SOME BOUNDS AND LIMITS IN THE THEORY OF RIEMANN'S ZETA FUNCTION

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ABSTRACT. For any real $a > 0$ we determine the supremum of the real σ such that $\zeta(\sigma+it) = a$ for some real t. For $0 < a < 1$, $a = 1$, and $a > 1$ the results turn out to be quite different.

We also determine the supremum E of the real parts of the 'turning points', that is points $\sigma + it$ where a curve $\operatorname{Im}\zeta(\sigma+it) = 0$ has a vertical tangent. This supremum E (also considered by Titchmarsh) coincides with the supremum of the real σ such that $\zeta'(\sigma+it) = 0$ for some real t.

We find a surprising connection between the three indicated problems: $\zeta(s) = 1$, $\zeta'(s) = 0$ and turning points of $\zeta(s)$. The almost extremal values for these three problems appear to be located at approximately the same height.

1. INTRODUCTION.

In this paper we study various bounds and limits related to the values of Riemann's $\zeta(s) = \zeta(\sigma+it)$ with s in the half-plane $\sigma > 1$. For example, in Titchmarsh [8, Theorem 11.5(C)] it is shown that $E :=$ the supremum of all σ such that $\zeta'(\sigma+it) = 0$ for some $t \in \mathbb{R}$, satisfies $2 < E < 3$. Also, one of us [3] proved that $\sigma_0 :=$ the unique solution to the equation $\sum_p \arcsin(p^{-\sigma}) = \frac{\pi}{2}$, is the supremum of all σ such that $\operatorname{Re}\zeta(\sigma+it) < 0$ for some $t \in \mathbb{R}$ and $\operatorname{Re}\zeta(\sigma_0+it) > 0$ for all $t \in \mathbb{R}$.

In [4] and [5] we encounter the question of the supremum $\sigma(1)$ of $\operatorname{Re}(s)$ for the solutions of $\zeta(s) = 1$. In Sections 3 and 4 we will solve this problem and also answer the same question for the solutions of $\zeta(s) = a$ for any given $a > 0$.

In Section 5 we give a more direct proof of Theorem 11.5(C) of Titchmarsh.

Our method is constructive so that it allowed us to find explicit roots of $\zeta(s) = 1$ with σ near the extremal value $\sigma(1)$ (by means of the Lenstra–Lenstra–Lovász lattice basis reduction algorithm), and analogously solutions of $\zeta'(s) = 0$ with $\operatorname{Re}(s)$ near E. We also found a relation between the two problems: Near every almost-extremal solution for $\zeta(s) = 1$ there is one for $\zeta'(\rho) = 0$ with $\rho - s \approx E - \sigma(1)$ (see Section 6 for a more precise formulation).

In Section 7 we will discuss some similar aspects of general Dirichlet functions $L(s, \chi)$.

There are two types of curves $\operatorname{Im}\zeta(\sigma + it) = 0$. One kind (the I_1 curves) is crossing the halfplane $\sigma > 0$ more or less horizontally whereas the other kind (the I_2 curves) has the form of a loop. These loops do not stick out arbitrarily far to the right. In Section 9 we determine exactly the limit of the I_2 curves $\operatorname{Im}\zeta(\sigma + it) = 0$. This problem was also mentioned in [3].

The somewhat surprising fact is that this limit of the I_2 curves is equal to the limit E of the zeros of $\zeta'(s)$ considered in Theorem 11.5 (C) of Titchmarsh.

2. The key lemmas.

We will use the following

Lemma 2.1. *There exists a sequence of real numbers (t_k) such that*

$$\lim_{k\to\infty} \zeta(s + it_k) = \frac{2^s - 1}{2^s + 1}\zeta(s)$$

uniformly on compact sets of the half plane $\sigma > 1$.

Proof. Since the numbers $\log p_n$ are linearly independent over $\mathbb{Q}$, there are (by Kronecker's theorem [1, Theorem 7.9, p. 150]) for each positive integer N and any $\eta > 0$ a real number t and integers $g_1, \dots, g_N$ such that

$$|-t\log 2 - \pi + 2\pi g_1| < \eta, \quad |-t\log p_j + 2\pi g_j| < \eta, \quad 2 \le j \le N$$

where p_n denotes the n-th prime number.

Taking η small enough we may obtain in this way a real t such that

$$|2^{-it} + 1| < \varepsilon, \quad |p_j^{-it} - 1| < \varepsilon, \quad 2 \le j \le N.$$

Repeating this construction we obtain a sequence of real numbers (t_k) such that

$$\lim_{k\to\infty} 2^{-it_k} = -1, \quad \lim_{k\to\infty} p^{-it_k} = 1 \quad \text{for any odd prime } p.$$

Now we prove that any such sequence satisfies the Lemma. For any natural number n let $\nu(n)$ be the exponent of 2 in the prime factorization of n. Let $n = 2^{\nu(n)} q_1^{a_1} \cdots q_r^{a_r}$ be the prime factorization of n. Then we will have $n^{-it_k} \to (-1)^{\nu(n)}$, and as we will show

$$\lim_{k\to\infty} \zeta(s+it_k) = \sum_{n=1}^\infty \frac{(-1)^{\nu(n)}}{n^s} \quad \text{uniformly for } \sigma \geq a > 1.$$

Given $a > 1$ and $\varepsilon > 0$ we first determine N such that

$$\sum_{n=N+1}^\infty \frac{1}{n^a} < \varepsilon.$$

For $1 \leq n \leq N$ we then have $n^{-it_k} \to (-1)^{\nu(n)}$, so that there exists a K such that

$$|n^{-it_k} - (-1)^{\nu(n)}| < \frac{\varepsilon}{N}, \qquad 1 \leq n \leq N, \quad k \geq K.$$

For $\operatorname{Re}(s) = \sigma \geq a$ and $k > K$ we will then have

$$\Big|\zeta(s+it_k) - \sum_{n=1}^\infty \frac{(-1)^{\nu(n)}}{n^s}\Big| \leq \Big|\sum_{n=1}^\infty \frac{n^{-it_k} - (-1)^{\nu(n)}}{n^s}\Big| \leq$$

$$\leq \sum_{n=1}^N |n^{-it_k} - (-1)^{\nu(n)}| n^{-a} + 2 \sum_{n=N+1}^\infty n^{-a} \leq 3\varepsilon.$$

Finally we check whether

$$\sum_{n=1}^\infty \frac{(-1)^{\nu(n)}}{n^s} = \Big(\sum_{j=0}^\infty \frac{(-1)^j}{2^{js}}\Big)\Big(\sum_{k=0}^\infty \frac{1}{(2k+1)^s}\Big) =$$

$$= \Big(1 + \frac{1}{2^s}\Big)^{-1} \prod_{p\geq 3}\Big(1 - \frac{1}{p^s}\Big)^{-1} = \frac{1 - \frac{1}{2^s}}{1 + \frac{1}{2^s}}\zeta(s) = \frac{2^s - 1}{2^s + 1}\zeta(s).$$

□

Lemma 2.2. *There exists a sequence of real numbers (t_k) such that*

$$\lim_{k\to\infty} \zeta(s+it_k) = \frac{\zeta(2s)}{\zeta(s)}$$

uniformly on compact sets of the half plane $\sigma > 1$.

Proof. The proof is similar to that of the previous Lemma. Applying Kronecker's theorem we get a sequence of real numbers (t_k) such that

$$\lim_{k\to\infty} p^{-it_k} = -1 \qquad \text{for all primes } p.$$

Similarly as in the proof of Lemma 2.1 we obtain

$$\lim_{k\to\infty} \zeta(s+it_k) = \sum_{n=1}^\infty \frac{(-1)^{\Omega(n)}}{n^s} \quad \text{uniformly for } \sigma \geq \sigma_0 > 1$$

where $\Omega(n)$ is the total number of prime factors of n counting multiplicities. It is well known that this series is equal to $\frac{\zeta(2s)}{\zeta(s)}$ (see Titchmarsh [8, formula (1.2.11)]). □

To apply these lemmas we will use a theorem of Hurwitz (see [7, Theorem 3.45, p. 119] or [2, Theorem 4.10d and Corollary 4.10e, p. 282–283]). We will use it in the following form:

Theorem 2.3 ((Hurwitz))**.** *Assume that a sequence (f_n) of holomorphic functions on a region Ω converges uniformly on compact sets of Ω to the function f which has an isolated zero $a \in \Omega$. Then for $n \geq n_0$ the functions f_n have a zero $a_n \in \Omega$ such that $\lim_n a_n = a$.*

3. The bound for $\zeta(s) = a$ (> 0) with $a \neq 1$.

For a positive real number a let $\sigma(a)$ denote the supremum of all real σ such that $\zeta(\sigma + it) = a$ for some $t \in \mathbb{R}$.

Theorem 3.1. *Let a be > 0 but $\neq 1$. If $a > 1$ then $\sigma(a)$ is the unique solution of $\zeta(\sigma) = a$ with $\sigma > 1$. If $0 < a < 1$ then $\sigma(a)$ is the unique solution of $\frac{\zeta(2\sigma)}{\zeta(\sigma)} = a$ with $\sigma > 1$.*

Proof. It will be convenient to define σ_a as the (unique) solution of the equations considered in the theorem.

The case $a > 1$. It is easily seen that in this case we have $\sigma(a) = \sigma_a$.

In the case $0 < a < 1$ we consider a solution to $\zeta(s) = a$. Then

$$a = |\zeta(s)| = \prod_p \frac{1}{\left|1 - \frac{1}{p^s}\right|} \geq \prod_p \frac{1}{1 + \frac{1}{p^\sigma}} = \frac{\zeta(2\sigma)}{\zeta(\sigma)}, \qquad (\sigma > 1).$$

It is clear from the last equality that $\frac{\zeta(2\sigma)}{\zeta(\sigma)}$ is strictly increasing (for $\sigma > 1$) from 0 to 1. Hence, there exists a unique solution σ_a to the equation $a = \frac{\zeta(2\sigma)}{\zeta(\sigma)}$. The inequality $a \geq \frac{\zeta(2\sigma)}{\zeta(\sigma)}$ is then equivalent to $\sigma \leq \sigma_a$. Taking the supremum of σ for all solutions of $\zeta(s) = a$ we obtain $\sigma(a) \leq \sigma_a$.

To prove the converse we apply Lemma 2.2: There exists a sequence of real numbers (t_k) such that $\zeta(s + it_k) - a$ converges uniformly on compact sets of $\sigma > 1$ to the function $\frac{\zeta(2s)}{\zeta(s)} - a$. The limit function has a zero at $s = \sigma_a$. So, by Hurwitz's theorem σ_a is a limit point of zeros b_k $(k \geq k_0)$ of $\zeta(s + it_k) - a$.

Therefore $\zeta(b_k + it_k) - a = 0$ and $\lim_k b_k = \sigma_a$. For $s_k := b_k + it_k$ we have $\zeta(s_k) = a$ and

$$\lim_k \operatorname{Re}(s_k) = \lim_k \operatorname{Re}(b_k) = \operatorname{Re}(\lim_k b_k) = \sigma_a.$$

It follows that

$$\sigma(a) = \sup\{\sigma : \zeta(s) = a\} \geq \lim_k \operatorname{Re}(s_k) = \sigma_a.$$

Therefore $\sigma(a) = \sigma_a$, proving our theorem.

□

4. The bound for $\zeta(s) = 1$.

Theorem 4.1. *The supremum $\sigma(1)$ of all real σ such that $\zeta(\sigma+it) = 1$ for some value of $t \in \mathbb{R}$, is equal to the unique solution $\sigma > 1$ of the equation*

$$\zeta(\sigma) = \frac{2^\sigma + 1}{2^\sigma - 1}. \tag{1}$$

Numerically we have

$$\sigma(1) = 1.94010\,16837\,43625\,28601\,74693\,90525\,54887\,82302\,47607\ldots$$

Proof. Assume that $\zeta(s) = 1$ with $\operatorname{Re}(s) = \sigma > 1$. Then by the Euler product formula

$$1 - \frac{1}{2^s} = \prod_{p\geq 3}\Bigl(1 - \frac{1}{p^s}\Bigr)^{-1} = \sum_{k=1}^\infty \frac{1}{(2k-1)^s}$$

or

$$-1 = \sum_{k=2}^\infty \Bigl(\frac{2}{2k-1}\Bigr)^s.$$

Therefore

$$1 = \Bigl|\sum_{k=2}^\infty \Bigl(\frac{2}{2k-1}\Bigr)^s\Bigr| \leq \sum_{k=2}^\infty \Bigl(\frac{2}{2k-1}\Bigr)^\sigma.$$

Since the right hand side is decreasing in σ, it follows that there is a unique solution σ_1 of the equation

$$1 = \sum_{k=2}^\infty \Bigl(\frac{2}{2k-1}\Bigr)^\sigma = (2^\sigma - 1)\zeta(\sigma) - 2^\sigma \tag{2}$$

and that $\sigma \leq \sigma_1$. Now observe that (2) is equivalent to (1). Therefore, $\zeta(s) = 1$ implies $\sigma \leq \sigma_1$ which is by definition the solution of equation (1). Taking the sup over all solutions of $\zeta(s) = 1$ we get $\sigma(1) \leq \sigma_1$.

For the converse inequality we apply Lemma 2.1 to get a sequence of real numbers (t_k) such that

$$\lim_k\{\zeta(s + it_k) - 1\} = \frac{2^s - 1}{2^s + 1}\zeta(s) - 1$$

uniformly on compact sets of $\sigma > 1$. By definition σ_1 is a zero of the limit function $\frac{2^s-1}{2^s+1}\zeta(s) - 1$, so that there exists a natural number n_0

and a sequence of complex numbers (z_k) such that $\zeta(z_k + it_k) - 1 = 0$ and $\lim_k z_k = \sigma_1$. For $s_k := z_k + it_k$ we then have $\zeta(s_k) = 1$ and $\lim_k \sigma_k = \sigma_1$ (with $\sigma_k := \operatorname{Re}(s_k)$).

It follows that $\sigma(1) = \sup_{\zeta(s)=1} \operatorname{Re} s \geq \sigma_1$, proving the theorem. □

5. The bound for $\zeta'(s) = 0$. A new proof of Titchmarsh's Theorem 11.5(C).

Theorem 11.5(C) in Titchmarsh [8] says that there exists a constant E between 2 and 3, such that $\zeta'(s) \neq 0$ for $\sigma > E$, while $\zeta'(s)$ has an infinity of zeros in every strip between $\sigma = 1$ and $\sigma = E$. In this section we give a more direct proof of this theorem and determine the precise value of E.

Theorem 5.1. *Let E be the unique solution of the equation*

$$\frac{2^{\sigma+1}}{4^\sigma - 1}\log 2 = -\frac{\zeta'(\sigma)}{\zeta(\sigma)}, \qquad (\sigma > 1). \tag{3}$$

Then $\zeta'(s) \neq 0$ for $\sigma > E$, while $\zeta'(s)$ has a sequence of zeros (s_k) with $\lim_k \operatorname{Re}(s_k) = E$.

The value of this constant is

$$E = 2.81301\,40202\,52898\,36752\,72554\,01216\,68696\,38461\,40560\ldots$$

Proof. Assuming that $\zeta'(s) = 0$ (for $\sigma > 1$) we have

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\log\zeta(s) = \frac{d}{ds}\sum_p -\log\Bigl(1 - \frac{1}{p^s}\Bigr) = -\sum_p \frac{\log p}{p^s - 1}$$

so that we may write the equation $\zeta'(s) = 0$ as

$$\sum_p \frac{\log p}{p^s - 1} = 0$$

or

$$-\frac{\log 2}{2^s - 1} = \sum_{p\geq 3}\frac{\log p}{p^s - 1}.$$

So, we must necessarily have

$$\frac{\log 2}{2^\sigma + 1} \leq \Bigl|-\frac{\log 2}{2^s - 1}\Bigr| = \Bigl|\sum_{p\geq 3}\frac{\log p}{p^s - 1}\Bigr| \leq \sum_{p\geq 3}\frac{\log p}{p^\sigma - 1}$$

and we may write this inequality as

$$\log 2 \leq \sum_{p\geq 3}(2^\sigma + 1)\Bigl(\frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} + \frac{1}{p^{3\sigma}} + \cdots\Bigr)\log p.$$

Since the right hand side is strictly decreasing in σ this is equivalent to $\sigma \leq E :=$ the unique solution of the equation

$$\frac{\log 2}{2^\sigma + 1} + \frac{\log 2}{2^\sigma - 1} = \sum_{p\geq 2} \frac{\log p}{p^\sigma - 1}$$

which is equivalent to (3).

This proves that there is no zero of $\zeta'(s)$ with $\sigma > E$.

Now we must find a sequence of complex numbers (s_k) with $\zeta'(s_k) = 0$ and $\lim_k \operatorname{Re}(s_k) = E$.

By Lemma 2.1 $\zeta'(s + it_k)$ converges uniformly on compact sets of $\sigma > 1$ to the function

$$\frac{d}{ds}\frac{2^s - 1}{2^s + 1}\zeta(s) = \Big(\frac{2^{s+1}}{4^s - 1}\log 2 + \frac{\zeta'(s)}{\zeta(s)}\Big) \cdot \frac{2^s - 1}{2^s + 1}\zeta(s).$$

This function has a zero at $s = E$ (see equation (3)), so that by Hurwitz's theorem, there exist for $k \geq k_0$ numbers z_k such that $z_k \to E$ and $\zeta'(z_k + it_k) = 0$. Taking $s_k = z_k + it_k$ we will have $\zeta'(s_k) = 0$ and

$$\lim_k \operatorname{Re}(s_k) = \lim_k \operatorname{Re}(z_k + it_k) = \lim_k \operatorname{Re}(z_k) = E$$

as we wanted to show.

With Mathematica we found that the solution to equation (3) is approximately the number given in the theorem. □

6. The connection between $\zeta(s) = 1$ and $\zeta'(s) = 0$.

We have seen that to get points with $\zeta(s) = 1$ and σ near $\sigma(1)$, and points ρ with $\zeta'(\rho) = 0$ and $\operatorname{Re}\rho$ near E, we have applied in both cases Lemma 2.1. The limit function $f(s) := \frac{2^s-1}{2^s+1}\zeta(s)$ satisfies $f(\sigma(1)) = 1$ and $f'(E) = 0$. Hence, from the approximate function $\zeta(s + it_k)$ we may obtain simultaneously points s and ρ with $\zeta(s) = 1$ and $\zeta'(\rho) = 0$ and more or less to the same height t_k.

We will say that a sequence of complex numbers (s_n) is *almost extremal* for $\zeta(s) = 1$ if $\zeta(s_n) = 1$ and $\lim_n \operatorname{Re}(s_n) = \sigma(1)$. Analogously (ρ_n) is said to be *almost extremal* for $\zeta'(s) = 0$ if $\zeta'(\rho_n) = 0$ and $\lim_n \operatorname{Re}(\rho_n) = E$.

First we prove that an almost extremal sequence is related to the situation of Lemma 2.1.

Theorem 6.1. *(a) If (s_n) is an almost extremal sequence for $\zeta(s) = 1$, then $t_n := \operatorname{Im}(s_n)$ satisfies*

(4) $$\lim_{n\to\infty} 2^{-it_n} = -1, \quad \lim_{n\to\infty} p^{-it_n} = 1 \quad \textit{for every odd prime } p.$$

(b) If (ρ_n) *is an almost extremal sequence for* $\zeta'(s) = 0$*, then* $t_n := \operatorname{Im}(\rho_n)$ *also satisfies* (4)*.*

Proof. (a) Let $s_n = \sigma_n + it_n$. Since $\lim_n \sigma_n = \sigma(1) > 1$ we may assume that $\sigma_n > 1$ for all n.

As in the proof of Theorem 4.1 the equation $\zeta(s_n) = 1$ may be written as

$$-1 = \sum_{k=2}^{\infty}\Bigl(\frac{2}{2k-1}\Bigr)^{\sigma_n+it_n}.$$

Since $\lim_n \sigma_n = \sigma(1)$, we see that σ_n converges to the unique solution to the equation

$$1 = \sum_{k=2}^{\infty}\Bigl(\frac{2}{2k-1}\Bigr)^{\sigma}.$$

Therefore

$$\sum_{k=2}^{\infty}\Bigl(\frac{2}{2k-1}\Bigr)^{\sigma_n+it_n} = -\sum_{k=2}^{\infty}\Bigl(\frac{2}{2k-1}\Bigr)^{\sigma(1)}$$

so that, for all $n \in \mathbb{N}$ we have

$$\sum_{k=2}^{\infty}\Bigl(\frac{2}{2k-1}\Bigr)^{\sigma(1)}\Bigl(1 + \Bigl(\frac{2}{2k-1}\Bigr)^{\sigma_n-\sigma(1)+it_n}\Bigr) = 0. \tag{5}$$

We now prove that for each $k \geq 2$ we must have

$$\lim_n \Bigl(\frac{2}{2k-1}\Bigr)^{it_n} = -1. \tag{6}$$

We proceed by contradiction and assume that (6) is not true for some k_0. Since the absolute value of $\bigl(\frac{2}{2k-1}\bigr)^{it_n}$ is 1, there must exist a subsequence n_j such that

$$\lim_j \Bigl(\frac{2}{2k_0-1}\Bigr)^{it_{n_j}} = a_{k_0} \neq -1, \qquad |a_{k_0}| = 1.$$

By a diagonal argument we may assume that for this subsequence we also have the limits

$$\lim_j \Bigl(\frac{2}{2k-1}\Bigr)^{it_{n_j}} = a_k, \qquad |a_k| = 1, \quad k \neq k_0.$$

Now consider the equation (5) for $n = n_j$ and take the limit for $j \to \infty$. Interchanging limit and sum we then obtain

$$\sum_{k=2}^{\infty}\Bigl(\frac{2}{2k-1}\Bigr)^{\sigma(1)}(1 + a_k) = 0.$$

Now take real parts in this equation. Since $\operatorname{Re}(1 + a_k) \geq 0$ but $\operatorname{Re}(1 + a_{k_0}) > 0$ we get a contradiction, proving (6).

Hence, for any k we have (6). Now if p is an odd prime we have $p = 2k+1$ and $p^2 = 2m+1$ so that

$$\lim_n \Big(\frac{2}{p}\Big)^{it_n} = -1, \quad \lim_n \Big(\frac{2}{p^2}\Big)^{it_n} = -1.$$

Hence

$$\lim_n p^{it_n} = \Big(\frac{2}{p}\Big)^{it_n} \cdot \Big(\frac{2}{p^2}\Big)^{-it_n} = 1$$

so that

$$\lim_n 2^{it_n} = \lim_n \Big(\frac{2}{p}\Big)^{it_n} p^{it_n} = -1.$$

(b) Assume now that (ρ_n) is an almost extremal sequence for $\zeta'(s) = 0$. Let $\rho_n = \sigma_n + it_n$. Since $\lim_n \sigma_n = E > 1$ we may assume that $\sigma_n > 1$ for all n.

As in the proof of Theorem 5.1 we will have

$$\frac{\log 2}{2^{\sigma_n}+1} \le \Big|-\frac{\log 2}{2^{\rho_n}-1}\Big| = \Big|\sum_{p\ge 3}\frac{\log p}{p^{\rho_n}-1}\Big| \le \sum_{p\ge 3}\frac{\log p}{p^{\sigma_n}-1}.$$

Since $\lim_n \sigma_n = E$ and E satisfies equation (3) we have

$$\lim_{n\to\infty}\frac{\log 2}{2^{\sigma_n}+1} = \lim_{n\to\infty}\sum_{p\ge 3}\frac{\log p}{p^{\sigma_n}-1}$$

so that

$$\lim_{n\to\infty}\Big|-\frac{\log 2}{2^{\rho_n}-1}\Big| = \frac{\log 2}{2^E+1} = \sum_{p\ge 3}\frac{\log p}{p^E-1} = \lim_{n\to\infty}\Big|\sum_{p\ge 3}\frac{\log p}{p^{\rho_n}-1}\Big|. \tag{7}$$

The first equality in (7) implies that $\lim_n |1-2^{\sigma_n+it_n}| = 1+2^E$. Let a be a limit point of the sequence (2^{it_n}). We may choose a sequence (n_k) such that $\lim_k 2^{it_{n_k}} = a$. Then $\lim_k |1-2^{\sigma_{n_k}+it_{n_k}}| = |1-2^E a| = 1+2^E$. Since $|a| = 1$ this is possible only if $a = -1$. Therefore, (2^{it_n}), beeing a bounded sequence with a unique limit point, is convergent and $\lim_n 2^{it_n} = -1$.

For each odd prime p the sequence (p^{it_n}) has 1 as unique limit point. Indeed, if not, then there is an odd prime q and a sequence (n_k) with

$$\lim_k q^{it_{n_k}} = a_q \ne 1.$$

By a diagonal argument we may assume that the limits $\lim_k p^{it_{n_k}} = a_p$ exist for each prime p. We will always have $|a_p| = 1$. Taking limits in the last equality of (7) (for the subsequence (n_k)) we obtain

$$\sum_{p\geq 3}\frac{\log p}{p^E-1}=\Bigl|\sum_{p\geq 3}\frac{\log p}{p^E a_p-1}\Bigr|.$$

We have $|p^E a_p-1|\geq p^E-1$, but the above equality is only possible if we have for all p the equality $|p^E a_p-1|=p^E-1$, which is in contradiction with our assumption $a_q\neq 1$. □

Now we can prove the connection between the two problems:

Theorem 6.2. *Let (s_n) be an almost extremal sequence for $\zeta(s)=1$. Then there exists an almost extremal sequence (ρ_n) for $\zeta'(s)=0$ such that*

$$\lim_n (\rho_n-s_n)=E-\sigma(1).$$

Analogously if (ρ_n) is an almost extremal sequence for $\zeta'(s)=0$, there exists an almost extremal sequence (s_n) for $\zeta(s)=1$ satisfying the same condition.

Proof. Let $s_n=\sigma_n+it_n$. By Theorem 6.1 we then have (4). In the proof of Lemma 2.1 we have seen that (4) implies

$$\lim_n \zeta(s+it_n)=\frac{2^s-1}{2^s+1}\zeta(s)\quad\text{uniformly on compact sets of } \sigma>1.$$

It follows that $\zeta'(s+it_n)$ also converges uniformly on compact sets of $\sigma>1$ to the derivative of $f(s):=\frac{2^s-1}{2^s+1}\zeta(s)$. In the proof of Theorem 5.1 we have seen that $f'(E)=0$. Hence, by Hurwitz's theorem for $n\geq n_0$ the function $\zeta'(s+it_n)$ has a zero $s=b_n$ such that $\lim b_n=E$. Writing $\rho_n:=b_n+it_n$ we have $\zeta'(\rho_n)=0$ and

$$\lim_n \operatorname{Re}(\rho_n)=\lim_n \operatorname{Re}(b_n+it_n)=\lim_n \operatorname{Re}(b_n)=\operatorname{Re}(\lim_n b_n)=E.$$

Hence (ρ_n) is almost extremal for $\zeta'(s)=0$ and

$$\lim_n(\rho_n-s_n)=\lim_n(b_n-\sigma_n)=E-\sigma(1).$$

The proof for the other case is similar. □

7. Some bounds for Dirichlet L-functions.

Our previous analysis may also be applied to general Dirichlet L-functions. We will give two typical examples.

For the modulus 4 the non-trivial Dirichlet character is given by $\chi(2n+1)=(-1)^n$, $\chi(2n)=0$, so that

$$L(s,\chi)=\prod_p\Bigl(1-\frac{\chi(p)}{p^s}\Bigr)^{-1}=\Bigl(1+\frac{1}{3^s}\Bigr)^{-1}\Bigl(1-\frac{1}{5^s}\Bigr)^{-1}\Bigl(1+\frac{1}{7^s}\Bigr)^{-1}\cdots$$

So, the equation $L(s,\chi)=1$ is equivalent to

$$\Big(1+\frac{1}{3^s}\Big)=\Big(1-\frac{1}{5^s}\Big)^{-1}\Big(1+\frac{1}{7^s}\Big)^{-1}\Big(1+\frac{1}{11^s}\Big)^{-1}\Big(1-\frac{1}{13^s}\Big)^{-1}\cdots$$

Now (similarly as in earlier sections) we let the factor $\big(1+\frac{1}{3^s}\big)$ "point strictly westward" and all other factors "strictly eastward" (Kronecker's theorem applies here just as well). As in Section 4 this leads to the equation

$$\Big(1+\frac{1}{3^\sigma}\Big)=\Big(1-\frac{1}{5^\sigma}\Big)^{-1}\Big(1-\frac{1}{7^\sigma}\Big)^{-1}\Big(1-\frac{1}{11^\sigma}\Big)^{-1}\Big(1-\frac{1}{13^\sigma}\Big)^{-1}\cdots$$

or

$$\frac{1+\frac{1}{3^\sigma}}{\big(1-\frac{1}{2^\sigma}\big)\big(1-\frac{1}{3^\sigma}\big)}=\zeta(\sigma).$$

(This kind of *trick* also works in the general case.)

Using Mathematica we found that in this case the supremum of all σ such that $L(\sigma+it,\chi)=1$ for some real t equals

$$1.88779\,09267\,08118\,92719\,63215\,42035\,11666\,82234\,70126\ldots$$

For $n=7$ we find (for *every charachter* χ mod 7) that $L(s,\chi)=1$ leads to the equation

$$\frac{1+\frac{1}{2^\sigma}}{\big(1-\frac{1}{2^\sigma}\big)\big(1-\frac{1}{7^\sigma}\big)}=\zeta(\sigma)$$

and the bound

$$1.83843\,45030\,97314\,94016\,69429\,96760\,82067\,80491\,61315\ldots$$

For $L(s,\chi)=a$ with $0<a<1$ we let all factors $\Big(1-\frac{\chi(p)}{p^s}\Big)^{-1}$ point "strictly westward". This leads to the equation

$$\prod_p\Big(1+\frac{|\chi(p)|}{p^s}\Big)^{-1}=a$$

and the *missing factors* are easily supplied. For the modulus 4 and $a=\frac{1}{2}$ this leads to the equation

$$\Big(1+\frac{1}{2^\sigma}\Big)\frac{\zeta(2\sigma)}{\zeta(\sigma)}=\frac{1}{2}$$

and the bound

$$1.33538\,71957\,45311\,13312\,01066\,99878\,57500\,83328\,78290\ldots$$

We leave the straightforward general formulation to the reader.

8. Application of the Lenstra–Lenstra–Lovász lattice basis reduction algorithm.

For various problems the existence of almost extremal sequences (σ_k+it_k) depends heavily on the existence of the limits $\lim_k p^{it_k} =: a_p$. Given a sequence of real numbers (θ_j), Kronecker's theorem guarantees the existence of a sequence of real numbers (t_k) such that

$$\lim_k p_j^{it_k} = e^{i\theta_j}, \qquad (j\in\mathbb{N}).$$

We want to find $t\in\mathbb{R}$ such that $\sigma+it$ is almost extremal for an adequate σ. To this end, given n we must find $t\in\mathbb{R}$ such that for certain $m_j\in\mathbb{Z}$

$$|t\log p_j-\theta_j-2m_p\pi|<\varepsilon, \qquad 1\le j\le n$$

for some small ε.

We will use the LLL algorithm similarly as Odlyzko and te Riele [6] in their disproof of the Mertens conjecture.

Given a basis for a lattice L contained in $\mathbb{Z}^N$, the LLL algorithm yields a reduced basis for L, usually consisting of short vectors.

So, we fix n, some weights $(w_j)_{j=1}^n$ (in practice we used $w_j=1.15^{40-j}$) and two natural numbers ν and r, and construct a lattice L in $\mathbb{Z}^{n+2}$ by means of $n+2$ vectors v_1, v_2, $\dots$, v_n, v and v' in $\mathbb{Z}^{n+2}$ (the method uses lattices in $\mathbb{Z}^N$):

$$\begin{array}{rccccccc}
v_1 = & (& \lfloor 2\pi w_1\cdot 2^\nu\rfloor, & 0, & 0, & \dots & 0, & 0, & 0)\\
v_2 = & (& 0, & \lfloor 2\pi w_2\cdot 2^\nu\rfloor, & 0, & \dots & 0, & 0, & 0)\\
\\
v_n = & (& 0, & 0, & 0, & \dots & \lfloor 2\pi w_n\cdot 2^\nu\rfloor, & 0, & 0)\\
v = & (& \lfloor w_1 2^{\nu-r}\lambda_1\rfloor, & \lfloor w_2 2^{\nu-r}\lambda_2\rfloor, & \lfloor w_3 2^{\nu-r}\lambda_3\rfloor, & \dots & \lfloor w_n 2^{\nu-r}\lambda_n\rfloor, & 0, & 1)\\
v' = & (& -\lfloor w_1\theta_1 2^\nu\rfloor, & -\lfloor w_2\theta_2 2^\nu\rfloor, & -\lfloor w_3\theta_3 2^\nu\rfloor, & \dots & -\lfloor w_n\theta_n 2^\nu\rfloor, & 2^\nu n^4, & 0)
\end{array}$$

where we have put $\lambda_j=\log p_j$.

Applying the LLL algorithm to these vectors we get a reduced basis v_1^*, v_2^*, $\dots v_{n+2}^*$ such that at least one of these vectors will have a non-null $(n+1)$-coordinate. But given that $2^\nu n^4$ is very large compared with all other entries of the original basis, in a reduced basis (with short vectors) we do not expect more than one large vector. Assuming that it is v_1^*, its $(n+1)$ coordinate will be $\pm 2^\nu n^4$, and without loss of generality we may assume that it is $2^\nu n^4$. Let x be the last coordinate of v_1^*. Then this vector will have coordinate j equal to (since it is a linear combination of the initial vectors)

$$x\lfloor w_j 2^{\nu-r}\log p_j\rfloor + m_j\lfloor 2\pi w_j 2^\nu\rfloor - \lfloor w_j\theta_j 2^\nu\rfloor$$

for some integers m_j. Since it is a reduced basis, we expect this coordinate to be small. Hence also the number

$$xw_j2^{\nu-r}\log p_j + m_j2\pi w_j2^\nu - w_j\theta_j2^\nu = 2^\nu w_j\Big(\frac{x}{2^r}\log p_j - \theta_j + 2\pi m_j\Big)$$

will be small and $t = \frac{x}{2^r}$ will have the property we are looking for: $t\log p_j - \theta_j + 2\pi m_j$ will be small for $1 \le j \le n$.

Figure 1 illustrate the results obtained. This figure (and others similar to it) is at the origin of our results in Section 6. We were searching for near extremal values for the problem $\zeta(s) = 1$, and the figure clearly shows that we also obtain a near extremal value for the problem $\zeta'(s) = 0$.

The figure represents the rectangle $(-2,4)\times(h-3,h+3)$ where $h = 156326000$. The solid curves are those points where $\zeta(s)$ takes real values. On the dotted curves $\zeta(s)$ is purely imaginary. For reference we have drawn the lines $\sigma = 0$ and $\sigma = 1$ limiting the critical strip.

The value $h = 156326000$ was given by the LLL algorithm as a candidate for a near extreme value of $\zeta(s) = 1$. This is the point labelled a. In fact $\operatorname{Re} a = 1.907825\ldots$ is near the limit $\sigma(1) = 1.94010\ldots$ We see also the connected extreme value for $\zeta'(s) = 0$. This is the point ρ whose real part is also near the corresponding limit value E. The role of the point b will be explained in the next Section.

9. Bound for the real loops.

Since $\zeta(s)$ is real for all real s, there is no interest in the question of the supremum of all σ such that $\zeta(\sigma+it)\in\mathbb{R}$ for some $t\in\mathbb{R}$. We now focus on the supremum of the real loops.

Since $u(s) := \operatorname{Im}\zeta(s)$ is a harmonic function the points where $u(s) = 0$ are arranged in a set of analytic curves. These curves are of two main types. Some of them traverse the entire plane from $\sigma = -\infty$ to $\sigma = +\infty$ (in [3] they are called I_1 curves). In figure 9 we have plotted one of these curves. All the other solid curves in this figure are I_2 curves, they form a loop starting at $\sigma = -\infty$ and ending again at $\sigma = -\infty$. Each such I_2 curve has a *turning point*, a point on the curve with σ maximal.

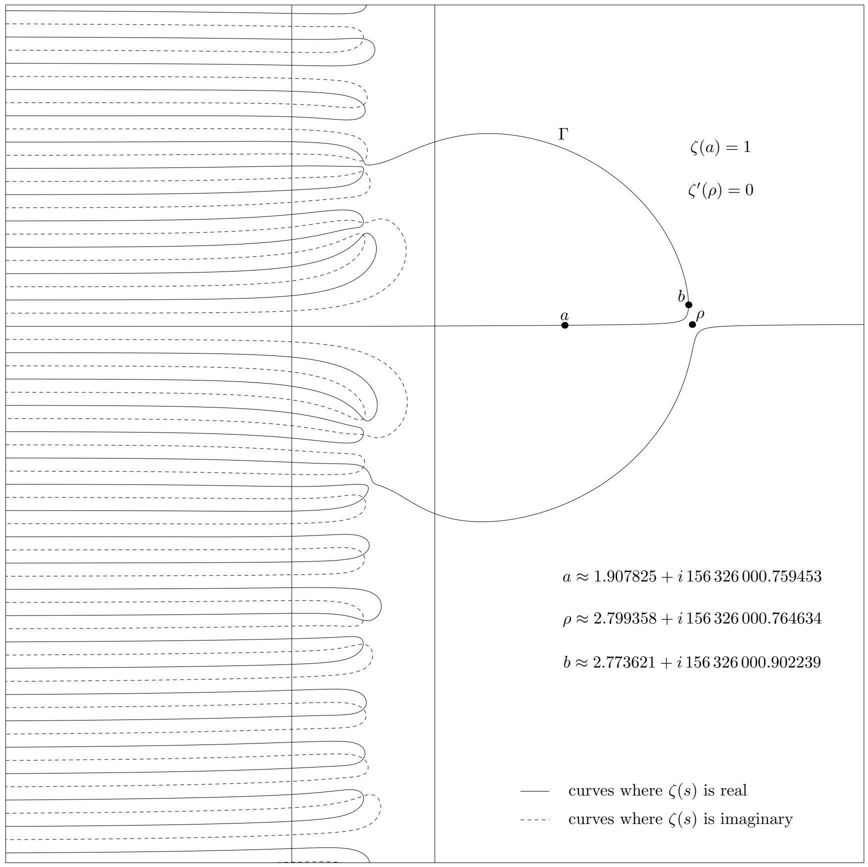

FIGURE 1. Curves $\operatorname{Re}\zeta(s) = 0$ and $\operatorname{Im}\zeta(s) = 0$ near $t = 156326000$.

In the case of the curve Γ in figure 9 this is the point labelled b. It is easy to see that at these points, since the curve $u(\sigma + it) = 0$ has a vertical tangent, we must have $u_\sigma(\sigma+it) = 0$. By the Cauchy-Riemann equations this is equivalent to $\operatorname{Re}\zeta'(\sigma + it) = 0$.

Hence we define a *turning point* as a point $b = \sigma + it$ such that

$$\operatorname{Im}\zeta(b) = 0 \quad \text{and} \quad \operatorname{Re}\zeta'(b) = 0.$$

The first equation says that b is on a real curve (i. e. a curve where the function $\zeta(s)$ is real), whereas the second equation means that at the point b the tangent to such a curve is vertical.

The question of the supremum T of all σ of turning points of the I_2 loops of $\zeta(s)$ was mentioned in [3]. Here we solve this problem.

Theorem 9.1. *Let $E = 2.813014\dots$ be the constant of Theorem 5.1. Then each turning point $b = \sigma + it$ for $\zeta(s)$ satisfies $\sigma \le E$, and there is a sequence of turning points (b_k) for $\zeta(s)$ with $\lim_k \operatorname{Re}(b_k) = E$.*

We will use the following theorem

Theorem 9.2. *Let A be the unique solution of the equation*

$$\sum_p \arcsin(p^{-\sigma}) = \frac{\pi}{2}, \qquad (\sigma > 1).$$

Then A is the supremum of the $\sigma \in \mathbb{R}$ such that there is a $t \in \mathbb{R}$ with $\operatorname{Re}\zeta(\sigma+it) < 0$. For $\sigma = A$ we have $\operatorname{Re}\zeta(\sigma+it) > 0$ for all $t \in \mathbb{R}$.

The value of the constant A is

$$A = 1.19234\,73371\,86193\,20289\,75044\,27425\,59788\,34011\,19230\ldots$$

The proof can be found in [3]. The constant A has been computed with high precision by R. P. Brent and J. van de Lune.

We break the proof of Theorem 9.1 in several lemmas.

Lemma 9.3. *The point $\sigma + it$ with $\sigma > A$ is a turning point for the function $\zeta(s)$ if and only if*

$$\sum_p \sum_{k=1}^{\infty} \frac{1}{k} \frac{\sin(kt\log p)}{p^{k\sigma}} = 0 \quad \textit{and} \quad \sum_p \sum_{k=1}^{\infty} \frac{\cos(kt\log p)}{p^{k\sigma}} \log p = 0. \tag{8}$$

Proof. By Theorem 9.2 for $\sigma > A = 1.192347\ldots$ we have $\operatorname{Re}\zeta(s) > 0$. In the sequel $\log z$ will be the main branch of the logarithm for $|\arg z| < \pi$, so that $\log\zeta(s)$ is well defined and analytic for $\sigma > A$.

In view of $\log z = \log|z| + i\arg z$ it should be clear that, for $\sigma > A$ the two functions $\zeta(s)$ and $\log\zeta(s)$ are real at the same points, so that also the turning points of the loops $\operatorname{Im}\zeta(s) = 0$ and $\operatorname{Im}\log\zeta(s) = 0$ are the same.

For s real and > 1 both functions $\zeta(s)$ and $\log\zeta(s)$ are real so that we may write

$$\log\zeta(s) = \sum_p \log\Bigl(1 - \frac{1}{p^s}\Bigr)^{-1} = \sum_p \sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{p^{ks}}, \qquad (\sigma > 1) \tag{9}$$

and this equality is true for $\sigma > A$ by analytic continuation.

Since the turning points for some function $f(s)$ are defined as the solutions of the system of equations $\operatorname{Im} f(s) = 0$, $\operatorname{Re} f'(s) = 0$, the turning points of $\log\zeta(s)$ with $\sigma > A$ are just those points satisfying equations (8). □

Now we introduce some notations. We may write equations (8) in the form

$$\tag{10} \begin{aligned} -\sum_{k=1}^{\infty}\frac{1}{k}\frac{\sin(kt\log 2)}{2^{k\sigma}} &= \sum_{p\ge 3}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\sin(kt\log p)}{p^{k\sigma}}\\ -\sum_{k=1}^{\infty}\frac{\cos(kt\log 2)}{2^{k\sigma}}\log 2 &= \sum_{p\ge 3}\sum_{k=1}^{\infty}\frac{\cos(kt\log p)}{p^{k\sigma}}\log p. \end{aligned}$$

For $\sigma>0$ and $t\in\mathbb{R}$ we now define

$$\tag{11} f(\sigma,t) := \sum_{k=1}^{\infty}\frac{1}{k}\frac{\sin(kt\log 2)}{2^{k\sigma}}$$

and

$$\tag{12} g(\sigma,t) := \frac{\partial}{\partial t}f(\sigma,t) = \sum_{k=1}^{\infty}\frac{\cos(kt\log 2)}{2^{k\sigma}}\log 2.$$

Note that f and g are periodic functions of t with period $2\pi/\log 2$.

So, a turning point $\sigma+it$ must satisfy

$$-f(\sigma,t) = \sum_{p\ge 3}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\sin(kt\log p)}{p^{k\sigma}} \quad \text{and}$$

$$-g(\sigma,t) = \sum_{p\ge 3}\sum_{k=1}^{\infty}\frac{\cos(kt\log p)}{p^{k\sigma}}\log p.$$

We now consider the function

$$U(\sigma,t) := 2^{2\sigma}f(\sigma,t)^2 + \Big(\frac{2^\sigma}{\log 2}\Big)^2 g(\sigma,t)^2$$

the choice of the coefficients $2^{2\sigma}$ and $(2^\sigma/\log 2)^2$ being motivated by (use (11) and (12))

$$\begin{aligned} &\lim_{\sigma\to+\infty}U(\sigma,t) = \\ &\lim_{\sigma\to+\infty}\Big\{2^{2\sigma}\Big(\sum_{k=1}^{\infty}\frac{1}{k}\frac{\sin(kt\log 2)}{2^{k\sigma}}\Big)^2 + \Big(\frac{2^\sigma}{\log 2}\Big)^2\Big(\sum_{k=1}^{\infty}\frac{\cos(kt\log 2)}{2^{k\sigma}}\log 2\Big)^2\Big\}\\ &\qquad = \sin^2(t\log 2)+\cos^2(t\log 2) = 1. \end{aligned}$$

Lemma 9.4. *Let a and b be arbitrary real numbers. Then there exist real numbers x and y such that*

$$ax+by = (a^2+b^2)^{1/2} \quad \textit{and} \quad x^2+y^2=1.$$

Proof. If $a^2 + b^2 = 0$ then $a = b = 0$ and we need only take x and y such that $x^2 + y^2 = 1$.

If $a^2 + b^2 \neq 0$ then we can take $x = \frac{a}{\sqrt{a^2+b^2}}$ and $y = \frac{b}{\sqrt{a^2+b^2}}$. □

Lemma 9.5. *If $\sigma + it$ is a turning point of $\zeta(s)$ with $\sigma > A$, then*

$$U(\sigma, t) < \Big(\frac{2^\sigma}{\log 2}\Big)^2 \Big(\sum_{p\geq 3}\sum_{k=1}^{\infty} \frac{\log p}{p^{ks}}\Big)^2.$$

Proof. We apply Lemma 9.4 to

$$a = -2^\sigma f(\sigma, t) \quad \text{and} \quad b = -\frac{2^\sigma}{\log 2} g(\sigma, t)$$

to get

$$\Big\{2^{2\sigma} f(\sigma,t)^2 + \Big(\frac{2^\sigma}{\log 2}\Big)^2 g(\sigma,t)^2\Big\}^{1/2} = -2^\sigma x f(\sigma, t) - \frac{2^\sigma}{\log 2} y g(\sigma, t)$$

which, by (10), may be written as

$$\begin{aligned} U(\sigma,t)^{1/2} = 2^\sigma \sum_{p\geq 3}\sum_{k=1}^{\infty} \frac{x}{k}\frac{\sin(kt\log p)}{p^{k\sigma}} &+ \\ + \frac{2^\sigma}{\log 2}\sum_{p\geq 3}\sum_{k=1}^{\infty} \frac{y\cos(kt\log p)}{p^{k\sigma}}\log p &= \\ = 2^\sigma \sum_{p\geq 3}\sum_{k=1}^{\infty}\Big(\frac{x}{k}\frac{\sin(kt\log p)}{p^{k\sigma}} + \frac{y\log p}{\log 2}\frac{\cos(kt\log p)}{p^{k\sigma}}\Big)&. \end{aligned}$$

Applying the Cauchy-Schwarz inequality to the right hand side we obtain the condition

$$(13)\quad U(\sigma,t)^{1/2} \leq$$
$$\leq 2^\sigma \sum_{p\geq 3}\sum_{k=1}^{\infty}\Big(\frac{x^2}{k^2} + \frac{y^2\log^2 p}{\log^2 2}\Big)^{1/2}\Big(\frac{\sin^2(kt\log p) + \cos^2(kt\log p)}{p^{2k\sigma}}\Big)^{1/2}.$$

Now observe that in (13) $\frac{1}{k^2} < \frac{\log^2 p}{\log^2 2}$ so that

$$\frac{x^2}{k^2} + \frac{y^2\log^2 p}{\log^2 2} < \frac{(x^2+y^2)\log^2 p}{\log^2 2} \leq \frac{\log^2 p}{\log^2 2}.$$

Using this we thus obtain the condition

$$U(\sigma,t)^{1/2} < 2^\sigma \sum_{p\geq 3}\sum_{k=1}^{\infty}\frac{\log p}{\log 2}\frac{1}{p^{k\sigma}}$$

or

$$U(\sigma,t) < \Big(\frac{2^\sigma}{\log 2}\Big)^2\Big(\sum_{p\ge3}\sum_{k=1}^\infty \frac{\log p}{p^{k\sigma}}\Big)^2$$

as we wanted to show. □

For $\sigma > 1$ we define

$$H(\sigma) := \Big(\frac{2^\sigma}{\log 2}\Big)^2\Big(\sum_{p\ge3}\sum_{k=1}^\infty \frac{\log p}{p^{k\sigma}}\Big)^2. \tag{14}$$

Lemma 9.6. *For each $t\in\mathbb{R}$ there exists a largest solution $u(t)$ to the equation in σ*

$$U(\sigma,t) = H(\sigma) \tag{15}$$

and

$$U(\sigma,t) > H(\sigma), \qquad (\sigma > u(t)).$$

Proof. By (14) it is easily seen that $H(\sigma)$ is continuous and strictly decreasing for $\sigma > 1$ from $+\infty$ to 0. In particular

$$\lim_{\sigma\to\infty} H(\sigma) = 0.$$

Since $U(\sigma,t)$ is continuous for $\sigma > 0$ and $t\in\mathbb{R}$, and

$$\lim_{\sigma\to+\infty} U(\sigma,t) = 1$$

we see that for every t the infimum $u(t)$ of the a such that $U(\sigma,t) > H(\sigma)$ for $\sigma > a$ exists and is larger than 1.

From this it is clear that $u(t)$ must be a solution of equation (15) in σ. □

Lemma 9.7. *We have the closed formulas*

$$f(\sigma,t) = \arctan\frac{\sin(t\log 2)}{2^\sigma - \cos(t\log 2)},$$

$$g(\sigma,t) = -\frac{(1-2^\sigma\cos(t\log 2))\log 2}{1+4^\sigma-2^{1+\sigma}\cos(t\log 2)}.$$

Proof. The first follows from the identity $f(\sigma,t) = \operatorname{Im}(\log(1-2^{-s}))$, and the second by differentiation. □

Lemma 9.8. *We have $u(\pi/\log 2) = E$.*

Proof. We have

$$\sum_{p\ge3}\sum_{k=1}^\infty\frac{\log p}{p^{k\sigma}} = \sum_p\sum_{k=1}^\infty\frac{\log p}{p^{k\sigma}} - \sum_{k=1}^\infty\frac{\log 2}{2^{k\sigma}} = -\frac{\zeta'(\sigma)}{\zeta(\sigma)} - \frac{\log 2}{2^\sigma-1}$$

so that

$$H(\sigma) = \Bigl(\frac{2^\sigma}{\log 2}\Bigr)^2\Bigl(\frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{\log 2}{2^\sigma - 1}\Bigr)^2.$$

By its definition $u(t)$ is the largest solution of the equation $U(\sigma, t) = H(\sigma)$.

For $t = \pi/\log 2$ we have

$$f(\sigma, t) = \sum_{k=1}^\infty \frac{1}{k}\frac{\sin(kt\log 2)}{2^{k\sigma}} = \sum_{k=1}^\infty \frac{1}{k}\frac{\sin(k\pi)}{2^{k\sigma}} = 0$$

and

$$g(\sigma, t) = \sum_{k=1}^\infty \frac{\cos(kt\log 2)}{2^{k\sigma}}\log 2 = \sum_{k=1}^\infty \frac{\cos(k\pi)}{2^{k\sigma}}\log 2 =$$

$$= \sum_{k=1}^\infty \frac{(-1)^k}{2^{k\sigma}}\log 2 = -\frac{\log 2}{2^\sigma + 1}$$

so that $u(\pi/2)$ satisfies the equation

$$\Bigl(\frac{2^\sigma}{\log 2}\Bigr)^2\Bigl(\frac{\log 2}{2^\sigma + 1}\Bigr)^2 = \Bigl(\frac{2^\sigma}{\log 2}\Bigr)^2\Bigl(\frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{\log 2}{2^\sigma - 1}\Bigr)^2.$$

Since $\frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{\log 2}{2^\sigma - 1} < 0$ this is equivalent to

$$\frac{\log 2}{2^\sigma + 1} = -\frac{\zeta'(\sigma)}{\zeta(\sigma)} - \frac{\log 2}{2^\sigma - 1}$$

or

$$\frac{2^{\sigma+1}}{4^\sigma - 1}\log 2 = -\frac{\zeta'(\sigma)}{\zeta(\sigma)}.$$

But E is the unique solution of this equation for $\sigma > 1$ (see Theorem 5.1).

Hence $u(\pi/\log 2) = E$. □

Lemma 9.9. *For all $\sigma > 1$ and all $t \in \mathbb{R}$ we have*

$$U(\sigma, t) \geq U(\sigma, \pi/\log 2). \tag{16}$$

Proof. We have computed $U(\sigma, \pi/\log 2)$ in the proof of Lemma 9.8. Substituting this value and the definition of $U(\sigma, t)$, (16) may be written

$$2^{2\sigma} f(\sigma, t)^2 + \Bigl(\frac{2^\sigma}{\log 2}\Bigr)^2 g(\sigma, t)^2 \geq \Bigl(\frac{2^\sigma}{\log 2}\Bigr)^2\Bigl(\frac{\log 2}{2^\sigma + 1}\Bigr)^2.$$

In view of Lemma 9.7 we thus need to prove

$$\arctan^2\Bigl(\frac{\sin(t\log 2)}{2^\sigma-\cos(t\log 2)}\Bigr)+\Bigl(\frac{(1-2^\sigma\cos(t\log 2))}{1+4^\sigma-2^{1+\sigma}\cos(t\log 2)}\Bigr)^2\ge \ge\Bigl(\frac{1}{2^\sigma+1}\Bigr)^2. \tag{17}$$

We change notations putting $t\log 2=\varphi$ and $2^\sigma=x^{-1}$, so that we have to prove for $0<x<1$ and $0<\varphi<2\pi$

$$u(x,\varphi):=\arctan^2\Bigl(\frac{x\sin\varphi}{1-x\cos\varphi}\Bigr)+\Bigl(\frac{x(x-\cos\varphi)}{1+x^2-2x\cos\varphi}\Bigr)^2\ge\Bigl(\frac{x}{1+x}\Bigr)^2. \tag{18}$$

The right hand side is the value for $\varphi=\pi$ of the left hand side.

So, we want to prove that $u(x,\varphi)$ has an absolute minimum at $\varphi=\pi$. It is easy to show that $u(x,\pi-\theta)=u(x,\pi+\theta)$. So, we only have to prove inequality (18) for $0<\varphi<\pi$. We will split the proof in two cases.

(1) Proof of (18) for $\frac{\pi}{2}<\varphi<\pi$.

If we differentiate $u(x,\varphi)$ with respect to φ and simplify we arrive at

$$u_\varphi(x,\varphi)=\frac{2x(x-\cos\varphi)}{(1+x^2-2x\cos\varphi)^3}\Bigl\{-\arctan\Bigl(\frac{x\sin\varphi}{1-x\cos\varphi}\Bigr)\times \times(1+x^2-2x\cos\varphi)^2+x(1-x^2)\sin\varphi\Bigr\}. \tag{19}$$

We will show that $u_\varphi(x,\varphi)<0$ for $\frac{\pi}{2}<\varphi<\pi$, so that (18) will follow.

In this interval $\cos\varphi<0$ and $\sin\varphi>0$. The first factor in the right hand side of (19) is positive, and we will show that the second is negative. That is we will show that

$$x(1-x^2)\sin\varphi\le\arctan\Bigl(\frac{x\sin\varphi}{1-x\cos\varphi}\Bigr)(1+x^2-2x\cos\varphi)^2. \tag{20}$$

Let

$$\alpha=\arctan\Bigl(\frac{x\sin\varphi}{1-x\cos\varphi}\Bigr),\qquad \tan\alpha=\frac{x\sin\varphi}{1-x\cos\varphi}, \tag{21}$$

$$\frac{1}{\cos^2\alpha}=1+\Bigl(\frac{x\sin\varphi}{1-x\cos\varphi}\Bigr)^2=\frac{1+x^2-2x\cos\varphi}{(1-x\cos\varphi)^2},$$

$$\cos^2\alpha=\frac{(1-x\cos\varphi)^2}{1+x^2-2x\cos\varphi},$$

(22)
$$\sin^2\alpha = 1-\frac{(1-x\cos\varphi)^2}{1+x^2-2x\cos\varphi} = \frac{x^2-x^2\cos^2\varphi}{1+x^2-2x\cos\varphi} = \frac{x^2\sin^2\varphi}{1+x^2-2x\cos\varphi}$$
so that
$$\sin\alpha = \frac{x\sin\varphi}{\sqrt{1+x^2-2x\cos\varphi}}$$
(with the sign + since certainly $\alpha\in(0,\pi/2)$, since $\tan\alpha>0$).

Now we have
$$1-x^2<1<(1+x^2-2x\cos\varphi)^{3/2}$$
so that
$$x(1-x^2)\sin\varphi \le x\sin\varphi(1+x^2-2x\cos\varphi)^{3/2}$$
and
$$x(1-x^2)\sin\varphi \le \sin\alpha(1+x^2-2x\cos\varphi)^2 \le \alpha(1+x^2-2x\cos\varphi)^2$$
which is equivalent to (20).

(2) Proof of (18) for $0<\varphi<\frac{\pi}{2}$.

Defining α as in (21), $\sin^2\alpha$ is still given by (22). Although in this case we do not know the sign of $\sin\alpha$, inequality (18) will still follow from

$$\frac{x^2\sin^2\varphi}{1+x^2-2x\cos\varphi}+\Big(\frac{x(x-\cos\varphi)}{1+x^2-2x\cos\varphi}\Big)^2 \ge \Big(\frac{x}{1+x}\Big)^2 \tag{23}$$

since $\sin^2\alpha<\alpha^2$.

To prove (23) we consider two cases.

(2a) Proof of (23) when $1+x^2-2x\cos\varphi>1$.

Then $(1+x^2-2x\cos\varphi)^2>1+x^2-2x\cos\varphi$, so that

$$\frac{\sin^2\varphi}{1+x^2-2x\cos\varphi}+\frac{(x-\cos\varphi)^2}{(1+x^2-2x\cos\varphi)^2}\ge$$
$$\ge \frac{\sin^2\varphi}{(1+x^2-2x\cos\varphi)^2}+\frac{(x-\cos\varphi)^2}{(1+x^2-2x\cos\varphi)^2}=$$
$$=\frac{1+x^2-2x\cos\varphi}{(1+x^2-2x\cos\varphi)^2}=\frac{1}{1+x^2-2x\cos\varphi}.$$

Recall that $0<\varphi<\frac{\pi}{2}$. Then $-2x\cos\varphi<2x$, so that $1+x^2-2x\cos\varphi<1+x^2+2x=(1+x)^2$, and we obtain
$$\frac{\sin^2\varphi}{1+x^2-2x\cos\varphi}+\frac{(x-\cos\varphi)^2}{(1+x^2-2x\cos\varphi)^2}>\frac{1}{(1+x)^2}.$$

(2b) Proof of (23) when $1+x^2-2x\cos\varphi\le 1$. In this case $(1+x^2-2x\cos\varphi)^2\le 1+x^2-2x\cos\varphi$ so that

$$\frac{\sin^2\varphi}{1+x^2-2x\cos\varphi}+\frac{(x-\cos\varphi)^2}{(1+x^2-2x\cos\varphi)^2}\ge$$
$$\ge\frac{\sin^2\varphi}{(1+x^2-2x\cos\varphi)}+\frac{(x-\cos\varphi)^2}{(1+x^2-2x\cos\varphi)}=$$
$$=\frac{1+x^2-2x\cos\varphi}{1+x^2-2x\cos\varphi}=1>\frac{1}{(1+x)^2}.$$

□

Lemma 9.10. *For each $t\in\mathbb{R}$ we have $u(t)\le u(\pi/\log 2)$.*

Proof. By Lemma 9.6

$$U(\sigma,\pi/\log 2)>H(\sigma)\quad\text{for}\quad \sigma>u(\pi/\log 2)$$

and by Lemma 9.9

$$U(\sigma,t)\ge U(\sigma,\pi/\log 2).$$

It follows that

$$U(\sigma,t)>H(\sigma),\qquad (\sigma>u(\pi/\log 2)).$$

By definition $U(\sigma,t)>H(\sigma)$ is not true for $\sigma=u(t)$, and it follows that $u(t)\le u(\pi/\log 2)$. □

Proof of the first half of Theorem 9.1. Let $\sigma+it$ be a turning point for $\zeta(s)$. It is clear that $\sigma\le A=1.192\dots$ implies $\sigma<E=2.813\dots$. For $\sigma>A$, by Lemma 9.5 we will have

$$U(\sigma,t)<H(\sigma)$$

so that Lemma 9.6 implies that

$$\sigma<u(t).$$

By Lemma 9.10

$$u(t)\le u(\pi/\log 2)$$

and by Lemma 9.8

$$u(\pi/\log 2)=E.$$

It follows that $\sigma<E$.

Therefore, the supremum T of the real parts of the turning points is less than or equal to E. We have even proved a little more: On the line $\sigma=E$ there is no turning point. □

We will now show that there is a sequence (b_n) of turning points for $\zeta(s)$ such that $\lim_n \operatorname{Re}(b_n) = E$. This will end the proof of Theorem 9.1.

By Lemma 2.1 there exists a sequence of real numbers (t_k) such that $\zeta(s+it_k)$ converges to $f(s) := \frac{2^s-1}{2^s+1}\zeta(s)$. Since

$$f(E) = 0.9\dots, \quad f'(E) = 0, \quad f''(E) = 0.07\dots, \quad f'''(E) = -0.17\dots$$

E is a turning point for $f(s)$.

We are going to show that the functions $\zeta(s+it_k)$ must have a turning point very near to E.

We prove a slightly more general result. We break the proof in several lemmas.

Given a holomorphic function f defined on a disc with center at 0 and radius R we define the associated (continuous) function

$$h(r,\varphi) = \operatorname{Im} f(re^{i\varphi}) + i\operatorname{Re} f'(re^{i\varphi})$$

so that $re^{i\varphi}$ will be a turning point for $f(z)$ if and only if $h(r,\varphi) = 0$.

For each $0 < r < R$ let γ_r be the curve $\varphi\colon [0, 2\pi) \mapsto h(r,\varphi)$.

Proposition 9.11. *Let $f(z) = a_0 + a_2z^2 + a_3z^3 + \cdots$ be a holomorphic function on $\Delta(0,R)$ the disc with center 0 and radius R. Assume that $a_0 > 0$, $a_2 > 0$ and $a_3 < 0$. Then there exists an $r_0 > 0$ such that for $0 < r < r_0$, the curve γ_r does not pass through $z = 0$ and the index (the winding number) of the curve γ_r with respect to 0 is $\omega(\gamma_r, 0) = 1$.*

To prove Proposition 9.11 we will use some lemmas.

Lemma 9.12. *Let f be as in Proposition 9.11 and define*

$$u(r,\varphi) := \operatorname{Im} f(re^{i\varphi}), \qquad v(r,\varphi) := \operatorname{Re} f'(re^{i\varphi}).$$

Then there exists r_0 such that for $0 < r < r_0$, $(r \to 0)$

$$u(r,\varphi) = a_2r^2\sin 2\varphi + a_3r^3\sin 3\varphi + \boldsymbol{\mathcal{O}}(r^4)$$
$$v(r,\varphi) = 2a_2r\cos\varphi + 3a_3r^2\cos 2\varphi + \boldsymbol{\mathcal{O}}(r^3)$$
$$u_\varphi(r,\varphi) = 2a_2r^2\cos 2\varphi + 3a_3r^3\cos 3\varphi + \boldsymbol{\mathcal{O}}(r^4)$$
$$v_\varphi(r,\varphi) = -2a_2r\sin\varphi - 6a_3r^2\sin 2\varphi + \boldsymbol{\mathcal{O}}(r^3)$$

where the implicit constants do not depend on φ.

Proof. Let $f(z) = \sum_{n=0}^\infty a_nz^n$ be the power series of f at 0, and take r_0 less than the radius of convergence. Then

$$f(z) = a_0 + a_2z^2 + a_3z^3 + \sum_{n=4}^\infty a_nz^n$$

so that

$$u(r,\varphi) = a_2 r^2 \sin 2\varphi + a_3 r^3 \sin 3\varphi + \sum_{n=4}^{\infty} r^n \operatorname{Im}(a_n e^{in\varphi})$$

and

$$u_\varphi(r,\varphi) = 2a_2 r^2 \cos 2\varphi + 3a_3 r^3 \cos 3\varphi + \sum_{n=4}^{\infty} r^n \operatorname{Im}(in a_n e^{in\varphi})$$

and for $0 < r < r_0$ we will have

$$\Bigl|\sum_{n=4}^{\infty} r^n \operatorname{Im}(a_n e^{in\varphi})\Bigr| \le r^4 \sum_{n=4}^{\infty} |a_n| r_0^{n-4},$$
$$\Bigl|\sum_{n=4}^{\infty} r^n \operatorname{Im}(in a_n e^{in\varphi})\Bigr| \le r^4 \sum_{n=4}^{\infty} n|a_n| r_0^{n-4}.$$

The last two sums converge and this proves our lemma for u and u_φ. For v and v_φ the proof is similar. □

We divide the interval $[-\frac{\pi}{8}, \frac{15\pi}{8}]$ of length 2π in 8 intervals

$$I_1 = [-\pi/8, \pi/8], \quad I_2 = [\pi/8, 3\pi/8], \quad I_3 = [3\pi/8, 5\pi/8],$$
$$I_4 = [5\pi/8, 7\pi/8], \quad I_5 = [7\pi/8, 9\pi/8], \quad I_6 = [9\pi/8, 11\pi/8],$$
$$I_7 = [11\pi/8, 13\pi/8], \quad I_8 = [13\pi/8, 15\pi/8].$$

Lemma 9.13. *There exists an $r_0 > 0$ such that for $0 < r < r_0$ the function u has exactly four zeros on $[-\pi/8, 15\pi/8]$, denoted by $\alpha_1 \in I_1$, $\alpha_3 \in I_3$, $\alpha_5 \in I_5$ and $\alpha_7 \in I_7$, so that u is positive on (α_1, α_3), negative on (α_3, α_5), positive on (α_5, α_7) and negative on $(\alpha_7, \alpha_1 + 2\pi)$*

Proof. By Lemma 9.12 for $r \to 0$

$$u(r,\varphi) = a_2 r^2(\sin 2\varphi + \mathcal{O}(r)), \quad u_\varphi(r,\varphi) = 2a_2 r^2(\cos 2\varphi + \mathcal{O}(r)).$$

On I_2 and I_6 $\sin 2\varphi > 2^{-1/2}$, whereas $\sin 2\varphi < -2^{-1/2}$ on I_4 and I_8. Then, if we take r_0 small enough, $u(r,\varphi) > 0$ on I_2 and I_6, and $u(r,\varphi) < 0$ on I_4 and I_8 (we only need to take the $\mathcal{O}(r)$ terms less than $2^{-1/2}$).

By continuity of $u(r,\varphi)$ this implies that for each $0 < r < r_0$ the function $u(r,\varphi)$ has at least one zero on each of the intervals I_1, I_3, I_5 and I_7. But $\cos 2\varphi > 2^{-1/2}$ on I_1 and I_5, and $\cos 2\varphi < -2^{-1/2}$ on I_3 and I_7, so that choosing r_0 small enough the sign of $u_\varphi(r,\varphi)$ will be negative on I_3 and I_7 and positive on I_1 and I_5. Therefore on each of these intervals the function $u(r,\varphi)$ is monotonic and has only one zero. □

There is an analogous result for $v(r,\varphi)$.

Lemma 9.14. *There exists an $r_0 > 0$ such that for $0 < r < r_0$ the function $v(r,\varphi)$ has exactly two zeros for $\varphi \in [-\pi/8, 15\pi/8]$, denoted by $\beta_3 \in I_3$ and $\beta_7 \in I_7$, so that $v(r,\varphi)$ is negative on (β_3, β_7), and positive on $(\beta_7, \beta_3 + 2\pi)$.*

Proof. Observing that $v(r,\varphi) = 2a_2 r(\cos\varphi + \mathcal{O}(r))$, the proof is similar to that of Lemma 9.13. □

Lemma 9.15. *There exists an $r_0 > 0$ such that for $0 < r < r_0$ the zeros of $u(r,\varphi)$ and $v(r,\varphi)$ satisfy the relation*

$$\alpha_3 < \beta_3, \qquad \beta_7 < \alpha_7.$$

Proof. Putting $a = -a_3/a_2 > 0$ we have for $0 < r < r_0$ (r_0 small enough to make the previous lemmas valid)

$$u(r,\varphi) = a_2 r^2(\sin 2\varphi - ar\sin 3\varphi + \mathcal{O}(r^2)$$
$$v(r,\varphi) = 2a_2 r(\cos\varphi - \frac{3a}{2} r\cos 2\varphi + \mathcal{O}(r^2)$$

with $\mathcal{O}$-constants independent of φ.

The two zeros α_3 and β_3 are on I_3 an interval with center at $\frac{\pi}{2}$. At the point $\frac{\pi}{2} + ar$ we have

$$\frac{u(r,\pi/2+ar)}{a_2 r^2} = ar\cos(3ar) - \sin(2ar) + \mathcal{O}(r^2)$$
$$\frac{v(r,\pi/2+ar)}{2a_2 r} = \frac{3ar}{2}\cos(2ar) - \sin(ar) + \mathcal{O}(r^2).$$

Expanding in Taylor series we get

$$\frac{u(r,\pi/2+ar)}{a_2 r^2} = -ar + \mathcal{O}(r^2)$$
$$\frac{v(r,\pi/2+ar)}{2a_2 r} = \frac{ar}{2} + \mathcal{O}(r^2).$$

Choosing r_0 small enough we obtain $u(r,\pi/2+ar) < 0 < v(r,\pi/2+ar)$ for $0 < r < r_0$. Since both $u(r,\varphi)$ and $v(r,\varphi)$ are decreasing on this interval, the zero of $u(r,\varphi)$ must come before $\frac{\pi}{2} + ar$ and the zero of $v(r,\varphi)$ must come after $\frac{\pi}{2} + ar$. That is

$$\alpha_3 < \frac{\pi}{2} + ar < \beta_3.$$

The center of I_7 is $\frac{3\pi}{2}$. We compute the functions at $\frac{3\pi}{2} - ar$. In the same way as before we find

$$\frac{u(r, 3\pi/2 - ar)}{a_2 r^2} = -ar\cos(3ar) + \sin(2ar) + \mathcal{O}(r^2) = ar + \mathcal{O}(r^2)$$
$$\frac{v(r, 3\pi/2 - ar)}{2a_2 r} = \frac{3ar}{2}\cos(2ar) - \sin(ar) + \mathcal{O}(r^2) = \frac{ar}{2} + \mathcal{O}(r^2).$$

On the interval I_7 the function $u(r, \varphi)$ is decreasing whereas $v(r, \varphi)$ is increasing, so that the above computation implies that for r_0 small enough, we will have that the zero of $u(r, \varphi)$ will come after $\frac{3\pi}{2} - ar$, and that the zero of $v(r, \varphi)$ will come before this value. That is

$$\beta_7 < \frac{3\pi}{2} - ar < \alpha_7.$$

□

Proof of Proposition 9.11. Taking r_0 small enough all previous lemmas will apply. We have seen that the zeros of $u(r, \varphi)$ and $v(r, \varphi)$ satisfy

$$\alpha_1 < \alpha_3 < \beta_3 < \alpha_5 < \beta_7 < \alpha_7 < \alpha_1 + 2\pi$$

so that in particular these functions do not vanish simultaneously. Therefore, the curve γ_r with equation

$$\varphi \mapsto h(r, \varphi) = u(r, \varphi) + iv(r\varphi)$$

does not pass through $z = 0$.

Since we know the sign of u and v on the intervals limited by the above zeros, we easily compute the index $\omega(\gamma_r, 0) = 1$. □

Theorem 9.16. *Let f be a holomorphic function in the conditions of Proposition 9.11. Let (f_n) be a sequence of holomorphic functions on the disc where f is defined and converging uniformly to f on compact sets of this disc. Then there exist n_0 and a sequence (b_n) of complex numbers such that for $n \geq n_0$, b_n is a turning point of f_n and $\lim_n b_n = 0$.*

Proof. Let r_0 be small enough to make all previous lemmas applicable to f. Put $u_n(r, \varphi) := \operatorname{Im} f_n(re^{i\varphi})$ and $v_n(r, \varphi) = \operatorname{Re} f_n'(re^{i\varphi})$. The uniform convergence implies that for each $0 < r < r_0$, $\lim_n u_n(r, \varphi) = u(r, \varphi)$ and $\lim_n v_n(r, \varphi) = v(r, \varphi)$ uniformly in φ. Finally put $h_n(r, \varphi) := u_n(r, \varphi) + iv_n(r, \varphi)$.

Let (r_n) be a decreasing sequence of real numbers with $0 < r_n < r_0$ and $\lim_n r_n = 0$.

In Proposition 9.11 $h(r_n, \varphi)$ does not vanish. Since it is continuous there exists a $\delta_n > 0$ such that $|h(r_n, \varphi)| > \delta_n$ for all φ. By the uniform convergence there exists N_n such that $|h(r_n,\varphi) - h_m(r_n,\varphi)| < \delta_n$ for each $m \geq N_n$ and all φ.

Let γ_n be the curve $\varphi \mapsto h(r_n,\varphi)$. We have seen in Proposition 9.11 that $\omega(\gamma_n, 0) = 1$. Let $\gamma_n^{(m)}$ be the curve $\varphi \mapsto h_m(r_n,\varphi)$. Since

$$|h(r_n,\varphi) - h_m(r_n,\varphi)| < \delta_n < |h(r_n,\varphi)|, \qquad (m \geq N_n)$$

we find that $\omega(\gamma_n^{(m)}, 0) = \omega(\gamma_n, 0) = 1$.

Since $\omega(\gamma_n^{(m)}, 0) = 1$ there is no homotopy of the curve to a point in $\mathbb{C} \setminus \{0\}$. The equation of this curve is

$$\varphi \mapsto h_m(r_n,\varphi).$$

The curves $\varphi \mapsto h_m(r,\varphi)$ for $0 \leq r \leq r_n$ will be a homotopy of $\gamma_n^{(m)}$ to the point $h_m(0,\varphi)$ if this function does not vanish for $(r,\varphi) \in [0,r_0] \times [0,2\pi]$. It follows that there is a point with $h_m(r,\varphi) = 0$. This makes $b_{n,m} := re^{i\varphi}$ a turning point of f_m with $|b_{n,m}| \leq r_n$

For each n we have found N_n such that for $m \geq N_n$ there exists a turning point $b_{n,m}$ of f_m with $|b_{n,m}| < r_n$. It is clear that we may take $N_1 < N_2 < N_3 < \cdots$.

Now define for $N_k \leq m < N_{k+1}$ the point $b_m := b_{k,m}$. This is a sequence defined for $m \geq N_1$.

The sequence (b_m) satisfies our theorem. Indeed, by construction b_m is a turning point for f_m and for each m there is a k with $|b_m| = |b_{k,m}| < r_k$ where $N_k \leq m < N_{k+1}$. Hence for $m > N_k$ we will have $|b_m| < r_j \leq r_k$, so that $\lim b_m = 0$. □

Now we can prove the last part of Theorem 9.1: *There is a sequence (b_n) of turning points for $\zeta(s)$ with $\lim_{n\to\infty} \operatorname{Re}(b_n) = E$.*

Proof of the second half of Theorem 9.1. Let $g(s) := \frac{2^s-1}{2^s+1}\zeta(s)$, and define $f(s) = g(s+E)$. We then have $f(0) = 0.933\ldots$, $f'(0) = 0$, $f''(0) = 0.070\ldots$, $f'''(0) = -0.178\ldots$.

By Lemma 2.1 there exists a sequence (t_n) of real numbers with

$$\lim_{n\to\infty} \zeta(s + it_n) = g(s) = f(s - E)$$

uniformly on compact sets of $\sigma > 1$.

It follows that the functions $\zeta(s+E+it_n)$ converge to $f(s)$ uniformly on the disc with center 0 and radius $E - 1$.

By Theorem 9.11 there exists a sequence (c_n) such that c_n is a turning point of $\zeta(s + E + it_n)$ and $\lim_n c_n = 0$.

Put $b_n = c_n + E + it_n$. It is clear that b_n is a turning point of $\zeta(s)$ and

$$\lim_{n\to\infty} \operatorname{Re}(b_n) = \lim_{n\to\infty} \operatorname{Re}(c_n + E + it_n) = \lim_{n\to\infty} \operatorname{Re}(c_n + E) = \\ = E + \lim_{n\to\infty} \operatorname{Re}(c_n) = E + \operatorname{Re}\big(\lim_{n\to\infty} c_n\big) = E.$$

□

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