Stability of some turbulent vertical models for the ocean mixing boundary layer

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Abstract

We consider four turbulent models to simulate the boundary mixing layer of the ocean. We show the existence of solutions to these models in the steady-state case then we study the mathematical stability of these solutions.

Key-words : oceanography, turbulence models, stability, partial differential equations

MSC classification : 35J60, 35K55, 76E20, 76F40

1 Introduction

The presence of an homogeneous layer near the surface of the ocean has been observed since a long time. The so called "mixed layer" presents almost constant profiles of temperature and salinity (or equivalently the density). The bottom of the mixed layer corresponds either to the top of the thermocline, zone of large gradients of temperature, or to the top of the zone where haline stratification is observed [8]. Some attempts to describe this phenomenon can be found for example in Defant [3] or Lewandowski [5]. The effect of the wind-stress acting on the sea-surface was then considered to be the main forcing of this boundary layer. Observations in situ were completed by laboratory experiments [2] and more recently by numerical modelizations of the mixed layer.

In this note, we consider four turbulent models to describe this homogeneous boundary layer. The first one is the Pacanowski-Philander model, and two of these models are new models. They aim to compute the velocity and the water density of a water column, are one space dimensional and the eddy viscosities depend on the Richardson number. For those model, we show the existence of a steady-state solution and we analyse the mathematical linear stability of these steady state solution, showing that only one of these model, the one we introduce in this note (model labelized as $R - 2 - 2 - 4$ below), has a unique staedy state solution with a large range of stabilty. Moreover, in [1] we have used these models to simulate the warm pool at the equator. Numerical results confirm that $R - 2 - 2 - 4$ is the most accurate parametrization.

2 The equations

We denote by (u, v) the horizontal water velocity and ρ its density. Since the numerical simulation performed in [1] concerns the equator zone, we do not take the Coriolis force into account. The closure equations are:

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$$(2.1) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial z}\left(\nu_1 \dfrac{\partial u}{\partial z}\right) = 0, \quad \dfrac{\partial v}{\partial t} - \dfrac{\partial}{\partial z}\left(\nu_1 \dfrac{\partial v}{\partial z}\right) = 0, \\[2ex] \dfrac{\partial \rho}{\partial t} - \dfrac{\partial}{\partial z}\left(\nu_2 \dfrac{\partial \rho}{\partial z}\right) = 0, \text{ for } t \geqslant 0 \text{ and } -h \leqslant z \leqslant 0, \\[2ex] u = u_b, \ v = v_b, \ \rho = \rho_b \ \text{ at the depth } z = -h, \\[2ex] \nu_1 \dfrac{\partial u}{\partial z} = \dfrac{\rho_a}{\rho_0} V_x, \ \nu_1 \dfrac{\partial v}{\partial z} = \dfrac{\rho_a}{\rho_0} V_y, \ \nu_2 \dfrac{\partial \rho}{\partial z} = Q \ \text{ at the surface } z = 0, \\[2ex] u = u_0, \ v = v_0, \ \rho = \rho_0 \ \text{ at initial time } t = 0. \end{cases}$$

In system (2.1), the coefficients ν_1 and ν_2 are the vertical eddy viscosity and diffusivity coefficients and will be expressed as functions of the Richardson number R defined as

$$R = \frac{-\dfrac{g}{\rho_0} \cdot \dfrac{\partial \rho}{\partial z}}{\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2}$$

where g is the gravitational acceleration and ρ_0 a reference density ($\rho_0 \simeq 1025\ kg.m^{-3}$).

The constant h denotes the thickness of the studied layer that must contain the mixing layer. Therefore the circulation for $z < -h$, under the boundary layer, is supposed to be known, either by observations or by a deep circulation numerical model. This justifies the choice of Dirichlet boundary conditions at $z = -h$, u_b, v_b and ρ_b being the values of horizontal velocity and density in the layer located below the mixed layer. The air-sea interactions are represented by the fluxes at the sea-surface : V_x and V_y are respectively the forcing exerced by the zonal wind-stress and the meridional wind-stress and Q represents the thermodynamical fluxes, heating or cooling, precipitations or evaporation. We have $V_x = C_D\,|u^a|^2$ and $V_y = C_D\,|v^a|^2$, where $U^a = (u_a, v_a)$ is the air velocity and C_D a friction coefficient.

We study hereafter four different formulations for the eddy coefficients $\nu_i = f_i(R)$, labeled as "$R-2-i$" and/or "$R-2-i-j$" . In all models, $f_1(R) = \alpha_1 + \dfrac{\beta_1}{(1+5R)^2}$, except in model $R-2-3$ below:

$$(2.2) \quad \begin{aligned} &R-2-1-3: && f_2(R) = \alpha_2 + \frac{f_1(R)}{1+5R} = \alpha_2 + \frac{\alpha_1}{1+5R} + \frac{\beta_1}{(1+5R)^3}. \\[1.5ex] &R-2-3 && f_1(R) = \alpha_1 + \frac{\beta_1}{(1+10R)^2}, \quad f_2(R) = \alpha_2 + \frac{\beta_2}{(1+10R)^3} \\[1.5ex] &R-2-2-4: && f_2(R) = \alpha_2 + \frac{f_1(R)}{(1+5R)^2} = \alpha_2 + \frac{\alpha_1}{(1+5R)^2} + \frac{\beta_1}{(1+5R)^4}, \\[1.5ex] &R-2-2 && f_2(R) = \alpha_2 + \frac{\beta_2}{(1+5R)^2}, \end{aligned}$$

Formulation $R-2-1-3$ corresponds to the modelization of the vertical mixing proposed by Pacanowski and Philander [7]. The coefficients α_1, β_1 and α_2 have the following values:

$\alpha_1 = 1.10^{-4}$, $\beta_1 = 1.10^{-2}$, $\alpha_2 = 1.10^{-5}$(units:m^2s^{-1}).This formulation has been used in the OPA code developed in Paris 6 University [6] with coefficients $\alpha_1 = 1.10^{-6}$, $\beta_1 = 1.10^{-2}$, $\alpha_2 = 1.10^{-7}$ (units: m^2s^{-1}).The selection criterion for the coefficients appearing in these formulas was the best agreement of numerical results with observations carried out in different tropical areas. Formulation $R - 2 - 3$ has been proposed by Gent [4]. Formulations $R - 2 - 2 - 4$ and $R - 2 - 2$ are new as far as we know. Notice that models $R - 2 - 1 - 3$ and $R - 2 - 3$ are no more physically valid respectively for $R \in (-3.13, -0.2)$ and $R \in (-2.25, -0.1)$ since the coefficient ν_2 becomes negative.

2.1 Steady-state solutions

Steady-state solutions to system (2.1) satisfy

$$(2.3) \qquad \frac{\partial}{\partial z}\left(f_1\left(R\right)\frac{\partial u}{\partial z}\right) = 0, \quad \frac{\partial}{\partial z}\left(f_1\left(R\right)\frac{\partial v}{\partial z}\right) = 0, \quad \frac{\partial}{\partial z}\left(f_2\left(R\right)\frac{\partial\rho}{\partial z}\right) = 0.$$

Theorem 2.1 *System (2.3) has at leat one smooth solution on $[0, -h]$ for each model in (2.2). In case of $R - 2 - 2 - 4$ the solution is unique.*

Proof. Integrating (2.3) with respect to z, yields

$$(2.4) \qquad f_1\left(R\right)\frac{\partial u}{\partial z} = \frac{V_x\rho_a}{\rho_0}, \quad f_1\left(R\right)\frac{\partial v}{\partial z} = \frac{V_y\rho_a}{\rho_0}, \quad f_2\left(R\right)\frac{\partial\rho}{\partial z} = Q.$$

and since $R = \dfrac{-\dfrac{g}{\rho_0}\cdot\dfrac{\partial\rho}{\partial z}}{\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2}$ we deduce from (2.4) that

$R = -\dfrac{gQ\rho_0}{\rho_a^2(V_x^2 + V_y^2)}\cdot\dfrac{(f_1\left(R\right))^2}{f_2\left(R\right)}$, which yields

$$(2.5) \qquad \frac{(f_1\left(R\right))^2}{f_2\left(R\right)} = -\frac{\rho_a^2(V_x^2 + V_y^2)}{gQ\rho_0}R$$

which is a fixed point equation for R. Any solution R to equation (2.5) yields a Richardson number R^e corres ponding to the fluxes V_x, V_y and Q and not on z as ν_1 and ν_2 are independent on the depth variable z as well asq the turbulent viscosities. The Richardson number R^e being known, steady-state profiles for velocity and density are obtained by integrating (2.4) with respect to z, taking into account the boundary conditions at $z = -h$:

$$(2.6) \qquad \begin{aligned} u^e\left(z\right) &= u_b + \frac{V_x\rho_a}{\rho_0 f_1\left(R^e\right)}\left(z + h\right), \quad v^e\left(z\right) = v_b + \frac{V_y\rho_a}{\rho_0 f_1\left(R^e\right)}\left(z + h\right), \\ \rho^e(z) &= \rho_b + \frac{Q}{f_2\left(R^e\right)}\left(z + h\right). \end{aligned}$$

It remains to analyse the existence of solutions of equation (2.5). These solutions can be interpreted as the intersection of the curves $k\left(R\right) = \dfrac{(f_1\left(R\right))^2}{f_2\left(R\right)}$ and $h\left(R\right) = CR$ with $C = -\dfrac{\rho_a^2(V_x^2 + V_y^2)}{gQ\rho_0}$. The existence and the number of solutions are controlled by the

constant C and then by the parameter $\dfrac{V^2}{Q}$, $V^2 = V_x^2 + V_y^2$, depending only on the surface fluxes. The graph of function k and h for $Q < 0$ and $Q > 0$ is plotted on Figures 1 and 2 below when f_1 and f_2 in case of R-2-2-4 and $R - 2 - 2$.
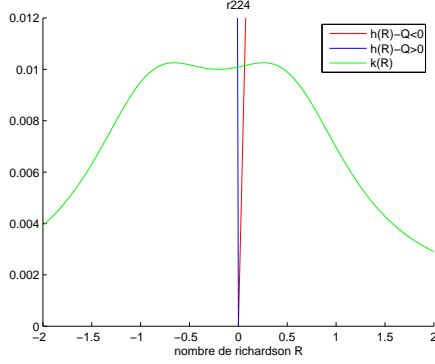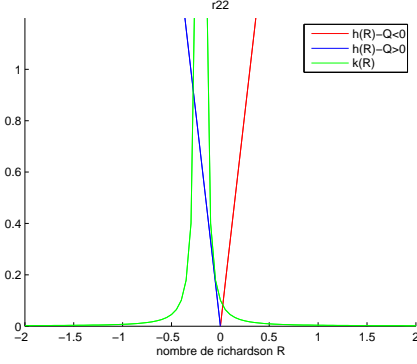
Formulation R-2-2-4	Formulation R-2-2
Figure A	**Figure B**

The qualitative behaviour obtained with formulation R-2-3 and R-2-1-3 is the same as R-2-2. The intersection of $k(R)$ and $h(R) = CR$ consists in one point for $Q < 0$ and several points for $Q > 0$. The number of points depends to the values of surface fluxes.

The graphs obtained for the R-2-2-4 modelization (Figure A) and its simplified version R-2-2 (Figure B) are very different. It is obvious in Figure A that any straight line $h\left(R\right) = CR$ meets k at only one point for $Q > 0$ and $Q < 0$. Therefore it exists one unique equilibrium Richardson number R^e whatever the values of the surface fluxes V_x, V_y and Q. In the case of the other models, we get several solutions. The proof is finished. Notice that in [1] we show that the most accurate model is $R - 2 - 2 - 4$ from the physical and numerical viewpoint.

2.2 Linear stability of the equilibrium solutions

In this section we analyse the time evolution of a small perturbation of one of the equilibrium states (u^e, v^e, ρ^e) described in the previous section.

At initial time $t = 0$ we set $(u_0, v_0, \rho_0) = (u^e, v^e, \rho^e) + (u_0', v_0', \rho_0')$ and we denote by

$$(u, v, \rho) = (u^e, v^e, \rho^e) + \left(u', v', \rho'\right)$$

the solution of equations (2.1) at time t where (u^e, v^e, ρ^e) are solution to the steady-state system (2.3), and $\nu_1^e = f_1\left(R^e\right)$ and $\nu_2^e = f_2\left(R^e\right)$ are two positive constants.

Introducing the new variables $\psi = \dfrac{\partial \rho}{\partial z}$, $\theta = \dfrac{\partial u}{\partial z}$ and $\beta = \dfrac{\partial v}{\partial z}$, the Richardson number can be expressed as

$$R = -\frac{g}{\rho_0} \frac{\psi}{(\theta^2 + \beta^2)} = R\left(\theta, \beta, \psi\right)$$

Applying the Taylor formula, we get

$$\mathcal{F} = (\theta - \theta^e) \frac{\partial \nu_1}{\partial \theta}\left(\theta^e, \beta^e, \psi^e\right) + (\beta - \beta^e) \frac{\partial \nu_1}{\partial \beta}\left(\theta^e, \beta^e, \psi^e\right) + (\psi - \psi^e) \frac{\partial \nu_1}{\partial \psi}\left(\theta^e, \beta^e, \psi^e\right) + \cdots$$

$$\mathcal{G} = (\theta - \theta^e) \frac{\partial \nu_2}{\partial \theta}\left(\theta^e, \beta^e, \psi^e\right) + (\beta - \beta^e) \frac{\partial \nu_2}{\partial \beta}\left(\theta^e, \beta^e, \psi^e\right) + (\psi - \psi^e) \frac{\partial \nu_2}{\partial \psi}\left(\theta^e, \beta^e, \psi^e\right) + \cdots$$

We set for $k = 1, 2$: $\mathcal{F} = \nu_1(\theta, \beta, \psi) - \nu_1(\theta^e, \beta^e, \psi^e)$, $\mathcal{G} = \nu_2(\theta, \beta, \psi) - \nu_2(\theta^e, \beta^e, \psi^e)$ and $\nu_k^e = \nu_k(\theta^e, \beta^e, \psi^e)$, $\theta' = \theta - \theta^e$, $\beta' = \beta - \beta^e$, $\psi' = \psi - \psi^e$,

$$\left(\frac{\partial \nu_k}{\partial \theta}\right)^e = \frac{\partial \nu_k}{\partial \theta}(\theta^e, \beta^e, \psi^e), \quad \left(\frac{\partial \nu_k}{\partial \beta}\right)^e = \frac{\partial \nu_k}{\partial \beta}(\theta^e, \beta^e, \psi^e), \quad \left(\frac{\partial \nu_k}{\partial \psi}\right)^e = \frac{\partial \nu_k}{\partial \psi}(\theta^e, \beta^e, \psi^e).$$

The equations satisfied by the perturbation (u', v', ρ') are deduced from equations (2.1):

(2.7)
$$\begin{cases} \dfrac{\partial u'}{\partial t} - \dfrac{\partial}{\partial z}(\nu_1(\theta, \beta, \psi)(\theta^e + \theta')) = 0, \quad \dfrac{\partial v'}{\partial t} - \dfrac{\partial}{\partial z}(\nu_1(\theta, \beta, \psi)(\beta^e + \beta')) = 0, \\ \dfrac{\partial \rho'}{\partial t} - \dfrac{\partial}{\partial z}(\nu_2(\theta, \beta, \psi)(\psi^e + \psi')) = 0. \end{cases}$$

We now replace ν_1 and ν_2 by expresions deuced from the Taylor's development and retain only the first order terms. The approximated equations for (u', v', ρ') then are

(2.8)
$$\begin{cases} \dfrac{\partial u'}{\partial t} - \dfrac{\partial}{\partial z}\left(\left(\nu_1^e + \theta^e\left(\dfrac{\partial \nu_1}{\partial \theta}\right)^e\right)\theta'\right) - \dfrac{\partial}{\partial z}\left(\theta^e\left(\dfrac{\partial \nu_1}{\partial \beta}\right)^e \beta'\right) - \dfrac{\partial}{\partial z}\left(\theta^e\left(\dfrac{\partial \nu_1}{\partial \psi}\right)^e \psi'\right) = 0, \\ \dfrac{\partial v'}{\partial t} - \dfrac{\partial}{\partial z}\left(\beta^e\left(\dfrac{\partial \nu_1}{\partial \theta}\right)^e \theta'\right) - \dfrac{\partial}{\partial z}\left(\left(\nu_1^e + \beta^e\left(\dfrac{\partial \nu_1}{\partial \beta}\right)^e\right)\beta'\right) - \dfrac{\partial}{\partial z}\left(\beta^e\left(\dfrac{\partial \nu_1}{\partial \psi}\right)^e \psi'\right) = 0, \\ \dfrac{\partial \rho'}{\partial t} - \dfrac{\partial}{\partial z}\left(\psi^e\left(\dfrac{\partial \nu_2}{\partial \theta}\right)^e \theta'\right) - \dfrac{\partial}{\partial z}\left(\psi^e\left(\dfrac{\partial \nu_2}{\partial \beta}\right)^e \beta'\right) - \dfrac{\partial}{\partial z}\left(\left(\nu_2^e + \psi^e\left(\dfrac{\partial \nu_2}{\partial \psi}\right)^e\right)\psi'\right) = 0. \end{cases}$$

We set

$$A = \begin{pmatrix} \nu_1^e + \theta^e\left(\dfrac{\partial \nu_1}{\partial \theta}\right)^e & \theta^e\left(\dfrac{\partial \nu_1}{\partial \beta}\right)^e & \theta^e\left(\dfrac{\partial \nu_1}{\partial \psi}\right)^e \\ \beta^e\left(\dfrac{\partial \nu_1}{\partial \theta}\right)^e & \nu_1^e + \beta^e\left(\dfrac{\partial \nu_1}{\partial \beta}\right)^e & \beta^e\left(\dfrac{\partial \nu_1}{\partial \psi}\right)^e \\ \psi^e\left(\dfrac{\partial \nu_2}{\partial \theta}\right)^e & \psi^e\left(\dfrac{\partial \nu_2}{\partial \beta}\right)^e & \nu_2^e + \psi^e\left(\dfrac{\partial \nu_2}{\partial \psi}\right)^e \end{pmatrix}, \quad V = \begin{pmatrix} u' \\ v' \\ \rho' \end{pmatrix},$$

Equations (2.8) can be written

(2.9)
$$\frac{\partial V}{\partial t} - \frac{\partial}{\partial z}\left(A\frac{\partial V}{\partial z}\right) = \frac{\partial V}{\partial t} - A\frac{\partial^2 V}{\partial z^2} = 0.$$

Let $(\lambda_1, \lambda_2, \lambda_3)$ be the eigenvalues of matrix A. Assuming the eigenvalues dinstincts, matrix A is equal to $P^{-1}DP$, where D is diagonal, and such that $d_{11} = \lambda_1$, $d_{22} = \lambda_2$ and $d_{33} = \lambda_3$. Set now $W = PV$. The vector W verifies the system $\dfrac{\partial W}{\partial t} - D\dfrac{\partial^2 W}{\partial z^2} = 0$, i.e.

(2.10)
$$\frac{\partial w_1}{\partial t} - \lambda_1\frac{\partial^2 w_1}{\partial z^2} = 0, \quad \frac{\partial w_2}{\partial t} - \lambda_2\frac{\partial^2 w_2}{\partial z^2} = 0, \quad \frac{\partial w_3}{\partial t} - \lambda_3\frac{\partial^2 w_3}{\partial z^2} = 0.$$

Stability of the equilibrium solution (u^e, v^e, ρ^e) means that any perturbation (u'_0, v'_0, ρ'_0) imposed at initial time $t = 0$ is damped as $t \to \infty$. This is verified if the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are such that $Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$ and $Re(\lambda_3) > 0$. These three conditions are equivalence to $\det(A) > 0$, $\mathrm{tr}(A) > 0$ and $\mathrm{tr}(Adj(A)) > 0$. From these conditions, we build the graph below (see figure 1), obtained thanks an analytical computation (we skip the technical details here):
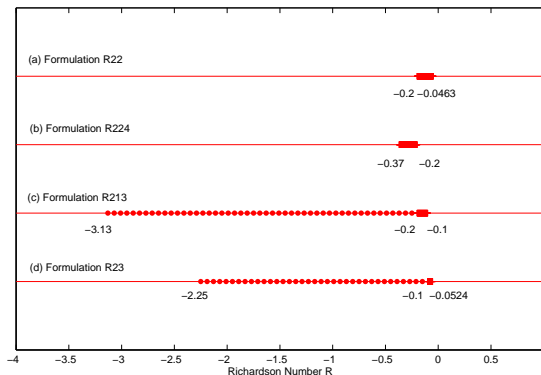
The results are summarized in Figure 1. The circle zone represents a zone where the solution is physically not valid. It is the case for the R-2-3 and R-2-1-3 formulation. The rectangular zone is a unstability zone. All formulations have a unstability zone. Nevertheles, one observes that for each model, mathematical stability holds for non negative R.

Figure 1: Numerical stability

3 Conclusion

All the models have a steady-state solution, unique in the case of R-2-2-4. Each one is linearly stable for non negative R, which corresponds to physical stability. All these models present a linear unstable zone, located in a region where R is non positive. They all presents a linear stability zone for some non positive values of R, situation that can arise in real situation, as reported in [1] (physical unstability). All these models have been tested in [1]. The simulation confirms the existence of stable linear steady-state solutions and the ability of these models to describe a boundary mixing layer. However, the numerical study in [1] confirms that $R - 2 - 2 - 4$ yields better numerical results.

References

[1] A. C. Bennis, T. Chacon Rebello, M. Gomez Marmol, R. Lewandowski, and F. Brossier, *Parametrization of the mixing layer: Comparison of four models depending on the richardson number*, To appear.

[2] J. W. Deardorff, G. Willis, and D. Lilly, *Laboratory investigation of nonsteady penetrative convection*, J. Fluid. Mech., 35 (1969), pp. 7–31.

[3] A. Defant, *Schichtung und zirculation des atlantischen ozeans*, Wiss. Ergebn. Deutsch. Atlant. Exp. Meteor, 6 (1936), pp. 289–411.

[4] P. R. Gent, *The heat budget of the toga-coare domain in an ocean model*, J. Geophys. Res., 96 (1991), pp. 3323–3330.

[5] R. Lewandowski, *Analyse mathématique et océanographie*, Masson, 1997.

[6] G. Madec, P. Delecluse, M. Imbard, and C. Levy, *O.p.a. version 8.0. ocean general circulation model, reference manual*, 1997. Technical report.

[7] R. C. Pacanowski and S. G. H. Philander, *Parametrization of vertical mixing in numericals models of the tropical oceans*, J. Phys. Oceanogr., 11 (1981), pp. 1443–1451.

[8] J. Vialard and P. Delecluse, *An ogcm study for the toga decade. part i: Role of salinity in the physics of the western pacific fresh pool*, J. Phys. Oceanogr., 28 (1998), pp. 1071–1088.