Two Weighted Inequalities for Potential and Fractional Type Maximal Operators

CARLOS PÉREZ

1. Introduction. In his survey paper [15], B. Muckenhoupt raised the general question of characterizing when the weighed norm inequality,

$$\left(\int_{\mathbb{R}^n} (w(y)\,|Tf(y)|)^q\,dy\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} (v(y)\,|f(y)|)^p\,dy\right)^{1/p} \tag{1}$$

holds for appropriate f. In the case of "one weight," i.e., $w = v$, $p = q$ and for many classical operators, inequality (1) can be characterized by remarkably simple conditions. The most important one is Muckenhoupt's A_p condition,

$$\left(\frac{1}{|Q|}\int_Q w(y)^p\,dy\right)^{1/p}\left(\frac{1}{|Q|}\int_Q w(y)^{-p'}\,dy\right)^{1/p'} \leq c \tag{2}$$

for all cubes Q. This condition is necessary and sufficient for (1) when T is the Hardy–Littlewood maximal operator or the Riesz transform. For a general introduction to the subject and for historical comments as well, we shall refer the reader to [10].

In this paper we shall be concerned with inequality (1) for general potential operators and related maximal operators. For a nonnegative, locally integrable function Φ, we define the potential operator T_Φ by

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy.$$

The study of weighted inequalities for general potential operators is of interest not only for its own sake but because it is relevant to many applications in partial differential equations and quantum mechanics. We shall refer the reader to [22] and to [9], Section 9, for further information.

Indiana University Mathematics Journal ©, Vol. 43, No. 2 (1994)

Although the basic example is provided by the Riesz potentials or fractional integrals I_α, defined by the kernel $\Phi(t) = |t|^{\alpha-n}$, $0 < \alpha < n$, there are other important examples such as the Bessel potentials (cf. [23]). They are denoted by $J_{\beta,\lambda}$, β, $\lambda > 0$, and the kernel $\Phi = K_{\beta,\lambda}$ is best defined by means of its Fourier transform,

$$\widehat{K_{\beta,\lambda}}(\xi) = (\lambda^2 + |\xi|^2)^{-\beta/2}.$$

In both cases, Φ is radial and decreasing. We shall not assume that Φ is radial, and we shall consider instead a wider class of kernels, those satisfying the following weak growth condition

There are constants δ, $c > 0$, $0 \leq \varepsilon < 1$ with the property that

$$\text{(D)} \qquad \sup_{2^k < |x| \leq 2^{k+1}} \Phi(x) \leq \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| \leq 2\delta(1+\varepsilon)2^k} \Phi(y)\, dy,$$

for all $k \in \mathbb{Z}$.

A similar but stronger condition was first considered in [11]. Condition (D) is very general since the cases of radial and nonincreasing or nondecreasing functions Φ are included. Also, if Φ is essentially constant on annuli, i.e., $\Phi(y) \leq c\Phi(x)$ for $|y|/2 \leq |x| \leq 2|y|$, then Φ satisfies condition (D).

In a recent paper [22], E. Sawyer and R. Wheeden have obtained Fefferman–Phong type conditions on the weights for (1) with $T = I_\alpha$. To be more precise, let $1 < p \leq q < \infty$. Sawyer–Wheeden's result establishes that, if any given couple of weights (w, v) satisfies for some r, $1 < r < \infty$, the following generalized Fefferman–Phong condition,

$$(3) \qquad |Q|^{\alpha/n+1/q-1/p} \left(\frac{1}{|Q|} \int_Q w(y)^{qr}\, dy \right)^{1/qr} \left(\frac{1}{|Q|} \int_Q v(y)^{-p'r}\, dy \right)^{1/p'r} \leq K,$$

for all cubes Q, then inequality (1) holds with $T = I_\alpha$. This is not longer true when $r = 1$ (cf. the end of Section 4). We observe that the Fefferman–Phong condition is a particular case of (3) by taking $v \equiv 1$, $\alpha = 1$ and $p = q = 2$.

On the other hand, Chang, Wilson, and Wolff in [5], and subsequently Chanillo and Wheeden in [6], improved the Fefferman–Phong condition for the trace inequality by replacing the power function $t \to t^r$ by weaker functions φ. The main purpose of this paper is to sharpen and unify all these results. Our main theorem contains the related work done in [17], [16] and [24].

We improve these results in two ways. First, we consider weaker norms than those in (3); these norms will be defined in terms of certain mapping properties of appropriate maximal operators associated to each norm (cf. (7)). Secondly, we consider more general potential operators, namely those with kernels satisfying condition (D). Our approach is a blend from ideas in [11], [17], and [22].

In this paper we also consider the corresponding problem for the maximal operator $M_{\tilde{\Phi}}$ (cf. Theorem 2.11). This operator, which is intimately related to T_Φ, is defined by

$$M_{\tilde{\Phi}}f(x) = \sup_{x\in Q} \frac{\tilde{\Phi}(\ell(Q))}{|Q|} \int_Q |f(y)|\, dy, \tag{4}$$

where

$$\tilde{\Phi}(t) = \int_{|z|\le 1} \Phi(z)\, dz.$$

$M_{\tilde{\Phi}}$ is the analogue for a general function $\tilde{\Phi}$ of the classical fractional maximal operators M_α, and it has previously been studied by Kerman and Sawyer in [12], and later by Jawerth, Pérez, and Welland in [11].

Let us summarize the paper by briefly describing the content of each of the sections. In the next section, Section 2, we state our results and some of their consequences and we make further comments and remarks. In Section 3 we collect some defintionx and auxilirary facts. In Section 4 we prove the main result concerning T_Φ and in Section 5 the corresponding one for $M_{\tilde{\Phi}}$.

Finally, we shall make some brief comments about the notation. Whenever Q is used as an index set, as in $\sum_{\mathbf{Q}}$, this means that Q runs over all dyadic cubes in $\mathbb{R}^n$, denoted by D. The sidelength of a cube Q is denoted $\ell(Q)$, and aQ, $a > 0$, denotes the cube concentric with Q and with sidelength $a\ell(Q)$. We consider only cubes with sides parallel to the axes, and use Φ to denote a nonnegative, locally integrable function. Given a weight w and a measurable set E, we will denote the weighted measure of E by $w(E)$, i.e., $w(E) = \int_E w(x)\, dx$. The weighted space $L^p(w)$, $0 < p \le \infty$, will be equipped with the norm $\|f\|_{L^p(w)} = \left(\int |f(x)|^p w(x)\, dx\right)^{1/p}$. As usual, the letter c will denote a constant, often different from time to time.

2. Statement of the results. Let X be a Banach function space over $\mathbb{R}^n$ with respect to the Lebesgue measure (cf. next section). Given any measurable function f and a cube Q, we define the X–average of f over Q by

$$\|f\|_{X,Q} = \left\|\tau_{\ell(Q)}(f\chi_Q)\right\|_X, \tag{5}$$

where τ_δ, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$, and χ_E is characteristic function of E. Observe that in the case $X = L^r$, the average equals

$$\left(\frac{1}{|Q|}\int_Q w(y)^r\, dy\right)^{1/r}.$$

More generally, let $X = L^B$ be the Orlicz space defined by the Young function B. Then the B–average is given by

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right) dy \le 1\right\}. \tag{6}$$

For any Banach function space X, we associate the following maximal operator defined for each locally integrable function f by

$$M_X f(x) = \sup_{x\in Q} \|f\|_{X,Q},$$

where the supremum is taken over all the cubes containing x. When X is an Orlicz space $X = L^B$, we denote its maximal operator by $M_X = M_B$.

As we mentioned in the introduction, we shall assume the following condition on the kernel Φ.

There are constants $\delta,\ c > 0,\ 0 \le \varepsilon < 1$ with the property that

$$\sup_{2^k<|x|\le 2^{k+1}} \Phi(x) \le \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k<|y|\le 2\delta(1+\varepsilon)2^k} \Phi(y)\, dy, \tag{D}$$

for all $k \in \mathbb{Z}$.

Associated to any kernel Φ we denote by $\tilde{\Phi}$ the positive function defined for $t \ge 0$,

$$\tilde{\Phi}(t) = \int_{|z|\le t} \Phi(z)\, dz.$$

Theorem 2.1. *Let $1 < p \le q < \infty$, and let Φ satisfy condition* (D) *above. Let X and Y be Banach function spaces such that*

$$\begin{cases} M_{X'} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n) \\ M_{Y'} : L^{p}(\mathbb{R}^n) \to L^{p}(\mathbb{R}^n) \end{cases} \tag{7}$$

Suppose that (w,v) is any couple of weights such that, for some positive constant K and for every cube Q,

$$\tilde{\Phi}(\ell(Q))|Q|^{1/q-1/p}\ \|w\|_{X,Q}\ \|v^{-1}\|_{Y,Q} \le K. \tag{8}$$

Then

$$T_\Phi : L^p(v^p) \to L^q(w^q). \tag{9}$$

Remark 2.2. This result contains Sawyer–Wheeden's theorem mentioned in the introduction. Indeed, let M_s, $0 < s < \infty$, denote the maximal operator $M_s f = M(f^s)^{1/s}$. Also, for $1 < r < \infty$, we let $X = L^{qr}$, $Y = L^{p'r}$. Then $M_{X'} = M_{(qr)'}$ and $M_{Y'} = M_{(p'r)'}$ are bounded on $L^{q'}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ respectively by the Hardy–Littlewood maximal theorem, satisfying the hypotheses of the theorem.

Remark 2.3. Observe that condition (8) is a two-sided condition in the sense that both weights w and v^{-1} are required to have stronger norms than L^p and L^q, respectively. This is not, however, required in the case of maximal operators as it is shown in Theorem 2.11, where only the norm of v^{-1} is stronger than L^q. In fact, this one–sided condition is false for Riesz potentials as shown by D. Adams in [2] and with explicit examples in [12], p. 208 and [21], p. 344.

Remark 2.4. Let $1 < p \le q < \infty$. Suppose that both w^q and $v^{-p'}$ are A_∞ weights. Suppose further that, for some constant K and for every cube Q,

$$\tilde{\Phi}(\ell(Q))|Q|^{1/q-1/p}\left(\frac{1}{|Q|}\int_Q w(y)^q\,dy\right)^{1/q}\left(\frac{1}{|Q|}\int_Q v(y)^{-p'}\,dy\right)^{1/p'} \le K. \tag{10}$$

Then (9) holds. To see this, we recall that a weight u belongs to the class of weights RH_t, $1 < t < \infty$, if it satisfies the reverse Hölder inequality of order t uniformly on each cube Q, that is, if

$$\left(\frac{1}{|Q|}\int_Q u(y)^t\,dy\right)^{1/t} \le \frac{c}{|Q|}\int_Q u(y)\,dy, \tag{11}$$

where C is a positive constant independent of each cube Q. Since it is well known that $u \in A_\infty$ implies that, for some $1 < t < \infty$, $u \in RH_t$, we have that $w^q \in RH_r$ and $v^{-p'} \in RH_s$, for some $1 < r, s < \infty$. This together with (10) implies condition (8) with $X = L^{qr}$ and $Y = L^{p's}$.

To sharpen Sawyer–Wheeden's condition (3) within the scale of Orlicz spaces, we introduce the following definition (cf. [18]).

Definition 2.5. *Let $1 < r < \infty$. We say that a Young function B belongs to the class B_r or that satisfies the B_r condition if there is a positive constant c for which*

$$\int_c^\infty \frac{B(t)}{t^r}\frac{dt}{t} < \infty.$$

Remark 2.6. Sometimes it is more convenient to deal with trhe complementary function $\bar{B}$ of B (cf. next section). It can be checked that $B \in B_r$ if there is a positive constant c for which

$$\int_c^\infty \left(\frac{t^{r'}}{\bar{B}(t)}\right)^{r-1} \frac{dt}{t} < \infty. \tag{12}$$

Corollary 2.7. *Let $1 < p \leq q < \infty$, and let Φ satisfy condition* (D) *above. Let A and B be Young functions such that $\bar{A} \in B_{q'}$ and $\bar{B} \in B_p$. Suppose that (w,v) is any couple of weights such that, for some positive constant K and for every cube Q,*

$$\tilde{\Phi}\big(\ell(Q)\big)\, |Q|^{1/q-1/p}\, \|w\|_{A,Q}\, \|v^{-1}\|_{B,Q} \leq K. \tag{13}$$

Then

$$T_\Phi : L^p(v^p) \to L^q(w^q). \tag{14}$$

We shall give the proof of the corollary in Section 4.

Example 2.8. Typical examples are given by considering Young functions A and B such that

$$\bar{A}(t) \cong t^q \log^{q-1+\delta}(1+t),$$

and

$$\bar{B}(t) \cong t^{p'} \log^{p'-1+\delta}(1+t),$$

$\delta > 0$. Also,

$$\bar{A}(t) \cong t^q \log^{q-1}(1+t)\big[\log \log(1+t)\big]^{q-1+\delta},$$

and

$$\bar{B}(t) \cong t^{p'} \log^{p'-1}(1+t)\big[\log \log(1+t)\big]^{p'-1+\delta},$$

$\delta > 0$. In both cases $\bar{A} \in B_{q'}$ and $\bar{B} \in B_p$ by the remark above.

For $0 < \alpha < n$ the case $\Phi(t) = |t|^{\alpha-n}$ is particularly important in many applications since it corresponds to the Riesz potential of order α. Observing that in this case $\tilde{\Phi}(t) \cong t^{\alpha/n}$, we have the following corollary:

Corollary 2.9. *Let $1 < p \le q < \infty$, and $0 < \alpha < n$. Let X and Y be Banach function spaces such that*

$$\begin{cases} M_{X'} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n) \\ M_{Y'} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n). \end{cases}$$

Suppose that the couple of weights (w,v) satisfies, for some constant K,

$$|Q|^{\alpha/n}\, |Q|^{1/q-1/p}\, \|w\|_{X,Q}\, \|v^{-1}\|_{Y,Q} \le K, \tag{15}$$

for all cubes Q. Then

$$I_\alpha : L^p(v^p) \to L^q(w^q). \tag{16}$$

In the next result we give a corresponding version of Theorem 2.1 for the maximal operator $M_{\tilde{\Phi}}$; however, it is not necessary to assume condition (D), and in fact it holds for a slightly more general maximal operator M_φ,

$$M_\varphi f(x) = \sup_{x\in Q} \frac{\varphi(|Q|)}{|Q|} \int_Q |f(y)|\, dy,$$

where $\varphi : (0,\infty) \to (0,\infty)$.

Before stating our result for M_φ we use this notation to observe that the weights $w = u^{1/p}$ and $v = (M_{\Phi^{pr}}(u^r))^{1/pr}$, $1 < p, r < \infty$ satisfies (8) with $p = q$, $X = L^{pr}$ and $Y = L^{p'r}$. This yields the following consequence of Theorem 2.1.

Corollary 2.10. *Let $1 < p < \infty$, and let Φ satisfy condition* (D) *above. For each $r > 1$ there is a positive constant c, such that for every nonnegative locally integrable function f and v,*

$$\int_{\mathbb{R}^n} T_\Phi f(y)^p u(y)\, dy \le c \int_{\mathbb{R}^n} f(y)^p \left(M_{\tilde{\Phi}^{pr}}(u^r)(y)\right)^{1/r} dy. \tag{17}$$

A corresponding inequality for classical singular nintegral operators was first proved by A. Córdoba and C. Fefferman in [7]. Also, D. Adams obtained (17) for Riesz potentials in [1].

We remark that inequality (17) does not hold for $r = 1$ as can be seen at the end of Section 4.

In the next theorem we shall assume that φ is essentially nondecreasing, i.e., there is a positive constant ρ for which

$$\varphi(t) \le \rho\varphi(s), \qquad t \le s,$$

and

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0. \tag{18}$$

A condition like (18) is necessary to rule out examples such as $\varphi(t) = t^m$, $m > 1$. In this case, $M_\varphi f(x) = \infty$ for all x, if we consider $f \cong \delta$, the point mass at the origin. This defeats any L^p inequality.

We state now our main result for M_φ.

Theorem 2.11. *Let $1 < p \leq q < \infty$, and let φ be as above. Let Y be any Banach function space such that $M_{Y'} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Suppose that (w,v) is any couple of weights such that for some positive constant K and for every cube Q,*

$$\varphi(|Q|)\,|Q|^{1/q-1/p}\left(\frac{1}{|Q|}\int_Q w(y)^q\,dy\right)^{1/q}\|v^{-1}\|_{Y,Q} \leq K. \tag{19}$$

Then

$$M_\varphi : L^p(v^p) \to L^q(w^q). \tag{20}$$

Part (i) of the following corollary is the analogue of Corollary 2.10. Observe that a simple duality argument shows that inequality (17) is equivalent to

$$\int_{\mathbb{R}^n} T_\Phi f(y)^p \frac{dy}{M_{\tilde{\Phi}^{ps}}(u^s)(y)^{1/s}} \leq c\int_{\mathbb{R}^n} f(y)^p \frac{dy}{u(y)}, \tag{21}$$

for each $s > p' - 1$. Part (ii) is the corresponding dual statement for M_φ which does not follow directly from Part (i) since maximal operators are not linear.

Corollary 2.12. *Let $1 < p < \infty$, and let φ be as above.*

(i) *There is a positive constant c such that, for every nonnegative locally integrable function f and v,*

$$\int_{\mathbb{R}^n} M_\varphi f(y)^p u(y)\,dy \leq c\int_{\mathbb{R}^n} f(y)^p M_{\varphi^p}u(y)\,dy. \tag{22}$$

(ii) *For each $s > p' - 1$ there is a positive constant c such that, for every nonnegative locally integrable function f and u,*

$$\int_{\mathbb{R}^n} M_\varphi f(y)^p \frac{dy}{M_{\varphi^{ps}}(u^s)(y)^{1/s}} \leq c\int_{\mathbb{R}^n} f(y)^p\,\frac{dy}{u(y)}. \tag{23}$$

Furthermore, this inequality does not hold for $s = p' - 1$.

Part (i) follows by applying Theorem 2.11 to the weights $w = u^{1/p}$ and $v = (M_{\varphi^p}u)^{1/p}$, since it is very easy to check that the couple (w,v) satisfies (19) with $p = q$ and $Y = L^{p'r}$, $1 < r < \infty$.

For part (ii) we consider the weights $w = \left(M_{\varphi^{ps}}(u^s)\right)^{-1/ps}$, $v = u^{-1/p}$. Then the couple (w,v) satisfies (19) with $Y = L^{p'r}$, $r = (p-1)s$. Observe that $r > 1$, and the theorem applies.

To see that (23) is false whenever $s = p' - 1$, consider for instance the Hardy–Littlewood maximal operator M, corresponding to $\varphi \equiv 1$. For any positive function $g \in L^1(\mathbb{R}^n)$ consider the weight $u = g^{p-1}$. Observe that, for $f = g$, the right–hand part of (23) is finite, while the left–hand part is infinite since $\int_{\mathbb{R}^n} Mg(y)\, dy = \infty$, for each $g \in L^1(\mathbb{R}^n)$.

The case $\varphi \equiv 1$ corresponds to the classical Fefferman-Stein inequality (cf. [10], p. 150). If $\varphi(t) = t^m$, $0 < m < 1$, M_φ is the fractional maximal operator of Marcinkiewicz, and the weighted inequality is due to E. Sawyer ([19]).

3. Preliminaries. In this brief section we shall provide some background from the theory of function spaces that will be used later. We begin by recalling some basic facts about the theory of Banach function spaces. We shall refer to [3] for a more complete account. Let (R, μ) be a measure space, and let $M^+(R)$ be the cone of μ–measurable functions on R whose values lie in $[0, \infty]$. A mapping $\rho : M^+(R) \to [0, oo]$ is called a Banach function norm if, for all f, g, f_n $(n = 1,2,3,\ldots)$ in $M^+(R)$, for all constants $a \geq 0$, and for all μ–measurable subsets E of R, the following properties hold:

(i) $\rho(f) = 0$ iff $f = 0$ μ–a.e.; $\rho(af) = a\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$;
(ii) $0 \leq g \leq f$ μ–a.e. implies $\rho(g) \leq \rho(f)$;
(iii) $0 \leq f_n \uparrow f$ μ–a.e. implies $\rho(f_n) \uparrow \rho(f)$;
(iv) $\mu(E) < \infty$ implies $\rho(\chi_E) < \infty$;
(v) $\mu(E) < \infty$ implies $\int_E f\, d\mu \leq C_E \rho(f)$,

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f. Let $M(R)$ denote the collection of all μ–measurable functions on R. The collection $X = X(\rho)$ of all functions $f \in M(R)$ for which $\rho(|f|) = \|f\|_X < \infty$ is called a Banach function space. The most important property of the Banach function spaces that we shall be using is as follows. Given a Banach function space X there is another Banach function space X', the associate space of X, for which the following generalized Hölder inequality holds:

$$\int_R |f(y)\, g(y)|\, d\mu(y) \leq \|f\|_X\ \|g\|_{X'}. \tag{24}$$

Examples of Banach function spaces include the Lebesgue L^p spaces, the Lorentz Λ, M, and $L^{p,q}$ spaces (cf. [3]), and the Orlicz spaces that we shall briefly describe next.

The Orlicz spaces are one of hte most relevant Banach function spaces. We shall provide some basic facts about these spaces, and refer to the classical reference [13] or to [14], Chapter 3, for a general account. A function B defined on $[0, \infty)$ is a Young function if it is continuous, convex and increasing satisfying $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. We shall assume that B is normalized so that $B(1) = 1$. We shall require that B satisfy the Δ_2 condition, namely, there are constants $C > 0, k \geq 0$ so that

$$B(2t) \leq CB(t), \qquad t \geq k \geq 0. \tag{25}$$

Each Young function B has associated a complementary Young function $\bar{B}$ that satisfies

$$t \leq B^{-1}(t)\,\bar{B}^{-1}(t) \leq 2t, \qquad t > 0. \tag{26}$$

Let (X, μ) be a measure space and let B be a Young function. The Orlicz space $L^B(\mu)$ consists of all μ–neasurable functions f such that

$$\int_X B\left(\frac{|f(y)|}{\lambda}\right) d\mu(y) < \infty,$$

for some $\lambda > 0$. $L^B(\mu)$ is equipped with the Luxemburg norm defined by

$$\|f\|_{B,\mu} = \inf\left\{\lambda > 0 : \int_X B\left(\frac{|f(y)|}{\lambda}\right)\, d\mu(y) \leq 1\right\}.$$

Proof of Theorem 2.1.

Proof of Theorem 2.1. Since T_Φ is a positive operator, and the set of bounded functions with compact support is dense in $L^p(v^p)$, it is enough to show that there is a constant C such that

$$\left(\int_{\mathbb{R}^n} \left(w(y)T_\Phi f(y)\right)^q\, dy\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} \left(v(y)f(y)\right)^p\, dy\right)^{1/p},$$

for each nonnegative bounded function with compact support f. This is in turn equivalent by duality to

$$\int_{\mathbb{R}^n} w(y)T_\Phi f(y)g(y)\, dy \leq C\left(\int_{\mathbb{R}^n} \left(v(y)f(y)\right)^p\, dy\right)^{1/p}\left(\int_{\mathbb{R}^n} g(y)^{q'}\, dy\right)^{q'}, \tag{27}$$

for all nonnegative bounded function with compact support f, g.

We set for each $t > 0$,

$$\bar{\Phi}(t) = \sup_{t<|x|\leq 2t} \Phi(x),$$

and

$$\underline{\Phi}(t) = \frac{1}{t^n} \int_{\delta(1-\varepsilon)t<|y|\leq 2\delta(1+\varepsilon)t} \Phi(y)\, dy,$$

where ε, δ are the numbers provided by condition (D). Following [11], we can discretize the operator T_Φ as follows:

$$\begin{aligned} T_\Phi f(t) &= \sum_{\nu\in\mathbb{Z}} \int_{2^{-\nu-1}<|z-y|\leq 2^{-\nu}} \Phi(y-z) f(z)\, dz \\ &\leq \sum_{\nu\in\mathbb{Z}} \bar{\Phi}(2^{-\nu-1}) \int_{|z-y|\leq 2^{-\nu}} f(z)\, dz \\ &= \sum_{\nu\in\mathbb{Z}} \sum_{Q:\ell(Q)=2^{-\nu}} \chi_Q(y) \bar{\Phi}(2^{-\nu-1}) \int_{|z-y|\leq 2^{-\nu}} f(z)\, dz. \end{aligned}$$

The ball $B(y, 2^{-\nu})$ is covered by the cube $3Q$ if $y \in Q$ and $\ell(Q) = 2^{-\nu}$. Hence

$$T_\Phi f(y) \leq \sum_Q \bar{\Phi}\left(\frac{\ell(Q)}{2}\right) \int_{3Q} f(z)\, dz\, \chi_Q(y).$$

We can then estimate

$$\begin{aligned} \int_{\mathbb{R}^n} w(y) T_\Phi f(y) g(y)\, dy &\leq \sum_Q \bar{\Phi}\left(\frac{\ell(Q)}{2}\right) \int_{3Q} f(z)\, dz \int_Q w(z) g(z)\, dz \\ &= \sum_Q \frac{1}{|Q|} \int_Q w(z) g(z)\, dz\, |Q| \bar{\Phi}\left(\frac{\ell(Q)}{2}\right) \int_{3Q} f(z)\, dz. \end{aligned}$$

The next task is to replace the sum over all dyadic cubes by the sum over more suitable dyadic cubes, namely Calderón–Zygmund cubes. To do this, we let a be a constant larger than 2^n that will be chosen later. Since g has compact support,

$$\frac{1}{|Q|} \int_Q w(z) g(z)\, dz \to 0$$

as $Q \uparrow \mathbb{R}^n$. Let k be an integer; it follows that, if for some dyadic cube Q,

$$a^k < \frac{1}{|Q|} \int_Q w(z) g(z)\, dz, \tag{28}$$

then Q is contained in dyadic cubes satisfying this condition, which are maximal with respect to the inclusion. Thus, for each integer k there is a family of maximal nonoverlapping dyadic cubes $\{Q_{k,j}\}$ satisfying (28). Furthermore, by maximality, we have

$$a^k < \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} w(z)\,g(z)\,dz \le 2^n\,a^k. \tag{29}$$

We adapt now some ideas from [22]. For each integer k we let

$$\mathcal{C}^k = \left\{Q \in \mathcal{D} : a^k < \frac{1}{|Q|}\int_Q w(z)\,g(z)\,dz \le a^{k+1}\right\}.$$

Every dyadic cube Q for which $\int_Q w(z)\,g(z)\,dz \neq 0$ belongs to exactly one $\mathcal{C}^k$. Furthermore, if $Q \in \mathcal{C}^k$, it follows that $Q \subset Q_{k,j}$ for some j. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^n} w(y)T_\Phi f(y)\,g(y)\,dy \\
&\le \sum_{k\in\mathbf{Z}}\sum_{Q\in\mathcal{C}^k}\frac{1}{|Q|}\int_Q w(z)\,g(z)\,dz\;|Q|\hat{\Phi}\left(\frac{\ell(Q)}{2}\right)\int_{3Q} f(z)\,dz \\
&\le a\sum_{k\in Z} a^k \sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{C}^k:Q\subset Q_{k,j}} |Q|\bar{\Phi}\left(\frac{\ell(Q)}{2}\right)\int_{3Q} f(z)\,dz \\
&\le a\sum_{k,j}\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} w(z)\,g(z)\,dz \sum_{Q:Q\subset Q_{k,j}} |Q|\bar{\Phi}\left(\frac{\ell(Q)}{2}\right)\int_{3Q} f(z)\,dz \\
&\le a\sum_{k,j}\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} w(z)\,g(z)\,dz\;\tilde{\Phi}\big(\delta(1+\varepsilon)\ell(Q_{k,j})\big)\int_{3Q_{k,j}} f(z)\,dz.
\end{aligned} \tag{30}$$

Recalling that $\tilde{\Phi}(t) = \int_{|z|\le t}\Phi(z)\,dz$, the last inequality will follow if we show that there is a constant C_Φ such that, for any dyadic cube Q_0,

$$\sum_{Q:Q\subset Q_0} |Q|\bar{\Phi}\left(\frac{\ell(Q)}{2}\right)\int_{3Q} f(z)\,dz \le C_\Phi\tilde{\Phi}\big(\delta(1+\varepsilon)\ell(Q_0)\big)\int_{3Q_0} f(z)\,dz. \tag{31}$$

However, if $\ell(Q_0) = 2^{-\nu_0}$,

$$\begin{aligned}
&\sum_{Q:Q\subset Q_0} |Q|\bar{\Phi}\left(\frac{\ell(Q)}{2}\right)\int_{3Q} f(z)\,dz \\
&= \sum_{\nu\ge\nu_0}\sum_{\substack{\ell(Q)=2^{-\nu}\\ Q\subset Q_0}} 2^{-\nu n}\bar{\Phi}(2^{-\nu-1})\int_{3Q} f(z)\,dz \\
&\le C_n\Big[\sum_{\nu\ge\nu_0} 2^{-\nu n}\bar{\Phi}(2^{-\nu-1})\Big]\int_{3Q_0} f(z)\,dz,
\end{aligned} \tag{32}$$

since the overlap is finite. Now, by condition (D), $\bar{\Phi}(2^{-\nu}) \leq C_\Phi \underline{\Phi}(2^{-\nu})$, $\nu \in \mathbb{Z}$, and then

$$\sum_{\nu \geq \nu_0} 2^{-\nu n} \bar{\Phi}(2^{-\nu-1}) \leq C \sum_{\nu \geq \nu_0} 2^{-\nu n} \underline{\Phi}(2^{-\nu-1})$$

$$= C \sum_{\nu \geq \nu_0} \int_{\delta(1-\varepsilon)2^{-\nu-1} < |y| \leq \delta(1+\varepsilon)2^{-\nu}} \Phi(y)\, dy$$

$$\leq C_{\delta,\varepsilon} \int_{|y| \leq \delta(1+\varepsilon)2^{-\nu_0}} \Phi(y)\, dy$$

$$= C\tilde{\Phi}\big(\delta(1+\varepsilon)\ell(Q_0)\big),$$

again because the overlap is finite. Combining this and (32) yields inequality (31). Let $\rho = \delta(1+\varepsilon)$, then estimate (30) can be rewritten as

$$\int_{\mathbb{R}^n} w(y) T_\Phi f(y)\, g(y)\, dy$$

$$\leq C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(y)\, g(y)\, dy\ \tilde{\Phi}\big(\rho\ell(Q_{k,j})\big) \frac{1}{|Q_{k,j}|} \int_{3Q_{k,j}} f(y)\, dy |Q_{k,j}|$$

$$\leq C \sum_{k,j} \frac{1}{|\gamma Q_{k,j}|} \int_{\gamma Q_{k,j}} w(y)\, g(y)\, dy$$

$$\frac{1}{|\gamma Q_{k,j}|} \int_{\gamma Q_{k,j}} f(y)\, v(y)\, v(y)^{-1} \tilde{\Phi}\big(\ell(\gamma Q_{k,j})\big) |Q_{k,j}|,$$

where $\gamma = \max\{3, \rho\}$. The generalized Hölder's inequality (24) together with hypothesis (8) allows to estimate the last sum by a multiple of

$$\sum_{k,j} \|g\|_{X',\gamma Q_{k,j}}\ \|w\|_{X,\gamma Q_{k,j}}\ \|fv\|_{Y',\gamma Q_{k,j}}\ \|v^{-1}\|_{Y,\gamma Q_{k,j}} \tilde{\Phi}\big(\ell(\gamma Q_{k,j})\big)\, |Q_{k,j}|$$

$$\leq K \sum_{k,j} \|g\|_{X',\gamma Q_{k,j}}\ \|fv\|_{Y',\gamma Q_{k,j}}\ |Q_{k,j}|^{1/p+1/q'}.$$

Now, by using Hölder's ionequality and then that $p \leq q$, we can follow our chain of inequalities with

$$\leq K \Big(\sum_{k,j} \|g\|^{p'}_{X',\gamma Q_{k,j}}\ |Q_{k,j}|^{p'/q'} \Big)^{1/p'} \Big(\sum_{k,j} \|fv\|^{p}_{Y',\gamma Q_{k,j}}\ |Q_{k,j}| \Big)^{1/p}$$

$$\leq K \Big(\sum_{k,j} \|g\|^{q'}_{X',\gamma Q_{k,j}}\ |Q_{k,j}| \Big)^{1/q'} \Big(\sum_{k,j} \|fv\|^{p}_{Y',\gamma Q_{k,j}}\ |Q_{k,j}| \Big)^{1/p}. \tag{33}$$

Now, we shall use the following properties of the Calderón–Zygmund cubes. For each k we let $D_k = \bigcup_j Q_{k,j}$. Observe that $D_{k+1} \subset D_k$, and we can consider for each k, j the set $E_{k,j} = Q_{k,j} - Q_{k,j} \cap D_{k+1}$. Then $\{E_{k,j}\}$ is a disjoint family of sets which satisfy

$$|Q_{k,j} \cap D_{k+1}| < \frac{2^n}{a} |Q_{k,j}|, \tag{34}$$

and

$$|Q_{k,j}| < \frac{1}{1 - \frac{2^n}{a}} |E_{k,j}|. \tag{35}$$

Deferring the proof of these inequalities for the moment, we can estimate (33) by a multiple of

$$\begin{aligned}
&\Big(\sum_{k,j} \|g\|^{q'}_{X',\gamma Q_{k,j}} |E_{k,j}|\Big)^{1/q'} \Big(\sum_{k,j} \|fv\|^{p}_{Y',\gamma Q_{k,j}} |E_{k,j}|\Big)^{1/p} \\
&= \Big(\sum_{k,j} \int_{E_{k,j}} \|g\|^{q'}_{X',\gamma Q_{k,j}}\, dy\Big)^{1/q'} \Big(\sum_{k,j} \int_{E_{k,j}} \|fv\|^{p}_{Y',\gamma Q_{k,j}}\, dy\Big)^{1/p} \\
&\le \Big(\sum_{k,j} \int_{E_{k,j}} M_{X'} g(y)^{q'}\, dy\Big)^{1/q'} \Big(\sum_{k,j} \int_{E_{k,j}} M_{Y'}(fv)(y)^{p}\, dy\Big)^{1/p} \\
&\le \Big(\int_{\mathbb{R}^n} M_{X'} g(y)^{q'}\, dy\Big)^{1/q'} \Big(\int_{\mathbb{R}^n} M_{Y'}(fv)(y)^{p}\, dy\Big)^{1/p} \\
&\le C\Big(\int_{\mathbb{R}^n} g(y)^{q'}\, dy\Big)^{1/q'} \Big(\int_{\mathbb{R}^n} (f(y)v(y))^{p}\, dy\Big)^{1/p},
\end{aligned}$$

concluding the proof of the Theorem, save for (34) and (35). The proof of these inequalities is a simple adaptation of arguments in [4] (cf. [10], p. 398). The family $\{E_{k,j}\}$ is clearly disjoint. Now, by standard properties of the dyadic cubes together with (29) we have

$$\begin{aligned}
|Q_{k,j} \cap D_{k+1}| &= \sum_i |Q_{k,j} \cap Q_{k+1,i}| \quad = \\
&= \sum_{i:Q_{k+1,i} \subset Q_{k,j}} |Q_{k+1,i}| \\
&< \sum_{i:Q_{k+1,i} \subset Q_{k,j}} \frac{1}{a^{k+1}} \int_{Q_{k+1,i}} w(y)\, g(y)\, dy
\end{aligned}$$

$$\leq \frac{1}{a^{k+1}} \int_{Q_{k,j} \cap \cup_i Q_{k+1,i}} w(y)\, g(y)\, dy$$

$$\leq \frac{2^n}{a} |Q_{k,j}|.$$

This gives (34). Finally,

$$\frac{|E_{k,j}|}{|Q_{k,j}|} > 1 - \frac{2^n}{a} > 0$$

yields (35), which concludes the proof of Theorem 2.1. □

Proof of Corollary 2.7. We recall that, for any Young function B, the maximal operator M_B is defined by

$$M_B f(x) = \sup_{x \in Q} \|f\|_{B,Q},$$

where

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1 \right\}.$$

By Theorem 2.1, we need to show that both

$$\begin{cases} M_{\bar{A}} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n), \\ M_{\bar{B}} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n). \end{cases} \tag{36}$$

This follows from part (i) and (ii) of the following theorem that can be found in [18].

Theorem 4.1. *Let $1 < r < \infty$. Suppose that B is a Young function. Then the following are equivalent:*

(i) *We have:*

$$B \in B_r\,; \tag{37}$$

(ii) *there is a constant c such that*

$$\int_{\mathbb{R}^n} M_B f(y)^r\, dy \leq c \int_{\mathbb{R}^n} f(y)^r\, dy \tag{38}$$

for all nonnegative, locally integrable functions f;

(iii) *there is a constant c such that*

$$\int_{\mathbb{R}^n} M_B f(y)^r w(y)\, dy \le c \int_{\mathbb{R}^n} f(y)^r MW(y)\, dy \tag{39}$$

for all nonnegative, locally integrable functions f and w;

(iv) *there is a constant c such that*

$$\int_{\mathbb{R}^n} Mf(y)^r \frac{w(y)}{[M_{\bar{B}}(u^{1/r})(y)]^r}\, dy \le c \int_{\mathbb{R}^n} f(y)^r \frac{Mw(y)}{u(y)}\, dy\,, \tag{40}$$

for all nonnegative, locally integrable functions f, w, and u.

5. Proof of Theorem 2.11. Recall that the maximal operator M_φ is defined by

$$M_\varphi f(x) = \sup_{x\in Q} \frac{\varphi(|Q|)}{|Q|} \int_Q f(y)\, dy\,.$$

M_φ^d denotes the corresponding dyadic version. Recall that we assume that $\varphi : (0,\infty) \to (0,\infty)$ is essentially nondecreasing, i.e., that $\varphi(t) \le \rho\varphi(s)$, $t \le s$, and also that

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0.$$

We need the following lemma:

Lemma 5.1. *Suppose that f is a nonnegative bounded function with compact support. For each $t > 0$, let $E_t = \{x \in \mathbb{R}^n : M_\varphi f(x) > t\}$. Then, if E_t is not empty, we have*

$$E_t \subset \bigcup_j 3Q_j,$$

where Q_j is the family of nonoverlapping maximal dyadic cubes satisfying

$$\frac{t}{4^n\rho} < \frac{\varphi(|Q_j|)}{|Q_j|} \int_{Q_j} f(y)\, dy \le \frac{t}{2^n} \tag{41}$$

for each integer j.

Furthermore, we have that

$$\left\{x \in \mathbb{R}^n : M_\varphi^d f(x) > \frac{t}{4^n\rho}\right\} = \bigcup_j Q_j.$$

Proof. Following [10], p. 160, we let $C_t = \{P_j\}$ be the family of the dyadic maximal nonoverlapping cubes satisfying the condition

$$t < \frac{\varphi(|P_j|)}{|P_j|} \int_{P_j} f(y)\, dy. \tag{42}$$

To show that there is such a family C_t, observe that

$$\frac{\varphi(|Q|)}{|Q|} \int_Q f(y)\, dy \to 0$$

as $Q \uparrow \mathbb{R}^n$, since f has compact support and since $\lim_{t\to\infty} \varphi(t)/t = 0$. If for some dyadic cube Q

$$t < \frac{\varphi(|Q|)}{|Q|} \int_Q f(y)\, dy, \tag{43}$$

then Q is contained in dyadic cubes satisfying this condition, which are maximal with respect to the inclusion. Thus, there is a family of maximal nonoverlapping dyadic cubes $\{P_j\}$ satisfying (43). The growth condition on φ and the maximality of the cubes P_j yield

$$t < \frac{\varphi(|P_j|)}{|P_j|} \int_{P_j} f(y)\, dy \le 2^n \rho \frac{\varphi(|P_j'|)}{|P_j'|} \int_{P_j'} f(y)\, dy \le 2^n \rho t, \tag{44}$$

where P_j' denotes the only dyadic cube containing P_j. From this discussion it is clear that

$$\{x \in \mathbb{R}^n : M_\varphi^d f(x) > t\} = \bigcup_j P_j.$$

Let $x \in E_t$; by definition, there is a cube R containing x such that

$$t < \frac{\varphi(|R|)}{|R|} \int_R f(y)\, dy. \tag{45}$$

Let k be the unique integer such that $2^{-(k+1)n} < |R| \le 2^{-kn}$. There is some dyadic cube with side length 2^{-k}, and at most 2^n of them, $\{J_i : i = 1, \dots, 2^n\}$ meeting the interior of R. It is easy to see that, for one of these cubes, say J_1,

$$\frac{t}{2^n} < \frac{\varphi(|R|)}{|R|} \int_{R\cap J_1} f(y)\, dy. \tag{46}$$

Now, since $|R| \leq |J_1| < 2^n|R|$, $\varphi(|R|) \leq \rho\varphi(|J_1|)$, and then

$$\frac{t}{4^n}|J_1| < \frac{t}{2^n}|R| < \varphi(|R|)\int_{R\cap J_1} f(y)\,dy \leq \rho\varphi(|J_1|)\int_{J_1} f(y)\,dy. \tag{47}$$

Hence,

$$\frac{t}{4^n\rho} < \frac{\varphi(|J_1|)}{|J_1|}\int_{J_1} f(y)\,dy. \tag{48}$$

By letting $C_{t/4^n\rho} = \{Q_j\}$, we see that $J_1 \subset Q_k$, for some k, and $R \subset 3J_1 \subset 3Q_k$. From this we conclude that

$$E_t \subset \bigcup_j 3Q_j.$$

Finally, it follows from (44) that

$$\frac{t}{4^n\rho} < \frac{\varphi(|Q_j|)}{|Q_j|}\int_{Q_j} f(y)\,dy \leq t2^n, \tag{49}$$

for each j, concluding the proof of the lemma. □

Remark 5.2. We point out that the lemma works also under the assumption that there is a positive constant ρ with

$$\varphi(t) \leq \rho\varphi(\lambda t) \tag{50}$$

for all $1 \leq \lambda \leq 2$, $t > 0$. The estimates are the same replacing ρ by ρ^n.

Proof of Theorem 2.11. As above, it is enough to show that there is a constant C such that

$$\left(\int_{\mathbb{R}^n} \left(w(y)M_\varphi f(y)\right)^q\,dy\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} \left(v(y)f(y)\right)^p\,dy\right)^{1/p},$$

for each nonnegative bounded function with compact support f.

This time we do not use any duality argument, but we discretize the operator in the following way. For each integer k, and for a constant $a > 2^n$ that will be chosen later, we let Ω_k and D_k be the sets

$$\Omega_k = \left\{x \in \mathbb{R}^n : a^k < M_\varphi f(x)\right\}$$

$$D_k = \left\{x \in \mathbb{R}^n : \frac{a^k}{4^n} < M_\varphi^d f(x)\right\},$$

respectively.

By using Lemma 5.1 with $t = 4^n \rho a^k$, there is a family of maximal nonoverlapping cubes $\{Q_{k,j}\}$ for which $\Omega_k \subset \bigcup_j 3Q_{k,j}$, $D_k = \bigcup_j Q_{k,j}$, and

$$a^k < \frac{\varphi(|Q_{k,j}|)}{|Q_{k,j}|} \int_{Q_{k,j}} f(y)\, dy \leq a^k 2^n \rho. \tag{51}$$

With this notation we have the following string of inequalities

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} (w(y) M_\varphi f(y))^q \, dy\Big)^{1/q} \\
&= \Big(\sum_k \int_{\Omega_k - \Omega_{k+1}} (w(y) M_\varphi f(y))^q \, dy\Big)^{1/q} \\
&\leq a\Big(\sum_k a^{kq} (w^q)(\Omega_k)\Big)^{1/q} \\
&\leq c\Big(\sum_{k,j} \left(\frac{\varphi(|Q_{k,j}|)}{|Q_{k,j}|} \int_{Q_{k,j}} f(y)\, dy\right)^q (w^q)(3Q_{k,j})\Big)^{1/q} \\
&\leq a\rho\Big(\sum_k a^{kq} (w^q)(\Omega_k)\Big)^{1/q} \\
&\leq \Big(\sum_{k,j} \left(\frac{\varphi(|3Q_{k,j}|)}{|Q_{k,j}|} \int_{Q_{k,j}} f(y) v(y) v(y)^{-1}\, dy\right)^q (w^q)(3Q_{k,j})\Big)^{1/q} \\
&\leq c\Big(\sum_{k,j} \varphi(|3Q_{k,j}|)^q \, \|fv\|^q_{Y',3Q_{k,j}} \, \|v^{-1}\|^q_{Y,3Q_{k,j}} \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} w(y)^q \, dy \, |Q_{k,j}|\Big)^{1/q} \\
&\leq c\Big(\sum_{k,j} \|fv\|^q_{Y',3Q_{k,j}} \, |Q_{k,j}|^{q/p}\Big)^{1/q} \\
&\leq c\Big(\sum_{k,j} \|fv\|^p_{Y',3Q_{k,j}} \, |Q_{k,j}|\Big)^{1/p},
\end{aligned}$$

since $p \leq q$. As in the proof of Theorem 2.1, we let for each k, j the set $E_{k,j} = Q_{k,j} - Q_{k,j} \cap D_{k+1}$. Then $\{E_{k,j}\}$ is a disjoint family of sets which satisfy with $a > 2^n p^2$ the estimates

$$|Q_{k,j} \cap D_{k+1}| < \frac{2^n \rho^2}{a} |Q_{k,j}|, \tag{52}$$

and

$$(53) \qquad |Q_{k,j}| < \frac{1}{1 - \dfrac{2^n \rho^2}{a}} |E_{k,j}|.$$

The proof of these inequalities is exactly the same as the proof of (34) and (35), using the growth assumption on φ. Hence, we shall conclude as in the proof of Theorem 2.1..

$$\begin{aligned}
\Big(\int_{\mathbb{R}^n} \big(w(y) M_\varphi f(y)\big)^q \, dy\Big)^{1/q} &\leq \Big(\sum_{k,j} \|fv\|_{Y',3Q_{k,j}} |E_{k,j}|\Big)^{1/p} \\
&\leq \Big(\sum_{k,j} \int_{E_{k,j}} \|fv\|^p_{Y',\gamma Q_{k,j}} \, dy\Big)^{1/p} \\
&\leq \Big(\int_{\mathbb{R}^n} M_{Y'}(fv)(y)^p \, dy\Big)^{1/p} \\
&\leq \Big(\int_{\mathbb{R}^n} \big(f(y)v(y)\big)^p \, dy\Big)^{1/p},
\end{aligned}$$

since, by hypothesis, $M_{Y'} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. This concludes the proof of the theorem. □

References

[1] D. ADAMS, *Weighted nonlinear potential theory*, Trans. Amer. Math. Soc. **297** (1986), 73–94.

[2] ———, *On the existence of the capacitary strong type estimate in* $\mathbb{R}^n$, Ark. Mat. **14** (1976), 125–140.

[3] C. BENNETT & R. SHARPLEY, *Interpolation of Operators*, Academic Press, New York, 1988.

[4] A. P. CALDERÓN, *Inequalities for the maximal function relative to a metric*, 57 (1976), 297–306.

[5] S. Y. A. CHANG, J. M. WILSON, AND T. H. WOLFF, *Some weighted norm inequalities concerning the Schrödinger operators*, Comment. Math. Helvetici **60** (1985), 217–286.

[6] S. CHANILLO, R. L. WHEEDEN, *L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators*, Comm. P. D. E. **10** (1985), 1077–1116.

[7] A. CÓRDOBA & C. FEFFERMAN, *A weighted inequality for singular integrals*, Studia Math. **57** (1976), 97–101.

[8] C. FEFFERMAN, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), 129–206.

[9] M. FRAZIER, B. JAWERTH & G. WEISS, *Littlewood–Paley Theory and the Study of Function Spaces*, Vol. 79, CBMS–AMS Regional Conference Series, 1990.

[10] J. GARCÍA–CUERVA & J. L. RUBIO DE FRANCIA, *Weighted Norm Inequalities and Related Topics*, North–Holland Math. Studies, Vol. 116, North–Holland, Amsterdam, 1985.

[11] B. JAWERTH, C. PÉREZ & G. WELLAND, *The positive cone in Triebel–Lizorkin spaces and the relation among potential and maximal operators*, Contemporary Mathematics Series, M. Milman, ed., Amer. Math. Soc., Providence, Rhode Island, 1989.

[12] R. KERMAN & E. SAWYER, *The trace inequality and eigenvalue estimates for Schrödinger operators*, Ann. Inst. Fourier **36** (1986), 207–228.

[13] M. A. KRASNOSEL'SKIĬ & J. B. RUTICKIĬ, *Convex Functions and Orlicz Spaces*, Noordhoff, Gröningen, 1961.

[14] A. KUFNER, J. OLDŘICH & F. SVATOPLUK, *Function Spaces*, Noordhoff, Leyden, 1977.

[15] B. MUCKENHOUPT, *Weighted norm inequalities for classical operators*, Proc. Symp. Pure Math. **35** (1977), 69–83.

[16] C. PÉREZ, *Two weighted norm inequalities for Riesz potentials and uniform L^p-weighted Sobolev inequalities*, Indiana Univ. Math. J. **39** (1990).

[17] ———, *On a theorem of Muckenhoupt and Wheeden and a weighted inequality related to Schrödinger operators*, Trans. Amer. Math. Soc. **340** (1993), 549–562.

[18] ———, *On sufficient conditions for the boundedness of the Hardy–Littlewood maximal operator between weighted L^p-spaces with different weights*, J. London Math. Soc. (to appear).

[19] E. SAWYER, *Weighted norm inequalities for fractional maximal operators*, Proc. C. M. S. **1** (1981), 283–309.

[20] ———, *A characterization of two weight norm inequalities for fractioanl and Poisson integrals*, 281 (1988), 533–545.

[21] ———, *A two weight weak type inequality for fractional integrals*, Trans. Amer. Math. Soc. **281** (1984), 339–345.

[22] E. SAWYER & R. L. WHEEDEN, *Weighted inequalities for fractional integrals on euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874.

[23] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.

[24] J. M. WILSON, *Weighted norm inequalities for the continuous square functions*, Trans. Amer. Math. Soc. **341** (1989), 661–692.

Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506

CURRENT ADDRESS:
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain

Received: August 21st, 1991; revised: July 5th, 1992.