A Bochev-Dohrmann-Gunzburger stabilization method for the Primitive Equations of the Ocean

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Abstract

We introduce a low-order stabilized discretization of the Primitive Equations of the Ocean, with a highly reduced computational complexity. We prove stability through a specific inf-sup condition, and weak convergence to a weak solution. We also perform some numerical test for relevant flows.

Keywords: primitive equations; finite elements; stabilized methods; inf-sup condition.
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1. Introduction

The Primitive Equations (PE) of the Ocean are a mathematical model for large space and time scales of the oceanic flow, extensively used for climatic, weather and ecological studies (Cf. [13], [11], [12]). Existence of weak solutions $(\mathbf{u}, p)$ (with $H^1 \times L^2$ regularity) is proved in [4] and [6], and existence and uniqueness of strong solutions (with $H^2 \times H^1$ regularity) is proved in [5] and [9]. Finite element discretizations are well suited to irregular oceanic bottoms. In this paper we introduce a stabilized discretization of the PE for first order finite elements. We adapt the Bochev-Dohrmann-Gunzburger stabilization technique introduced in [1] to a reduced model of PE, that retains only the (3D) horizontal velocity and the (2D) surface pressure as unknowns. This yields a solver with highly reduced computational complexity. We introduce the reduced model of PE in Section 2, and the numerical discretization in Section 3. We prove the stability and convergence of the discretization based upon a specific inf-sup condition in Section 4. In Section 5 we describe some numerical tests for relevant flows.

2. Primitive equations of the ocean

Let be ω a bounded domain in $\mathbb{R}^{d-1}$ ($d = 2$ o $d = 3$) that represents a piece of the ocean surface, and $D : \overline{\omega} \to \mathbb{R}$ a depth function. We consider the ocean domain $\Omega = \{(\mathbf{x}, z) \in \mathbb{R}^d$ such that $\mathbf{x} \in \omega,\ -D(\mathbf{x}) \le z \le 0\}$. For simplicity we assume that ω is polygonal and D is piecewise affine on some triangulation of ω, so that Ω is a polyhedron with flat surface. We suppose that the boundary of Ω is split as $\partial\Omega = \Gamma_s \cup \Gamma_b$, with $\Gamma_s = \{(\mathbf{x}, 0) \in \mathbb{R}^d;\ \mathbf{x} \in \omega\}$ representing the ocean surface, and $\Gamma_b = \partial\Omega - \Gamma_s$, the ocean bottom and, eventually, sidewalls. We consider the following steady reduced Primitive Equations model:

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Find a horizontal velocity field $\mathbf{u} : \overline{\Omega} \mapsto \mathbb{R}^{d-1}$ and a surface pressure $p : \overline{\omega} \mapsto \mathbb{R}$ such that

$$\begin{cases} (\mathbf{u}, u_z) \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla_H p + \varphi\, \mathbf{u}^\perp = \mathbf{f}, \quad u_z(\mathbf{x}, z) = \displaystyle\int_z^0 \nabla_H \cdot \mathbf{u}(\mathbf{x}, s)\, ds \qquad \text{in } \Omega; \\ \nabla_H \cdot \left(\displaystyle\int_{-D(\mathbf{x})}^0 \mathbf{u}(\mathbf{x}, s)\, ds \right) = 0 \qquad \text{in } \omega; \\ \mathbf{u}|_{\Gamma_b} = 0, \quad \mu \partial_z \mathbf{u}|_{\Gamma_s} = \tau. \end{cases} \tag{1}$$

Here, μ is the viscosity coefficient and $\nabla_H = (\partial_x, \partial_y)$ denotes the horizontal gradient. The term $\varphi \mathbf{u}^\perp$ stands for the Coriolis aceleration, that only appears when $d = 3$. In this case, if $\mathbf{u} = (u_1, u_2)$, $\mathbf{u}^\perp = (-u_2, u_1)$. Thus, we define $\varphi = 0$ for $d = 2$ and $\varphi = 2\theta \sin \phi$, with θ the angular rotation rate of the earth and ϕ the latitude, for $d = 3$. The source term $\mathbf{f}$ takes into account variable density effects, due to variations of temperature and salinity and τ is the wind tension at the surface.

This model is an approximation of the Navier-Stokes equations for thin domains (Cf. [10]). In particular the pressure is assumed to be hydrostatic. The surface pressure p may be interpreted as the pressure that must be exerced at the flow surface to keep it flat. It is the Lagrange multiplier associated to the second equation in (1), that represents the mass conservation. Observe that the 3D velocity field $(\mathbf{u}, u_z)$ is incompressible. Also, that $u_z = 0$ on Γ_s. This is the rigid lid assumption.

Let us consider the following spaces for the velocities and pressures,

$$\mathbf{W}_b^{1,k}(\Omega) = \{\mathbf{v} \in W^{1,k}(\Omega)^{d-1} :\ \mathbf{v}|_{\Gamma_b} = 0\},\ k \geq 1, \quad \mathbf{H}_b^1(\Omega) = \mathbf{W}_b^{1,2}(\Omega),$$

$$L^r_D(\omega) = \{q : \omega \mapsto \mathbb{R} \text{ measurable such that } \int_\omega D(\mathbf{x}) |q(\mathbf{x})|^r\, dx < \infty\}, \quad L^r_{D,0}(\omega) = L^r_D(\omega)/\mathbb{R}.$$

We define the weak solutions of problem (1) as the solutions of the following variational formulation: Given $\mathbf{f} \in [\mathbf{H}_b^1(\Omega)]'$ and $\tau \in \mathbf{H}^{-1/2}(\Gamma_s)$, find $(\mathbf{u}, p) \in \mathbf{H}_b^1(\Omega) \times L^{3/2}_{D,0}(\omega)$ such that

$$B(\mathbf{u}; (\mathbf{u}, p), (\mathbf{v}, q)) = L(\mathbf{v}), \quad \forall\, (\mathbf{v}, q) \in \mathbf{W}_b^{1,3}(\Omega) \times L^2_{D,0}(\omega), \quad \text{where} \tag{2}$$

$$B(\mathbf{a}; (\mathbf{u}, p), (\mathbf{v}, q)) = < \vec{\mathbf{a}} \cdot \nabla \mathbf{u}, \mathbf{v} > + \mu\, (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla_H \cdot \mathbf{v}) + (\nabla_H \cdot \mathbf{u}, q) + (\varphi \mathbf{u}^\perp, \mathbf{v}),$$

$$L(v) = < \mathbf{f}, \mathbf{v} >_\Omega + < \tau, \mathbf{v} >_{\Gamma_s} .$$

Here $\vec{\mathbf{a}} = (\mathbf{a}, a_z)$ for some $\mathbf{a} \in \mathbf{H}_b^1(\Omega)$ with a_z defined from $\mathbf{a}$ as in (1). The convection term is defined by duality as $< \vec{\mathbf{a}} \cdot \nabla \mathbf{u}, \mathbf{v} > = -\int_\Omega (\vec{\mathbf{a}} \cdot \nabla \mathbf{v})\, \mathbf{u}$. This Petrov-Galerkin formulation is needed when $d = 3$ (not when $d = 2$) because the vertical velocity a_z has only L^2 regularity, and then the convection operator has not H^{-1} regularity. Problem (2) is studied in [6].

3. Numerical scheme

Consider a family of triangulations $\{\mathcal{C}_h\}_{h>0}$ of $\bar{\omega}$. For each $T \in \mathcal{C}_h$ we define the prism $P_T = \{(\mathbf{x}, z) \in \mathbb{R}^d$, such that $\mathbf{x} \in T, -D(\mathbf{x}) \leq z \leq 0\}$. Consider a triangulation $\mathcal{T}_h$ of Ω associated to $\mathcal{C}_h$ by subdividing each prism P_T into triangles (when d=2) or tetrahedra (d=3), in such a way that the projection of any $K \in \mathcal{T}_h$ on Γ_s (that we identify with ω) is an element of $\mathcal{C}_h$. Let the finite element spaces $\mathcal{U}_h = \{\mathbf{v}_h \in \mathcal{C}^0(\bar{\Omega})^{d-1} : \mathbf{v}_h|_K \in \mathbb{P}_1(K)^{d-1},\ \forall\, K \in \mathcal{T}_h;\ \mathbf{v}_h|_{\Gamma_b} = 0\ \}$, $Q_h = \{q_h \in \mathcal{C}^0(\bar{\omega}) :\ q_h|_T \in \mathbb{P}_1(T),\ \forall\, T \in \mathcal{C}_h\}$, $\mathcal{P}_h = Q_h/\mathbb{R}$, $\mathcal{R}_h = \{\phi \in L^2(\omega) : \phi|_T \in \mathbb{P}_0(T),\ \forall\, T \in \mathcal{C}_h\}$, where $\mathbb{P}_m(K)$ is the space of polynomials on K of degree smaller than or equal to m and similarly $\mathbb{P}_l(T)$ For all $T \in \mathcal{C}_h$ we denote b_T the barycenter of T and we define the interpolation operator $\Pi_h : \mathcal{C}^0(\bar{\omega}) \mapsto \mathcal{R}_h$ such that $\Pi_h \phi|_T = \phi(b_T),\ \forall\, T \in \mathcal{C}_h$. We discretize problem (2) by: Find $(\mathbf{u}_h, p_h) \in \mathcal{U}_h \times \mathcal{P}_h$ such that

$$B_h(\mathbf{u}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = L(\mathbf{v}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{U}_h \times \mathcal{P}_h, \tag{3}$$

where $B_h(\mathbf{a};(\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)) = B(\mathbf{a};(\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)) + s_h(p_h,q_h)$. Here, the stabilization term s_h, denoting $\Pi_h^* = Id - \Pi_h$, is defined as

$$s_h(p_h,q_h) = h^\sigma\,(D\,\Pi_h^* p_h, \Pi_h^* q_h)_\omega \quad \text{with } \sigma=0 \text{ if } d=2 \text{ and } \sigma=1 \text{ if } d=3. \tag{4}$$

The analysis that follows shows that the term s_h yields the stability of the discretization of the pressure, in the natural norms associated to the formulation (2).

4. Stability and convergence analysis

The stability of discretization (3) follows from the following discrete inf-sup condition:

Lemma 1. *Assume that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is uniformly regular. Then for any $r\in(1,+\infty)$ there exists a constant $\gamma_r>0$ independent of h such that $\forall q_h\in\mathcal{P}_h$,*

$$\gamma_r\,\|q_h\|_{L^r_{D,0}(\omega)} \le \sup_{\mathbf{v}_h\in\mathcal{U}_h-\{0\}} \frac{(\nabla_H\cdot\mathbf{v}_h,q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + h^{\frac{d}{r}-\frac{d}{2}}\,\|\Pi_h^* q_h\|_{L^2_D(\omega)},\ \textit{where } \frac{1}{s}+\frac{1}{r}=1. \tag{5}$$

Proof: Given $q_h\in\mathcal{P}_h$, we consider its extension $\tilde q_h$ to $\overline{\Omega}$ defined by $\tilde q_h(\mathbf{x},z)=q_h(\mathbf{x})$, $\forall\mathbf{x}\in\omega$, $-D(\mathbf{x})\le z\le 0$. By Amrouche and Girault (Cf. [2]), there exists a constant $\tilde\gamma_r>0$ such that $\tilde\gamma_r\,\|\tilde q_h\|_{0,r,\Omega} \le \sup\limits_{\vec{\mathbf{v}}\in[W_0^{1,s}(\Omega)]^d-\{0\}} (\nabla\cdot\vec{\mathbf{v}},\tilde q_h)/|\vec{\mathbf{v}}|_{1,s,\Omega}$. If we denote $\vec{\mathbf{v}}=(\mathbf{v},v_z)$, observe that $(\partial_z v_z,\tilde q_h)=0$, because $\partial_z\tilde q_h=0$ and $v_z=0$ in $\partial\Omega$. As $\|\tilde q_h\|_{0,r,\Omega}=\|q_h\|_{L^r_{D,0}(\omega)}$, it follows $\tilde\gamma_r\,\|q_h\|_{L^r_{D,0}(\omega)} \le \sup\limits_{\mathbf{v}\in[W_0^{1,s}(\Omega)]^{d-1}-\{0\}} (\nabla_H\cdot\mathbf{v},\tilde q_h)/|\mathbf{v}|_{1,s,\Omega}$. So there exists $\mathbf{v}\in[W_0^{1,s}(\Omega)]^{d-1}$ such that

$$\tilde\gamma_r\,\|q_h\|_{L^r_{D,0}(\omega)} \le \frac{(\nabla_H\cdot\mathbf{v},\tilde q_h)}{|\mathbf{v}|_{1,s,\Omega}}. \tag{6}$$

We use an adaptation of Verfürth's trick (Cf. [15]): There exists $\mathbf{v}$, $\mathbf{v}_h\in\mathcal{U}_h\cap[H_0^1(\Omega)]^{d-1}$ such that

$$|\mathbf{v}_h|_{1,s,\Omega}\le c\,|\mathbf{v}|_{1,s,\Omega},\quad \|\mathbf{v}-\mathbf{v}_h\|_{0,s,K}\le c\,h_K\,|\mathbf{v}|_{1,s,K}. \tag{7}$$

for some constant c independent of h. Using the first inequality in (7),

$$\frac{(\nabla_H\cdot\mathbf{v},\tilde q_h)}{|\mathbf{v}|_{1,s,\Omega}} \le c\,\frac{(\nabla_H\cdot\mathbf{v}_h,\tilde q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + \frac{(\nabla_H\cdot(\mathbf{v}-\mathbf{v}_h),\tilde q_h)}{|\mathbf{v}|_{1,s,\Omega}}. \tag{8}$$

As $(\nabla_H\cdot(\mathbf{v}-\mathbf{v}_h),\tilde q_h) = -(\mathbf{v}-\mathbf{v}_h,\nabla_H\tilde q_h) \le \left(\sum\limits_{K\in\mathcal{T}_{h>0}}\|\mathbf{v}-\mathbf{v}_h\|_{0,s,K}^s\,h_K^{-s}\right)^{1/s}\left(\sum\limits_{K\in\mathcal{T}_{h>0}} h_K^r\,\|\nabla_H\tilde q_h\|_{0,r,K}^r\right)^{1/r}$ $\le c\,|\mathbf{v}|_{1,s,\Omega}\,h\,\|\nabla_H\tilde q_h\|_{0,r,\Omega}$, using (7). Then from (6) and (8),

$$\tilde\gamma_r\,\|q_h\|_{L^r_{D,0}(\omega)} \le c\sup_{\mathbf{v}_h\in\mathcal{U}_h-\{0\}}\frac{(\nabla_H\cdot\mathbf{v}_h,q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + c\,h\,\|\nabla_H\tilde q_h\|_{0,r,\Omega}. \tag{9}$$

Consider the finite element space $\tilde{\mathcal{R}}_h=\{\Phi\in L^2(\Omega):\phi|_K\in\mathbb{P}_0(K),\ \forall K\in\mathcal{T}_h\}$. Define the interpolation operator $\tilde\Pi_h:\mathcal{C}^0(\bar\Omega)\mapsto\tilde{\mathcal{R}}_h$ by $\tilde\Pi_h\Phi|_K=\Phi(b_{T,K})$, $\forall K\in\mathcal{T}_h$, where $b_{T,K}$ is some node located in K whose projection on Γ_s is b_T. Then,
$\|(Id-\tilde\Pi_h)\tilde q_h\|_{0,\Omega}^2 = \sum\limits_{T\in\mathcal{C}_h}\int_{P_T}|q_h(\mathbf{x})-q_h(b_T)|^2\,d\mathbf{x}\,dz = \sum\limits_{T\in\mathcal{C}_h}\int_T D(\mathbf{x})|q_h(\mathbf{x})-q_h(b_T)|^2\,d\mathbf{x} = \|(Id-\Pi_h)q_h\|_{L^2_D(\omega)}^2.$
Using an inverse inequality between polynomial spaces (Cf. [3]) and the regularity of the grids,

$$
\begin{aligned}
\|\nabla_H \tilde{q}_h\|_{0,r,\Omega}^r &= \sum_{K\in\mathcal{T}_h} \|\nabla_H(\tilde{q}_h - \tilde{\Pi}_h\tilde{q}_h)\|_{0,r,K}^r \le c_I \sum_{K\in\mathcal{T}_h} h_K^{r(-1+\frac{d}{r}-\frac{d}{2})}\|\tilde{q}_h - \tilde{\Pi}_h\tilde{q}_h\|_{0,K}^r \\
&\le c_I\, h^{r(-1+\frac{d}{r}-\frac{d}{2})}\|(Id-\tilde{\Pi}_h)\tilde{q}_h\|_{0,\Omega}^r = c_I\, h^{r(-1+\frac{d}{r}-\frac{d}{2})}\|(Id-\Pi_h)q_h\|_{L^2_D(\omega)}^r.
\end{aligned}
$$

Then, from (9), (5) follows.

We next prove the stability of the discretization (3).

Theorem 1. *Assume that the family of grids $\{\mathcal{T}_h\}_{h>0}$ is uniformly regular. Then the discrete problem (3) admits a solution $(\mathbf{u}_h, p_h) \in \mathcal{U}_h \times \mathcal{P}_h$ which is bounded in $\mathbf{H}^1_b(\Omega) \times L^r_{D,0}(\omega)$, satisfying*

$$
|\mathbf{u}_h|_{1,\Omega} \le \frac{C}{\mu}\, l; \quad h^{\sigma/2}\|\Pi_h^* p_h\|_{L^2_D(\omega)} \le \frac{C}{\sqrt{\mu}}\, l; \quad \|p_h\|_{L^r_{D,0}(\omega)} \le C\,(\frac{l}{\mu^2} + \frac{1}{\mu} + 1); \tag{10}
$$

where C is a constant independent of h, $l = \|L\|_{-1,\Omega}$, σ is defined in (4) and $r = 2$ when $d = 2$ or $r = \frac{3}{2}$ when $d = 3$.

Proof: The existence of solutions of problem (3) follows from a standard compactness argument in finite dimension lying on the linearization of the convection term. The base of this proof is estimate (10), whose deduction we describe next. Assume that $(\mathbf{u}_h, p_h)$ is a solution of this problem. Set $\mathbf{v}_h = \mathbf{u}_h$, $q_h = p_h$ in (3) and denote $\vec{\mathbf{u}}_h = (\mathbf{u}_h, u_{hz})$. Then, as $\nabla\cdot\vec{\mathbf{u}}_h = 0$ and $\mathbf{u}_h|_{\Gamma_b} = 0$, $B(\mathbf{u}_h; (\mathbf{u}_h, p_h); (\mathbf{u}_h, p_h)) = \mu\,\|\nabla\mathbf{u}_h\|_{0,\Omega}^2$. Thus $\frac{\mu}{2}|\mathbf{u}_h|_{1,\Omega}^2 + h^\sigma\|\Pi_h^* p_h\|_{L^2_D(\omega)}^2 \le \frac{1}{2\mu}\|L\|_{-1,\Omega}^2$. This yields the two first estimates in (10). To estimate the pressure we use the inf-sup condition (5). Taking $q_h = 0$ in (3), using Sobolev injections and the two first estimates in (10),

$$
\begin{aligned}
(\nabla_H\cdot\mathbf{v}_h, p_h) &= (\vec{\mathbf{u}}_h\cdot\nabla\mathbf{u}_h, \mathbf{v}_h) + \mu(\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) + (\varphi\,\mathbf{u}_h^\perp, \mathbf{v}_h) - L(\mathbf{v}_h) \\
&\le C\,\left(|\mathbf{u}_h|_{1,\Omega}^2 + \mu|\mathbf{u}_h|_{1,\Omega} + \|\varphi\|_{0,\infty,\Omega}|\mathbf{u}_h|_{1,\Omega} + \|L\|_{-1,\Omega}\right)|\mathbf{v}_h|_{1,s,\Omega} \\
&\le C\,\left(l/\mu^2 + 1 + 1/\mu\right) l\,|\mathbf{v}_h|_{1,s,\Omega}
\end{aligned}
$$

As the second summand in (5) is estimated in (10), we obtain the pressure estimate in (10).

We finally prove the convergence of the discretization (3).

Theorem 2. *Assume that the family of grids $\{\mathcal{T}_h\}_{h>0}$ is uniformly regular. Then the sequence $\{(\mathbf{u}_h, p_h)\}_{h>0}$ of solutions of discrete problem (3) contains a subsequence which is weakly convergent in $\mathbf{H}^1_b(\Omega) \times L^r_{D,0}(\omega)$ (with r as in Theorem 1) to a solution of the continouos problem (2). If this solution is the strong solution, then the whole sequence strongly converges to it.*

Proof: By Theorem 1 the sequence $\{(\mathbf{u}_h, p_h)\}_{h>0}$ is bounded in $\mathbf{H}^1_b(\Omega)\times L^r_{D,0}(\omega)$, that is a reflexive space. Then, it contains a subsequence, that we still denote in the same way, weakly convergent in that space to a pair $(\mathbf{u}, p)$. Consider a pair of test functions $(\mathbf{v}, q) \in \mathbf{W}^{1,3}_b(\Omega) \times L^2_{D,0}(\omega)$. By the interpolation theory by finite elements (Cf. [3]) there exists a sequence $\{(\mathbf{v}_h, q_h)\}_{h>0}$ in $\mathcal{U}_h \times \mathcal{P}_h$ which is strongly convergent to a $(\mathbf{v}, q)$ in $\mathbf{W}^{1,3}_b(\Omega) \times L^2(\omega)$ and also in $\mathbf{W}^{1,3}_b(\Omega) \times L^2_{D,0}(\omega)$, as $\|q_h - q\|_{L^2_{D,0}(\omega)} \le \|q_h - q\|_{L^2_D(\omega)} \le \|D\|_{0,\infty,\omega}^{1/2}\|q_h - q\|_{0,\omega}$. Moreover, $\int_{-D(\mathbf{x})}^0 \mathbf{v}_h(\mathbf{x}, s)\,ds$ strongly converges to $\int_{-D(\mathbf{x})}^0 \mathbf{v}(\mathbf{x}, s)\,ds$ in $\mathbf{W}^{1,3}_b(\omega)$. All these convergence allow to pass to the limit in all terms of formulation (3), as in [7]. This proves that $\lim\limits_{h\to 0} B(\mathbf{u}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = B(\mathbf{u}; (\mathbf{u}, p), (\mathbf{v}, q))$. To analyze the convergence of stabilization term, let $q \in \mathcal{D}(\omega)$ we may suppose that q_h strongly converges to q in $L^\infty(\omega)$. Then, $\|\Pi_h^* q_h\|_{L^2_D(\omega)} \le C$. Using (10),

$$
|s_h(p_h, q_h)| = |h^\sigma\,(\Pi_h^* p_h, \Pi_h^* q_h)_{D,\omega}| \le C\, h^{\sigma/2}\,\|\Pi_h^* q_h\|_{L^2_D(\omega)} \le C\, h^{\sigma/2}.
$$

Thus, $\lim\limits_{h\to 0} s_h(p_h,q_h)=0$. We deduce that the limit $(\mathbf{u},p)$ is a solution of the continuos problem (2) but with test functions $(\mathbf{v},q)\in \mathbf{W}_b^{1,3}(\Omega)\times\mathcal{D}(\omega)$. As $\mathcal{D}(\omega)$ is dense in $L^2_{D,0}(\omega)$ this holds for all $q\in L^2_{D,0}(\omega)$. If the solution is strong, then it is unique by [5], and the whole sequence converges to it by a standard compactness argument. Furthermore, in this case $(\mathbf{u},p)\in H^2(\Omega)^{d-1}\times H^1(\omega)$, and then $(\mathbf{u},p)$ may be taken as test function in problem (2). Then $\lim\limits_{h\to 0}\|\nabla\mathbf{u}_h\|_{0,2,\Omega}=\|\nabla\mathbf{u}\|_{0,2,\Omega}$ and the convergence is strong. A standard argument using the inf-sup condition also proves that the pressures p_h strongly converge to p in $L^2_{D,0}(\omega)$.

We have assumed in our analysis that the grids are uniformly regular for brevity. This is not an essential hypothesis, that may be dropped if the discrete inf-sup condition (5) is changed into a more general condition for standard regular grids. This work shall appear in short.

5. Numerical tests

We have solved the 3D steady Primitive Equations (1) as the steady state of the evolution ones, by a semi-implicit Euler method: Set $\mathbf{u}_h^0=0$. For $n\geq 0$, given $\mathbf{u}_h^n\in\mathcal{U}_h$, find $(\mathbf{u}_h^{n+1},p_h^{n+1})\in\mathcal{U}_h\times\mathcal{P}_h$ such that $\forall(\mathbf{v}_h,q_h)\in\mathcal{U}_h\times\mathcal{P}_h$

$$\frac{1}{\Delta t}(\mathbf{u}_h^{n+1},\mathbf{v}_h)+B_h(\mathbf{u}_h^n;(\mathbf{u}_h^{n+1},p_h^{n+1}),(\mathbf{v}_h,q_h))=L(\mathbf{v}_h)+\frac{1}{\Delta t}(\mathbf{u}_h^n,\mathbf{v}_h).$$

This problem have been solved using the application FreeFem++ (Cf. [14]).

Test 1: Convergence rate. We have set $\Omega=(0,1)^3$, $\mu=0{,}5$ and the source terms $\mathbf{f}$ and τ such that $p=exp(x+y)-2{,}95$, $\mathbf{u}=((2z(z-1)+z^2)x^2(x-1)y(y-1),(2z(z-1)+z^2)x(x-1)y^2(y-1))$. Table 1 shows the estimated convergence orders for the horizontal velocity (in $H^1(\Omega)$ norm) and surface pressure (in $L^{\frac{3}{2}}(\omega)$ norm) using unstructured regular grids. We recover first order for pressure and a somewhat higher order for velocity that decreases as h tends to zero.

h	Horizontal velocity	Order	Pressure	Order
0.141	0.00450		0.02317	
0.070	0.00145	1.64	0.00965	1.26
0.047	0.00078	1.52	0.00609	1.13
0.035	0.00051	1.46	0.00447	1.07

Table 1: Estimated convergence orders.

Test 2: Upwelling flow. In this case we have considered a swimming-pool domain $\omega\times(-D(x),0)$ shown in Figure 1, where $\omega=(0,10000)\times(0,5000)$ and

$$D(x)=\begin{cases}50 & \text{if}\quad 0\leq x\leq 4000\\ 0{,}05x-150 & \text{if}\quad 4000\leq x\leq 5000\\ 100 & \text{if}\quad 5000\leq x\leq 10000\end{cases}$$

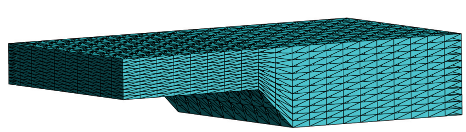

Figura 1: Domain and grid for Test 2

We have set the data $\mu_H=10^2$, $\mu_z=10^{-2}$ (m^2/s), $\mathbf{f}=0$, $\tau=\alpha\mathbf{v}|\mathbf{v}|$, with $\alpha=9{,}27\cdot 10^{-7}$ and $\mathbf{v}=(7{,}5,0)$ (m/s), and $\varphi=2\theta\sin 45^{\mathrm{O}}N$, with $\theta=7{,}3\cdot 10^{-5}$. In our results the velocity at the surface points $\pi/4$ degrees to the right of the wind, according to the Eckman theory (Figure 2, left). Also the pressure increases in the direction of the wind and to its right, due to Coriolis force (Figure 2, right). Figure 3, left shows a span-wise recirculation induced by the wind. Finally, in Figure 3, right we shows the upwelling and downwellings in a cross-wind plane induced by the interaction between wind and Coriolis forces. All these effects agree with the physics of the flow and with preceding numerical results (Cf. [8]).

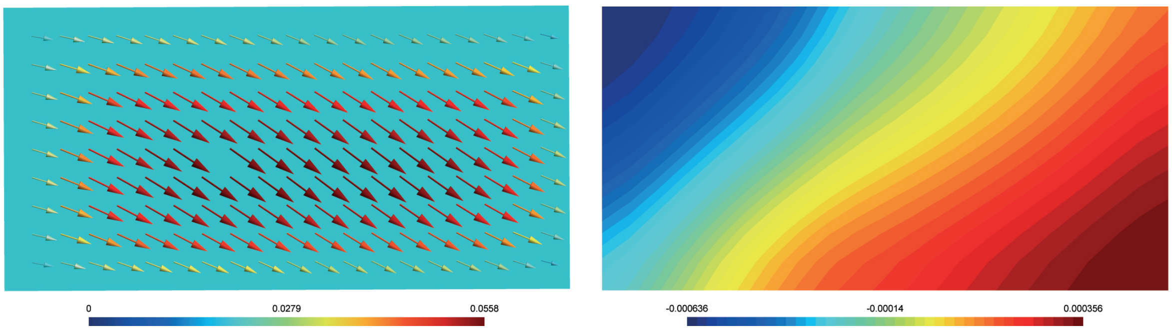

Figura 2: Surface horizontal velocity and pressure.

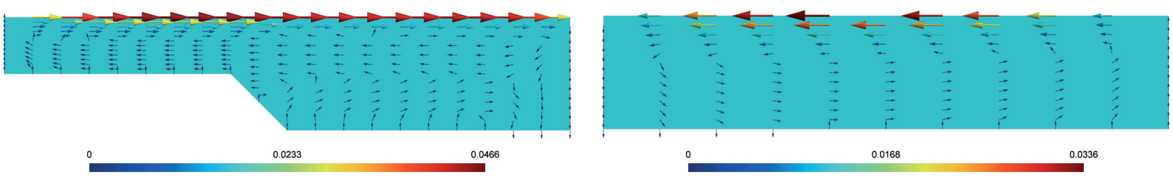

Figura 3: Velocity of the flow on planes $y = 2500$ (left) and $x = 6000$ (right).

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