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Periodic Boundary Value Problem for Nonlinear First Order Ordinary Differential Equations with Impulses at Fixed Moments †

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1. INTRODUCTION

Many evolution processes are characterized by the fact that they are subject to short-time perturbations implying a sudden change of state. For example, when a hammer hits a string, it experiences a rapid change of velocity; heavy harvesting or epidemics result in a drastic decreasing of a population density of a species; a pendulum of a clock undergoes a sudden change of momentum when it crosses its equilibrium position, etc.

Since in the modeling of a process the short-time perturbations are usually assumed to have the form of instantaneous impulses, differential equations with impulses provide a natural description of such evolution processes. The theory of impulsive differential equations is rich enough, and, unlikely to differential equations without impulse perturbation, the impulsive differential equations exhibit several new phenomena and pose a number of specific problems that cannot be treated with the usual techniques within the standard framework of ordinary differential equations.

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Though the qualitative theory of impulsive differential equations is of interest in itself, the recent progress in its development has been to great extent stimulated by numerous applications to problems arising in medicine, biology, optimal control, economics, etc. We mention here the model of a single species population with abrupt changes of important biological parameters [2, 13], the competition model with abrupt harvesting [5], the Kruger-Thiemer model for drug distribution [7], impulsive stabilization of a state which may not be an equilibrium point of the system (stabilization of an inverted pendulum) [8], the model for the vibration of a string with energy dissipation where impulsive "pumping" occurs at moments when total energy of system falls down to a preassigned critical level [10], and this list is yet to be completed.

In this paper, we present some new comparison principles for impulsive differential equations which improve known results. This technique is one of the basic tools in the qualitative theory of differential equations. We use it to study the existence of solutions for the first order periodic impulsive problem

$$\begin{aligned} u'(t) &= f(t, u(t)), && \text{a.e. } t \in J' = J \setminus \{t_1, t_2, \dots, t_p\}, \\ u(t_k^+) &= I_k(u(t_k)), && k = 1, 2, \dots, p, \\ u(0) &= u(T), \end{aligned} \tag{1.1}$$

where $J = [0, T]$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $I_k \colon \mathbb{R} \to \mathbb{R}$ is continuous for $k = 1, 2, \dots, p$ and $f \colon J \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., $f(\cdot, u)$ is measurable for every $u \in \mathbb{R}$, $f(t, \cdot)$ is continuous for *a.e.* $t \in J$, and for every $R > 0$ there exits a function $h_R \in L^1(J)$ such that $|f(t, u)| \le h_R(t)$ for *a.e.* $t \in J$ and for every $u \in \mathbb{R}$ with $|u| \le R$.

When f is continuous, this problem has been studied, for instance, in [7] where the authors introduced the concepts of lower and upper solutions for (1.1), i.e., functions $\alpha, \beta : J \to \mathbb{R}$ such that $\alpha, \beta \in C^1(J')$, $\alpha, \beta, \alpha', \beta'$ have discontinuities of the first order at the moments of impulses, α and β are left continuous on J, and

$$\begin{aligned} \alpha'(t) &\le f(t, \alpha(t)), && t \in J', \\ \alpha(t_k^+) &\le I_k(\alpha(t_k)), && k = 1, 2, \dots, p, \\ \beta'(t) &\ge f(t, \beta(t)), && t \in J', \\ \beta(t_k^+) &\ge I_k(\beta(t_k)), && k = 1, 2, \dots, p, \\ \alpha(0) &\le \alpha(T), && \beta(0) \ge \beta(T). \end{aligned} \tag{1.2}$$

Then, they proved the validity of the method of upper and lower solutions. Some existence results are also known for the case when f is a Carathéodory function. For example, existence of solution for a problem with nonlinear boundary conditions that includes (1.1) as a particular case has been proved in [9].

Following new directions in the development of the qualitative theory of ordinary differential equations [1, 6], various authors have recently discussed existence of solutions for (1.1) with more general definitions of lower and upper solutions. More precise, new concepts of lower and upper solutions that admit violation of some inequalities in (1.2) have been proposed. The case when the inequalities at the boundary fail, i.e.

$$\alpha(0) > \alpha(T) \quad \text{or/and} \quad \beta(0) < \beta(T),$$

has been studied in [3, 12].

The case when the impulsive inequalities are violated, i.e. for some $j \in \{1, 2, \dots, p\}$

$$\alpha(t_j^+) > I_j(\alpha(t_j)) \quad \text{or/and} \quad \beta(t_j^+) < I_j(\beta(t_j)),$$

has been considered in [12].

We note that the latter case is intrinsic for the impulsive differential equations, on the contrary to other situations that may be considered for many types of differential equations.

In this paper, we study the problem (1.1) from this point of view, paying more attention to the case when the impulsive inequalities are violated. The results that we present extend and complement those in [1, 3, 4, 6, 11, 12].

2. Comparison Principles

In order to define the concept of solution of (1.1), we introduce the following space

$$\Omega_1^1(J) = \Big\{u : J \to \mathbb{R} \,:\, u_{|(t_k,t_{k+1})} \in W^{1,1}(t_k, t_{k+1}), \\ u(t_{k+1}) = u(t_{k+1}^-),\ k = 0, 1, \dots, p,\ u(0) = u(0^+)\Big\}.$$

We note that Ω_1^1 is a Banach space with the norm

$$\|u\|_{\Omega_1^1} = \sum_{k=0}^{p} \|u_k\|_{W^{1,1}(t_k,t_{k+1})},$$

where $u_k : [t_k, t_{k+1}] \to \mathbb{R}$ is defined by $u_k(t_k) = u(t_k^+)$ and $u_k(t) = u(t)$, $t \in (t_k, t_{k+1}]$, $k = 0, \dots, p$.

We recall that for $v \in W^{1,1}(t_k, t_{k+1})$, the norm is defined as follows

$$\|v\|_{W^{1,1}(t_k,t_{k+1})} = \|v\|_{L^1(t_k,t_{k+1})} + \|v'\|_{L^1(t_k,t_{k+1})} .$$

DEFINITION 2.1. We say that a function u is a solution of (1.1) if $u \in \Omega_1^1(J)$,

$$u(t) = u(t_k^+) + \int_{t_k}^t f(s, u(s))ds , \qquad t \in (t_k, t_{k+1}] , \quad k = 0, 1, \dots, p ,$$

$$u(t_k^+) = I_k(u(t_k)) , \qquad k = 1, 2, \dots, p ,$$

and

$$u(0) = u(T) .$$

The following result is a new version of Theorem 1.4.1 in [7] for the space $\Omega_1^1(J)$. It has been already proved in [3] but we include it for the sake of completeness.

In what follows, by $\sum\limits_{s<t_i<t}$ and $\prod\limits_{s<t_i<t}$ we mean, as usualy, $\sum\limits_{\{i:s<t_i<t\}}$ and $\prod\limits_{\{i:s<t_i<t\}}$.

LEMMA 2.1. *Let $a, b \in L^1(J)$, and $c_k \geq 0$, d_k, $k = 1, 2, \dots, p$ be constants. Assume that a function $v \in \Omega_1^1(J)$ is such that satisfies the inequalities*

$$v'(t) \leq b(t)v(t) + a(t) , \qquad \text{a.e. } t \in J', \tag{2.1}$$

$$v(t_k^+) \leq c_k v(t_k) + d_k , \qquad k = 1, 2, \dots, p . \tag{2.2}$$

Then for every $t \in J$ the following inequality holds:

$$\begin{aligned} v(t) \leq v(0) \prod_{0<t_k<t} c_k e^{B(t)} &+ \int_0^t \prod_{s<t_k<t} c_k e^{B(t)-B(s)} a(s)ds \\ &+ \sum_{0<t_k<t} \prod_{t_k<t_i<t} c_i e^{B(t)-B(t_k)} d_k , \end{aligned} \tag{2.3}$$

where $B(t) = \int_0^t b(r)dr$.

Proof. From (2.1), we have that

$$v(t) \leq v(t_0)e^{B(t)} + \int_{t_0}^{t} a(s)e^{B(t)-B(s)}ds, \ t \in [t_0, t_1]. \tag{2.4}$$

Hence, the inequality (2.3) holds on $[t_0, t_1]$. Assume now that (2.3) holds for $t \in [t_0, t_n]$, where n, $1 < n \leq p$, is an integer. Then, it follows from (2.1) and (2.4) that

$$v(t) \leq v(t_n^+)e^{B(t)-B(t_n)} + \int_{t_n}^{t} a(s)e^{B(t)-B(s)}ds, \quad t \in [t_n, t_{n+1}].$$

Now, using (2.2), we conclude that

$$v(t) \leq (c_n v(t_n) + d_n)e^{B(t)-B(t_n)} + \int_{t_n}^{t} a(s)e^{B(t)-B(s)}ds.$$

Therefore,

$$v(t) \leq d_n e^{B(t)-B(t_n)} + \int_{t_n}^{t} a(s)e^{B(t)-B(s)}ds + \Bigg[v(t_0) \prod_{t_0<t_k<t_n} c_k e^{B(t_n)} +$$

$$\sum_{t_0<t_k<t_n} \Bigg(\prod_{t_k<t_i<t_n} c_i e^{B(t_n)-B(t_k)} \Bigg) d_k + \int_{t_0}^{t_n} \prod_{s<t_k<t_n} e^{B(t_n)-B(s)} a(s)ds \Bigg] c_n e^{B(t)-B(t_n)},$$

which gives the desired result. ∎

THEOREM 2.1. *Let the functions* $u \in \Omega_1^1(J)$, $a \in L^1(J)$, $a(t) \geq 0$ *for a.e.* $t \in J$, *and let* $d_k, c_k \in \mathbb{R}$, $c_k \geq 0$ *be constants such that*

$$\begin{aligned} u'(t) + Mu(t) + a(t) &\leq 0, && \text{a.e. } t \in J', \\ u(t_k^+) &\leq c_k u(t_k) + d_k, && k = 1, 2, \ldots, p, \\ u(0) &\leq 0. \end{aligned}$$

Assume that for every $k \in \{1, 2, \ldots, p\}$

$$d_k \leq c_k \int_{t_{k-1}}^{t_k} e^{-M(t_k - s)} a(s)ds. \tag{2.5}$$

Then $u(t) \leq 0$ *for* $t \in J$.

Proof. Let $\overline{k} = \max\{k : t_k < t, k = 0, 1, \dots, p\}$. By Lemma 2.1, for every $t \in J$ we have that

$$\begin{aligned} u(t)e^{Mt} &\le -\int_0^t \prod_{s<t_k<t} c_k e^{Ms} a(s)ds + \sum_{0<t_k<t} \prod_{t_k<t_i<t} c_i e^{Mt_k} d_k \\ &= \sum_{k=1}^{\overline{k}} \left\{ \prod_{i=k+1}^{\overline{k}} c_i e^{Mt_k} d_k - \int_{t_{k-1}}^{t_k} \prod_{i=k}^{\overline{k}} c_i e^{Ms} a(s)ds \right\} - \int_{t_{\overline{k}}}^t e^{Ms} a(s)ds \\ &\le \sum_{k=1}^{\overline{k}} \left\{ (e^{Mt_k} d_k - \int_{t_{k-1}}^{t_k} c_k e^{Ms} a(s)ds) \prod_{i=k+1}^{\overline{k}} c_i \right\} \le 0 . \end{aligned}$$

This concludes the proof. ∎

Remark 2.1. If $d_j \le 0$ for some $j \in \{1, 2, \dots, p\}$, then inequality (2.5) is automatically verified. Let $K_d = \{j \in \{1, 2, \dots, p\} : d_j > 0\}$. Then condition (2.5) can be written as

$$d_k \le c_k \int_{t_{k-1}}^{t_k} e^{-M(t_k - s)} a(s)ds , \qquad k \in K_d .$$

For the particular case $c_k = 1$, $k = 1, 2, \dots, p$, we obtain the following result which improves Theorem 2.1.

COROLLARY 2.1. *Let $c_k = 1$, for $k = 1, 2, \dots, p$, and let the hypothesis* (2.5) *be replaced by*

$$\sum_{j \in K_d, t_j \le t_k} e^{Mt_j} d_j \le \int_0^{t_k} e^{Ms} a(s)ds , \qquad \text{for } k \in K_d . \tag{2.6}$$

Then the conclusion of Theorem 2.1 *holds.*

Proof. It suffices to put $c_k = 1$ in the proof of Theorem 2.1. ∎

Note that we do not impose any restrictions on the sign of M both in Theorem 2.1 and Corollary 2.1. Hence, the results are valid independently of the sign of M.

The case when at the last impulsive instant t_p the constants c_p and d_p satisfy conditions $c_p = 0$ and $d_p \le 0$ is very interesting, since it permits us to give a comparison principle based on an *a priori* inequality on the boundary values of the function instead of the inequality on the initial data as in the results presented above.

THEOREM 2.2. *Let the functions* $u \in \Omega_1^1(J)$, $a \in L^1(J)$, $a(t) \geq 0$ *a.e.* $t \in J$, $\lambda \in \mathbb{R}$ *and let the constants* $d_k, c_k \in \mathbb{R}$, $c_k \geq 0$, $k = 1, 2, \dots, p$ *be such that* $c_p = 0$, $d_p \leq 0$, *and*

$$\begin{aligned} u'(t) + Mu(t) + a(t) \leq 0\,, \qquad & \text{a.e. } t \in J', \\ u(t_k^+) \leq c_k u(t_k) + d_k\,, \qquad & k = 1, 2, \dots, p\,, \\ u(0) \leq u(T) + \lambda\,. & \end{aligned}$$

Assume also that (2.5) *holds and*

$$\int_{t_p}^{T} e^{-M(T-s)} a(s) ds \geq \lambda\,.$$

Then $u(t) \leq 0$ *for each* $t \in J$.

Proof. We show that $u(0) \leq 0$. Since $u(t_p^+) \leq d_p \leq 0$, by Lemma 2.1, we have that

$$u(T) \leq -\int_{t_p}^{T} e^{-M(T-s)} a(s) ds\,.$$

Then, by the hypothesis of the theorem

$$u(0) \leq u(T) + \lambda \leq -\int_{t_p}^{T} e^{-M(T-s)} a(s) ds + \lambda \leq 0\,.$$

Thus, $u(0) \leq 0$, and we can apply Theorem 2.1 to conclude that $u \leq 0$ on J. ∎

We note that if $c_k = d_k = 0$, $k = 1, 2, \dots, p$, Theorem 2.2 reduces to Corollary 2.2 in [3], and thus it presents the generalization of the latter result.

If $d_p > 0$ it is also possible to show that the function u is non-positive, but we need to assume that $M > 0$ and $c_k = 1$, $k = 1, 2, \dots, p$. We prove this result for the particular case $a(t) = Mr$, with $r \in \mathbb{R}$, $r > 0$, in order to compare it with those known in the literature.

THEOREM 2.3. *Let the function* $u \in \Omega_1^1(J)$, *and let* $d_k \in \mathbb{R}$, $k = 1, 2, \dots, p$ *and* $r, M \in \mathbb{R}$, $r, M > 0$ *be constants such that*

$$\begin{aligned} u'(t) + Mu(t) + Mr \leq 0\,, \qquad & \text{a.e. } t \in J', \\ u(t_k^+) \leq u(t_k) + d_k\,, \qquad & k = 1, 2, \dots, p\,, \\ u(0) \leq u(T)\,. & \end{aligned}$$

Assume that

$$\sum_{k\in K_d} \frac{d_k}{(1-e^{-MT})} \le r\,. \tag{2.7}$$

Then $u(t) \le 0$ *for* $t \in J$.

Proof. Using again Lemma 2.1, we obtain

$$u(t) \le u(0)e^{-Mt} - \int_0^t e^{-M(t-s)}Mrds + \sum_{k\in K_d, t_k<t} e^{-M(t-t_k)}d_k\,. \tag{2.8}$$

For $t = T$,

$$u(0) \le u(T) \le u(0)e^{-MT} - \int_0^T e^{-M(T-s)}Mrds + \sum_{k\in K_d} e^{-M(T-t_k)}d_k\,.$$

By the latter inequality, we conclude that

$$(1-e^{-MT})u(0) \le e^{-MT}\Big(\sum_{k\in K_d} e^{Mt_k}d_k - \int_0^T e^{Ms}Mrds\Big)\,, \tag{2.9}$$

and, substituting (2.9) in (2.8), we have

$$u(t) \le \sum_{k\in K_d} g(t,t_k)d_k - \int_0^T g(t,s)Mrds\,, \tag{2.10}$$

where $g : J \times J \to \mathbb{R}$ is the Green's function for the boundary-value problem

$$\begin{aligned} u'(t) + Mu(t) &= 0\,, \qquad t \in J\,,\\ u(0) &= u(T)\,, \end{aligned}$$

that is,

$$g(t,s) = \frac{1}{1-e^{-MT}} \cdot \begin{cases} e^{-M(t-s)} & \text{if } 0 \le s < t \le T\,,\\ e^{-M(T+t-s)} & \text{if } 0 \le t \le s \le T\,. \end{cases}$$

We note that g has the following properties:

$$\begin{aligned} \frac{e^{-MT}}{1-e^{-MT}} \le g(t,s) \le \frac{1}{1-e^{-MT}}\,, &\qquad t,s \in J\,,\\ \int_0^T g(t,s)ds = \frac{1}{M}\,, &\qquad t \in J\,. \end{aligned} \tag{2.11}$$

From (2.7), (2.10) and (2.11), for each $t \in J$, we obtain

$$u(t) \leq \sum_{k \in K_d} g(t, t_k) d_k - \int_0^T g(t,s) Mr ds \leq \sum_{k \in K_d} \frac{d_k}{1 - e^{-MT}} - r \leq 0 .$$

The proof is complete. ∎

Remark 2.2. If we consider an arbitrary positive function from $L^1(J)$ instead of $a(t) = Mr$, we cannot calculate explicitly the integral

$$\int_0^T g(t,s) a(s) ds .$$

Nevertheless, using (2.11), it is possible to obtain the following estimate

$$-\int_0^T g(t,s) a(s) ds \leq \frac{e^{-MT}}{1 - e^{-MT}} \int_0^T a(s) ds ,$$

which permits us to establish the result similar to Theorem 2.3 for the general case.

3. Existence Results

In this section, we introduce new concepts of upper and lower solutions for the problem (1.1) and prove the validity of the method of upper and lower solutions.

Definition 3.1. We say that $\alpha \in \Omega_1^1(J)$ is a lower solution for the problem (1.1) if

$$\begin{aligned} &\alpha'(t) \leq f(t, \alpha(t)) , && \text{a.e. } t \in J', \\ &\alpha(t_k^+) \leq I_k(\alpha(t_k)) , && k \in \{1, 2, \ldots, p\} \setminus K_\alpha , \\ &\alpha(0) \leq \alpha(T) , \end{aligned}$$

where $K_\alpha \subset \{1, 2, \ldots, p\}$, and if for each $k \in K_\alpha$ and for some constant $M_\alpha \in \mathbb{R}$ there exists a constant $c_k^\alpha > 0$ such that

$$c_k^\alpha \int_{t_{k-1}}^{t_k} e^{-M_\alpha (t_k - s)} [f(s, \alpha(s)) - \alpha'(s)] ds \geq \alpha(t_k^+) - I_k(\alpha(t_k)) . \tag{3.1}$$

Analogously, $\beta \in \Omega_1^1(J)$ is an upper solution for (1.1) if

$$\begin{aligned} &\beta'(t) \geq f(t,\beta(t)), && \text{a.e. } t \in J',\\ &\beta(t_k^+) \geq I_k(\beta(t_k)), && k \in \{1,2,\dots,p\} \setminus K_\beta,\\ &\beta(0) \geq \beta(T), \end{aligned}$$

where $K_\beta \subset \{1,2,\dots,p\}$, and if for each $k \in K_\beta$ and for some constant $M_\beta \in \mathbb{R}$ there exists a constant $c_k^\beta > 0$ such that

$$c_k^\beta \int_{t_{k-1}}^{t_k} e^{-M_\beta(t_k-s)}[\beta'(s) - f(s,\beta(s))]ds \geq I_k(\beta(t_k)) - \beta(t_k^+). \tag{3.2}$$

To present the main result of this section, we need the following hypotheses:

(H1) There exists a constant $M \in \mathbb{R}$ such that

$$f(t,u) - f(t,v) \geq -M(u-v)$$

for a.e. $t \in J$ and for $u, v \in \mathbb{R}$, $\alpha(t) \leq v \leq u \leq \beta(t)$.

(H2) The functions I_k are continuous for $k = 1,2,\dots,p$. Moreover, for $k \notin K_\alpha \cup K_\beta$, I_k are nondecreasing, and if $c_k = \max\{c_k^\alpha, c_k^\beta\} > 0$ then for $k \in K_\alpha \cup K_\beta$ and for $\alpha(t_k) \leq y \leq x \leq \beta(t_k)$ we have that

$$I_k(y) - I_k(x) \leq c_k(y-x).$$

Remark 3.1. We note that (H2) implies, in particular, that I_k is nondecreasing, for $k = 1,2,\dots,p$.

THEOREM 3.1. *Let α, $\beta \in \Omega_1^1(J)$ be lower and upper solutions for* (1.1) *such that $\alpha \leq \beta$ on J. Suppose that* (H1)-(H2) *hold with $M = \min\{M_\alpha, M_\beta\}$. Then the problem* (1.1) *has at least one solution $u \in [\alpha,\beta]$.*

Proof. We define the following modified problem

$$\begin{aligned} &u'(t) = F(t,u(t)) - Mu(t), && \text{a.e. } t \in J',\\ &u(t_k^+) = I_k(\overline{u}(t_k)), && k \notin K_\alpha \cup K_\beta,\\ &u(t_k^+) = I_k(\overline{u}(t_k)) - c_k(\overline{u}(t_k) - u(t_k)), && k \in K_\alpha \cup K_\beta,\\ &u(0) = \overline{u}(T), \end{aligned} \tag{3.3}$$

where $F(t,u(t)) = f(t,\overline{u}(t)) + M\overline{u}(t)$, $\overline{u}(t) = \gamma(t,u(t))$ and $\gamma : J \times \mathbb{R} \to \mathbb{R}$, $\gamma(t,x) = \min\{\beta(t), \max\{x, \alpha(t)\}\}$.

We will show that there exists at least one solution of (3.3); then we prove that it satisfies inequalities $\alpha(t) \le u(t) \le \beta(t)$ on J. Thus, it is a solution of (1.1) too.

For $\lambda \in [0,1]$, we define the following problem

$$\begin{aligned} u'(t) + Mu(t) &= \lambda F(t, u(t)), && \text{a.e. } t \in J', \\ u(t_k^+) &= \lambda I_k(\overline{u}(t_k)), && k \notin K_\alpha \cup K_\beta, \\ u(t_k^+) - c_k u(t_k) &= \lambda (I_k(\overline{u}(t_k)) - c_k \overline{u}(t_k)), && k \in K_\alpha \cup K_\beta, \\ u(0) &= \lambda \overline{u}(T). \end{aligned} \tag{3.4}$$

Let $X = L^1(J_0) \times L^1(J_1) \times \cdots \times L^1(J_p)$, $Y = W^{1,1}(J_0) \times W^{1,1}(J_1) \times \cdots \times W^{1,1}(J_p)$ with $J_0 = [0, t_1]$, $J_1 = [t_1, t_2]$, ..., $J_p = [t_p, T]$. We define the operators

$$\mathcal{L} : Y \to X \times \mathbb{R}^{p+1} \qquad \text{and} \qquad \mathcal{N} : X \to Y \times \mathbb{R}^{p+1}$$

by

$$\begin{aligned} \mathcal{L}(u_0, u_1, \ldots, u_p) = \Big(&\mathcal{L}_0(u_0), \mathcal{L}_1(u_1), \ldots, \mathcal{L}_p(u_p), \\ &\overline{\mathcal{L}}_1(u_0, u_1), \ldots, \overline{\mathcal{L}}_p(u_{p-1}, u_p), u(0)\Big) \end{aligned}$$

$$\begin{aligned} \mathcal{L}_k(u) &= u'(\cdot) + Mu(\cdot), && k = 0, 1, \ldots, p, \\ \overline{\mathcal{L}}_k(u, v) &= v(t_k), && k \notin K_\alpha \cup K_\beta, \\ \overline{\mathcal{L}}_k(u, v) &= v(t_k) - c_k u(t_k), && k \in K_\alpha \cup K_\beta, \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}(u_0, u_1, \ldots, u_p) = \Big(&\mathcal{N}_0(u_0), \mathcal{N}_1(u_1), \ldots, \mathcal{N}_p(u_p), \\ &\overline{\mathcal{N}}_1(u_0), \ldots, \overline{\mathcal{N}}_p(u_{p-1}), \overline{v}(T)\Big) \end{aligned}$$

$$\begin{aligned} \mathcal{N}_k(u) &= F(\cdot, u(\cdot)), && k = 0, 1, \ldots, p, \\ \overline{\mathcal{N}}_k(u) &= I_k(\overline{u}(t_k)), && k \notin K_\alpha \cup K_\beta, \\ \overline{\mathcal{N}}_k(u) &= I_k(\overline{u}(t_k)) - c_k \overline{u}(t_k), && k \in K_\alpha \cup K_\beta. \end{aligned}$$

Then (3.4) is equivalent to the abstract equation

$$\mathcal{L}(u_0, u_1, \ldots, u_p) = \lambda \mathcal{N} \circ C(u_0, u_1, \ldots, u_p), \quad (u_0, u_1, \ldots, u_p) \in Y, \tag{3.5}$$

where $C : Y \to X$ is the compact embedding of Y onto X.

The operator $\mathcal{L}$ is one-to-one since the problem

$$\begin{aligned} u'(t) + Mu(t) &= \sigma_k(t)\,, && \text{a.e. } t \in J_k\,,\ k = 0,1,\dots,p\,, \\ u(t_k^+) &= B_k\,, && k \notin K_\alpha \cup K_\beta\,, \\ u(t_k^+) &= c_k u(t_k) + B_k\,, && k \notin K_\alpha \cup K_\beta\,, \\ u(0) &= B_0\,, \end{aligned}$$

with $\sigma_k \in L^1(J_k)$, $B_k \in \mathbb{R}$, $k = 0,1,\dots,p$, has the unique solution $u \in \Omega_1^1(J)$.

So, we can consider the inverse operator $\mathcal{L}^{-1}$. Then (3.5) is equivalent to

$$(u,v) = \lambda \mathcal{L}^{-1} \circ \mathcal{N} \circ C(u,v) = \lambda H(u,v)\,, \quad (u,v) \in Y\,,$$

where H is a compact operator. Now, since f satisfies the conditions of Carathéodory and F is bounded, the set of the solutions of $x = \lambda Hx$ is bounded in Y. By Schaefer's Theorem (see [14]) there exists at least one solution of (3.4) for $\lambda = 1$.

To show that $\alpha(t) \le u(t) \le \beta(t)$ on J, we define the function $v = \alpha - u$, which, by hypotheses and Remark 3.1, satisfies

$$\begin{aligned} v'(t) + Mv(t) &= \alpha'(t) - f(t,\alpha(t)) + f(t,\alpha(t)) + M\alpha(t) - f(t,\overline{u}(t)) - M\overline{u}(t) \\ &\le \alpha'(t) - f(t,\alpha(t))\,, \qquad \text{a.e. } t \in J\,, \end{aligned}$$

$$v(t_k^+) = \alpha(t_k^+) - u(t_k^+) \le I_k(\alpha(t_k)) - I_k(\overline{u}(t_k)) \le 0\,, \qquad k \notin K_\alpha\,,$$

$$\begin{aligned} v(t_k^+) &= \alpha(t_k^+) - u(t_k^+) \\ &= \alpha(t_k^+) - I_k(\alpha(t_k)) + I_k(\alpha(t_k)) - I_k(\overline{u}(t_k)) + c_k\overline{u}(t_k) - c_k u(t_k) \\ &\le c_k(\alpha(t_k) - u(t_k)) + \alpha(t_k^+) - I_k(\alpha(t_k)) \\ &= c_k v(t_k) + \alpha(t_k^+) - I_k(\alpha(t_k))\,, \qquad k \in K_\alpha \subset K_\alpha \cup K_\beta\,, \end{aligned}$$

$$v(0) = \alpha(0) - \overline{u}(T) \le 0\,.$$

We apply Theorem 2.1 to show that $v \le 0$ in J. Consequently, $\alpha \le u$ in J. Analogously, we can conclude that $u \le \beta$ in J, which completes the proof. ∎

Remark 3.2. We point out the tight relation between the functions f and I_k. We note also that it is easier to find the upper or lower solution if the constant M is small and the constants c_k are large. From this point of view it is useful to have the possibility to consider constants M with negative sign.

We note that, in certain sense, this result is more general that Theorem 5.3 in [12], since the constants M_α and M_β in our Theorem 3.1 can be non-positive and the hypotheses on the functions I_k are less restrictive.

In order to give a result for the case when $M_\alpha > 0$ and $M_\beta > 0$, we introduce the following new definitions of lower and upper solutions.

DEFINITION 3.2. We say that $\alpha \in \Omega_1^1(J)$ is a lower solution for (1.1) if there exists a constant $M_\alpha > 0$ such that

$$\begin{aligned} &\alpha'(t) \le f(t, \alpha(t)) - M_\alpha r_\alpha\,, && \text{a.e. } t \in J',\\ &\alpha(t_k^+) \le I_k(\alpha(t_k))\,, && k \in \{1, 2, \dots, p\} \setminus K_\alpha\,,\\ &\alpha(0) \le \alpha(T)\,, \end{aligned}$$

where $K_\alpha \subset \{1, 2, \dots, p\}$ and the following inequality holds:

$$\sum_{k \in K_\alpha} \frac{\alpha(t_k^+) - I_k(\alpha(t_k))}{(1 - e^{-M_\alpha T})} \le r_\alpha\,. \tag{3.6}$$

An upper solution is defined similarly by reversing the inequalities with (3.6) changed for

$$\sum_{k \in K_\beta} \frac{I_k(\beta(t_k)) - \beta(t_k^+)}{(1 - e^{-M_\beta T})} \le r_\beta\,.$$

THEOREM 3.2. *Let α, $\beta \in \Omega_1^1(J)$ be lower and upper solutions for (1.1) in the sense of Definition 3.2 such that $\alpha \le \beta$ on J. Suppose that (H1) holds with $M = \min\{M_\alpha, M_\beta\}$, and the functions $I_k^*(x) = I_k(x) - x$, $x \in \mathbb{R}$ are non-decreasing for $k = 1, 2, \dots, p$.*

Then the problem (1.1) has at least one solution in $[\alpha, \beta]$.

Proof. The only difference with the proof of Theorem 3.1 concerns the definition of the modified problem which is set up as follows:

$$\begin{aligned} u'(t) &= F(t, u(t)) - Mu(t)\,, && \text{a.e. } t \in J',\\ u(t_k^+) &= I_k(\overline{u}(t_k)) - \overline{u}(t_k) + u(t_k)\,, && k = 1, 2, \dots, p\,,\\ u(0) - u(T) &= 0\,. \end{aligned} \tag{3.7}$$

Futhermore, in order to show that the solution of this problem is between α and β, we use Theorem 2.3. ∎

Since we do not assume the functions I_k^* to be bounded for $k \in K_\alpha \cup K_\beta$, Theorem 3.2 provides existence of solution with definition of upper and lower solutions equivalent to that used in Theorem 5.3 of [12].

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