

**LOCAL PATHWISE SOLUTIONS TO STOCHASTIC EVOLUTION
EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS
WITH HURST PARAMETERS $H \in (1/3, 1/2]$**

MARÍA J. GARRIDO-ATIENZA, KENING LU, AND BJÖRN SCHMALFUSS

ABSTRACT. In this article we are concerned with the study of the existence and uniqueness of pathwise mild solutions to evolutions equations driven by a Hölder continuous function with Hölder exponent in $(1/3, 1/2)$. Our stochastic integral is a generalization of the well-known Young integral. To be more precise, the integral is defined by using a fractional integration by parts formula and it involves a tensor for which we need to formulate a new equation. From this it turns out that we have to solve a system consisting in a path and an area equations. In this paper we prove the existence of a unique *local* solution of the system of equations. The results can be applied to stochastic evolution equations with a non-linear diffusion coefficient driven by a fractional Brownian motion with Hurst parameter in $(1/3, 1/2]$, which in particular includes white noise.

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1. INTRODUCTION

In this article, we shall focus on the study of a local solution for the following kind of stochastic evolution equations

$$(1) \quad \begin{cases} du(t) &= Au(t)dt + G(u(t))d\omega(t), \\ u(0) &= u_0, \end{cases}$$

in a Hilbert-space V , where the noise input ω is a Hölder continuous function with Hölder exponent in the interval $(1/3, 1/2)$, A is the infinitesimal generator of an analytic semigroup $S(\cdot)$ on V and G is a nonlinear term satisfying certain assumptions which will be described in the next sections. As a particular case of driving noises we can consider a fractional Brownian motion B^H with Hurst parameter $H \in (1/3, 1/2]$. To be more precise, we will study (1) in the sense of mild solutions given by

$$(2) \quad u(t) = S(t)u_0 + \int_0^t S(t-r)G(u(r))d\omega.$$

Our interpretation of pathwise is that we obtain a solution of these stochastic equations which does not produce exceptional sets depending on the initial conditions. In the classical theory of stochastic evolution equations, i.e., stochastic evolution

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equations (SEEs) driven by Brownian motion $B^{1/2}$, stochastic Ito integrals are constructed to be a limit in probability of particular random variables defined only almost surely, where the exceptional sets may depend on the initial conditions, which is in contradiction with the cocycle property needed to define a random dynamical system. Pathwise results for that classical theory are only available for the white noise case ($G = \text{id}$) and a few special cases when $u \mapsto G(u)$ is linear.

During the last two decades different integration theories have been developed to treat more general noise inputs, and in particular, for tackling the fractional Brownian motion B^H . One of these attempts is given by the *Rough Path Theory*, and we refer to Lyons and Qian [18] and Friz and Victoir [8] for a comprehensive presentation of this theory. Some interesting papers dealing with the study of SEEs by using the rough path theory are [2], [3], [7], [13], [15], [6] and [14] among others. In particular, in this last paper the authors proved the existence of local mild solutions of stochastic SEEs driven by rough paths for β -Hölder-continuous paths ($\beta \in (1/3, 1/2]$) with a special quadratic nonlinearity.

A different technique called *Fractional Calculus* was developed by Zähle [24], who considered for a fractional Brownian motion with $H > 1/2$ the well-known Young integral. In contrast to the Ito-or Stratonovich integral, that integral can be defined in a *pathwise sense*, given by fractional derivatives, which allows a pathwise estimate of the integrals in terms of the integrand and the integrator using special norms. In [20] it is shown the existence and uniqueness of the solution of a finite-dimensional stochastic differential equation driven by a fractional Brownian motion for $H > 1/2$. These results were extended in [19] to show the existence of mild solutions for SEEs driven by fractional Brownian motion for $H > 1/2$.

Recently Hu and Nualart [16] have proved an existence and uniqueness result for finite-dimensional stochastic differential equations having coefficients which are sufficiently smooth and driven by a fractional Brownian motion B^H with $H \in (1/3, 1/2]$, for which they needed to formulate a second equation for the so-called *area* in the space of tensors. In our article we adapt the techniques in [16] to obtain a mild solution for (1). However, there are significant differences between our setting and the one in [16], as for instance that in order to define the area equation in the infinite-dimensional setting we have to construct an area object $\omega \otimes_S \omega$, depending on the noise path ω as well as on the semigroup S , satisfying useful properties as the Chen-equality.

Under general hypothesis on the nonlinearity G we derive the existence and uniqueness of a *local* pathwise mild solution u to (1). However, global existence is missing in this general context. Under some more restrictive conditions on G , in a forthcoming paper we will obtain the existence and uniqueness of a *global* mild pathwise solution, which in particular will guarantee that stochastic evolution equations like (1) and driven by an fBm B^H with $H \in (1/3, 1/2]$ generate random dynamical systems, a challenging and rather open problem to the best of our knowledge.

The article is organized as follows. In Section 2 we give the analytical background to present our theory. In Section 3 we present the so called fractional integration by parts method. Using this technique we can introduce pathwise stochastic integrals allowing us to formulate pathwise stochastic differential equations. In Section 4 we introduce mild path-area solutions. In addition, we formulate and solve a fixed-point equation having two components, a path- and an area-component. The role of the semigroup S in the area equation will be given in terms of a particular tensor

object $\omega \otimes_S \omega$. We also present an example to show a nonlinearity G that matches the abstract theory. The appendix section contains the proofs of some technical results.

Finally, we want to stress that two different constructions of the key tensor object $\omega \otimes_S \omega$ by using an approximation of the noise path by smooth paths can be found in [12]. One construction considers as driving noise an infinite-dimensional fractional Brownian motion B^H with $H \in (1/3, 1/2]$, while, in a less restrictive setting, the second one considers a Hilbert-valued trace-class Brownian motion $B^{1/2}$.

Furthermore, we refer to [10] for a short and recent announcement of our results.

2. PRELIMINARIES

Let $V = (V, (\cdot, \cdot), |\cdot|)$ be a separable Hilbert-space. On V we define A to be the negative and symmetric generator of an analytic semigroup S . We suppose that $-A$ has a point spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots$ tending to infinite where the associated eigenlements $(e_i)_{i \in \mathbb{N}}$ form a complete orthonormal system on V . $D((-A)^\kappa) = V_\kappa$ denotes the domain of $(-A)^\kappa$ for $\kappa \in \mathbb{R}$, and as usual, $L(V_\kappa, V_\zeta)$ denotes the space of continuous linear operators from V_κ into V_ζ , for $\kappa, \zeta \in \mathbb{R}$. We then have the following estimates for the semigroup S :

$$(3) \quad \|S(t)\|_{L(V_\kappa, V_\gamma)} = \|(-A)^\gamma S(t)\|_{L(V_\kappa, V)} \leq ct^{\kappa-\gamma}, \quad \text{for } \gamma \geq \kappa,$$

$$(4) \quad \|S(t) - \text{id}\|_{L(V_\sigma, V_\theta)} \leq ct^{\sigma-\theta}, \quad \text{for } \sigma - \theta \in [0, 1].$$

From these properties we can derive easily the following result:

Lemma 1. *For any $\nu, \eta, \mu \in [0, 1]$, $\kappa, \gamma, \rho \in \mathbb{R}$ such that $\kappa \leq \gamma + \mu$, there exists a constant $c > 0$ such that for $0 < q < r < s < t$ we have that*

$$\|S(t-r) - S(t-q)\|_{L(V_\kappa, V_\gamma)} \leq c(r-q)^\mu (t-r)^{-\mu-\gamma+\kappa},$$

$$\begin{aligned} & \|S(t-r) - S(s-r) - S(t-q) + S(s-q)\|_{L(V_\rho, V_\rho)} \\ & \leq c(t-s)^\eta (r-q)^\nu (s-r)^{-(\nu+\eta)}. \end{aligned}$$

Throughout the whole paper we will write very often a constant c . This constant can change from line to line. However this constant is always chosen independent of time parameters contained in a fixed interval $[0, T]$.

Let $V \times V$ and $V \otimes V$ be the cartesian product and the tensor product of V , see [17]. The norm of $V \otimes V$ is denoted by $\|\cdot\|$. For $x, y \in V$ we denote by $x \otimes_V y$ the rank-one tensor of $V \otimes V$. Then $(e_i \otimes_V e_j)_{i,j \in \mathbb{N}}$ is a complete orthonormal system of $V \otimes V$ where $(e_i)_{i \in \mathbb{N}}$ can be any complete orthonormal system for V , although for the following we choose the orthonormal system given at the beginning of this section. Let $(\hat{V}, |\cdot|_{\hat{V}}, (\cdot, \cdot)_{\hat{V}})$ be another separable Hilbert-space. By $L_2(V, \hat{V})$ ($L_2(V \times V, \hat{V})$) we denote the Hilbert-Schmidt operators from V ($V \times V$) to \hat{V} . In particular $G \in L_2(V \times V, \hat{V})$ if and only if

$$\sum_{i,j} |G(e_i, e_j)|_{\hat{V}}^2 < \infty.$$

We note that $G \in L_2(V \times V, \hat{V})$ can be *extended* to a linear operator \hat{G} defined on $V \otimes V$ such that $\hat{G} \in L_2(V \otimes V, \hat{V})$, see [17] Chapter 2.6. More precisely, we can construct a weak Hilbert-Schmidt mapping $p : V \times V \rightarrow V \otimes V$ where

$p(e_i, e_j) = e_i \otimes_V e_j$ for $i, j \in \mathbb{N}$. Then \hat{G} on $V \otimes V$ is determined by factorization such that $G = \hat{G}p$. In addition, we have

$$\|\hat{G}\|_{L_2(V \otimes V, \hat{V})}^2 := \sum_{i,j} |\hat{G}(e_i \otimes_V e_j)|_{\hat{V}}^2 = \sum_{i,j} |G(e_i, e_j)|_{\hat{V}}^2 = \|G\|_{L_2(V \times V, \hat{V})}^2.$$

In the following we will write for \hat{G} also the symbol G .

Let us now describe the coefficient of the evolution equation that we have in mind.

Lemma 2. *Let \hat{V} be a subspace of V . Assume that the mapping $G : V \rightarrow L_2(V, \hat{V})$ is three times continuously Fréchet-differentiable with bounded first, second and third derivatives $DG(u)$, $D^2G(u)$ and $D^3G(u)$, for $u \in V$. Let us denote, respectively, by c_{DG} , c_{D^2G} and c_{D^3G} the bounds for DG , D^2G and D^3G , and let $c_G = \|G(0)\|_{L_2(V, \hat{V})}$. Then, for $u_1, u_2, v_1, v_2 \in V$, we have*

- $\|G(u_1)\|_{L_2(V, \hat{V})} \leq c_G + c_{DG}|u_1|,$
- $\|G(u_1) - G(v_1)\|_{L_2(V, \hat{V})} \leq c_{DG}|u_1 - v_1|,$
- $\|DG(u_1) - DG(v_1)\|_{L_2(V \times V, \hat{V})} \leq c_{D^2G}|u_1 - v_1|,$
- $\|G(u_1) - G(u_2) - DG(u_2)(u_1 - u_2)\|_{L_2(V, \hat{V})} \leq c_{D^2G}|u_1 - u_2|^2,$
- $\|G(u_1) - G(v_1) - (G(u_2) - G(v_2))\|_{L_2(V, \hat{V})} \leq c_{DG}|u_1 - v_1 - (u_2 - v_2)|$
 $+ c_{D^2G}|u_1 - u_2|(|u_1 - v_1| + |u_2 - v_2|),$
- $\|DG(u_1) - DG(v_1) - (DG(u_2) - DG(v_2))\|_{L_2(V \times V, \hat{V})}$
 $\leq c_{D^2G}|u_1 - v_1 - (u_2 - v_2)| + c_{D^3G}|u_1 - u_2|(|u_1 - v_1| + |u_2 - v_2|).$
- $\|G(u_1) - G(u_2) - DG(u_2)(u_1 - u_2) - (G(v_1) - G(v_2) - DG(v_2)(v_1 - v_2))\|_{L_2(V, \hat{V})}$
 $\leq c_{D^2G}(|u_1 - u_2| + |v_1 - v_2|)|u_1 - v_1 - (u_2 - v_2)|$
 $+ c_{D^3G}|v_1 - v_2||u_2 - v_2|(|u_1 - u_2| + |u_1 - v_1 - (u_2 - v_2)|).$

These estimates follow by the mean value theorem; for a proof of the last one see [20].

Notice that, in particular, $DG : V \rightarrow L_2(V, L_2(V, \hat{V}))$ (or equivalently, $DG : V \rightarrow L_2(V \times V, \hat{V})$) is a bilinear map, that can be extended to $DG : V \rightarrow L_2(V \otimes V, \hat{V})$, and $D^2G(u)$ is a trilinear map.

Next we introduce some function spaces. Let $T > 0$. For $\beta \in (0, 1]$, we consider the Banach-space of β -Hölder-continuous functions on $[0, T]$ with values in V , denoted by $C_\beta([0, T]; V)$, with the seminorm

$$\|u\|_\beta = \sup_{0 \leq t \leq T} |u(t)| + \|u\|_\beta, \quad \|u\|_\beta = \sup_{0 \leq s < t \leq T} \frac{|u(t) - u(s)|}{(t - s)^\beta}.$$

If $\beta = 1$ we call these functions Lipschitz-continuous. Let $C_{\beta, \sim}([0, T]; V)$ be the space of functions on $[0, T]$ with values in V and with norm

$$\|u\|_{\beta, \sim} = \sup_{0 \leq t \leq T} |u(t)| + \sup_{0 < s < t \leq T} s^\beta \frac{|u(t) - u(s)|}{(t - s)^\beta}.$$

Lemma 3. $C_{\beta, \sim}([0, T], V)$ is a Banach-space.

The proof of this result can be found in Chen *et al.* [4].

Let $\Delta_{0, T}$ be the triangle $\{(s, t) : 0 < s \leq t \leq T\}$. For $\beta + \beta' < 1$, $\beta \leq \beta'$ we introduce the space $C_{\beta + \beta', \sim}(\Delta_{0, T}, V \otimes V)$ of continuous functions v defined on $\Delta_{0, T}$, which are zero for $0 < s = t$, such that

$$\|v\|_{\beta + \beta', \sim} = \sup_{0 < s < t \leq T} s^\beta \frac{\|v(s, t)\|}{(t - s)^{\beta + \beta'}} < \infty.$$

These functions may not be defined for $s = 0$ and can have a singularity for (s, t) , $s = 0$.

Lemma 4. The space $C_{\beta + \beta', \sim}(\Delta_{0, T}; V \otimes V)$ is a Banach-space.

The proof is similar to the proof of Lemma 3 and therefore we omit here.

Let us define $\bar{\Delta}_{0, T} = \{(s, t) : 0 \leq s \leq t \leq T\}$ and consider the Banach-space $C_{\beta + \beta'}(\bar{\Delta}_{0, T}; V \otimes V)$ of continuous functions v defined on $\bar{\Delta}_{0, T}$, which are zero for $s = t$, equipped with the norm

$$\|v\|_{\beta + \beta'} = \sup_{0 \leq s < t \leq T} \frac{\|v(s, t)\|}{(t - s)^{\beta + \beta'}} < \infty.$$

We often use the following integral formula: for every $s < t$, $\mu, \nu > -1$

$$(5) \quad \int_s^t (r - s)^\mu (t - r)^\nu dr = c(t - s)^{\mu + \nu + 1}$$

where c only depends on μ, ν . This property follows by the definition of the Beta function simply by performing a suitable change of variable.

3. FRACTIONAL CALCULUS

In this paper the main instrument to treat (2) is fractional calculus. In this section we present the main features of this theory. We are going to assume that for some $T > 0$ we have that $\omega \in C_{\beta'}([0, T]; V)$, $u \in C_{\beta, \sim}([0, T]; V)$ and $v \in C_{\beta + \beta', \sim}(\Delta_{0, T}; V \otimes V)$ for $1/3 < \beta < \beta' < 1/2$, and that this triple of elements satisfies the Chen-equality given by

$$(6) \quad v(s, r) + v(r, t) + (u(r) - u(s)) \otimes_V (\omega(t) - \omega(r)) = v(s, t)$$

for $0 < s \leq r \leq t \leq T$. We would like to emphasize that when ω is smooth an example for v is given by

$$(7) \quad (u \otimes \omega)(r, t) = \int_r^t (u(q) - u(r)) \otimes_V d\omega(q).$$

This tensor area is clearly well defined and satisfies, for $0 < r < t$,

$$(8) \quad \|(u \otimes \omega)(r, t)\| \leq \frac{c}{r^\beta} \|u\|_{\beta, \sim} \|\omega\|_{C_1} (t - r)^{1 + \beta} \leq \frac{c}{r^\beta} (t - r)^{\beta + \beta'},$$

and therefore $(u \otimes \omega) \in C_{\beta + \beta', \sim}(\Delta_{0, T}; V \otimes V)$. Moreover, the Chen-equality easily follows in this case.

Let $\alpha \in (0, 1)$. We define the right hand side fractional derivative of order α of u and the left hand side fractional derivative of order $1 - \alpha$ of $\omega_{t-}(\cdot) := \omega(\cdot) - \omega(t)$, given for $0 < s \leq r \leq t$ by the expressions

$$\begin{aligned} D_{s+}^{\alpha} u[r] &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{u(r)}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{u(r) - u(q)}{(r-q)^{1+\alpha}} dq \right) \\ D_{t-}^{1-\alpha} \omega_{t-}[r] &= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\omega(r) - \omega(t)}{(t-r)^{1-\alpha}} + (1-\alpha) \int_r^t \frac{\omega(r) - \omega(q)}{(q-r)^{2-\alpha}} dq \right), \end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function. For tensor valued elements v and for $0 < r < t$ we define

$$\mathcal{D}_{t-}^{1-\alpha} v[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{v(r, t)}{(t-r)^{1-\alpha}} + (1-\alpha) \int_r^t \frac{v(r, q)}{(q-r)^{2-\alpha}} dq \right).$$

Lemma 5. *Suppose that $\beta > \alpha$. Then there exists a constant $c > 0$ such that for $0 \leq s < r \leq t \leq T$*

$$|D_{s+}^{\alpha} u(\cdot)[r]| \leq \frac{c \|u\|_{\beta, \sim}}{(r-s)^{\alpha}}, \quad |D_{t-}^{1-\alpha} \omega_{t-}[r]| \leq c \|\omega\|_{\beta'} (t-r)^{\alpha+\beta'-1},$$

and for $0 < r < q < t$

$$(9) \quad \|\mathcal{D}_{t-}^{1-\alpha} v[q] - \mathcal{D}_{t-}^{1-\alpha} v[r]\| \leq \frac{c}{r^{\beta}} (\|u\|_{\beta, \sim} \|\omega\|_{\beta'} + \|v\|_{\beta+\beta', \sim}) (q-r)^{\alpha+\beta+\beta'-1}.$$

The proof of the two first inequalities follows straightforwardly, and the proof of (9) is similar to the one of Lemma 6.3 in [16] with the difference that in that paper the authors work in different function spaces. We refer the reader to Lemma 21 and Corollary 29 in the Appendix section.

As an extension of the fractional derivative of order α , for the mapping $G : V \mapsto L_2(V, \hat{V})$ and $s < r$ we introduce the so-called compensated fractional derivative of order α given by

$$\begin{aligned} \hat{D}_{s+}^{\alpha} G(u(\cdot))[r] &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{G(u(r))}{(r-s)^{\alpha}} \right. \\ &\quad \left. + \alpha \int_s^r \frac{G(u(r)) - G(u(q)) - DG(u(q))(u(r) - u(q))}{(r-q)^{1+\alpha}} dq \right). \end{aligned}$$

It is immediate to prove the following result:

Lemma 6. *Suppose that $\alpha < 2\beta$ and G satisfies the assumptions of Lemma 2. Then there exists a positive constant c such that for every $0 \leq s < r \leq T$*

$$|\hat{D}_{s+}^{\alpha} G(u(\cdot))[r]| \leq \frac{c(1 + \|u\|_{\beta, \sim}^2)}{(r-s)^{\alpha}}.$$

Let us assume for a while that $V, \hat{V} = \mathbb{R}$. We first recall the following useful property which is an integration by parts formula

$$(10) \quad (-1)^{\alpha} \int_s^t D_{s+}^{\alpha} u[r] \omega(r) dr = \int_s^t u(r) D_{t-}^{\alpha} \omega[r] dr,$$

see Zähle [24], formula (21). For $\beta > \alpha$ and $\alpha + \beta' > 1$ the fractional integral is given by

$$\int_s^t u d\omega := (-1)^{\alpha} \int_s^t D_{s+}^{\alpha} u[r] D_{t-}^{1-\alpha} \omega_{t-}[r] dr,$$

see again [24], which is a version of the Young integral. By Lemma 5 and the property (5), for the above integral it is easy to derive that

$$\left| \int_s^t u d\omega \right| \leq c \|u\|_{\beta, \sim} \|\omega\|_{\beta'} (t-s)^{\beta'}.$$

For Hölder-continuous u and ω this kind of integral was defined by Young [23]. However, our function u is not Hölder-continuous in the strong sense but $u \in C_{\beta, \sim}([0, T]; \mathbb{R})$, in which case that integral is also well defined in the above sense since, according to [24], what we need is that $u \in I_{s+}^{\alpha}(L^p((s, t); \mathbb{R}))$, $u(s+)$ bounded and $\omega_{t-} \in I_{t-}^{1-\alpha}(L^q((s, t); \mathbb{R}))$, with $\alpha p < 1$, $p^{-1} + q^{-1} \leq 1$ (for the definition of these spaces we refer to Samko *et al.* [22]). In particular, under our conditions on α , β and β' , we know that $\omega_{t-} \in I_{t-}^{1-\alpha}(L^q((s, t); \mathbb{R}))$ for any $q > 1$ and $u \in I_{s+}^{\alpha}(L^p((s, t); \mathbb{R}))$ when $\alpha p < 1$, see Theorem 13.2 of [22].

Next we introduce integrals of fractional type with values in a separable Hilbert-space. To do that, we need a new separable Hilbert-space $(\tilde{V}, |\cdot|_{\tilde{V}}, (\cdot, \cdot)_{\tilde{V}})$.

Lemma 7. *Assume $\beta > \alpha$ and $\alpha + \beta' > 1$. Let \hat{V}, \tilde{V} be two separable Hilbert-spaces, being $(\tilde{e}_i)_{i \in \mathbb{N}}$ and $(f_j)_{j \in \mathbb{N}}$ complete orthonormal basis of \tilde{V} and \hat{V} resp., and let*

$$[s, t] \ni r \mapsto F(r) \in L_2(\tilde{V}, \hat{V}), \quad [s, t] \ni r \mapsto \xi(r) \in \tilde{V}$$

be measurable functions such that $F \in C_{\beta, \sim}([0, T]; L_2(\tilde{V}, \hat{V}))$, $\xi \in C_{\beta'}([0, T]; \tilde{V})$ and $r \mapsto \|D_{s+}^{\alpha} F[r]\|_{L_2(\tilde{V}, \hat{V})} |D_{t-}^{1-\alpha} \xi[r]|_{\tilde{V}}$ is Lebesgue-integrable. Then for $0 \leq s \leq r \leq t \leq T$ we can define

$$\int_s^t F(r) d\xi(r) := (-1)^{\alpha} \sum_j \left(\sum_i \int_s^t D_{s+}^{\alpha}(f_j, F(\cdot) \tilde{e}_i)_{\tilde{V}}[r] D_{t-}^{1-\alpha}(\tilde{e}_i, \xi(\cdot))_{\tilde{V}}[r] dr \right) f_j.$$

That this expression is well defined follows by

$$\begin{aligned} & \left| \int_s^t F(r) d\xi(r) \right|_{\tilde{V}} \\ &= \left(\sum_j \left(\sum_i \int_s^t D_{s+}^{\alpha}(f_j, F(\cdot) \tilde{e}_i)_{\tilde{V}}[r] D_{t-}^{1-\alpha}(\tilde{e}_i, \xi(\cdot))_{\tilde{V}}[r] dr \right)^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_j \left(\int_s^t \left(\sum_i (D_{s+}^{\alpha}(f_j, F(\cdot) \tilde{e}_i)_{\tilde{V}}[r])^2 \sum_i (D_{t-}^{1-\alpha}(\tilde{e}_i, \xi(\cdot))_{\tilde{V}}[r])^2 \right)^{\frac{1}{2}} dr \right)^2 \right)^{\frac{1}{2}} \\ &\leq \int_s^t \left(\sum_{j,i} (D_{s+}^{\alpha}(f_j, F(\cdot) \tilde{e}_i)_{\tilde{V}}[r])^2 \sum_i (D_{t-}^{1-\alpha}(\tilde{e}_i, \xi(\cdot))_{\tilde{V}}[r])^2 \right)^{\frac{1}{2}} dr \\ &\leq \int_s^t \left(\sum_{j,i} (D_{s+}^{\alpha}(f_j, F(\cdot) \tilde{e}_i)_{\tilde{V}}[r])^2 \right)^{\frac{1}{2}} \left(\sum_i (D_{t-}^{1-\alpha}(\tilde{e}_i, \xi(\cdot))_{\tilde{V}}[r])^2 \right)^{\frac{1}{2}} dr \\ (11) \quad &= \int_s^t \|D_{s+}^{\alpha} F[r]\|_{L_2(\tilde{V}, \hat{V})} |D_{t-}^{1-\alpha} \xi[r]|_{\tilde{V}} dr < \infty. \end{aligned}$$

Observe that this last integral is finite since

$$\begin{aligned}
& \|D_{s+}^\alpha F[r]\|_{L_2(\tilde{V}, \hat{V})} = \left(\sum_{j,i} (D_{s+}^\alpha (f_j, F(\cdot)\tilde{e}_i)_{\hat{V}}[r])^2 \right)^{\frac{1}{2}} \\
& = \left(\sum_{j,i} \left(\frac{1}{\Gamma(1-\alpha)} \left(\frac{(f_j, F(r)\tilde{e}_i)_{\hat{V}}}{(r-s)^\alpha} + \alpha \int_s^r \frac{(f_j, F(r)\tilde{e}_i)_{\hat{V}} - (f_j, F(q)\tilde{e}_i)_{\hat{V}}}{(r-q)^{1+\alpha}} dq \right) \right)^2 \right)^{\frac{1}{2}} \\
& \leq \sqrt{2}c \left(\frac{(\sum_{j,i} (f_j, F(r)\tilde{e}_i)_{\hat{V}}^2)^{\frac{1}{2}}}{(r-s)^\alpha} + \left(\sum_{j,i} \left(\int_s^r \frac{(f_j, F(r)\tilde{e}_i)_{\hat{V}} - (f_j, F(q)\tilde{e}_i)_{\hat{V}}}{(r-q)^{1+\alpha}} dq \right)^2 \right)^{\frac{1}{2}} \right) \\
& \leq \sqrt{2}c \left(\frac{\|F(r)\|_{L_2(\tilde{V}, \hat{V})}}{(r-s)^\alpha} + \int_s^r \frac{\|F(r) - F(q)\|_{L_2(\tilde{V}, \hat{V})}}{(r-q)^{1+\alpha}} dq \right) \\
& \leq qc(r-s)^{-\alpha} \|F\|_{C_{\beta, \sim}([0, T]; L_2(\tilde{V}, \hat{V}))},
\end{aligned}$$

and $|D_{t-}^{1-\alpha} \xi[r]|_{\hat{V}} \leq c \|\xi\|_{\beta'} (t-r)^{\alpha+\beta'-1}$ (see Lemma 5), so it suffices to apply (5).

Now we can apply Lemma 7 to define integrals of fractional type with values in a separable Hilbert-space, as well as to consider integrators with values in V or $V \otimes V$. For example, consider an integrand of the type $G(u(r))$ where $u \in C_{\beta, \sim}([0, T]; V)$, and $\beta < \alpha < 2\beta$, $\alpha + \beta' > 1$, $\beta + 1 > 2\alpha$. Note that if DG is bounded then $G(u) \in C_{\beta, \sim}([0, T], V)$, but since we do not assume that $\beta > \alpha$, then $D_{s+}^\alpha G(u(\cdot))$ is not well-defined. However, we can apply Lemma 7 in the following way

$$\begin{aligned}
(12) \quad & \int_s^t G(u(\cdot)) d\omega = (-1)^\alpha \int_s^t D_{s+}^\alpha DG(u(\cdot))(u(\cdot) - u(s), \cdot)[r] D_{t-}^{1-\alpha} \omega_{t-}[r] dr \\
& + (-1)^\alpha \int_s^t D_{s+}^\alpha (G(u(\cdot)) - DG(u(\cdot))(u(\cdot) - u(s), \cdot))[r] D_{t-}^{1-\alpha} \omega_{t-}[r] dr.
\end{aligned}$$

Since u, ω, v are coupled by the Chen-equality (6), we get

$$\begin{aligned}
(13) \quad & D_{t-}^{1-\alpha} v(s, \cdot)_{t-}[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{v(s, r) - v(s, t)}{(t-r)^\alpha} + (1-\alpha) \int_r^t \frac{v(s, r) - v(s, q)}{(q-r)^{2-\alpha}} dq \right) \\
& = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{-v(r, t) - (u(r) - u(s)) \otimes_V (\omega(t) - \omega(r))}{(t-r)^\alpha} \right. \\
& \quad \left. + (1-\alpha) \int_r^t \frac{-v(r, q) - (u(r) - u(s)) \otimes_V (\omega(q) - \omega(r))}{(q-r)^{2-\alpha}} dq \right) \\
& = -D_{t-}^{1-\alpha} v[r] + (u(r) - u(s)) \otimes_V D_{t-}^{1-\alpha} \omega_{t-}[r].
\end{aligned}$$

Similarly, it is easy to derive that

$$\begin{aligned}
& D_{s+}^\alpha (G(u(\cdot)) - DG(u(\cdot))(u(\cdot) - u(s), \cdot))[r] \\
& = \hat{D}_{s+}^\alpha G(u(\cdot))[r] - D_{s+}^\alpha DG(u(\cdot))[r](u(r) - u(s), \cdot),
\end{aligned}$$

and thus, coming back to (12) we obtain that

$$\begin{aligned}
(14) \quad \int_s^t G(u(\cdot))d\omega &= (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha} \omega_{t-}[r] dr \\
&\quad - (-1)^\alpha \int_s^t D_{s+}^\alpha DG(u(\cdot))[r] \mathcal{D}_{t-}^{1-\alpha} v(\cdot, t)[r] dr \\
&= (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha} \omega_{t-}[r] dr \\
&\quad - (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} DG(u(\cdot))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} v(\cdot, t)[r] dr,
\end{aligned}$$

where the last equality is true thanks to the fractional integration by parts formula (10). The last integral on the right hand side of (14) is well-defined due to, for $0 \leq s \leq r \leq t \leq T$,

$$(15) \quad \|D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} v[r]\| \leq c(\|v\|_{\beta+\beta', \sim} + \|u\|_{\beta, \sim} \|\omega\|_{\beta'}) (r-s)^{-\beta} (t-r)^{\beta+\beta'+2\alpha-2}$$

(see Lemma 21 above) and hence this integral can be defined by using Lemma 7.

Remark 8. *It may be checked that the integral given in Lemma 7 can be also defined for an integrator like $F(r) = S(t-r)G(u(r))$, which is not Hölder-continuous at zero. The reason is because the f_j -modes of this expression are Hölder continuous. Furthermore, the integrability condition (11) is satisfied by the regularity properties of the analytic semigroup S and the regularity of G and u .*

Finally, on account of (14) and the estimates of the different fractional derivatives that take part in that expression, we can establish the following result:

Lemma 9. *Suppose u, v, ω satisfy the assumptions from the beginning of this section and G satisfies the assumptions of Lemma 2. Let $\beta < \alpha < 2\beta$, $\alpha + \beta' > 1$, $\beta + 1 > 2\alpha$. Then for $0 \leq s \leq t \leq T$ we have*

$$\left| \int_s^t G(u)d\omega \right| \leq c((1 + \|u\|_{\beta, \sim}^2) \|\omega\|_{\beta'} + \|u\|_{\beta, \sim} \|v\|_{\beta+\beta', \sim}) (t-s)^{\beta'}.$$

4. PATH-AREA-SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS

In this section we give the definition of a mild solution to (2) and establish a result about the *local* existence and uniqueness of such solution. In order to understand the notion of mild solution to (2), in a first step we consider the case in which the driving noise is regular, to later on consider the Hölder case in which we are interested in.

Note that we could also add a nonlinear diffusion term F on the right-hand side of the equation in (1). Nevertheless, to simplify the whole presentation we have not considered it since the dt -nonlinearity is not the interesting problem to be treated in the paper.

4.1. System (2) driven by smooth paths ω . In this situation, since A has the properties of Section 2 and DG is bounded, we remind that for any $u_0 \in V$ there exists a unique solution $u \in C([0, T]; V)$ of (2) for any $T > 0$, see [21]. In addition, applying the properties (3), (4) we obtain that this solution is in $C_{\gamma, \sim}([0, T]; V)$ for any $\gamma \in (0, 1)$. However, we do not obtain that the solution is in $C_\gamma([0, T]; V)$, due to the fact that $t \mapsto S(t)u_0$ is not Hölder-continuous in general but in $C_{\gamma, \sim}([0, T]; V)$.

Since we want to find the appropriate definition of mild solution to (2) when the driving noise is *only* Hölder continuous, in what follows we will use the fractional integration techniques of the previous section to rewrite (2) in a way that allows us to shed light on that issue.

Assuming that G satisfies the conditions of Lemma 2, similarly to the expression (14) of last section, we can rewrite (2) as

$$(16) \quad \begin{aligned} u(t) &= S(t)u_0 + (-1)^\alpha \int_0^t \hat{D}_{0+}^\alpha (S(t-\cdot)G(u(\cdot)))[r] D_{t-}^{1-\alpha} \omega_{t-}[r] dr \\ &\quad - (-1)^{2\alpha-1} \int_0^t D_{0+}^{2\alpha-1} (S(t-\cdot)DG(u(\cdot)))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} (u \otimes \omega)[r] dr, \end{aligned}$$

where the tensor $(u \otimes \omega)$ is given by (7). Looking at (16) we realize that we also need an equation to determine the corresponding counterpart of $(u \otimes \omega)$. In order to get such an equation, let us introduce the mapping $(\omega \otimes_S \omega)(s, t)$ given by

$$(17) \quad \begin{aligned} L_2(V, \hat{V}) \ni E &\mapsto E(\omega \otimes_S \omega)(s, t) = \int_s^t \int_s^\xi S(\xi-r) E \omega'(r) dr \otimes_V \omega'(\xi) d\xi \\ &= \int_s^t \int_r^t S(\xi-r) E \omega'(r) d\xi \otimes_V \omega'(\xi) dr \end{aligned}$$

Under weak conditions $(\omega \otimes_S \omega)$ is at least in $C_{2\beta'}(\bar{\Delta}_{0,T}, L_2(L_2(V, \hat{V}), V \otimes V))$, for which it suffices to execute the corresponding integrations. In addition, we set

$$(18) \quad \begin{aligned} e \in V &\mapsto \omega_S(s, t)e := (-1)^{-\alpha} \int_s^t (S(\xi-s)e) \otimes_V \omega'(\xi) d\xi, \\ E \in L_2(V, \hat{V}) &\mapsto S_\omega(s, t)E := \int_s^t S(t-r) E \omega'(r) dr. \end{aligned}$$

It is an easy exercise to check that $(\omega \otimes_S \omega)$ satisfies the Chen–equality

$$\begin{aligned} E(\omega \otimes_S \omega)(s, r) + E(\omega \otimes_S \omega)(r, t) + (-1)^{-\alpha} \omega_S(r, t) S_\omega(s, r) E \\ = E(\omega \otimes_S \omega)(s, t), \end{aligned}$$

which follows by joining the integrals given in (17) over $\bar{\Delta}_{s,r}$ and $\bar{\Delta}_{r,t}$ and the double integral over the rectangle $[s, r] \times [r, t]$ given by the composition of the expressions in (18).

Since ω is regular, the tensor $(u \otimes \omega)$ given by (7) can be expressed as

$$(19) \quad \begin{aligned} (u \otimes \omega)(s, t) &= \int_s^t (S(\xi-s) - \text{id})u(s) \otimes_V \omega'(\xi) d\xi \\ &\quad + \int_s^t \int_s^\xi S(\xi-r) G(u(r)) \omega'(r) dr \otimes_V \omega'(\xi) d\xi, \end{aligned}$$

and, exchanging the order of integration, the last integral of (19) can be written as

$$(20) \quad \int_s^t G(u(r)) D_1(\omega \otimes_S \omega)(r, t) dr.$$

Interpreting (20) in the sense of (14) and coming back to (19), it gives us

$$\begin{aligned}
(u \otimes \omega)(s, t) &= (-1)^\alpha \int_s^t D_{s+}^\alpha ((S(\cdot - s) - \text{id})u(s))[\xi] \otimes_V D_{t-}^{1-\alpha} \omega[\xi] d\xi \\
(21) \quad &- (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha} (\omega \otimes_S \omega)(\cdot, t)_{t-}[r] dr \\
&+ (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} DG(u(\cdot))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} (u \otimes (\omega \otimes_S \omega)(t))[r] dr,
\end{aligned}$$

where the element $w = (u \otimes (\omega \otimes_S \omega))$ is defined by

$$\begin{aligned}
(22) \quad \tilde{E}w(t, s, q) &:= \int_s^q \tilde{E}(u(r) - u(s), \cdot) D_1(\omega \otimes_S \omega)(r, t) dr \\
&= - \int_s^q \int_r^t S(\xi - r) \tilde{E}(u(r) - u(s), \omega'(r)) \otimes_V \omega'(\xi) d\xi dr \\
&= - (-1)^{-\alpha} \int_s^q \omega_S(r, t) \tilde{E}(u(r) - u(s), \omega'(r)) dr
\end{aligned}$$

for $\tilde{E} \in L_2(V \otimes V, \hat{V})$. Therefore, to evaluate the last integral of (21) we have to give a meaning to (22).

Lemma 10. *Let $\tilde{E} \in L_2(V \otimes V, \hat{V})$. Then for $0 < s \leq q \leq t \leq T$*

$$\begin{aligned}
\tilde{E}w(t, s, q) &= - \int_s^q \hat{D}_{s+}^\alpha \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \omega_{q-}[r] dr \\
&+ (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} (\omega_S(t) \otimes \omega)[r] dr \\
&+ (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] \tilde{E} D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} (u \otimes \omega)(\cdot, q)[r] dr,
\end{aligned}$$

where, for $s \leq \tau \leq t$ and $E \in L_2(V, \hat{V})$, $(\omega_S(t) \otimes \omega)$ satisfies

$$\begin{aligned}
E(\omega_S(t) \otimes \omega)(s, \tau) &= \int_s^\tau (\omega_S(r, t) - \omega_S(s, t)) E d\omega(r) \\
&= \int_s^\tau (\omega_S(r, t) - \omega_S(r, \tau)) E d\omega(r) + \int_s^\tau \omega_S(r, \tau) E d\omega(r) - \int_s^\tau \omega_S(s, t) E d\omega(r) \\
&= \omega_S(\tau, t) \int_s^\tau (S(\tau - r) - \text{id}) E d\omega(r) + (-1)^{-\alpha} E(\omega \otimes_S \omega)(s, \tau) \\
&\quad + (\omega_S(\tau, t) - \omega_S(s, t)) E(\omega(\tau) - \omega(s)).
\end{aligned}$$

The proof of this result is in the Appendix section. Note that joining the integrals on the right hand side of (22) with respect their domain of integration we obtain

$$\begin{aligned}
(23) \quad \tilde{E}w(t, s, r) + \tilde{E}w(t, r, q) - \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, q) \\
= \tilde{E}w(t, s, q) + \omega_S(q, t) S_\omega(q, r) \tilde{E}(u(r) - u(s), \cdot),
\end{aligned}$$

that for $q = t$ becomes

$$(24) \quad \tilde{E}w(t, s, r) + \tilde{E}w(t, r, t) - \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t) = \tilde{E}w(t, s, t)$$

since $\omega_S(t, t) = 0$, and this last formula corresponds to the Chen-equality for w .

We have finally arrived at the following definition:

Definition 11. *Under the described conditions on A and G , for any $u_0 \in V$, a mild solution of (2) is given by the pair $(u, u \otimes \omega)$ satisfying, respectively, the equations (16) and (21).*

We stress once more that in order to reach the previous definition we have considered a regular driving path ω , although in that situation the solution is simply given by the path component, as stated at the beginning of this subsection. Definition 11 has to be seen as the guide to establish the definition of a mild solution in the general case of dealing with driving functions that are *only* Hölder continuous.

4.2. System (2) driven by $\omega \in C_{\beta'}([0, T]; V)$. We start by standing several hypothesis denoted by **(H)**:

(H1) Let $H \in (1/3, 1/2]$ and let $1/3 < \beta \leq \beta' < H$. Suppose that there is an α such that $1 - \beta < \alpha < 2\beta$, $\alpha < \frac{\beta+1}{2}$.

(H2) Let A be the generator of an analytic semigroup S given at the beginning of Section 2 and let $G : V \rightarrow L_2(V, \hat{V})$ be a non-linear mapping satisfying the assumptions of Lemma 2, with $\hat{V} \subset V$ such that the embedding operator is Hilbert-Schmidt with norm $c_{V, \hat{V}}$. In particular, we could choose $\hat{V} = V_\kappa$ assuming that $c_{V, \hat{V}}^2 = \sum_i \lambda_i^{-2\kappa} < \infty$.

(H3) Let $(\omega^n)_{n \in \mathbb{N}}$ be a sequence of piecewise linear functions with values in V such that $((\omega^n \otimes_S \omega^n))_{n \in \mathbb{N}}$ is defined by (17). Assume then that for any $\beta' < H$ the sequence $((\omega^n, (\omega^n \otimes_S \omega^n)))_{n \in \mathbb{N}}$ converges to $(\omega, (\omega \otimes_S \omega))$ in $C_{\beta'}([0, T]; V) \times C_{2\beta'}(\bar{\Delta}_{0, T}; L_2(L_2(V, \hat{V}), V \otimes V))$ on a set of full measure.

We would like to mention that in Garrido-Atienza *et al.* [12] we propose two different settings where assumption **(H3)** is satisfied. In particular, in that paper we focus on the case of considering as driving noise a fractional Brownian-motion B^H with $H \in (1/3, 1/2]$ and values in a Hilbert-space, and, by another less restrictive method, on the infinite-dimensional trace-class Brownian-motion $B^{1/2}$. Both constructions are based on some results by Deya *et al.* [5].

Take a fixed $\omega \in C_{\beta'}([0, T], V)$ and consider γ such that $\alpha < \gamma < 1$. We denote by $(W_{0, T}, \|\cdot\|_W)$ the subspace of elements $U = (u, v)$ of the Banach-space $C_{\beta, \sim}([0, T]; V) \times C_{\beta+\beta', \sim}(\Delta_{0, T}; V \otimes V)$ such that the Chen-equality holds. Let us also consider a subset $\hat{W}_{0, T}$ of this space given by the limit points in this space of the set

$$(25) \quad \{(u^n, (u^n \otimes \omega^n)) : n \in \mathbb{N}, u^n \in C_\gamma([0, T]; V), u^n(0) \in D((-A)), \\ (\omega^n, \omega^n \otimes_S \omega^n) \text{ satisfies } \mathbf{(H3)}\}.$$

Here $(u^n \otimes \omega^n)$ is well defined by the integral (7). Note that this set of limit points is a subspace of $C_{\beta, \sim}([0, T]; V) \times C_{\beta+\beta', \sim}(\Delta_{0, T}; V \otimes V)$ which is closed. Hence $\hat{W}_{0, T}$ itself is a complete metric space depending on ω with a metric generated by the norm of $C_{\beta, \sim}([0, T]; V) \times C_{\beta+\beta', \sim}(\Delta_{0, T}; V \otimes V)$:

$$d_{\hat{W}}(U^1, U^2) := \|U^1 - U^2\|_W = \|u^1 - u^2\|_{\beta, \sim} + \|v^1 - v^2\|_{\beta+\beta', \sim}.$$

In addition, elements $U \in \hat{W}_{0, T}$ satisfy the Chen-equality (6) with respect to ω . Furthermore, for the elements $u^n, (u^n \otimes \omega^n)$ we can reformulate the right hand side

of (16) as in (2). Indeed, all expressions that appear in this procedure (even those which cancel) are well defined.

Furthermore, the additivity of integrals follows, and in particular:

Lemma 12. *Let $(u, v) \in \hat{W}_{0,T}$, $0 \leq s \leq t \leq T$ then*

$$S(t-s) \int_0^s S(s-r)G(u(r))d\omega(r) + \int_s^t S(t-r)G(u(r))d\omega(r) = \int_0^t S(t-r)G(u(r))d\omega(r)$$

where these integrals are defined in the sense of (16).

Proof. Replacing (u, v) by an approximating sequence $(u^n, (u^n \otimes \omega^n))$ then the above formula holds which follows because for $F(\cdot) = G(u^n(\cdot))$ the integral can be defined in the sense of Lemma 7 and $(u^n \otimes \omega^n)$ can be defined by (7). Then the additivity of that integral follows by the method in Zähle [24]. Now it suffices to rewrite these integrals according to (16) and consider their V -limit for $n \rightarrow \infty$. \square

Remark 13. *Let us stress that for smooth elements ω both integrals of (14) are additive for elements $(u, v) \in C_{\beta, \sim}([0, T]; V) \times C_{\beta+\beta', \sim}(\Delta_{0,T}, V \otimes V)$ satisfying (6). But for the sake of conciseness we did not to include the proof in the paper.*

We also would like to emphasize that some of the results that we are going to establish below still remain true when considering instead $\hat{W}_{0,T}$ the bigger space $W_{0,T}$. Nevertheless, for uniqueness of presentation we always keep the space $\hat{W}_{0,T}$.

In order to study the existence of solutions to (2) we consider the operator

$$(26) \quad \mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0) = (\mathcal{T}_1(U, \omega, u_0), \mathcal{T}_2(U, \omega, (\omega \otimes_S \omega), u_0))$$

defined for $U = (u, v) \in \hat{W}_{0,T}$ by the expressions

$$\begin{aligned} \mathcal{T}_1(U, \omega, u_0)(t) &:= S(t)u_0 + (-1)^\alpha \int_0^t \hat{D}_{0+}^\alpha (S(t-\cdot)G(u(\cdot)))[r] D_{t-}^{1-\alpha} \omega[r] dr \\ &\quad - (-1)^{2\alpha-1} \int_0^t D_{0+}^{2\alpha-1} (S(t-\cdot)DG(u(\cdot)))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} v[r] dr, \\ \mathcal{T}_2(U, \omega, (\omega \otimes_S \omega), u_0)(s, t) &:= (-1)^\alpha \int_s^t D_{s+}^\alpha ((S(\cdot-s) - \text{id})u(s))[\xi] \otimes_V D_{t-}^{1-\alpha} \omega[\xi] d\xi \\ &\quad - (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha} (\omega \otimes_S \omega)(\cdot, t)_{t-}[r] dr \\ &\quad + (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} DG(u(\cdot))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r] dr, \end{aligned}$$

where w is given, for $0 < s \leq q \leq t \leq T$ and $\tilde{E} \in L_2(V \otimes V, \hat{V})$, by

$$\begin{aligned} \tilde{E}w(t, s, q) &= - \int_s^q \hat{D}_{s+}^\alpha \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \omega_{q-}[r] dr \\ (27) \quad &+ (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} (\omega_S(t) \otimes \omega)[r] dr \\ &+ (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] \tilde{E} D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} v(\cdot, q)[r] dr, \end{aligned}$$

being $(\omega_S(t) \otimes \omega)$ defined, for $s \leq \tau \leq t$ and $E \in L_2(V, \hat{V})$, by

$$(28) \quad \begin{aligned} E(\omega_S(t) \otimes \omega)(s, \tau) &= \omega_S(\tau, t) \int_s^\tau (S(\tau - r) - \text{id}) E d\omega(r) \\ &+ (-1)^{-\alpha} E(\omega \otimes_S \omega)(s, \tau) + (\omega_S(\tau, t) - \omega_S(s, t)) E(\omega(\tau) - \omega(s)), \end{aligned}$$

where the last integral is defined by fractional derivatives due to the regularity of the semigroup S .

Having in mind the Definition 11, the corresponding definition of a mild solution of (2) is given as follows:

Definition 14. $U \in \hat{W}_{0,T}$ such that $U = \mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0)$ is called a mild path-area solution to (1).

In what follows we want to study the existence of a fixed point for the operator \mathcal{T} .

Lemma 15. Assume that $u_0 \in V$. There exists a $c > 0$ independent of $s < t$, u_0 and ω such that for $U \in \hat{W}_{0,T}$

$$\begin{aligned} \|\mathcal{T}_1(U, \omega, u_0)\|_{\beta, \sim} &\leq c(|u_0| + T^{\beta'}((1 + \|u\|_{\beta, \sim}^2)\|\omega\|_{\beta'} + \|u\|_{\beta, \sim}\|v\|_{\beta+\beta', \sim})) \\ &\leq c(|u_0| + T^{\beta'}(1 + \|\omega\|_{\beta'})(1 + \|U\|_{\hat{W}}^2)). \end{aligned}$$

Proof. By (3) and (4), for $0 < s < t \leq T$ we have that $|(S(t) - S(s))u_0| \leq c \frac{(t-s)^\beta}{s^\beta} |u_0|$, and then $\|S(\cdot)u_0\|_{\beta, \sim} \leq c|u_0|$. We now use the following abbreviations

$$\begin{aligned} C_{11}(s, t) &:= (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha (S(t - \cdot)G(u(\cdot))) [r] D_{t-}^{1-\alpha} \omega_{t-} [r] dr, \\ C_{12}(s, t) &:= -(-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} (S(t - \cdot)DG(u(\cdot))) [r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} v [r] dr, \\ C_{21}(0, s, t) &:= (-1)^\alpha \int_0^s \hat{D}_{0+}^\alpha ((S(t - \cdot) - S(s - \cdot))G(u(\cdot))) [r] D_{s-}^{1-\alpha} \omega_{s-} [r] dr, \\ C_{22}(0, s, t) &:= -(-1)^{2\alpha-1} \int_0^s D_{0+}^{2\alpha-1} ((S(t - \cdot) - S(s - \cdot))DG(u(\cdot))) [r] D_{s-}^{1-\alpha} \mathcal{D}_{s-}^{1-\alpha} v [r] dr. \end{aligned}$$

We are going to estimate the V -norm of the following expression:

$$\begin{aligned} &\int_0^t S(t-r)G(u(r))d\omega(r) - \int_0^s S(s-r)G(u(r))d\omega(r) \\ &= \int_s^t S(t-r)G(u(r))d\omega(r) + \int_0^s (S(t-r) - S(s-r))G(u(r))d\omega(r) \\ &= C_{11}(s, t) + C_{12}(s, t) + C_{21}(0, s, t) + C_{22}(0, s, t) \end{aligned}$$

where this equality holds by Lemma 12. For the first expression, for an α' with $\alpha' - \alpha > 0$ sufficiently small we obtain

$$\begin{aligned} |C_{11}(s, t)| &\leq \frac{1}{\Gamma(1-\alpha)} \int_s^t \left(\frac{|S(t-r)G(u(r))|}{(r-s)^\alpha} + \alpha \int_s^r \frac{|(S(t-r) - S(t-q))G(u(r))|}{(r-q)^{1+\alpha}} dq \right. \\ &\quad \left. + \alpha \int_s^r \frac{|S(t-q)(G(u(r)) - G(u(q)) - DG(u(q))(u(r) - u(q)))|}{(r-q)^{1+\alpha}} dq \right) \|\omega\|_{\beta'} (t-r)^{\alpha+\beta'-1} dr \\ &\leq c \int_s^t \left(\frac{c_G + c_{DG}|u(r)|}{(r-s)^\alpha} + \int_s^r \frac{(r-q)^{\alpha'} (c_G + c_{DG}|u(r)|)}{(t-r)^{\alpha'} (r-q)^{1+\alpha}} dq \right. \\ &\quad \left. + \int_s^r \frac{c_{D^2G} \|u\|_{\beta, \sim}^2 (r-q)^{2\beta}}{q^{2\beta} (r-q)^{1+\alpha}} dq \right) \|\omega\|_{\beta'} (t-r)^{\alpha+\beta'-1} dr. \end{aligned}$$

Now evaluating these integrals by (5) we see that

$$\sup_{0 < s < t \leq T} s^\beta \frac{|C_{11}(s, t)|}{(t-s)^\beta} \leq c \|\omega\|_{\beta'} T^{\beta'} (1 + \|U\|_W^2).$$

Now let us consider $C_{22}(0, s, t)$. By using Lemma 1 and (15) we have that

$$\begin{aligned} |C_{22}(0, s, t)| &\leq \frac{c(\|v\|_{\beta+\beta', \sim} + \|u\|_{\beta, \sim} \|\omega\|_{\beta'})}{\Gamma(2-2\alpha)} \int_0^s \left(\frac{|(S(t-r) - S(s-r))DG(u(r))|}{r^{2\alpha-1}} \right. \\ &\quad \left. + (2\alpha-1) \int_0^r \frac{|(S(t-r) - S(t-q) - (S(s-r) - S(s-q)))DG(u(r))|}{(r-q)^{2\alpha}} dq \right. \\ &\quad \left. + (2\alpha-1) \int_0^r \frac{|(S(t-q) - S(s-q))(DG(u(r)) - DG(u(q)))|}{(r-q)^{2\alpha}} dq \right) \frac{(s-r)^{2\alpha+\beta+\beta'-2}}{r^\beta} dr \\ &\leq c(\|v\|_{\beta+\beta', \sim} + \|u\|_{\beta, \sim} \|\omega\|_{\beta'}) \int_0^s \left(\frac{c_{DG}(t-s)^\beta}{(s-r)^\beta r^{2\alpha-1}} + \int_0^r \frac{c_{DG}(r-q)^{2\alpha'-1}(t-s)^\beta}{(s-r)^{2\alpha'-1+\beta}(r-q)^{2\alpha}} dq \right. \\ &\quad \left. + \int_0^r \frac{c_{D^2G}(t-s)^\beta (r-q)^\beta \|u\|_{\beta, \sim}}{q^\beta (s-r)^\beta (r-q)^{2\alpha}} dq \right) \frac{(s-r)^{2\alpha+\beta+\beta'-2}}{r^\beta} dr \end{aligned}$$

and by (H1) and (5) we obtain that

$$\sup_{0 < s < t \leq T} s^\beta \frac{|C_{22}(0, s, t)|}{(t-s)^\beta} \leq c T^{\beta'} (1 + \|\omega\|_{\beta'}) (1 + \|U\|_W^2).$$

The remaining terms can be estimated in a similar manner. Setting $s = 0$ and considering $C_{11}(0, t) + C_{12}(0, t)$ we obtain the same estimate for the norm of $C([0, T]; V)$, hence the proof is complete. \square

The following conclusion with respect to the regularity of the above integrals holds true:

Corollary 16. *Let $\beta < \beta' < H$, then there exists a $c > 0$ independent of $s < t$ and ω such that for $U \in \dot{W}_{0, T}$*

$$\left| (-A)^\beta \int_0^t S(t-r)G(u(r))d\omega \right| \leq c T^{\beta'-\beta} (1 + \|\omega\|_{\beta'}) (1 + \|U\|_W^2).$$

We omit the proof of this result since we only would need to estimate $|(-A)^\beta C_{11}(0, t) + (-A)^\beta C_{12}(0, t)|$, where C_{11} and C_{12} were defined in the proof of Lemma 15. Hence, it suffices to follow the estimates of that result together with the properties of the semigroup S .

Lemma 17. *Suppose that the conditions of Lemma 15 hold. Then there exists a $c > 0$ depending on ω such that for $U^1, U^2 \in \hat{W}_{0,T}$*

$$\|\mathcal{T}_1(U^1) - \mathcal{T}_1(U^2)\|_{\beta, \sim} \leq cT^{\beta'}(1 + \|U^1\|_W^2 + \|U^2\|_W^2)\|U^1 - U^2\|_W + c|u_0^1 - u_0^2|.$$

Proof. Let us denote $\Delta u = u^1 - u^2$, $\Delta U = (u^1 - u^2, v^1 - v^2)$. Then for $r \in [0, T]$ by Lemma 2 we have

$$\|G(u^1(r)) - G(u^2(r))\|_{L_2(V, \hat{V})} \leq c_{DG}\|\Delta u\|_{\beta, \sim}$$

and

$$\begin{aligned} & \|G(u^1(r)) - G(u^1(q)) - DG(u^1(q))(u^1(r) - u^1(q)) \\ & - (G(u^2(r)) - G(u^2(q)) - DG(u^2(q))(u^2(r) - u^2(q)))\|_{L_2(V, \hat{V})} \\ & \leq c_{D^2G}(\|u^1\|_{\beta, \sim} + \|u^2\|_{\beta, \sim})\|\Delta u\|_{\beta, \sim} \frac{(r-q)^{2\beta}}{q^{2\beta}} \\ & + c_{D^3G}\|u^2\|_{\beta, \sim} \frac{(r-q)^\beta}{q^\beta} \sup_{r \in [0, T]} |\Delta u(r)| (2\|u^1\|_{\beta, \sim} + \|u^2\|_{\beta, \sim}) \frac{(r-q)^\beta}{q^\beta} \\ & \leq c(\|u^1\|_{\beta, \sim} + \|u^2\|_{\beta, \sim})\|\Delta u\|_{\beta, \sim} \frac{(r-q)^{2\beta}}{q^{2\beta}} (1 + \|u^2\|_{\beta, \sim}). \end{aligned}$$

Now we can follow the proof of Lemma 15. \square

Up to now we have obtained appropriate estimates for the first component \mathcal{T}_1 of the operator \mathcal{T} given by (26). Now we aim at getting the corresponding estimates for \mathcal{T}_2 , the second component of \mathcal{T} . To this end, we need the two following Lemmata, dealing with estimates of some of the terms appearing in the expression of \mathcal{T}_2 .

Lemma 18. *Under the Hypothesis (H) the following statements hold:*

(i) *For the mapping*

$$e \in V \mapsto \omega_S(s, t)e = (-1)^{-\alpha} \int_s^t (S(\xi - s)e) \otimes_V d\omega(\xi)$$

the following properties hold true: for $0 \leq s \leq r \leq t \leq T$, $e \in V$ and $1/3 < \beta' < \beta'' < H$,

$$\begin{aligned} \|\omega_S(r, t)e - \omega_S(s, t)e\| & \leq c(r-s)^{\beta'} (\|\omega\|_{\beta'} + \|\omega\|_{\beta''})|e|, \\ \|\omega_S(s, t)e\| & \leq c(t-s)^{\beta'} \|\omega\|_{\beta'}|e|, \\ \|\omega_S(s, t)(-A)^{\beta'}e\| & \leq c\|\omega\|_{\beta''}|e|. \end{aligned}$$

(ii) *The mapping*

$$E \in L_2(V, \hat{V}) \mapsto S_\omega(s, t)E = \int_s^t S(t-r)Ed\omega(r)$$

is in $L_2(L_2(V, \hat{V}), V)$, with norm bounded by $c\|\omega\|_{\beta'}(t-s)^{\beta'}\|E\|_{L_2(V, \hat{V})}$.

Proof. Let $\beta' < \beta'' < H$ such that for $\alpha < \alpha' < 1$ we have $\beta' + \alpha' < 1 < \beta'' + \alpha$. Then

$$(29) \quad \begin{aligned} \omega_S(r, t)e - \omega_S(s, t)e &= (-1)^\alpha \int_r^t (S(\xi - r)e - S(\xi - s)e) \otimes_V d\omega(\xi) \\ &\quad - (-1)^\alpha \int_s^r S(\xi - s)e \otimes_V d\omega(\xi). \end{aligned}$$

Interpreting these integrals in a fractional sense and using Lemma 1 we obtain

$$\begin{aligned} |D_{r+}^\alpha (S(\cdot - r)e - S(\cdot - s)e)[\xi]| &\leq c \left(\frac{(r-s)^{\beta'}}{(\xi-r)^{\alpha+\beta'}} + \alpha \int_r^\xi \frac{(r-s)^{\beta'} (\xi-q)^{\alpha'}}{(\xi-q)^{1+\alpha} (q-r)^{\alpha'+\beta'}} dq \right) |e| \\ &\leq c(r-s)^{\beta'} (\xi-r)^{-\alpha-\beta'} |e|. \end{aligned}$$

Moreover, since $\beta'' < H$, due to (H3) $\omega \in C_{\beta''}([0, T]; V)$ (and in particular $\omega \in C_{\beta'}([0, T], V)$), then by Lemma 5

$$|D_{t-}^{1-\alpha} \omega_{t-}[\xi]| \leq c \|\omega\|_{\beta''} (t-\xi)^{\alpha+\beta''-1}.$$

Hence, by applying (5), the first integral on the right-hand side of (29) is bounded in particular by $c(r-s)^{\beta'} \|\omega\|_{\beta''} |e|$.

Furthermore, for the last term in (29), thanks to (H1) and the regularity properties of the semigroup we obtain

$$\int_s^r \|D_{s+}^\alpha S(\cdot - s)e[\xi] \otimes_V D_{r-}^{1-\alpha} \omega_{r-}[\xi]\| d\xi \leq c \|\omega\|_{\beta'} (r-s)^{\beta'} |e|.$$

The second statement of (i) follows directly from the first one taking $r = t$. For the last conclusion of (i), for the parameters chosen at the beginning of the proof,

$$\begin{aligned} |D_{s+}^\alpha (S(\cdot - s)(-A)^{\beta'} e)[\xi]| &\leq c \left(\frac{|e|}{(\xi-s)^{\alpha+\beta'}} \right. \\ &\quad \left. + \int_s^\xi \frac{(\xi-q)^{\alpha'} |e|}{(\xi-q)^{1+\alpha} (q-s)^{\alpha'+\beta'}} dq \right) \leq c |e| (\xi-s)^{-\alpha-\beta'}, \end{aligned}$$

and therefore, since in particular $\beta' + \alpha < 1$,

$$\begin{aligned} &\int_s^t \|D_{s+}^\alpha (S(\cdot - s)(-A)^{\beta'} e)[\xi] \otimes_V D_{t-}^{1-\alpha} \omega_{t-}[\xi]\| d\xi \\ &\leq c \|\omega\|_{\beta''} |e| \int_s^t (\xi-s)^{-\alpha-\beta'} (t-\xi)^{\alpha+\beta''-1} d\xi \\ &\leq c \|\omega\|_{\beta''} (t-s)^{\beta''-\beta'} |e| \leq c \|\omega\|_{\beta''} |e|. \end{aligned}$$

Now we prove (ii). First of all, note that S_ω is well-defined since, thanks to Remark 8, it suffices the Hölder-continuity of $r \mapsto S(t-r)E$ on any interval $[\epsilon, t]$. In addition by the embedding $\hat{V} \subset V$ the mapping

$$L_2(V, \hat{V}) \ni E \mapsto ED_{t-}^{1-\alpha} \omega_{t-}[r] \in V$$

is in $L_2(L_2(V, \hat{V}), V)$, and the norm can be estimated $c_{V, \hat{V}} |D_{t-}^{1-\alpha} \omega_{t-}[r]|$, see Lemma 28 (ii). Furthermore, the integrand $D_{s+}^\alpha S(t-\cdot)[r] \cdot D_{t-}^{1-\alpha} \omega_{t-}[r]$ is weakly measurable with respect to $L_2(L_2(V, \hat{V}), V)$ such that by Pettis' theorem the integrand is

measurable. Because of $D_{t-}^{1-\alpha} E\omega_{t-}[r] = ED_{t-}^{1-\alpha}\omega_{t-}[r]$, we get

$$\begin{aligned} \|S_\omega(s, t) \cdot\|_{L_2(L_2(V, \hat{V}), V)} &= \left\| \int_s^t S(t-r) \cdot d\omega(r) \right\|_{L_2(L_2(V, \hat{V}), V)} \\ &\leq \int_s^t \|D_{s+}^\alpha S(t-\cdot)[r]\|_{L(V)} \|D_{t-}^{1-\alpha}\omega_{t-}[r]\|_{L_2(L_2(V, \hat{V}), V)} dr \\ &\leq c\|\omega\|_{\beta'}(t-s)^{\beta'}. \end{aligned}$$

□

Corollary 19. *Let $(\omega^n)_{n \in \mathbb{N}}$ be a sequence converging to ω in $C_{\beta'}([0, T]; V)$. Then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\substack{0 \leq s < t \leq T \\ |e| = 1}} \frac{\|(\omega - \omega^n)_S(s, t)e\|}{(t-s)^{\beta'}} &= 0, \\ \lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq T} \frac{\|S_{\omega - \omega^n}(s, t) \cdot\|_{L_2(L_2(V, \hat{V}), V)}}{(t-s)^{\beta'}} &= 0. \end{aligned}$$

From the previous result we also obtain that

$$D_{s+}^\alpha(S(\cdot - s)(S_\omega(q, s)\cdot))[\xi] \otimes_V D_{r-}^{1-\alpha}\omega_{r-}[\xi]$$

is weakly $L_2(L_2(V, \hat{V}), V \otimes V)$ -measurable.

Lemma 20. *Suppose that the hypothesis **(H)** holds. For $0 < s \leq q \leq t \leq T$, $\tilde{E} \in L_2(V \otimes V, \hat{V})$ and $U = (u, v) \in \hat{W}_{0, T}$ consider the mapping*

$$w(t, s, q) : L_2(V \otimes V, \hat{V}) \ni \tilde{E} \mapsto \tilde{E}w(t, s, q) \in V \otimes V$$

given by (27) and (28). Then it is well-defined and satisfies, for $\beta' < \beta''$ the estimate

$$\|w(t, s, q)\|_{L_2(L_2(V \otimes V, \hat{V}), V \otimes V)} \leq c\|U\|_W s^{-\beta}(q-s)^{\beta+\beta'}(t-s)^{\beta'},$$

where the constant c depends on $\|\omega\|_{\beta''}$ and $\|(\omega \otimes_S \omega)\|_{2\beta'}$. In particular

$$c \sim (1 + \|\omega\|_{\beta''}^2)\|(\omega \otimes_S \omega)\|_{2\beta'}.$$

For the proof, see the Appendix.

We would like to point out that w satisfies the generalized Chen–equality

$$\begin{aligned} (30) \quad &\tilde{E}w(t, s, r) + \tilde{E}w(t, r, q) - \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, q) \\ &= \tilde{E}w(t, s, q) + \omega_S(q, t)S_\omega(q, r)\tilde{E}(u(r) - u(s), \cdot), \end{aligned}$$

for $0 < s \leq r \leq q \leq t \leq T$, and the Chen–equality

$$(31) \quad \tilde{E}w(t, s, r) + \tilde{E}w(t, r, t) - \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t) = \tilde{E}w(t, s, t)$$

where the latter one is obtained from (30) taking $q = t$. In order to prove these two properties we only need to follow an approximation argument, considering $(\omega^n, (\omega^n \otimes_S \omega^n))$ satisfying **(H3)**, and therefore converging to $(\omega, (\omega \otimes_S \omega))$ in $C_{\beta'}([0, T], V) \times C_{2\beta'}(\bar{\Delta}_{0, T}; L_2(L_2(V, \hat{V}), V \otimes V))$ and take into account that the approximating elements $w^n := (u \otimes (\omega^n \otimes_S \omega^n))$ given in Lemma 10 satisfy the Chen–equalities (23) and (24). In particular, the terms $(\omega_S^n \otimes \omega^n)$ converge to the

corresponding term $(\omega_S \otimes \omega)$, see the proof of Lemma 20 in the Appendix section.

The following result gives estimates for the fractional derivatives of v and w .

Lemma 21. *Let $U = (u, v) \in \hat{W}_{0,T}$. Then for $0 < r < t \leq T$ we have*

$$\begin{aligned} \|D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} v[r]\|_{\beta+\beta', \sim} &\leq c \|U\|_W r^{-\beta} (t-r)^{2\alpha+\beta+\beta'-2}, \\ \|D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r]\| &\leq c \|U\|_W r^{-\beta} (t-r)^{2\alpha+\beta+2\beta'-2}, \end{aligned}$$

where the first constant c depends on $\|\omega\|_{\beta''}$, and the second one on $\|\omega\|_{\beta''}$ and $\|(\omega \otimes_S \omega)\|_{2\beta'}$.

We have also shifted the proof of this result to the Appendix section.

Lemma 22. *Assume that $u_0 \in V$. There exists positive constants \tilde{c} and c such that for $U \in \hat{W}_{0,T}$*

$$\|\mathcal{T}_2(U, \omega, (\omega \otimes_S \omega), u_0)\|_{\beta+\beta', \sim} \leq \tilde{c}|u_0| + cT^{\beta'}(1 + \|U\|_W^2)$$

and, in addition, for two elements $U^1, U^2 \in \hat{W}_{0,T}$:

$$\begin{aligned} \|\mathcal{T}_2(U^1, \omega, (\omega \otimes_S \omega), u_0) - \mathcal{T}_2(U^2, \omega, (\omega \otimes_S \omega), u_0)\|_{\beta+\beta', \sim} \\ \leq \tilde{c}|u_0^1 - u_0^2| + cT^{\beta'}(1 + \|U^1\|_W^2 + \|U^2\|_W^2)\|U^1 - U^2\|_W. \end{aligned}$$

The constant c depends on $\|\omega\|_{\beta''}$, $\|\omega\|_{\beta'}$ and $\|(\omega \otimes_S \omega)\|_{2\beta'}$, while \tilde{c} on $\|\omega\|_{\beta'}$.

Proof. Let us denote $\mathcal{T}_2(U)(s, t) =: B_1(s, t) + B_2(s, t) + B_3(s, t)$, corresponding to the three different addends of \mathcal{T}_2 .

For B_1 we can consider the following splitting:

$$\begin{aligned} B_1(s, t) &= \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V d\omega(\xi) \\ &= \int_s^t (S(\xi) - S(s))u_0 \otimes_V d\omega(\xi) \\ &\quad + \int_s^t (S(\xi - s) - \text{id}) \left(\int_0^s S(s-r)G(u(r))d\omega(r) \right) \otimes_V d\omega(\xi) \\ &=: B_{11}(s, t) + B_{12}(s, t). \end{aligned}$$

B_1 can be interpreted in the fractional sense thanks to the regularity of its integrand, which means that

$$B_{11}(s, t) = (-1)^\alpha \int_s^t D_{s+}^\alpha ((S(\cdot) - S(s))u_0)[\xi] \otimes_V D_{t-}^{1-\alpha} \omega_{t-}[\xi] d\xi.$$

For $\alpha < \alpha'$ where α' is sufficiently close to α and $s > 0$, applying (3) and (4),

$$\begin{aligned} |D_{s+}^\alpha ((S(\cdot) - S(s))u_0)[\xi]| &\leq c \left(\frac{|(S(\xi) - S(s))u_0|}{(\xi - s)^\alpha} + \int_s^\xi \frac{|(S(\xi) - S(q))u_0|}{(\xi - q)^{1+\alpha}} dq \right) \\ &\leq c \left(\frac{(\xi - s)^{\beta-\alpha}}{s^\beta} + \int_s^\xi \frac{(\xi - q)^{\alpha'} q^{-\beta}}{(\xi - q)^{1+\alpha} q^{\alpha'-\beta}} dq \right) |u_0| \\ &\leq c \left(\frac{(\xi - s)^{\beta-\alpha}}{s^\beta} + \frac{1}{s^\beta} \int_s^\xi \frac{(\xi - q)^{\alpha'}}{(\xi - q)^{1+\alpha} (q - s)^{\alpha'-\beta}} dq \right) |u_0| \leq c \frac{(\xi - s)^{\beta-\alpha}}{s^\beta} |u_0|, \end{aligned}$$

hence, for $s > 0$,

$$|B_{11}(s, t)| \leq c \frac{\|\omega\|_{\beta'} |u_0|}{s^\beta} \int_s^t (\xi - s)^{\beta - \alpha} (t - \xi)^{\beta' + \alpha - 1} d\xi \leq c \|\omega\|_{\beta'} |u_0| s^{-\beta} (t - s)^{\beta' + \beta},$$

which implies $\|B_{11}\|_{\beta + \beta', \sim} \leq \tilde{c} |u_0|$. Moreover, note that

$$\begin{aligned} & \left| D_{s+}^\alpha \left((S(\cdot - s) - \text{id}) \int_0^s S(s - r) G(u(r)) d\omega(r) \right) [\xi] \right| \\ & \leq c \frac{|S(\xi - s) - \text{id}| \int_0^s S(s - r) G(u(r)) d\omega(r)|}{(\xi - s)^\alpha} \\ & \quad + \int_s^\xi \frac{|\int_0^s S(\xi - r) - S(q - r) G(u(r)) d\omega(r)|}{(\xi - q)^{1 + \alpha}} dq. \end{aligned}$$

To deal with B_{12} on account of Corollary 16 and thanks to the fact that $\beta' > \beta$ we have

$$\begin{aligned} & \int_s^\xi \frac{|\int_0^s (S(\xi - q) - \text{id}) S(q - r) G(u(r)) d\omega(r)|}{(\xi - q)^{1 + \alpha}} dq \\ & \leq \int_s^\xi \frac{(\xi - q)^{\alpha'}}{(q - s)^{\alpha' - \beta} (\xi - q)^{1 + \alpha}} \left| (-A)^\beta \int_0^s S(s - r) G(u(r)) d\omega(r) \right| dq \\ & \leq c(1 + \|\omega\|_{\beta'}) (1 + \|U\|_W^2) (\xi - s)^{\beta - \alpha} s^{\beta' - \beta} \end{aligned}$$

and by (5)

$$\|B_{12}\|_{\beta + \beta', \sim} \leq cT^{\beta'} \|\omega\|_{\beta'}^2 (1 + \|U\|_W^2).$$

Finally, a similar estimate follows for B_2 and B_3 . In order to see this, note that B_2 and B_3 can be considered in a similar way to C_{11} and C_{12} in the proof of Lemma 15, with the difference that now we have to estimate $D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_t[r]$ and $D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r]$, for which we use the $2\beta'$ -Hölder continuity of $(\omega \otimes_S \omega)$ together with Lemma 21, arriving at

$$\begin{aligned} \|B_2\|_{\beta + \beta', \sim} & \leq cT^{\beta'} \|(\omega \otimes_S \omega)\|_{2\beta'} (1 + \|U\|_W^2), \\ \|B_3\|_{\beta + \beta', \sim} & \leq cT^{\beta'} (1 + \|\omega\|_{\beta'}^2) \|(\omega \otimes_S \omega)\|_{2\beta'} (1 + \|U\|_W^2). \end{aligned}$$

The second part of this lemma can be proven similarly and thus we omit it here. \square

Now we approximate $\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0)$ by piecewise linear noise.

Lemma 23. *Let $U \in \hat{W}_{0,T}$ and assume that $(\omega^n, (\omega^n \otimes_S \omega^n))$ satisfies **(H3)**. Then*

$$\lim_{n \rightarrow \infty} \|\mathcal{T}(U, \omega^n, (\omega^n \otimes_S \omega^n), u_0) - \mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0)\|_W = 0.$$

In addition, $\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0) \in \hat{W}_{0,T}$.

Proof. Note that if a term of \mathcal{T} contains ω or $(\omega \otimes_S \omega)$ then this term depends on these expressions linearly or bi-linearly (see the definition of $\mathcal{T}_1, \mathcal{T}_2$ together with (27)) which just gives the convergence conclusion. To see the second part of the statement we take $U = (u, v) \in \hat{W}_{0,T}$. For this element we choose an approximating sequence $(u^n, (u^n \otimes \omega^n))$ from (25). We note that $\mathcal{T}_1((u^n, (u^n \otimes \omega^n)), \omega^n, u_0^n)(0) \in D((-A))$ for any $\gamma \in (0, 1)$, see Pazy [21] Theorem 4.3.1 and (3)-(4). Therefore $\mathcal{T}_2((u^n, (u^n \otimes \omega^n)), \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)$ can be defined as $(\mathcal{T}_1((u^n, (u^n, \omega^n)), \omega^n, u_0^n) \otimes \omega^n)$ given

by (7). By definition of the space $\hat{W}_{0,T}$, by Lemmas 17, 22, and the first part of this lemma we have that $\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0) \in \hat{W}_{0,T}$. \square

Let us now prove the uniqueness of the path-area-solution in $\hat{W}_{0,T}$ if such a solution exists.

Theorem 24. *Suppose that $U^1 = (u^1, v^1)$, $U^2 = (u^2, v^2) \in \hat{W}_{0,T}$ are two path-area solutions related to the initial condition $u_0 \in V$. Then we have $U^1 = U^2$.*

Proof. Let $T_1 \in [0, T)$ such that $[0, T_1]$ represents the maximal interval of uniqueness. Here $T_1 = 0$ means that two different solutions are just branching from 0. Note that the restrictions $\tilde{U}^i = (u^i_{[T_1, \tilde{T}]}, v^i_{\Delta_{[T_1, \tilde{T}]}})$ are path-area solutions too with respect to the interval $[T_1, \tilde{T}]$, $\tilde{T} \leq T$ with initial condition $u(T_1)$. This follows because for \mathcal{T}_1 we can apply Lemma 12. On the other hand, \mathcal{T}_2 restricted to $T_1 < s < t \leq \tilde{T}$ keeps the same structure as the original \mathcal{T}_2 . Then, if the constant C is an estimate of $\|\tilde{U}^i\|_{W_{T_1, \tilde{T}}}^2$, similar to the estimates that we obtained in Lemmas 17 and 22, we obtain

$$0 \neq \|\tilde{U}^1 - \tilde{U}^2\|_{W_{T_1, \tilde{T}}} \leq c(\tilde{T} - T_1)^{\beta'} (1 + 2C) \|\tilde{U}^1 - \tilde{U}^2\|_{W_{T_1, \tilde{T}}}$$

or equivalently

$$1 \leq c(\tilde{T} - T_1)^{\beta'} (1 + 2C)$$

for any $\tilde{T} > T_1$, which is a contradiction if $\tilde{T} - T_1$ is sufficiently small. \square

Now we present the main theorem of the paper.

Theorem 25. *Suppose that the standing conditions **(H)** are satisfied and suppose that $T > 0$ is chosen sufficiently small depending on the parameters of the problem. Then \mathcal{T} has a fixed point in $\hat{W}_{0,T} \subset W_{0,T}$ that defines a mild path-area solution to (2) which is unique.*

Proof. Lemmas 15, 22 and 23 prove that \mathcal{T} maps a closed centered ball from $\hat{W}_{0,T}$ into itself if $T > 0$ is sufficiently small and by Lemma 17 and Lemma 22 we obtain that this mapping is a contraction if $T > 0$ is chosen sufficiently small. \square

Corollary 26. *Let $U \in \hat{W}_{0,T}$ be a mild path-area solution given by Theorem 25 and let U^n be the path-area solution corresponding to the equations (2) and (21) and driven by a smooth noise ω^n . Then we have*

$$\lim_{n \rightarrow \infty} \|U^n - U\|_W = 0.$$

In fact, for large enough n we can find a ball in $\hat{W}_{0,T}$ which is mapped into itself by $\mathcal{T}(\cdot, \omega^n, (\omega^n \otimes_S \omega^n), u_0)$ and $\mathcal{T}(\cdot, \omega, (\omega \otimes_S \omega), u_0)$, and where in addition these mappings are contractions with uniform contraction condition. Then the conclusion follows by the parameter version of the Banach-fixed point Theorem together with Lemma 23.

Remark 27. *An application of the main results of this article is to prove that the (pathwise) mild path-area solutions are global, see the forthcoming paper [11]. In addition, the results presented in this paper are the basement to prove that SEEs with non-trivial diffusion coefficients G and driven by fractional Brownian-motions B^H with $H \in (1/3, 1/2]$ generate a random dynamical system, see also [11]. That property has been established recently by the same authors when dealing with a more regular fBm, namely B^H with $H \in (1/2, 1)$, see [9].*

5. EXAMPLE

Let $V = l_2$ be the space of square additive sequences with values in \mathbb{R} . In addition, let A be a negative symmetric operator defined on $D(-A) \subset l_2$ with compact inverse. In particular, we can assume that $-A$ has a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots \rightarrow \infty$ where the associated eigenlements $(e_i)_{i \in \mathbb{N}}$ form a complete orthonormal system in l_2 . The spaces $D((-A)^\nu) = V_\nu$ are then defined by

$$\{u = (u_i)_{i \in \mathbb{N}} \in l_2 : \sum_i \lambda_i^{2\nu} u_i^2 =: |u|_{V_\nu}^2 < \infty\}.$$

Assume that there exists $\kappa > 0$ such that the Hilbert-Schmidt embedding $V_\kappa \subset V$ holds true, and take $\hat{V} = V_\kappa$. Consider a sequence of functions $(g_{ij})_{i,j \in \mathbb{N}}$, with $g_{ij} : V \rightarrow \mathbb{R}$, and define $G(u)$ for $u \in V$ by

$$G(u)v = \left(\sum_j g_{ij}(u)v_j \right)_{i \in \mathbb{N}} \quad \text{for all } v \in V.$$

We assume that

$$\|G(u)\|_{L_2(V, V_\kappa)}^2 = \sum_j |G(u)e_j|_{V_\kappa}^2 = \sum_j \left(\sum_i \lambda_i^{2\kappa} (G(u)e_j)_i^2 \right) = \sum_{i,j} \lambda_i^{2\kappa} g_{ij}^2(u) \leq c$$

uniformly with respect to $u \in V$.

In addition, assume that g_{ij} are four times differentiable and their derivatives are uniformly bounded in the following way

$$\begin{aligned} |Dg_{ij}(u)(e_k)| &\leq c_{g,1}^{ijk}, & |D^2g_{ij}(u)(e_k, h_1)| &\leq c_{g,2}^{ijk}|h_1|, & |D^3g_{ij}(u)(e_k, h_1, h_2)| &\leq c_{g,3}^{ijk}|h_1||h_2|, \\ |D^4g_{ij}(u)(e_k, h_1, h_2, h_3)| &\leq c_{g,4}^{ijk}|h_1||h_2||h_3| & \text{for any } u \in V, \end{aligned}$$

and these bounds satisfy

$$\sum_{ijk} \lambda_i^{2\kappa} (c_{g,1}^{ijk})^2 < \infty, \quad \sum_{ijk} \lambda_i^{2\kappa} (c_{g,2}^{ijk})^2 < \infty, \quad \sum_{ijk} \lambda_i^{2\kappa} (c_{g,3}^{ijk})^2 < \infty, \quad \sum_{ijk} \lambda_i^{2\kappa} (c_{g,4}^{ijk})^2 < \infty.$$

To see for instance that DG exists, note that by Taylor expansion

$$|g_{ij}(u+h) - g_{ij}(u) - Dg_{ij}(u)(h)|^2 \leq \frac{1}{2} |D^2g_{ij}(u+\eta h)(h, h)|^2 \leq (c_{g,2}^{ijk})^2 |h|^4$$

where $u, h \in V$ and $\eta \in [0, 1]$. In particular, we also note that

$$\sum_{jk} \left| DG(u)(e_k, e_j) \right|_{V_\kappa}^2 = \sum_{ijk} \lambda_i^{2\kappa} |Dg_{i,j}(u)(e_k)|^2 \leq \sum_{ijk} \lambda_i^{2\kappa} (c_{g,1}^{ijk})^2 =: c_{DG}^2 < \infty.$$

This condition ensures the Lipschitz continuity of G as well as the Hilbert-Schmidt property of DG . Similarly, we obtain that DG is also Lipschitz with respect to the Hilbert-Schmidt norm. We also obtain the existence of the second and third derivative. Hence the conditions on G in Hypothesis **(H)** hold.

The developed theory can be also applied to other examples of G , like kernel integrals, see [11].

6. APPENDIX

In the appendix we prove some technical estimates related to $w = (u \otimes (\omega \otimes_S \omega))$.

Proof of Lemma 10

Proof. For $\tilde{E} \in L_2(V \otimes V, \hat{V})$ we define $f_{\tilde{E}} : L_2(V, V \otimes V) \times V \rightarrow L_2(V, V \otimes V)$ given by

$$f_{\tilde{E}}(Q, u) = Q(\tilde{E}(u, \cdot)).$$

From (17) for smooth ω we have that

$$\begin{aligned} \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, q) &= -(-1)^\alpha \int_s^q \omega_S(r, t) \tilde{E}(u(r) - u(s), \omega'(r)) dr \\ &= -(-1)^\alpha \int_s^q f_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) \omega'(r) dr. \end{aligned}$$

Following Theorem 3.3 in [16], we have

$$\begin{aligned} &\int_s^q f_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) \omega'(r) dr \\ &= (-1)^\alpha \int_s^q \hat{D}_{s+}^\alpha f_{\tilde{E}}(\omega_S(\cdot, t), u(\cdot) - u(s)) [r] D_{q-}^{1-\alpha} \omega_{q-} [r] dr \\ (32) \quad &- (-1)^\alpha \int_s^q (\omega_S(r, t) - \omega_S(s, t)) D_{s+}^\alpha \tilde{E}(u(\cdot) - u(s), \cdot) [r] D_{q-}^{1-\alpha} \omega_{q-} [r] dr \\ &- (-1)^\alpha \int_s^q D_{s+}^\alpha \omega_S(\cdot, t) [r] \tilde{E}(u(r) - u(s), \cdot) D_{q-}^{1-\alpha} \omega_{q-} [r] dr \\ &+ \int_s^q Df_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) (\omega_S(r, t) - \omega_S(s, t), u(r) - u(s)) \omega'(r) dr. \end{aligned}$$

Now we calculate the derivative of $f_{\tilde{E}}$:

$$\begin{aligned} Df_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) (\omega_S(r, t) - \omega_S(s, t), u(r) - u(s)) \omega'(r) \\ = (\omega_S(r, t) - \omega_S(s, t)) \tilde{E}(u(r) - u(s), \omega'(r)) + \omega_S(r, t) \tilde{E}(u(r) - u(s), \omega'(r)) \\ = \tilde{E}(u(r) - u(s), \cdot) D_2(\omega_S(t) \otimes \omega)(s, r) + \omega_S(r, t) \tilde{E} D_2(u \otimes \omega)(s, r). \end{aligned}$$

Substituting the above expression in (32), after applying fractional integration (10) to the last two terms, we have to calculate $ED_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega)_{q-}(s, \cdot) [r]$, for $E \in L_2(V, \hat{V})$, and $\tilde{E} D_{q-}^{1-\alpha}(u \otimes \omega)(t)(s, \cdot) [r]$. First, we have

$$\begin{aligned} ED_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega)_{q-}(s, \cdot) [r] &= D_{q-}^{1-\alpha} E(\omega_S(t) \otimes \omega)_{q-}(s, \cdot) [r] \\ &= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{E(\omega_S(t) \otimes \omega)(s, r) - E(\omega_S(t) \otimes \omega)(s, q)}{(q-r)^{1-\alpha}} \right. \\ &\quad \left. + (1-\alpha) \int_r^q \frac{E(\omega_S(t) \otimes \omega)(s, r) - E(\omega_S(t) \otimes \omega)(s, \theta)}{(\theta-r)^{2-\alpha}} d\theta \right) \\ &= -\frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{E(\omega_S(t) \otimes \omega)(r, q) + (\omega_S(r, t) - \omega_S(s, t)) E(\omega(q) - \omega(r))}{(q-r)^{1-\alpha}} \right. \\ &\quad \left. + (1-\alpha) \int_r^q \frac{E(\omega_S(t) \otimes \omega)(r, \theta) + (\omega_S(r, t) - \omega_S(s, t)) E(\omega(\theta) - \omega(r))}{(\theta-r)^{2-\alpha}} d\theta \right) \\ &= -ED_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega) [r] + (\omega_S(r, t) - \omega_S(s, t)) ED_{q-}^{1-\alpha} \omega_{q-}(r). \end{aligned}$$

Secondly, similar to (13) we obtain that

$$\tilde{E}D_{q-}^{1-\alpha}(u \otimes \omega)(t)(s, \cdot)_{q-}[r] = -\tilde{E}\mathcal{D}_{q-}^{1-\alpha}(u \otimes \omega)(t)[r] + \tilde{E}(u(r) - u(s), D_{q-}^{1-\alpha}\omega_{q-}(r)),$$

and substituting the last two expressions into (32) we obtain the conclusion. \square

In the following we want to prove that w is well-defined for $U \in \hat{W}_{0,T}$. To this end we will consider a mapping

$$w(t, s, q) = w(U, \omega, (\omega \otimes_S \omega))(t, s, q)$$

which coincides for smooth ω with the expression introduced in Lemma 10.

In order to prove the regularity of w stated in Lemma 20 we need several properties that we collect and prove in the following result.

Lemma 28. (i) Let $\tilde{E} \in L_2(V \otimes V, \hat{V})$ and let $v \in V \otimes V$ be fixed. Then the mapping

$$\tilde{E} \mapsto \tilde{E}v$$

is in $L_2(L_2(V \otimes V, \hat{V}), V)$.

(ii) Let $E \in L_2(V, \hat{V})$ and let $u \in V$ be fixed. Then the mapping

$$E \mapsto Eu$$

is in $L_2(L_2(V, \hat{V}), V)$.

(iii) Let $\tilde{E} \in L_2(V \otimes V, \hat{V})$ and $u \in V$. Then

$$\|\tilde{E}(u, \cdot)\|_{L_2(V, \hat{V})} = \|\tilde{E}(u \otimes_V \cdot)\|_{L_2(V, \hat{V})} \leq |u| \|\tilde{E}\|_{L_2(V \otimes V, \hat{V})}.$$

Proof. Consider the separable Hilbert-space $L_2(V \otimes V, \hat{V})$ equipped with the complete orthonormal basis $(\tilde{E}_{ijk})_{i,j,k \in \mathbb{N}}$ given by

$$\tilde{E}_{ijk}(e_l \otimes_V e_m) = \tilde{E}_{ijk}(e_l, e_m) = \begin{cases} 0 & : j \neq l \text{ or } k \neq m \\ f_i & : j = l \text{ and } k = m. \end{cases}$$

We remind that $(e_i)_{i \in \mathbb{N}}$ and $(f_i)_{i \in \mathbb{N}}$ are, respectively, complete orthonormal basis of V and \hat{V} . Then for $v \in V \otimes V$,

$$\sqrt{\sum_{ijk} (\tilde{E}_{ijk}v)^2} = \sqrt{\sum_i |f_i|^2 \sum_{jk} v_{jk}^2} = c_{V, \hat{V}} \|v\|$$

where v_{jk} is the mode of v with respect to $e_j \otimes_V e_k$. This completes (i).

The second statement can be proven similarly and therefore we omit its proof.

Finally, we have

$$\begin{aligned} \|\tilde{E}(u, \cdot)\|_{L_2(V, \hat{V})}^2 &= \sum_i |\tilde{E}(u, e_i)|_{\hat{V}}^2 \\ &= \sum_i \left| \sum_k u_k \tilde{E}(e_k, e_i) \right|_{\hat{V}}^2 \leq \sum_i \left(\sum_k |u_k| |\tilde{E}(e_k, e_i)|_{\hat{V}} \right)^2 \\ &\leq \sum_i \left(\left(\sum_k |u_k|^2 \right)^{\frac{1}{2}} \left(\sum_k |\tilde{E}(e_k, e_i)|_{\hat{V}}^2 \right)^{\frac{1}{2}} \right)^2 = |u|^2 \|\tilde{E}\|_{L_2(V \otimes V, \hat{V})}^2. \end{aligned}$$

\square

In what follows we abbreviate the notation in the following way: let us denote

$$L_{2,\otimes} = L_2(L_2(V, \hat{V}), V \otimes V), \quad L_{2,\otimes,\otimes} = L_2(L_2(V \otimes V, \hat{V}), V \otimes V).$$

Proof of Lemma 20.

Proof. Let us consider separately the three terms of $w(t, s, q)$ given by (27). Precisely, we start estimating the third term

$$I_3(\tilde{E}) := \int_s^q D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \tilde{E}v[r] dr.$$

As we have seen in Lemma 28 (i), for a fixed $v \in V \otimes V$ the mapping $L_2(V \otimes V, \hat{V}) \ni \tilde{E} \mapsto \tilde{E}v$ is in $L_2(L_2(V \otimes V, \hat{V}), V)$ where an estimate of the norm of this operator is given by $c_{V, \hat{V}} \|v\|$. Then, since Lemma 18 (i) in particular implies that $D_{s+}^{2\alpha-1} \omega_S(r, t)$ is in $L(V, V \otimes V)$, the mapping $\tilde{E} \mapsto I_3(\tilde{E})$ is in $L_{2,\otimes,\otimes}$. We have

$$\begin{aligned} \|I_3(\cdot)\|_{L_{2,\otimes,\otimes}} &\leq \int_s^q \|D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] \cdot D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} v[r]\|_{L_{2,\otimes,\otimes}} dr \\ &\leq \int_s^q \|D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r]\|_{L(V, V \otimes V)} \cdot \|D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} v[r]\|_{L_2(L_2(V \otimes V, \hat{V}), V)} dr. \end{aligned}$$

In order to estimate the second factor in the integrand of I_3 , note that due to Lemma 21, for $r \in (s, q)$ and $U \in \hat{W}_{0,T}$, we obtain

$$\|D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} v[r]\| \leq c \|U\|_W r^{-\beta} (q-r)^{\beta+\beta'+2\alpha-2}$$

and as we have said at the beginning of this proof

$$\|D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \cdot v[r]\|_{L_2(L_2(V \otimes V, \hat{V}), V)} \leq c c_{V, \hat{V}} \|U\|_W r^{-\beta} (q-r)^{\beta+\beta'+2\alpha-2}.$$

On the other hand, by Lemma 18 (i),

$$\|D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r]\|_{L(V, V \otimes V)} \leq c \left(\frac{(t-r)^{\beta'}}{(r-s)^{2\alpha-1}} + \int_s^r \frac{(r-\xi)^{\beta'}}{(r-\xi)^{2\alpha}} d\xi \right) (\|\omega\|_{\beta'} + \|\omega\|_{\beta''}).$$

Combining the previous estimates we can conclude

$$\|I_3(\cdot)\|_{L_{2,\otimes,\otimes}} \leq c \|U\|_W s^{-\beta} (t-s)^{\beta'} (q-s)^{\beta+\beta'},$$

where c depends on $\|\omega\|_{\beta''}$ and $\|\omega\|_{\beta'}$. Next we deal with

$$I_2(\tilde{E}) := \int_s^q \hat{D}_{s+}^\alpha \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \omega_{q-}[r] dr.$$

Observe that

$$\begin{aligned} &\|\hat{D}_{s+}^\alpha \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \omega_{q-}[r]\|_{L_{2,\otimes,\otimes}} \\ &\leq \frac{\|\omega_S(r, t) \tilde{E}(u(r) - u(s), D_{q-}^{1-\alpha} \omega_{q-}[r])\|_{L_{2,\otimes,\otimes}}}{(r-s)^\alpha} \\ &\quad + c \int_s^r \frac{\|(\omega_S(r, t) - \omega_S(\theta, t)) \tilde{E}(u(r) - u(\theta), D_{q-}^{1-\alpha} \omega_{q-}[r])\|_{L_{2,\otimes,\otimes}}}{(r-\theta)^{1+\alpha}} d\theta \\ &\leq c c_{V, \hat{V}} (q-r)^{\alpha+\beta'-1} s^{-\beta} \|\tilde{E}\|_{L_2(V \otimes V, \hat{V})} \|u\|_{\beta, \sim} \|\omega\|_{\beta'} \\ &\quad \times (\|\omega\|_{\beta'} + \|\omega\|_{\beta''}) ((t-r)^{\beta'} (r-s)^{\beta-\alpha} + (r-s)^{\beta'+\beta-\alpha}), \end{aligned}$$

which follows by Lemma 18. Hence, integrating the right hand side of the previous expression between s and q we get

$$\|I_2(\cdot)\|_{L_2, \otimes, \otimes} \leq c \|U\|_W s^{-\beta} (t-s)^{\beta'} (q-s)^{\beta+\beta'},$$

with c depending on $\|\omega\|_{\beta''}$ and $\|\omega\|_{\beta'}$. Now we estimate

$$I_1(\tilde{E}) := \int_s^q D_{s+}^{2\alpha-1} \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} (\omega_S(t) \otimes \omega)[r] dr.$$

We emphasize that the expression $(\omega_S(t) \otimes \omega)$ is not well defined by an integral similar to (7) for nonregular ω . Nevertheless, splitting this expression as in Lemma 10 we can express it in terms of $(\omega \otimes_S \omega)$ which is well defined by **(H3)**. In addition, we can work with an approximation argument by the assumption **(H3)**. Now we split the previous integral into three integrals due to the definition (28). To treat the corresponding first expression let us write down the following estimate for $\alpha < \gamma < 1$, $\beta' < \gamma$:

$$\begin{aligned} |D_{s+}^\alpha ((S(\tau - \cdot) - \text{id})(-A)^{-\beta'} e)[r]| &\leq c \left(\frac{|(S(\tau - r) - \text{id})(-A)^{-\beta'} e|}{(r-s)^\alpha} \right. \\ &\quad \left. + \int_s^r \frac{|(S(\tau - r) - S(\tau - q))(-A)^{-\beta'} e|}{(r-q)^{1+\alpha}} dq \right) \\ &\leq c \left(\frac{(\tau - r)^{\beta'} |e|}{(r-s)^\alpha} + \int_s^r \frac{|(S(r - q) - \text{id})(-A)^{-\beta'} S(\tau - r) e|}{(r-q)^{1+\alpha}} dq \right) \\ &\leq c \left(\frac{(\tau - r)^{\beta'}}{(r-s)^\alpha} + \frac{(\tau - r)^{\beta' - \gamma}}{(r-s)^{\alpha - \gamma}} \right) |e|, \end{aligned}$$

for $e \in V$, which follows by (3) and (4). Note we have

$$\omega_S(\tau, t) \int_s^\tau (S(\tau - r) - \text{id}) E d\omega(r) = \omega_S(\tau, t) (-A)^{\beta'} \int_s^\tau (S(\tau - r) - \text{id})(-A)^{-\beta'} E d\omega(r),$$

hence, by the third statement of Lemma 18 (i) and Lemma 28 (ii), we conclude that

$$\begin{aligned} &\|\omega_S(\tau, t) \int_s^\tau (S(\tau - r) - \text{id}) \cdot d\omega(r)\|_{L_2, \otimes} \\ &\leq c \|\omega\|_{\beta''} \left\| \int_s^\tau (S(\tau - r) - \text{id})(-A)^{-\beta'} \cdot d\omega(r) \right\|_{L_2(L_2(V, \hat{V}), V)} \\ &\leq c c_{V, \hat{V}} \|\omega\|_{\beta''} \int_s^\tau |D_{s+}^\alpha ((S(\tau - \cdot) - \text{id})(-A)^{-\beta'} [r])| |D_{\tau-}^{1-\alpha} \omega_{\tau-}[r]| dr \\ &\leq c \|\omega\|_{\beta'} \|\omega\|_{\beta''} \int_s^\tau \left(\frac{(\tau - r)^{\beta'}}{(r-s)^\alpha} + \frac{(\tau - r)^{\beta' - \gamma}}{(r-s)^{\alpha - \gamma}} \right) (\tau - r)^{\beta' + \alpha - 1} dr \\ &\leq c \|\omega\|_{\beta''} \|\omega\|_{\beta'} (\tau - s)^{2\beta'}. \end{aligned}$$

For the other terms of the right-hand side of $(\omega_S \otimes \omega)$, by Hypothesis **(H3)** and Lemma 18 we have that

$$\|(-1)^{-\alpha} \cdot (\omega \otimes_S \omega)(s, \tau) + (\omega_S(\tau, t) - \omega_S(s, t)) \cdot (\omega(\tau) - \omega(s))\|_{L_2, \otimes} \leq c(\tau - s)^{2\beta'},$$

where c depends on $\|(\omega \otimes_S \omega)\|_{2\beta'}$ and $\|\omega\|_{\beta''}$. Since ω^n is smooth, the expression $\omega_S^n(t) \otimes \omega^n$ has the same structure as the integral (7) and satisfies the Chen-equality. Moreover, we have the convergence of $(\omega_S^n(t) \otimes \omega^n)$ to $(\omega_S(t) \otimes \omega)$ in L_2, \otimes such that

the latter term satisfies the Chen–equality too. This convergence holds because all expressions in $(\omega_S \otimes \omega)$ depend linearly or bilinearly on ω or $(\omega \otimes_S \omega)$. Then the regularity of $(\omega_S(t) \otimes \omega)$ yields

$$(33) \quad \|\cdot D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega)\|_{L_{2,\otimes}} \leq c(q-r)^{2\beta'+2\alpha-2}.$$

To establish the previous inequality we have to use that $(\omega_S \otimes \omega)$ is $2\beta'$ –Hölder continuous as well as the Chen–equality (in fact (33) looks similar to the first property of Lemma 21, but it is easier to derive, and thus its complete proof is left to the reader). Finally, (33) allows us to treat the integral $I_1(\tilde{E})$, obtaining a similar estimate to the ones we already have for I_2 and I_3 above. \square

Proof of Lemma 21.

Proof. We focus on proving the second estimate, since the first one is easier. Hence, we want to calculate $\|D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r]\|_{L_{2,\otimes,\otimes}}$ for which we take into account the expression:

$$(34) \quad \begin{aligned} D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r] &= \frac{(-1)^\alpha}{\Gamma(\alpha)} \left(\frac{\mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r]}{(t-r)^{1-\alpha}} \right. \\ &+ (1-\alpha) \int_r^t \frac{\mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r] - \mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[\theta]}{(\theta-r)^{2-\alpha}} d\theta \Big) \\ &= \frac{(-1)^\alpha}{\Gamma(\alpha)} \left(\frac{\mathcal{D}_{t-}^{1-\alpha} w(t, \cdot, \cdot)[r]}{(t-r)^{1-\alpha}} \right. \\ &+ (1-\alpha) \int_r^t \frac{\frac{w(t,r,t)}{(t-r)^{1-\alpha}} - \frac{w(t,\theta,t)}{(t-\theta)^{1-\alpha}} + (1-\alpha) \int_r^t \frac{w(t,r,\zeta)}{(\zeta-r)^{2-\alpha}} d\zeta - (1-\alpha) \int_\theta^t \frac{w(t,\theta,\zeta)}{(\zeta-\theta)^{2-\alpha}} d\zeta}{(\theta-r)^{2-\alpha}} d\theta \Big). \end{aligned}$$

We start by estimating the non-integral terms of the last expression. We obtain

$$\begin{aligned} \left\| \frac{w(t,r,t)}{(t-r)^{1-\alpha}} - \frac{w(t,\theta,t)}{(t-\theta)^{1-\alpha}} \right\|_{L_{2,\otimes,\otimes}} &\leq \frac{\|w(t,\theta,t)\|_{L_{2,\otimes,\otimes}} |(t-r)^{1-\alpha} - (t-\theta)^{1-\alpha}|}{(t-r)^{1-\alpha} (t-\theta)^{1-\alpha}} \\ &+ \frac{\|w(t,r,t) - w(t,\theta,t)\|_{L_{2,\otimes,\otimes}}}{(t-r)^{1-\alpha}} \\ &\leq cr^{-\beta} \|U\|_W ((t-\theta)^{\beta'+\beta} (t-\theta)^{\beta'} (\theta-r)^\beta (t-\theta)^{1-\alpha-\beta} (t-r)^{\alpha-1} (t-\theta)^{\alpha-1} \\ &+ (t-r)^{\beta'} (\theta-r)^{\beta'+\beta} (t-r)^{\alpha-1} + (\theta-r)^\beta (t-\theta)^{2\beta'} (t-r)^{\alpha-1}). \end{aligned}$$

To get these estimates we have used Lemma 20 and in addition, for the first expression on the right hand side we have used the trivial inequality

$$(35) \quad y^{1-\alpha} - x^{1-\alpha} \leq (y-x)^\beta x^{1-\alpha-\beta}$$

for any $x < y$, given in Lemma 6.1 of [16], while we have managed the second one by using the Chen–equality (31). Therefore, we obtain

$$\left\| \frac{w(t,r,t)}{(t-r)^{1-\alpha}} - \frac{w(t,\theta,t)}{(t-\theta)^{1-\alpha}} \right\|_{L_{2,\otimes,\otimes}} \leq cr^{-\beta} \|U\|_W (\theta-r)^\beta (t-r)^{2\beta'+\alpha-1}.$$

In addition, for the integral terms of the last expression of (34) we have

$$\begin{aligned} & \left\| \int_r^t \frac{w(t, r, \zeta)}{(\zeta - r)^{2-\alpha}} d\zeta - \int_\theta^t \frac{w(t, \theta, \zeta)}{(\zeta - \theta)^{2-\alpha}} d\zeta \right\|_{L_{2, \otimes, \otimes}} \\ & \leq \int_r^\theta \frac{\|w(t, r, \zeta)\|_{L_{2, \otimes, \otimes}}}{(\zeta - r)^{2-\alpha}} d\zeta + \left(\int_\theta^t \left\| \frac{w(t, r, \zeta)}{(\zeta - r)^{2-\alpha}} - \frac{w(t, \theta, \zeta)}{(\zeta - \theta)^{2-\alpha}} \right\|_{L_{2, \otimes, \otimes}} d\zeta \right) =: B_1 + B_2. \end{aligned}$$

On account of Lemma 15, due to $\beta + \beta' + \alpha > 1$, for B_1 we have

$$B_1 \leq \frac{c}{r^\beta} \|U\|_W \int_r^\theta \frac{(\zeta - r)^{\beta+\beta'} (t - r)^{\beta'}}{(\zeta - r)^{2-\alpha}} d\zeta \leq \frac{c}{r^\beta} \|U\|_W (t - r)^{\beta'} (\theta - r)^{\beta+\beta'+\alpha-1},$$

and for B_2

$$\begin{aligned} B_2 & \leq \int_\theta^t \frac{\|w(t, r, \zeta) - w(t, \theta, \zeta)\|_{L_{2, \otimes, \otimes}}}{(\zeta - r)^{2-\alpha}} d\zeta \\ & \quad + \int_\theta^t \frac{\|w(t, \theta, \zeta)\|_{L_{2, \otimes, \otimes}} |(\zeta - \theta)^{2-\alpha} - (\zeta - r)^{2-\alpha}|}{(\zeta - r)^{2-\alpha} (\zeta - \theta)^{2-\alpha}} d\zeta =: B_{21} + B_{22}. \end{aligned}$$

For B_{22} , using Lemma 15 and the estimate (35) we get:

$$\begin{aligned} B_{22} & \leq \frac{c}{r^\beta} \|U\|_W (t - \theta)^{\beta'} (\theta - r)^\beta \int_\theta^t (\zeta - r)^{\alpha-2} (\zeta - \theta)^{\beta'} d\zeta \\ & \leq cr^{-\beta} \|U\|_W (\theta - r)^\beta (t - r)^{2\beta'+\alpha-1}. \end{aligned}$$

To estimate B_{21} we need the generalized Chen–equality (30), giving us

$$\begin{aligned} B_{21} & \leq \frac{c}{r^\beta} \|U\|_W \int_\theta^t \left(\frac{(\theta - r)^{\beta+\beta'} (t - r)^{\beta'}}{(\zeta - r)^{2-\alpha}} + \frac{(\theta - r)^\beta (\zeta - \theta)^{2\beta'}}{(\zeta - r)^{2-\alpha}} \right. \\ & \quad \left. + \frac{(\theta - r)^\beta (t - \zeta)^{\beta'} (\zeta - \theta)^{\beta'}}{(\zeta - r)^{2-\alpha}} \right) d\zeta \\ & \leq cr^{-\beta} \|U\|_W \left((\theta - r)^{\beta+\beta'-1+\alpha} (t - r)^{\beta'} + (\theta - r)^\beta (t - \theta)^{2\beta'+\alpha-1} \right. \\ & \quad \left. + (t - \theta)^{\beta'} (t - \theta)^{\beta'+\alpha-1} (\theta - r)^\beta \right) \\ & \leq cr^{-\beta} \|U\|_W (\theta - r)^\beta (t - r)^{2\beta'+\alpha-1}. \end{aligned}$$

Indeed, taking into account (30) we have to estimate in particular the expression

$$\|\omega_S(\zeta, t) S_\omega(\zeta, \theta) \cdot (u(\theta) - u(r), \cdot)\| \leq \frac{c}{r^\beta} \|\omega\|_{\beta''}^2 (t - \zeta)^{\beta'} (\zeta - \theta)^{\beta'} (\theta - r)^\beta \|U\|_W$$

which can be easily done by Lemma 18 (i) and (ii). Therefore, the last expression of (34) gives us as estimate

$$cr^{-\beta} (t - r)^{2\beta'+\alpha-1} \|U\|_W \int_r^t \frac{(\theta - r)^\beta}{(\theta - r)^{2-\alpha}} d\theta \leq cr^{-\beta} (t - r)^{2\beta'+\beta+2\alpha-2}.$$

Similar we obtain an estimate for the first expression on the right side of (34) setting $\theta = t$.

Finally, note that the appearing constant c depends on $\|\omega\|_{\beta''}$ and $\|(\omega \otimes_S \omega)\|_{2\beta'}$. \square

Corollary 29. *The proof of (9) follows the same steps than the last proof, with the difference that we do not have to use (23) but the Chen–equality (6).*

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(María J. Garrido-Atienza) DPTO. ECUACIONES DIFERENCIALES Y ANÁLISIS NUMÉRICO, UNIVERSIDAD DE SEVILLA, APDO. DE CORREOS 1160, 41080-SEVILLA, SPAIN

E-mail address, María J. Garrido-Atienza: mgarrido@us.es

(Kening Lu) 346 TMCB, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602, USA

E-mail address, Kening Lu: klu@math.byu.edu

(Björn Schmalfuß) INSTITUT FÜR STOCHASTIK, FRIEDRICH SCHILLER UNIVERSITÄT JENA, ERNST ABBE PLATZ 2, 77043, JENA, GERMANY,

E-mail address, Björn Schmalfuß: bjoern.schmalfuss@uni-jena.de