\right|^2\right)h\,d\Gamma,\quad h\in M,$$

determine $(m\cdot n)|_{\partial D}$. Indeed, under the hypothesis (23), the bilinear form $(\ell,h)\mapsto\int_{\partial D}\ell\left(\left|\frac{\partial u}{\partial n}\right|^2+\left|\frac{\partial\theta}{\partial n}\right|^2\right)h\,d\Gamma$ is a scalar product in M. Therefore, our goal will be to write these integrals in terms of $\sigma(u',p')\cdot n|_\gamma$ and $\kappa\frac{\partial\theta'}{\partial n}|_\gamma$.

To this end, we will argue as follows. For the moment, let us assume that for each $h\in M$ we are able to solve the following exact finite dimensional control problem: find a control (w^1,w^2) such that $(w^1 1_\gamma,w^2 1_\gamma)\in H^{1/2}(\partial\Omega)^N\times H^{1/2}(\partial\Omega)$ and the corresponding weak solution $(y,q,z)\in H^1(\Omega\setminus\overline{D})^N\times L^2(\Omega\setminus\overline{D})\times H^1(\Omega\setminus\overline{D})$ of

$$\begin{cases}-\nu\Delta y-(\nabla y)^t\,u-(u\cdot\nabla)y+\nabla q=-z\nabla\theta,\quad\nabla\cdot y=0 & \text{in }\Omega\setminus\overline{D},\\ -\kappa\Delta z-u\cdot\nabla z=g\cdot y & \text{in }\Omega\setminus\overline{D},\\ y=w^1 1_\gamma,\quad z=w^2 1_\gamma & \text{on }\partial\Omega,\\ \sigma(y,q)\cdot n=\dfrac{\partial u}{\partial n}h,\quad\kappa\dfrac{\partial z}{\partial n}=\dfrac{\partial\theta}{\partial n}h & \text{on }\partial D\end{cases}\tag{72}$$

satisfies

$$\langle(\Phi,\Psi),(y|_{\partial D},z|_{\partial D})\rangle_{\partial D}=0\quad\forall(\Phi,\Psi)\in E,\tag{73}$$

where $\langle\cdot\,,\cdot\rangle_{\partial D}$ stands for the usual duality coupling for $H^{-1/2}(\partial D)^N\times H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)^N\times H^{1/2}(\partial D)$.

Then, using that (u',p',θ') is the solution of (71) and (73), we see that

$$\begin{aligned}-\int_{\partial D}(m\cdot n)\left(\left|\frac{\partial u}{\partial n}\right|^2+\left|\frac{\partial\theta}{\partial n}\right|^2\right)h\,d\Gamma&=\langle\sigma(y,q)\cdot n,u'\rangle_{\partial D\cup\partial\Omega}+\kappa\langle\frac{\partial z}{\partial n},\theta'\rangle_{\partial D\cup\partial\Omega}\\&=\langle\sigma(u',p')\cdot n,y\rangle_{\partial D\cup\partial\Omega}+\kappa\langle\frac{\partial\theta'}{\partial n},z\rangle_{\partial D\cup\partial\Omega}\\&=\langle\sigma(u',p')\cdot n,\,w^1\,1_\gamma\rangle_{\partial\Omega}+\kappa\langle\frac{\partial\theta'}{\partial n}\,,w^2\,1_\gamma\rangle_{\partial\Omega}\,,\end{aligned}$$

where $\langle\cdot\,,\cdot\rangle_\Sigma$ stands for the usual duality product in $H^{-1/2}(\Sigma)^N$ and $H^{-1/2}(\Sigma)$.

Notice that this allows us compute $(m\cdot n)|_{\partial D}$ (up to second-order perturbations) from $\sigma(u',p')\cdot n|_\gamma$ and $\kappa\frac{\partial\theta'}{\partial n}|_\gamma$. As we have seen in the previous step, this also allows us compute $(m\cdot n)|_{\partial D}$ from the known observations (α^m,β^m) and (α,β).

The conclusion of this step is that the proof of Theorem 4 will be achieved if we are able to solve (72)–(73).

Remark 6. From the practical viewpoint, what we have to do is the following. Let $\{\ell, \dots, \ell_I\}$ be a basis of M and let us put

$$(m \cdot n)|_{\partial D} = \sum_{i=1}^{I} a_i \ell_i \, .$$

Let (w_i^1, w_i^2) be, for each $i = 1, \dots, I$ a control solving the problem (72)–(73) with $h = \ell_i$. Then the coefficients a_i are given by the unique solution of the following linear system:

$$\sum_{i=1}^{I} \left(\int_{\partial D} \ell_i \left(\left| \frac{\partial u}{\partial n} \right|^2 + \left| \frac{\partial \theta}{\partial n} \right|^2 \right) \ell_j \, d\Gamma \right) a_i = q_j, \quad 1 \le j \le I,$$

where we have set $q_j = -\langle \sigma(u', p') \cdot n, \, w_j^1 \, 1_\gamma \rangle_{\partial\Omega} - \kappa \langle \frac{\partial \theta'}{\partial n} \, , w_j^2 \, 1_\gamma \rangle_{\partial\Omega} \, .$

• Step 3: *Resolution of the exact finite dimensional control problem* (72)–(73). We have the following result:

Lemma 4. *Let $E \subset H^{-1/2}(\partial D)^N \times H^{-1/2}(\partial D)$ be a finite dimensional space. Let us assume that $h \in L^\infty(\partial D)$ and $(u, \theta) \in H^2(\Omega \setminus \overline{D})^N \times H^2(\Omega \setminus \overline{D})$ satisfies $\nabla \cdot u = 0$ in $\Omega \setminus \overline{D}$ and* (11). *Then, there exist controls (w^1, w^2) such that $(w^1 1_\gamma, w^2 1_\gamma) \in H^{1/2}(\partial\Omega)^N \times H^{1/2}(\partial\Omega)$ and the associated solution (y, q, z) of* (72) *satisfies* (73). *Furthermore, for each $\varepsilon > 0$, we can choose w^1 and w^2 such that*

$$\|(y|_{\partial D}, z|_{\partial D})\|_{L^2(\partial D)} \le \varepsilon. \tag{74}$$

Proof. This result is a consequence of the unique continuation properties we have presented in Section 2 and the fact that E has finite dimension. There are several ways to prove it. Here, we follow the approach in [25] that, for each $\varepsilon > 0$, provides a control satisfying (74).

Let G be a bounded open set with boundary ∂G of class C^2 such that $\Omega \subset G$ and $\partial\Omega \cap G = \gamma$. Let ω be a nonempty open subset of $G \setminus \overline{\Omega}$. We consider the following distributed control problem: find controls $(f, k) \in L^2(\omega)^N \times L^2(\omega)$ such that the corresponding solution $(y, q, z) \in H^1(G \setminus \overline{D})^N \times L^2(G \setminus \overline{D}) \times H^1(G \setminus \overline{D})$ of

$$\begin{cases} -\nu \Delta y - (\nabla y)^t \, u - (u \cdot \nabla) y + \nabla q = -z \nabla \theta + f 1_\omega, \ \nabla \cdot y = 0 & \text{in } G \setminus \overline{D}, \\ -\kappa \Delta z - u \cdot \nabla z = g \cdot y + k 1_\omega & \text{in } G \setminus \overline{D}, \\ y = 0, \quad z = 0 & \text{on } \partial G, \\ \sigma(y, q) \cdot n = \dfrac{\partial u}{\partial n} h, \quad \kappa \dfrac{\partial z}{\partial n} = \dfrac{\partial \theta}{\partial n} h & \text{on } \partial D \end{cases} \tag{75}$$

satisfies (73).

For any $(a, b) \in H^{-1/2}(\partial D)^N \times H^{-1/2}(\partial D)$, let us consider the adjoint system

$$\begin{cases} -\nu \Delta \xi + (u \cdot \nabla) \xi + (\xi \cdot \nabla) u + \nabla \chi = \rho g, \quad \nabla \cdot \xi = 0 & \text{in } G \setminus \overline{D}, \\ -\kappa \Delta \rho + u \cdot \nabla \rho + \xi \cdot \nabla \theta = 0 & \text{in } G \setminus \overline{D}, \\ \xi = 0, \quad \rho = 0 & \text{on } \partial G, \\ \sigma(\xi, \chi) \cdot n = a, \quad \kappa \dfrac{\partial \rho}{\partial n} = b & \text{on } \partial D. \end{cases} \tag{76}$$

As a consequence of (11), for each $(a, b) \in H^{-1/2}(\partial D)^N \times H^{-1/2}(\partial D)$ system (76) possesses a unique weak solution $(\xi, \chi, \rho) \in H^1(G \setminus \overline{D})^N \times L^2(G \setminus \overline{D}) \times H^1(G \setminus \overline{D})$ that satisfies

$$\|\xi\|_{H^1(G \setminus \overline{D})} + \|\chi\|_{L^2(G \setminus \overline{D})} + \|\rho\|_{H^1(G \setminus \overline{D})} \le C \|(a, b)\|_{H^{-1/2}(\partial D)} \, ,$$

where C is a positive constant only depending on G, D, and u. In addition, we have $\sigma(\xi,\chi)\cdot n \in H^{-1/2}(\partial G)^N$ and $\frac{\partial\rho}{\partial n} \in H^{-1/2}(\partial G)$, with similar estimates.

On the other hand, if we multiply the first equation of (72) by ξ and the second one by z and we make appropriate integrations by parts, we get:

$$\int_\omega (f\cdot\xi + k\rho)\,dx = \int_{\partial D} h\left(\frac{\partial u}{\partial n}\cdot\xi + \frac{\partial\theta}{\partial n}\rho\right)d\Gamma - \langle (a,b)\,,(y|_{\partial D}\,, z|_{\partial D})\rangle_{\partial D}\,. \tag{77}$$

Let $\mathcal{L} : H^{-1/2}(\partial D) \mapsto H^{1/2}(\partial D)$ be the canonical identification operator, with

$$\begin{cases} \|\mathcal{L}\eta\|_{H^{1/2}(\partial D)} = \|\eta\|_{H^{-1/2}(\partial D)} & \forall\,\eta\in H^{-1/2}(\partial D), \\ (\mathcal{L}\eta\,,v)_{H^{1/2}(\partial D)} = \langle\eta\,,v\rangle_{\partial D} & \forall\,\eta\in H^{-1/2}(\partial D), \quad \forall\,v\in H^{1/2}(\partial D). \end{cases}$$

Let $\widetilde{E}$ be the finite dimensional space $\widetilde{E} = \mathcal{L}E \subset H^{1/2}(\partial D)^N \times H^{1/2}(\partial D)$ and let $P_{\widetilde{E}}$ be the associated orthogonal projector in $H^{1/2}(\partial D)^N\times H^{1/2}(\partial D)$.

For any $\varepsilon>0$, let us consider the functional $J_\varepsilon(a,b)$

$$\begin{cases} J_\varepsilon(a,b) = \dfrac{1}{2}\displaystyle\int_\omega \left(|\xi|^2+|\rho|^2\right)\,dx + \varepsilon\|(I-P_{\widetilde{E}})\mathcal{L}(a,b)\|_{H^{1/2}(\partial D)} \\ \quad -\displaystyle\int_{\partial D} h\left(\frac{\partial u}{\partial n}\cdot\xi + \frac{\partial\theta}{\partial n}\rho\right)d\Gamma \quad \forall (a,b)\in H^{-1/2}(\partial D)^N\times H^{-1/2}(\partial D), \end{cases}$$

where (ξ,χ,ρ) is the solution of (76).

Let us assume that there exists a minimizer $(\hat a,\hat b)\in H^{-1/2}(\partial D)^N\times H^{-1/2}(\partial D)$ of J_ε (we will justify this temporary assumption at the end of this proof). Let us denote by $(\hat\xi,\hat\chi,\hat\rho)$ the associated solution of (76). Then

$$\begin{aligned} &\int_\omega \left(\hat\xi\cdot\xi + \hat\rho\rho\right)dx + \frac{\varepsilon}{\|(I-P_{\widetilde{E}})\mathcal{L}(\hat a,\hat b)\|_{H^{1/2}(\partial D)}}\langle (I-P_{\widetilde{E}})\mathcal{L}(\hat a,\hat b)\,,(a,b)\rangle_{\partial D} \\ &\quad - \int_{\partial D} h\left(\frac{\partial u}{\partial n}\cdot\xi + \frac{\partial\theta}{\partial n}\rho\right)d\Gamma = 0 \end{aligned} \tag{78}$$

for any $(a,b)\in H^{-1/2}(\partial D)^N\times H^{-1/2}(\partial D)$.

Let us take the controls f and k given by

$$f = \hat\xi 1_\omega, \quad k = \hat\rho 1_\omega\,.$$

Then, we deduce from (78) and (77) that the associate state (y,q,z) satisfies

$$(y|_{\partial D}, z|_{\partial D}) = \frac{\varepsilon}{\|(I-P_{\widetilde{E}})\mathcal{L}(\hat a,\hat b)\|_{H^{1/2}(\partial D)}}(I-P_{\widetilde{E}})\mathcal{L}(\hat a,\hat b)$$

for all $(a,b)\in L^2(\partial D)^N\times L^2(\partial D)$.

Consequently, the controls $(w^1,w^2) = (y|_{\partial\Omega}\,, z|_{\partial\Omega})$ fulfill the statement of the lemma.

In order to end the proof, let us see that there exists a unique minimizer $(\hat a,\hat b)$ of J_ε. But this is a trivial consequence of the following properties of J_ε:

- J_ε is lower semi-continuous and strictly convex.
- J_ε is coercive. More precisely,

$$\liminf_{\|(a,b)\|_{H^{-1/2}(\partial D)}\to\infty} \frac{J_\varepsilon(a,b)}{\|(a,b)\|_{H^{-1/2}(\partial D)}} \geq \varepsilon.$$

This is a consequence of the following *unique continuation property* (given in Theorem 5): If the solution (ξ,χ,ρ) of (76) verifies $\xi = 0$ and $\rho = 0$ in ω, then $(\xi,\chi,\rho)\equiv(0,0,0)$ in $G\setminus\overline{D}$. This ends the proof. □

6. **Some technical results.** For completeness, in this section we will present a sketch of the proof of Theorem 1, which provides existence, uniqueness and regularity properties of the solution of (1). For the proof we will use the standard Galerkin's method and some properties of Sobolev spaces.

1 – EXISTENCE: Assume that $D \in \mathcal{D}$ and $(\varphi, \psi) \in H^{1/2}(\partial\Omega)^N \times H^{1/2}(\partial\Omega)$ is such that $\int_{\partial\Omega} \varphi \cdot n \, ds = 0$. For simplicity, the usual norms in the spaces $L^2(\Omega \setminus \overline{D})$, $H^1(\Omega \setminus \overline{D})$, ... will be respectively denoted by $\| \cdot \|_{L^2}$, $\| \cdot \|_{H^1}$, ... Let us set

$$V(\mathcal{O}) = \{ \, v \in H^1_0(\mathcal{O})^N \, : \, \nabla \cdot v = 0 \, \},$$

where $\mathcal{O} \subset \mathbb{R}^N$ is a given regular domain. Then, for every $\alpha_1, \alpha_2 > 0$, there exists $(\Phi^*_{\alpha_1}, \Psi^*_{\alpha_2}) \in H^1(\Omega \setminus \overline{D^*})^N \times H^1(\Omega \setminus \overline{D^*})$, that satisfies

$$\begin{cases} \nabla \cdot \Phi^*_{\alpha_1} = 0 & \text{in } \Omega \setminus \overline{D^*}, \\ \Phi^*_{\alpha_1} = \varphi, \quad \Psi^*_{\alpha_2} = \psi & \text{on } \partial\Omega, \\ \Phi^*_{\alpha_1} = 0, \quad \Psi^*_{\alpha_2} = 0 & \text{on } \partial D^*, \end{cases}$$

$$\|(\Phi^*_{\alpha_1}, \Psi^*_{\alpha_2})\|_{H^1(\Omega \setminus \overline{D^*})} \leq C(\Omega, D^*) \|(\varphi, \psi)\|_{H^{1/2}(\partial\Omega)}$$

and

$$\begin{cases} \left| \displaystyle\int_{\Omega \setminus \overline{D^*}} (u \cdot \nabla) \Phi^*_{\alpha_1} \cdot u \, dx \right| \leq \alpha_1 \|u\|^2_{H^1(\Omega \setminus \overline{D^*})}, \\ \left| \displaystyle\int_{\Omega \setminus \overline{D^*}} (v \cdot \nabla \Psi^*_{\alpha_2}) w \, dx \right| \leq \alpha_2 \|v\|_{H^1(\Omega \setminus \overline{D^*})} \|w\|_{H^1(\Omega \setminus \overline{D^*})}, \end{cases}$$

for every $u, v \in V(\Omega \setminus \overline{D^*})$ and every $w \in H^1_0(\Omega \setminus \overline{D^*})$ (see lemmas III.6.2 and VIII.4.2 in [16]). Let us denote by Φ_{α_1} and Ψ_{α_2} the extensions by zero of the functions $\Phi^*_{\alpha_1}$ and $\Psi^*_{\alpha_2}$ to the whole set Ω. Then the couple $(\Phi_{\alpha_1}, \Psi_{\alpha_2})$ belongs to $H^1(\Omega \setminus \overline{D})^N \times H^1(\Omega \setminus \overline{D})$ and satisfies

$$\begin{cases} \left| \displaystyle\int_{\Omega} (u \cdot \nabla) \Phi_{\alpha_1} \cdot u \, dx \right| \leq \alpha_1 \|u\|^2_{H^1} \quad \forall \, u \in V(\Omega), \\ \left| \displaystyle\int_{\Omega} (v \cdot \nabla \Psi_{\alpha_1}) w \, dx \right| \leq \alpha_2 \|v\|_{H^1} \|w\|_{H^1} \quad \forall \, v \in V(\Omega), \quad \forall \, w \in H^1_0(\Omega). \end{cases} \tag{79}$$

and

$$\|(\Phi_{\alpha_1}, \Psi_{\alpha_2})\|_{H^1} \leq C(\Omega, D^*) \|(\varphi, \psi)\|_{H^{1/2}(\partial\Omega)} \, .$$

Let us introduce F and G, with

$$F = \Psi_{\alpha_2} g + \nu \Delta \Phi_{\alpha_1} - (\Phi_{\alpha_1} \cdot \nabla) \Phi_{\alpha_1}, \quad G = \kappa \Delta \Psi_{\alpha_2} - \Phi_{\alpha_1} \cdot \nabla \Psi_{\alpha_2}.$$

Then we have

$$\begin{cases} \|F\|_{H^{-1}(\Omega \setminus \overline{D})} \leq C(\Omega, D^*) \left(\|\psi\|_{H^{1/2}} + \nu \|\varphi\|_{H^{1/2}} + \|\varphi\|^2_{H^{1/2}} \right), \\ \|G\|_{H^{-1}(\Omega \setminus \overline{D})} \leq C(\Omega, D^*) \left(\kappa \|\psi\|_{H^{1/2}} + \|\varphi\|_{H^{1/2}} \|\psi\|_{H^{1/2}} \right). \end{cases} \tag{80}$$

We will look for a solution (u, p, θ) of (1). Let us put $u = w + \Phi_{\alpha_1}$ and $\theta = \eta + \Psi_{\alpha_2}$. Then (w, p, η) must satisfy

$$\begin{cases} -\nu \Delta w + (w \cdot \nabla) \Phi_{\alpha_1} + (\Phi_{\alpha_1} \cdot \nabla) w + (w \cdot \nabla) w + \nabla p = \eta g + F & \text{in } \Omega \setminus \overline{D}, \\ \nabla \cdot w = 0 & \text{in } \Omega \setminus \overline{D}, \\ -\kappa \Delta \eta + \Phi_{\alpha_1} \cdot \nabla \eta + w \cdot \nabla \eta + w \cdot \nabla \Psi_{\alpha_2} = G & \text{in } \Omega \setminus \overline{D}, \\ (w, \eta) = (0, 0) & \text{on } \partial\Omega \cup \partial D. \end{cases} \tag{81}$$

It will be sufficient to show that there exist positive constants α_1, α_2 such that the nonlinear system (81) possesses at least one weak solution, more precisely, a couple (w,p,η) that belongs to $V(\Omega\setminus\overline{D})^N\times L^2(\Omega\setminus\overline{D})\times H^1_0(\Omega\setminus\overline{D})$ and satisfies the previous partial differential equations in the weak or distributional sense.

To this end, a standard Galerkin's method can be used. As usual, in order to obtain the existence result, the key point is to prove appropriate "a priori" estimates on the approximated solutions $\{(w_n,\eta_n)\}_{n\geq 1}$. We have:

$$\begin{cases} \nu\|\nabla w_n\|^2_{L^2} = -\displaystyle\int_{\Omega\setminus\overline{D}}(w_n\cdot\nabla)\,\Phi_{\alpha_1}\cdot w_n + \int_{\Omega\setminus\overline{D}}\eta_n g\cdot w_n + \langle F\,,w_n\rangle_{H^{-1}}\,,\\ \kappa\|\nabla\eta_n\|^2_{L^2} = -\displaystyle\int_{\Omega\setminus\overline{D}}(w_n\cdot\nabla\Psi_{\alpha_2})\eta_n + \langle G\,,\eta_n\rangle_{H^{-1}}\,. \end{cases}$$

Taking into account (79), we deduce that

$$\begin{cases} \nu\|\nabla w_n\|^2_{L^2} \leq \alpha_1\|\nabla w_n\|^2_{L^2} + \dfrac{C}{\nu}\|\nabla\eta_n\|^2_{L^2} + \dfrac{\nu}{2}\|\nabla w_n\|^2_{L^2} + \dfrac{1}{\nu}\|F\|^2_{H^{-1}}\,,\\ \kappa\|\nabla\eta_n\|^2_{L^2} \leq \alpha_2\|\nabla w_n\|_{L^2}\|\nabla\eta_n\|_{L^2} + \dfrac{\kappa}{4}\|\nabla\eta_n\|^2_{L^2} + \dfrac{1}{\kappa}\|G\|^2_{H^{-1}}\\ \qquad\qquad\quad \leq \dfrac{\alpha_2^2}{\kappa}\|\nabla w_n\|^2_{L^2} + \dfrac{\kappa}{2}\|\nabla\eta_n\|^2_{L^2} + \dfrac{1}{\kappa}\|G\|^2_{H^{-1}}\,, \end{cases}$$

where C is a positive constant only depending on Ω and g.

Now, let us take $\alpha_1=\nu/4$. It is then easy to deduce that

$$\begin{cases} \|\nabla w_n\|^2_{L^2} \leq \dfrac{4C}{\nu^2}\|\nabla\eta_n\|^2_{L^2} + \dfrac{4}{\nu^2}\|F\|^2_{H^{-1}}\,,\\ \|\nabla\eta_n\|^2_{L^2} \leq \dfrac{2\alpha_2^2}{\kappa^2}\|\nabla w_n\|^2_{L^2} + \dfrac{2}{\kappa^2}\|G\|^2_{H^{-1}}\,. \end{cases}$$

On the other hand, we can set $\alpha_2^2=\dfrac{\nu^2\kappa^2}{16C}$. From this last inequality, we see that

$$\begin{cases} \|\nabla w_n\|^2_{L^2} \leq \dfrac{16C}{\nu^2\kappa^2}\|G\|^2_{H^{-1}} + \dfrac{8}{\nu^2}\|F\|^2_{H^{-1}},\\ \|\nabla\eta_n\|^2_{L^2} \leq \dfrac{4}{\kappa^2}\|G\|^2_{H^{-1}} + \dfrac{1}{C}\|F\|^2_{H^{-1}} \end{cases} \tag{82}$$

and, therefore, (w_n,η_n) is uniformly bounded in $V(\Omega\setminus\overline{D})\times H^1_0(\Omega\setminus\overline{D})$.

In a classical way, this can be used to prove the existence of a weak solution (w,p,η) of (81) that belongs to $V(\Omega\setminus\overline{D})\times L^2(\Omega\setminus\overline{D})\times H^1_0(\Omega\setminus\overline{D})$. Finally, from (80) and (82) we deduce that

$$\|\nabla w\|_{L^2}\leq\frac{C(\Omega,D^*)}{\nu}K(\nu,\kappa,\varphi,\psi) \ \text{ and } \ \|\nabla\eta\|_{L^2}\leq C(\Omega,D^*)K(\nu,\kappa,\varphi,\psi),$$

where $K(\nu,\kappa,\varphi,\psi)$ is given by (7). Obviously, this proves (6). Therefore, (81) possesses at least one weak solution with the desired estimates.

Let us now see that $\sigma(u,p)\cdot n\in H^{-1/2}(\partial\Omega)^N$. In view of well known results, it suffices to prove that $\sigma(u,p)\in L^2(\Omega\setminus\overline{D})^{N\times N}$ and $\nabla\cdot\sigma(u,p)\in L^r(\Omega\setminus\overline{D})^N$ for some $r>1$ if $N=2$ and $r=6/5$ if $N=3$. But this is very easy to check. Indeed, we have $(u,p,\theta)\in H^1(\Omega\setminus\overline{D})^N\times L^2(\Omega\setminus\overline{D})\times H^1(\Omega\setminus\overline{D})$ and, consequently, $\sigma(u,p)\in L^2(\Omega\setminus\overline{D})^{N\times N}$. On the other hand, $\nabla\cdot\sigma(u,p)=(u\cdot\nabla)u+\theta\,g$, whence we also have $\nabla\cdot\sigma(u,p)\in L^\beta(\Omega\setminus\overline{D})^N$ for all $\beta<2$ if $N=2$ and $\beta=3/2$ if $N=3$.

In a very similar way, it can be proved that $\frac{\partial\theta}{\partial n}\in H^{-1/2}(\partial\Omega)$.

2 – Uniqueness: Let us assume that there exist two solutions (u^1,p^1,θ^1) and (u^2,p^2,θ^2) of (1) that verify (6) and let us set $u=u^1-u^2$, $p=p^1-p^2$ and

$\theta = \theta^1 - \theta^2$. We assume now that (7) is satisfied. We have that (u, p, θ) satisfies

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u^1 + (u^2\cdot\nabla)u + \nabla p = \theta g, \quad \nabla\cdot u = 0 & \text{in } \Omega\setminus\overline{D}, \\ -\kappa\Delta\theta + u^1\cdot\nabla\theta + u\cdot\nabla\theta^2 = 0 & \text{in } \Omega\setminus\overline{D}, \\ u = 0, \quad \theta = 0 & \text{on } \partial\Omega\cup\partial D, \end{cases} \tag{83}$$

By multiplying the first equation of (83) by u and integrating in $\Omega\setminus\overline{D}$, we get

$$\nu\|\nabla u\|_{L^2}^2 = \int_{\Omega\setminus\overline{D}} \theta g\cdot u\,dx - \int_{\Omega\setminus\overline{D}} (u\cdot\nabla)u^1\cdot u\,dx.$$

Consequently,

$$\begin{cases} \nu\|\nabla u\|_{L^2}^2 & \leq & C(\Omega)\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u^1\|_{L^2}\|u\|_{L^3}\|u\|_{L^6} \\ & \leq & C(\Omega)\left(\|\nabla\theta\|_{L^2} + \|\nabla u^1\|_{L^2}\|\nabla u\|_{L^2}\right)\|\nabla u\|_{L^2}\,. \end{cases} \tag{84}$$

We multiply the second equation of (83) by θ and we integrate in $\Omega\setminus\overline{D}$. We obtain

$$\kappa\|\nabla\theta\|_{L^2}^2 = -\int_{\Omega\setminus\overline{D}} u\cdot\nabla\theta^2\,\theta\,dx \leq C(\Omega, D^*)\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\|\nabla\theta^2\|_{L^2}\,.$$

Thus, we have

$$\kappa\|\nabla\theta\|_{L^2} \leq C(\Omega)\|\nabla u\|_{L^2}\|\nabla\theta^2\|_{L^2}\,.$$

Coming back to (84), we find that

$$\nu\|\nabla u\|_{L^2}^2 \leq C(\Omega)\left(\frac{1}{\kappa}\|\nabla\theta^2\|_{L^2} + \|\nabla u^1\|_{L^2}\right)\|\nabla u\|_{L^2}^2.$$

But, in view of the estimates (6) satisfied by u^i and θ^i, we have

$$\|\nabla u^1\|_{L^2} + \frac{1}{\kappa}\|\nabla\theta^2\|_{L^2} \leq C(\Omega, D^*)\frac{\nu+\kappa}{\nu\kappa}\,K(\nu,\kappa,\varphi,\psi),$$

whence we also obtain

$$\nu\|\nabla u\|_{L^2}^2 \leq C(\Omega, D^*)\frac{\nu+\kappa}{\nu\kappa}\,K(\nu,\kappa,\varphi,\psi)\nu\|\nabla u\|_{L^2}^2.$$

Thus, there exists $K_1(\Omega, D^*)$ such that, if the couple (φ,ψ) satisfies (7), then

$$C(\Omega, D^*)\frac{\nu+\kappa}{\nu\kappa}\,K(\nu,\kappa,\varphi,\psi) < \nu$$

and we necessary have $u \equiv 0$. This proves the uniqueness of (u, p, θ) (p is unique up to a constant).

3 – Regularity of u, p and θ: For simplicity, let us sketch the argument for $r = 2$. Let us now assume that $\varphi \in H^{3/2}(\partial\Omega)^N$ and $\psi \in H^{3/2}(\partial\Omega)$. Then the weak solutions of (1) are in fact strong solutions, that is to say, they satisfy $(u, p, \theta) \in H^2(\Omega\setminus\overline{D})^N \times H^1(\Omega\setminus\overline{D}) \times H^2(\Omega\setminus\overline{D})$, with appropriate estimates. This fact can be deduced as a consequence of the $W^{2,r}$-regularity theory for the Poisson equation and the Stokes problems (see for instance [8] and the references therein).

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E-mail address: `doubova@us.es; cara@us.es; manoloburgos@us.es; jortega@dim.uchile.cl`