

ON IRREGULAR BINOMIAL  $D$ -MODULES

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ABSTRACT. We prove that a holonomic binomial  $D$ -module  $M_A(I, \beta)$  is regular if and only if certain associated primes of  $I$  determined by the parameter vector  $\beta \in \mathbb{C}^d$  are homogeneous. We further describe the slopes of  $M_A(I, \beta)$  along a coordinate subspace in terms of the known slopes of some related hypergeometric  $D$ -modules that also depend on  $\beta$ . When the parameter  $\beta$  is generic, we also compute the dimension of the generic stalk of the irregularity of  $M_A(I, \beta)$  along a coordinate hyperplane and provide some remarks about the construction of its Gevrey solutions.

## 1. INTRODUCTION

Binomial  $D$ -modules have been introduced by A. Dickenstein, L.F. Matusevich and E. Miller in [DMM10]. These objects generalize both GKZ hypergeometric  $D$ -modules [GGZ87, GZK89] and (binomial) Horn systems, as treated in [DMM10] and [Sai02].

Here  $D$  stands for the complex Weyl algebra of order  $n$ , where  $n \geq 0$  is an integer. Elements in  $D$  are linear partial differential operators; such an operator  $P$  can be written as a finite sum

$$P = \sum_{\alpha, \gamma} p_{\alpha\gamma} x^\alpha \partial^\gamma$$

where  $p_{\alpha\gamma} \in \mathbb{C}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$  and  $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ ,  $\partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$ . The partial derivative  $\frac{\partial}{\partial x_i}$  is just denoted by  $\partial_i$ .

Our input is a pair  $(A, \beta)$  where  $\beta$  is a vector in  $\mathbb{C}^d$  and  $A = (a_{ij}) \in \mathbb{Z}^{d \times n}$  is a matrix whose columns  $a_1, \dots, a_n$  span the  $\mathbb{Z}$ -module  $\mathbb{Z}^d$ . We also assume that all  $a_i \neq 0$  and that the cone generated by the columns in  $\mathbb{R}^n$  contains no lines (one says in this case that this cone is *pointed*). The polynomial ring  $\mathbb{C}[\partial] := \mathbb{C}[\partial_1, \dots, \partial_n]$  is a subring of the Weyl algebra  $D$ . The matrix  $A$  induces a  $\mathbb{Z}^d$ -grading on  $\mathbb{C}[\partial]$  (also called the  $A$ -grading) by defining  $\deg(\partial_i) = -a_i$ .

A binomial in  $\mathbb{C}[\partial]$  is a polynomial with at most two monomial terms. An ideal  $I$  in  $\mathbb{C}[\partial]$  is said to be binomial if it is generated by binomials. We also say that the ideal  $I$  is an  $A$ -graded ideal if it is generated by  $A$ -homogenous elements (equivalently if for every polynomial in  $I$  all its  $A$ -graded components are also in  $I$ ).

The matrix  $A$  also induces a  $\mathbb{Z}^d$ -grading on the Weyl algebra  $D$  (also called the  $A$ -grading) by defining  $\deg(\partial_i) = -a_i$  and  $\deg(x_i) = a_i$ .

To the matrix  $A$  one associates the toric ideal  $I_A \subset \mathbb{C}[\partial]$  generated by the family of binomials  $\partial^u - \partial^v$  where  $u, v \in \mathbb{N}^n$  and  $Au = Av$ . The ideal  $I_A$  is a prime  $A$ -graded ideal.

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Recall that to the pair  $(A, \beta)$  one can associate the GKZ hypergeometric ideal

$$H_A(\beta) = DI_A + D(E_1 - \beta_1, \dots, E_d - \beta_d)$$

where  $E_i = \sum_{j=1}^n a_{ij} x_j \partial_j$  is the  $i^{\text{th}}$  Euler operator associated with  $A$ . The corresponding GKZ hypergeometric  $D$ -module is nothing but the quotient (left)  $D$ -module  $M_A(\beta) := \frac{D}{H_A(\beta)}$ , [GGZ87], [GZK89].

Following [DMM10], for any  $A$ -graded binomial ideal  $I \subset \mathbb{C}[\partial]$  we denote by  $H_A(I, \beta)$  the  $A$ -graded left ideal in  $D$  defined by

$$H_A(I, \beta) = DI + D(E_1 - \beta_1, \dots, E_d - \beta_d).$$

The binomial  $D$ -module associated with the triple  $(A, \beta, I)$  is, by definition, the quotient  $M_A(I, \beta) := \frac{D}{H_A(I, \beta)}$ . Notice that the ideal  $H_A(I, \beta)$  is nothing but the GKZ hypergeometric ideal  $H_A(\beta)$ .

In [DMM10] the authors have answered essential questions about binomial  $D$ -modules. The main treated questions are related to the holonomicity of the systems and to the dimension of their holomorphic solution space around a non singular point. In particular, in [DMM10, Theorem 6.3] they prove that the holonomicity of  $M_A(I, \beta)$  is equivalent to regular holonomicity when  $I$  is standard  $\mathbb{Z}$ -graded (i.e., the row-span of  $A$  contains the vector  $(1, \dots, 1)$ ). However, it turns out that the final sentence in [DMM10, Theorem 6.3], stating that the regular holonomicity of  $M_A(I, \beta)$  for a given parameter  $\beta$  implies standard homogeneity of the ideal  $I$ , is true for binomial Horn systems but it is not for general binomial  $D$ -modules. This is shown by Examples 3.10 and 3.11.

These two Examples are different in nature. More precisely, the system  $M_A(I, \beta)$  considered in Example 3.10 is regular holonomic for parameters  $\beta$  outside a certain line in the affine complex plane and irregular otherwise, while the system considered in Example 3.11 is regular holonomic for all parameters despite the fact that the binomial ideal  $I$  is not homogeneous with respect to the standard  $\mathbb{Z}$ -grading. This is a surprising phenomenon since it is not allowed neither for GKZ hypergeometric systems nor for binomial Horn systems.

We further provide, in Theorem 3.7, a characterization of the regular holonomicity of a system  $M_A(I, \beta)$  that improves the above mentioned result of [DMM10, Th. 6.3].

A central question in the study of the irregularity of a holonomic  $D$ -module  $M$  is the computation of its slopes along smooth hypersurfaces (see [Meb90] and [LM99]). On the other hand, the Gevrey solutions of  $M$  along smooth hypersurfaces are closely related with the irregularity and the slopes of  $M$ . More precisely, the classes of these Gevrey series solutions of  $M$  modulo convergent series define the 0-th cohomology group of the irregularity of  $M$  [Meb90, Définition 6.3.1].

In Section 4 we describe the  $L$ -characteristic variety and the slopes of  $M_A(I, \beta)$  along coordinate subspaces in terms of the same objects of the binomial  $D$ -modules associated with some of the *toral* primes of the ideal  $I$  determined by  $\beta$  (see Theorem 4.3). The binomial  $D$ -module associated with a toral prime is essentially a GKZ hypergeometric system and the  $L$ -characteristic variety and the slopes along coordinate subspaces of such a system are completely described in [SW08] in a combinatorial way (see also [CT03] and [Har03, Har04] for the cases  $d = 1$  and  $n = d + 1$ ).

Gevrey solutions of hypergeometric systems along coordinate subspaces are described in [Fer10] (see also [FC11], [FC08]). In Section 5 we compute the dimension of the generic stalk of the

irregularity of binomial  $D$ -modules when the parameter is generic (see Theorem 5.1). We finally give a procedure to compute Gevrey solutions of  $M_A(I, \beta)$  by using known results in the hypergeometric case ([GZK89], [SST00] and [Fer10]).

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## 2. PRELIMINARIES ON EULER–KOSZUL HOMOLOGY, BINOMIAL PRIMARY DECOMPOSITION AND TORAL AND ANDEAN MODULES

We review here some definitions, notations and results of [ES96], [MMW05], [DMM10] and [DMM<sub>2</sub>10] that will be used in the sequel.

We will denote  $R = \mathbb{C}[\partial]$ . Recall that the  $A$ -grading on the ring  $R$  is defined by  $\deg(\partial_j) = -a_j$  where  $a_j$  is the  $j^{\text{th}}$ -column of  $A$ . This  $A$ -grading on  $R$  can be extended to the ring  $D$  by setting  $\deg(x_j) = a_j$ .

**Definition 2.1.** [DMM10, Definition 2.4] Let  $V = \bigoplus_{\alpha \in \mathbb{Z}^d} V_\alpha$  be an  $A$ -graded  $R$ -module. The set of true degrees of  $V$  is

$$\text{tdeg}(V) = \{\alpha \in \mathbb{Z}^d : V_\alpha \neq 0\}$$

The set of quasidegrees of  $V$  is the Zariski closure in  $\mathbb{C}^d$  of  $\text{tdeg}(V)$ .

*Euler-Koszul complex  $\mathcal{K}_\bullet(E - \beta; V)$  associated with an  $A$ -graded  $R$ -module  $V$ .*

For any  $A$ -graded left  $D$ -module  $N = \bigoplus_{\alpha \in \mathbb{Z}^d} N_\alpha$  we denote  $\deg_i(y) = \alpha_i$  if  $y \in N_\alpha$ .

The map  $E_i - \beta_i : N_\alpha \rightarrow N_\alpha$  defined by  $(E_i - \beta_i)(y) = (E_i - \beta_i - \alpha_i)y$  can be extended (by  $\mathbb{C}$ -linearity) to a morphism of left  $D$ -modules  $E_i - \beta_i : N \rightarrow N$ . We denote by  $E - \beta$  the sequence of commuting endomorphisms  $E_1 - \beta_1, \dots, E_d - \beta_d$ . This allows us to consider the Koszul complex  $K_\bullet(E - \beta, N)$  which is concentrated in homological degrees  $d$  to  $0$ .

**Definition 2.2.** [MMW05, Definition 4.2] For any  $\beta \in \mathbb{C}^d$  and any  $A$ -graded  $R$ -module  $V$ , the Euler-Koszul complex  $\mathcal{K}_\bullet(E - \beta, V)$  is the Koszul complex  $K_\bullet(E - \beta, D \otimes_R V)$ . The  $i^{\text{th}}$  Euler-Koszul homology of  $V$ , denoted by  $\mathcal{H}_i(E - \beta, V)$ , is the homology  $H_i(\mathcal{K}_\bullet(E - \beta, V))$ .

**Remark 2.3.** Recall that we have the  $A$ -graded isomorphism  $\mathcal{H}_i(E - \beta, V)(\alpha) \simeq \mathcal{H}_i(E - \beta + \alpha, V)(\alpha)$  for all  $\alpha \in \mathbb{Z}^d$  [MMW05]. Here  $V(\alpha)$  is nothing but  $V$  with the shifted  $A$ -grading  $V(\alpha)_\gamma = V_{\alpha+\gamma}$  for all  $\gamma \in \mathbb{Z}^d$ .

*Binomial primary decomposition for binomial ideals.*

We recall from [ES96] that for any sublattice  $\Lambda \subset \mathbb{Z}^n$  and any partial character  $\rho : \Lambda \rightarrow \mathbb{C}^*$ , the corresponding associated binomial ideal is

$$I_\rho = \langle \partial^{u_+} - \rho(u) \partial^{u_-} \mid u = u_+ - u_- \in \Lambda \rangle$$

where  $u_+$  and  $u_-$  are in  $\mathbb{N}^n$  and they have disjoint supports. The ideal  $I_\rho$  is prime if and only if  $\Lambda$  is a saturated sublattice of  $\mathbb{Z}^n$  (i.e.  $\Lambda = \mathbb{Q}\Lambda \cap \mathbb{Z}^n$ ). We know from [ES96, Corollary 2.6] that any binomial prime ideal in  $R$  has the form  $I_{\rho, J} := I_\rho + \mathfrak{m}_J$  (where  $\mathfrak{m}_J = \langle \partial_j \mid j \notin J \rangle$ ) for some partial character  $\rho$  whose domain is a saturated sublattice of  $\mathbb{Z}^J$  and some  $J \subset \{1, \dots, n\}$ .

For any  $J \subset \{1, \dots, n\}$  we denote by  $\partial_J$  the monomial  $\prod_{j \in J} \partial_j$ .

**Theorem 2.4.** [DMM<sub>2</sub>10, Theorem 3.2] *Fix a binomial ideal  $I$  in  $R$ . Each associated binomial prime  $I_{\rho, J}$  has an explicitly defined monomial ideal  $U_{\rho, J}$  such that*

$$I = \bigcap_{I_{\rho, J} \in \text{Ass}(I)} \mathcal{C}_{\rho, J}$$

for  $\mathcal{C}_{\rho,J} = ((I + I_\rho) : \partial_J^\infty) + U_{\rho,J}$ , is a primary decomposition of  $I$  as an intersection of  $A$ -graded primary binomial ideals.

*Toral and Andean modules.*

In [DMM<sub>2</sub>10, Definition 4.3] a finitely generated  $A$ -graded  $R$ -module  $V = \bigoplus V_\alpha$  is said to be *toral* if its Hilbert function  $H_V$  (defined by  $H_V(\alpha) = \dim_{\mathbb{C}} V_\alpha$  for  $\alpha \in \mathbb{Z}^d$ ) is bounded above.

With the notations above, a  $R$ -module of type  $R/I_{\rho,J}$  is toral if and only if its Krull dimension equals the rank of the matrix  $A_J$  (see [DMM10, Lemma 3.4]). Here  $A_J$  is the submatrix of  $A$  whose columns are indexed by  $J$ . In this case the module  $R/\mathcal{C}_{\rho,J}$  is toral and we say that the ideal  $I_{\rho,J}$  is a toral prime and  $\mathcal{C}_{\rho,J}$  is a toral primary component.

If  $\dim(R/I_{\rho,J}) \neq \text{rank}(A_J)$  then the module  $R/\mathcal{C}_{\rho,J}$  is said to be *Andean*, the ideal  $I_{\rho,J}$  is an *Andean* prime and  $\mathcal{C}_{\rho,J}$  is an *Andean* primary component.

An  $A$ -graded  $R$ -module  $V$  is said to be *natively toral* if there exist a binomial toral prime ideal  $I_{\rho,J}$  and an element  $\alpha \in \mathbb{Z}^d$  such that  $V(\alpha)$  is isomorphic to  $R/I_{\rho,J}$  as  $A$ -graded modules (see [DMM10, Definition 4.1]).

**Proposition 2.5.** [DMM10, Proposition 4.2] *An  $A$ -graded  $R$ -module  $V$  is toral if and only if it has a filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$$

whose successive quotients  $V_k/V_{k-1}$  are all natively toral.

Such a filtration on  $V$  is called a *toral filtration*.

Following [DMM10, Definition 5.1] an  $A$ -graded  $R$ -module  $V$  is said to be *natively Andean* if there is an  $\alpha \in \mathbb{Z}^d$  and an Andean quotient ring  $R/I_{\rho,J}$  over which  $V(\alpha)$  is torsion-free of rank 1 and admits a  $\mathbb{Z}^J/\Lambda$ -grading that refines the  $A$ -grading via  $\mathbb{Z}^J/\Lambda \rightarrow \mathbb{Z}^d = \mathbb{Z}A$ , where  $\rho$  is defined on  $\Lambda \subset \mathbb{Z}^J$ . Moreover, if  $V$  has a finite filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$$

whose successive quotients  $V_k/V_{k-1}$  are all natively Andean, then  $V$  is Andean (see [DMM10, Section 5]).

In [DMM<sub>2</sub>10, Example 4.6] it is proven that the quotient  $R/\mathcal{C}_{\rho,J}$  is Andean for any Andean primary component  $\mathcal{C}_{\rho,J}$  of any  $A$ -graded binomial ideal.

We finish this section with the definition and a result about the so-called *Andean arrangement* associated with an  $A$ -graded binomial ideal  $I$  in  $R$ . Let us fix an irredundant primary decomposition

$$I = \bigcap_{I_{\rho,J} \in \text{Ass}(I)} \mathcal{C}_{\rho,J}$$

as in Theorem 2.4.

**Definition 2.6.** [DMM10, Definition 6.1] The Andean arrangement  $\mathcal{Z}_{\text{Andean}}(I)$  is the union of the quasidegree sets  $\text{qdeg}(R/\mathcal{C}_{\rho,J})$  for the Andean primary components  $\mathcal{C}_{\rho,J}$  of  $I$ .

From [DMM10, Lemma 6.2] the Andean arrangement  $\mathcal{Z}_{\text{Andean}}(I)$  is a union of finitely many integer translates of the subspaces  $\mathbb{C}A_J \subset \mathbb{C}^n$  for which there is an Andean associated prime  $I_{\rho,J}$ .

From [DMM10, Theorem 6.3] we have that the binomial  $D$ -module  $M_A(I, \beta)$  is holonomic if and only if  $-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$ .

3. CHARACTERIZING REGULAR HOLONOMIC BINOMIAL  $D$ -MODULES

Let  $I$  be an  $A$ -graded binomial ideal and fix a binomial primary decomposition  $I = \bigcap_{\rho, J} C_{\rho, J}$  where  $C_{\rho, J}$  is a  $I_{\rho, J}$ -primary binomial ideal.

Let us consider the ideal

$$I_\beta := \bigcap_{-\beta \in \text{qdeg}(R/C_{\rho, J})} C_{\rho, J}$$

i.e., the intersection of all the primary components  $C_{\rho, J}$  of  $I$  such that  $-\beta$  lies in the quasidegrees set of the module  $R/C_{\rho, J}$ .

**Remark 3.1.** Notice that if  $-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$  then  $R/I_\beta$  is contained in the toral direct sum

$$\bigoplus_{-\beta \in \text{qdeg}(R/C_{\rho, J})} R/C_{\rho, J}$$

and so it is a toral module.

The following result generalizes [DMM10, Proposition 6.4].

**Proposition 3.2.** *If  $-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$  then the natural surjection  $R/I \twoheadrightarrow R/I_\beta$  induces an isomorphism in Euler–Koszul homology*

$$\mathcal{H}_i(E - \beta, R/I) \simeq \mathcal{H}_i(E - \beta, R/I_\beta)$$

for all  $i$ . In particular,  $M_A(I, \beta) \simeq M_A(I_\beta, \beta)$ .

*Proof.* By [DMM10, Proposition 6.4] we have that

$$\mathcal{H}_i(E - \beta, R/I) \simeq \mathcal{H}_i(E - \beta, R/I_{\text{toral}})$$

for all  $i$ , where  $I_{\text{toral}}$  denotes the intersection of all the toral primary components of  $I$ . Thus, we can assume without loss of generality that all the primary components of  $I$  are toral. The rest of the proof is now analogous to the proof of [DMM10, Proposition 6.4] if we substitute the ideals  $I_{\text{toral}}$  and  $I_{\text{Andean}}$  there by the ideals  $I_\beta$  and  $\overline{I}_\beta$  respectively, where

$$\overline{I}_\beta = \bigcap_{-\beta \notin \text{qdeg}(R/C_{\rho, J})} C_{\rho, J},$$

and the Andean direct sum  $\bigoplus_{I_{\rho, J} \text{Andean}} R/C_{\rho, J}$  there by the toral direct sum

$$\bigoplus_{-\beta \notin \text{qdeg}(R/C_{\rho, J})} R/C_{\rho, J}$$

Finally, we can use Lemma 4.3 and Theorem 4.5 in [DMM10] in a similar way as [DMM10, Lemma 5.4] is used in the proof of Proposition 6.4 of [DMM10].  $\square$

The following Lemma gives a description of the quasidegrees set of a toral module of type  $R/C_{\rho, J}$ . E. Miller has pointed out that this result follows from Proposition 2.13 and Theorem 2.15 in [DMM210]. We will include here a slightly different proof of this Lemma.

**Lemma 3.3.** *For any  $I_{\rho, J}$ -primary toral ideal  $C_{\rho, J}$  the quasidegrees set of  $M = R/C_{\rho, J}$  equals the union of at most  $\mu_{\rho, J} \mathbb{Z}^d$ -graded translates of  $\mathbb{C}A_J$ , where  $\mu_{\rho, J}$  is the multiplicity of  $I_{\rho, J}$  in  $C_{\rho, J}$ . More precisely, for any toral filtration  $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M$  we have that*

the quasidegrees set of  $M$  is the union of the quasidegrees set of all the successive quotients  $M_i/M_{i-1}$  that are isomorphic to  $\mathbb{Z}^d$ -graded translates of  $R/I_{\rho,J}$ .

*Proof.* Since  $M$  is toral we have by [DMM10, Lemma 4.7] that  $\dim(\text{qdeg}(M)) = \dim M = \text{rank } A_J$ . Since  $C_{\rho,J}$  is primary, any zero-divisor of  $M$  is nilpotent. For all  $j \in J$  we have that  $\partial_j^m \notin C_{\rho,J} \subseteq I_{\rho} + \mathfrak{m}_J$  and so  $\partial_j$  is not a zero-divisor in  $M$  for all  $j \in J$ . Thus, the true degrees set of  $M$  verifies  $\text{tdeg}(M) = \text{tddeg}(M) - \mathbb{N}A_J$ . This and the fact that  $\dim(\text{qdeg}(M)) = \text{rank } A_J$  imply that there exists  $\alpha_1, \dots, \alpha_r \in \mathbb{Z}^d$  such that  $\text{tdeg}(M) = \cup_{i=1}^r (\alpha_i - \mathbb{N}A_J)$  and

$$(3.1) \quad \text{qdeg}(M) = \bigcup_{i=1}^r (\alpha_i + \mathbb{C}A_J)$$

Consider now a toral filtration  $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M$ . We know that there are exactly  $\mu_{\rho,J}$  different values of  $i$  such that  $M_i/M_{i-1} \simeq R/I_{\rho,J}(\gamma_i)$  for some  $\gamma_i \in \mathbb{Z}^d$ . For the other successive quotients  $M_l/M_{l-1} \simeq R/I_{\rho_l,J_l}(\gamma_l)$  we have that  $I_{\rho_l,J_l}$  is a toral prime which properly contains  $I_{\rho,J}$ . In particular, we have that  $\text{rank } A_{J_l} = \dim R/I_{\rho_l,J_l} < \dim R/I_{\rho,J} = \text{rank } A_J$ . Since  $\text{qdeg}(R/I_{\rho_l,J_l}) = \mathbb{C}A_{J_l}$  has dimension  $\text{rank } A_{J_l} < \text{rank } A_J$  and  $\text{qdeg}(M) = \bigcup_i \text{qdeg}(M_i/M_{i-1})$  we have by (3.1) that the quasidegrees set of any  $M_i/M_{i-1}$  is contained in the quasidegrees set of some  $M_j/M_{j-1} \simeq R/I_{\rho,J}(\gamma_j)$ . In particular  $r \leq \mu_{\rho,J}$  and each affine subspace  $(\alpha_i + \mathbb{C}A_J)$  in (3.1) is the quasidegrees set of some  $M_j/M_{j-1} \simeq R/I_{\rho,J}(\gamma_j)$ .  $\square$

**Remark 3.4.** Notice that  $H_A(I_{\rho,J}, \beta) = DH_{A_J}(I_{\rho}, \beta) + D(\partial_j : j \notin J)$ . In addition, if  $I_{\rho,J}$  is toral then the  $D_J$ -module  $M_{A_J}(I_{\rho}, \beta)$  is isomorphic to the hypergeometric system  $M_{A_J}(\beta)$  via an  $A$ -graded isomorphism of  $D_J$ -modules induced by rescaling the variables  $x_j$ ,  $j \in J$ , using the character  $\rho$ . Thus we can apply most of the well-known results for hypergeometric systems to  $M_A(I_{\rho,J}, \beta)$  (with  $I_{\rho,J}$  a toral prime) in an appropriated form.

**Lemma 3.5.** *If  $I_{\rho,J}$  is toral and  $-\beta \in \text{qdeg}(R/I_{\rho,J})$  the following conditions are equivalent:*

- i)  $\mathcal{H}_i(E - \beta, R/I_{\rho,J})$  is regular holonomic for all  $i$ .
- ii)  $\mathcal{H}_0(E - \beta, R/I_{\rho,J})$  is regular holonomic.
- iii)  $I_{\rho,J}$  is homogeneous (equivalently  $A_J$  is homogeneous).

*Proof.*  $i) \Rightarrow ii)$  is obvious,  $ii) \Rightarrow iii)$  follows straightforward from [SW08, Corollary 3.16] and  $iii) \Rightarrow i)$  is a particular case of the last statement in [DMM10, Theorem 4.5] and it also follows from [Hot98, Ch. II, 6.2, Thm.].  $\square$

**Remark 3.6.** Recall from [DMM10, Theorem 4.5] that for any toral module  $V$  we have that  $-\beta \notin \text{qdeg}V$  if and only if  $\mathcal{H}_0(E - \beta, V) = 0$  if and only if  $\mathcal{H}_i(E - \beta, V) = 0$  for all  $i$ . In particular, since the  $D$ -module  $0$  is regular holonomic it follows that conditions i) and ii) in Lemma 3.5 are also equivalent without the condition  $-\beta \in \text{qdeg}(R/I_{\rho,J})$ .

**Theorem 3.7.** *Let  $I \subseteq R$  be an  $A$ -graded binomial ideal such that  $M_A(I, \beta)$  is holonomic (equivalently,  $-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$ ). The following conditions are equivalent:*

- i)  $\mathcal{H}_i(E - \beta, R/I)$  is regular holonomic for all  $i$ .
- ii)  $M_A(I, \beta)$  is regular holonomic.
- iii) All the associated toral primes  $I_{\rho,J}$  of  $I$  such that  $-\beta \in \text{qdeg}(R/C_{\rho,J})$  are homogeneous.

*Proof.* The implication  $i) \Rightarrow ii)$  is obvious. Let us prove  $ii) \Rightarrow iii)$ . For any toral primary component  $C_{\rho,J}$  of  $I$  we have  $I \subseteq C_{\rho,J}$  and so there is a natural epimorphism  $M_A(I, \beta) \rightarrow M_A(C_{\rho,J}, \beta)$ . Since  $M_A(I, \beta)$  is regular holonomic then  $M_A(C_{\rho,J}, \beta)$  is also regular holonomic. Take a toral filtration of  $M = R/C_{\rho,J}$ ,  $0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ . We claim that

$$(3.2) \quad \mathcal{H}_j(E - \beta, M_i/M_{i-1}) \text{ and } \mathcal{H}_0(E - \beta, M_{i-1}) \text{ are regular holonomic}$$

for all  $i, j$ .

Let us prove (3.2) by decreasing induction on  $i$ . For  $i = r$ , we have a surjection from the regular holonomic  $D$ -module  $\mathcal{H}_0(E - \beta, M_r) = M_A(C_{\rho,J}, \beta)$  to  $\mathcal{H}_0(E - \beta, M_r/M_{r-1})$  and so it is regular holonomic too. By Remark 2.3, Lemma 3.5 and Remark 3.6 we have that the  $D$ -module  $\mathcal{H}_j(E - \beta, M_r/M_{r-1})$  is regular holonomic for all  $j$ . Since

$$\mathcal{H}_1(E - \beta, M_r/M_{r-1}) \rightarrow \mathcal{H}_0(E - \beta, M_{r-1}) \rightarrow \mathcal{H}_0(E - \beta, M_r)$$

is exact we have that  $\mathcal{H}_0(E - \beta, M_{r-1})$  is regular holonomic.

Assume that (3.2) holds for some  $i = k + 1 \leq r$  and for all  $j$ . We consider the exact sequence

$$0 \rightarrow M_{k-1} \rightarrow M_k \rightarrow M_k/M_{k-1} \rightarrow 0$$

and the following part of the long exact sequence of Euler-Koszul homology

$$(3.3) \quad \cdots \mathcal{H}_1(E - \beta, M_k/M_{k-1}) \rightarrow \mathcal{H}_0(E - \beta, M_{k-1}) \rightarrow \mathcal{H}_0(E - \beta, M_k) \rightarrow \mathcal{H}_0(E - \beta, M_k/M_{k-1}).$$

By induction hypothesis  $\mathcal{H}_0(E - \beta, M_k)$  is regular holonomic. This implies that  $\mathcal{H}_0(E - \beta, M_k/M_{k-1})$  is regular holonomic by (3.3). Applying Remark 2.3, Lemma 3.5 and Remark 3.6 we have that  $\mathcal{H}_j(E - \beta, M_k/M_{k-1})$  is regular holonomic for all  $j$ . Thus, by (3.3) we have that  $\mathcal{H}_0(E - \beta, M_{k-1})$  is regular holonomic too and we have finished the induction proof of (3.2).

Assume that  $-\beta \in \text{qdeg}(R/C_{\rho,J})$ . By Lemma 3.3 there exists  $i$  such that  $-\beta$  lies in the quasidegrees set of  $M_i/M_{i-1} \simeq R/I_{\rho,J}(\gamma_i)$  and we also have by (3.2) that

$$\mathcal{H}_0(E - \beta, M_i/M_{i-1}) \simeq \mathcal{H}_0(E - \beta + \gamma_i, R/I_{\rho,J})(\gamma_i)$$

is a nonzero regular holonomic  $D$ -module. Thus, by Lemma 3.5 we have that  $I_{\rho,J}$  is homogeneous.

Let us prove  $iii) \Rightarrow i)$ . By Proposition 3.2 we just need to prove that  $M_A(I_\beta, \beta)$  is regular holonomic. We have that all the associated primes of  $I_\beta$  are toral and homogeneous. In particular  $M = R/I_\beta$  is a toral module and for any toral filtration of  $M$  the successive quotients  $M_i/M_{i-1}$  are isomorphic to some  $\mathbb{Z}^d$ -graded translate of a quotient  $R/I_{\rho_i, J_i}$  where  $I_{\rho_i, J_i}$  is toral and contains a minimal prime  $I_{\rho, J}$  of  $I_\beta$ . Such minimal prime is homogeneous by assumption and so  $A_J$  is homogeneous. Since  $J_i \subseteq J$  we have that  $A_{J_i}$  and  $I_{\rho_i, J_i}$  are homogeneous too. Now, we just point out that the proof of the last statement in [DMM10, Theorem 4.5] still holds for  $V = M$  if we don't require  $A$  to be homogenous but all the primes occurring in a toral filtration of  $M$  to be homogeneous. □

**Remark 3.8.** Theorem 3.7 shows in particular that the property of a binomial  $D$ -module  $M_A(I, \beta)$  of being regular (holonomic) can fail to be constant when  $-\beta$  runs outside the Andean arrangement. This phenomenon is forbidden to binomial Horn systems  $M_A(I(B), \beta)$  (see [DMM10,

Definition 1.5]) since the inclusion  $I(B) \subseteq I_A$  induces a surjective morphism

$$\mathcal{H}_0(E - \beta, I(B)) \twoheadrightarrow M_A(\beta)$$

and then regular holonomicity of  $\mathcal{H}_0(E - \beta, R/I(B))$  implies regular holonomicity of  $M_A(\beta)$ , which is equivalent to the standard homogeneity of  $I_A$  by [Hot98, SST00, SW08].

**Definition 3.9.** The non-regular arrangement of  $I$  (denoted by  $\mathcal{Z}_{\text{non-regular}}(I)$ ) is the union of the Andean arrangement of  $I$  and the union of quasidegrees sets of the quotients of  $R$  by primary components  $C_{\rho,J}$  of  $I$  such that  $I_{\rho,J}$  is not homogeneous with respect to the standard grading. So, we have

$$\mathcal{Z}_{\text{non-regular}}(I) = \mathcal{Z}_{\text{Andean}}(I) \cup \left( \bigcup_{I_{\rho,J} \text{ non homogeneous}} \text{qdeg}(R/C_{\rho,J}) \right).$$

**Example 3.10.** Consider the ideal  $I = \langle \partial_1^2 \partial_2 - \partial_2^2, \partial_2 \partial_3, \partial_2 \partial_4, \partial_1^2 \partial_3 - \partial_3^2 \partial_4, \partial_1^2 \partial_4 - \partial_3 \partial_4^2 \rangle$ . It is  $A$ -graded for the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

but  $I$  is not standard  $\mathbb{Z}$ -graded. We have the prime decomposition  $I = I_1 \cap I_2 \cap I_3$  where  $I_1 = \langle \partial_2, \partial_3, \partial_4 \rangle$ ,  $I_2 = \langle \partial_1^2 - \partial_2, \partial_3, \partial_4 \rangle$  and  $I_3 = \langle \partial_2, \partial_1^2 - \partial_3 \partial_4 \rangle$  are toral primes of  $I$ . In particular  $\mathcal{Z}_{\text{Andean}}(I) = \emptyset$  and by the proof of [DMM10, Proposition 6.6] we have that  $\mathcal{Z}_{\text{primary}}(I) = \{0\}$  (see [DMM10, Definition 6.5] for the definition of the primary arrangement  $\mathcal{Z}_{\text{primary}}(I)$ ).

Using [DMM10, Theorem 6.8] we have that  $M_A(I, \beta)$  is isomorphic to the direct sum of  $M_A(I_j, \beta)$  for  $j = 1, 2, 3$  if  $\beta \neq 0$ . Moreover,  $\text{qdeg}(R/I_j) = \mathbb{C} \binom{1}{1}$  for  $j = 1, 2$  and  $\text{qdeg}(R/I_3) = \mathbb{C}^2$ . Thus, for generic parameters (more precisely for  $\beta \in \mathbb{C}^2 \setminus \mathbb{C} \binom{1}{1}$ ) we have that  $M_A(I, \beta)$  is isomorphic to  $M_A(I_3, \beta)$  that is a regular holonomic by Lemma 3.5.

On the other hand, there is a surjective morphism from  $M_A(I, \beta)$  to  $M_A(I_2, \beta)$  and if  $\beta \in \mathbb{C} \binom{1}{1}$  we have that  $M_A(I_2, \beta)$  is an irregular  $D$ -module because  $s = 2$  is a slope along  $x_2 = 0$ . Thus we conclude that  $M_A(I, \beta)$  is regular holonomic if  $\beta \in \mathbb{C}^2 \setminus \mathbb{C} \binom{1}{1}$  and it is an irregular holonomic  $D$ -module when  $\beta \in \mathbb{C} \binom{1}{1}$ . In particular,  $\mathcal{Z}_{\text{non-regular}}(I) = \mathbb{C} \binom{1}{1} \subset \mathbb{C}^2$ . It can also be checked that the singular locus of  $M_A(I, \beta)$  is  $\{x_1 x_2 x_3 x_4 (x_1^2 - 4x_3 x_4) = 0\}$  when  $\beta \in \mathbb{C} \binom{1}{1}$  and  $\{x_3 x_4 (x_1^2 - 4x_3 x_4) = 0\}$  otherwise.

**Example 3.11.** The primary binomial ideal  $I = \langle \partial_1 - \partial_2, \partial_3^4, \partial_4^3, \partial_3^3 - \partial_4^2 \rangle$  is  $A$ -graded with respect to the matrix  $A = (1 \ 1 \ 2 \ 3)$ . Note that  $I$  is not homogeneous with respect to the standard  $\mathbb{Z}$ -grading. However, its radical ideal  $\sqrt{I} = \langle \partial_1 - \partial_2, \partial_3, \partial_4 \rangle$  is homogeneous. Thus, by Theorem 3.7 we have that  $M_A(I, \beta)$  is regular holonomic.

#### 4. $L$ -CHARACTERISTIC VARIETY AND SLOPES OF BINOMIAL $D$ -MODULES

Let  $L$  be the filtration on  $D$  defined by a weight vector  $(u, v) \in \mathbb{R}^{2n}$  with  $u_i + v_i = c > 0$  for some constant  $c > 0$ .

This includes in particular the intermediate filtrations  $pF + qV$  between the filtration  $F$  by the order of the linear differential operators and the Kashiwara-Malgrange filtration  $V$  along



a coordinate subspace. The filtrations  $pF + qV$  are the ones considered when studying the algebraic slopes of a coherent  $D$ -module along a coordinate subspace [LM99].

We will consider the  $L$ -characteristic variety  $\text{Ch}^L(N)$  of a finitely generated  $D$ -module  $N$  on  $\mathbb{C}^n$  defined as the support of  $\text{gr}^L N$  in  $T^*\mathbb{C}^n$  (see e.g. [Lau87], [SW08, Definition 3.1]). We recall that in fact for  $L = pF + qV$  this is a global algebraic version of Laurent's microcharacteristic variety of type  $s = p/q + 1$  [Lau87, §3.2] (see also [SW08, Remark 3.3]).

The  $L$ -characteristic variety and the slopes of a hypergeometric  $D$ -module  $M_A(\beta)$  are controlled by the so-called  $(A, L)$ -umbrella [SW08]. Let us recall its definition in the special case when  $v_i > 0$  for all  $i$ . We denote by  $\Delta_A^L$  the convex hull of  $\{0, a_1^L, \dots, a_n^L\}$  where  $a_j^L = \frac{1}{v_j} a_j$ .

The  $(A, L)$ -umbrella is the set  $\Phi_A^L$  of faces of  $\Delta_A^L$  which do not contain 0. The empty face is in  $\Phi_A^L$ . One identifies  $\tau \in \Phi_A^L$  with  $\{j | a_j^L \in \tau\}$ , or with  $\{a_j | a_j^L \in \tau\}$ , or with the corresponding submatrix  $A_\tau$  of  $A$ .

By [SW08, Corollary 4.17] the  $L$ -characteristic variety of a hypergeometric  $D$ -module  $M_A(\beta)$  is

$$(4.1) \quad \text{Ch}^L(M_A(\beta)) = \bigcup_{\tau \in \Phi_A^L} \overline{C_A^\tau}$$

where  $\overline{C_A^\tau}$  is the Zariski closure in  $T^*\mathbb{C}^n$  of the conormal space to the orbit  $O_A^\tau \subset T_0^*\mathbb{C}^n = \mathbb{C}^n$  corresponding to the face  $\tau$ . In particular  $\text{Ch}^L(M_A(\beta))$  is independent of  $\beta$ . By definition we have the equality  $O_A^\tau := (\mathbb{C}^*)^d \cdot \mathbf{1}_A^\tau$  where  $\mathbf{1}_A^\tau \in \mathbb{N}^n$  is defined by  $(\mathbf{1}_A^\tau)_j = 1$  if  $j \in \tau$  and  $(\mathbf{1}_A^\tau)_j = 0$  otherwise. The action of the torus is given with respect to the matrix  $A$ . If the filtration given by  $L$  equals the  $F$ -filtration (i.e. the order filtration) then this description of the  $F$ -characteristic variety coincides with a result of [Ado94, Lemmas 3.1 and 3.2].

**Proposition 4.1.** *If  $M$  is a  $I_{\rho, J}$ -coprimary toral module and  $-\beta \in \text{qdeg}(M)$  then the  $L$ -characteristic variety of  $\mathcal{H}_0(E - \beta, M)$  is the  $L$ -characteristic variety of  $M_A(I_{\rho, J}, 0)$ . In particular, the set of slopes of  $\mathcal{H}_0(E - \beta, M)$  along a coordinate subspace in  $\mathbb{C}^n$  coincide with the ones of  $M_A(I_{\rho, J}, 0)$ .*

*Proof.* Since  $M$  is  $I_{\rho, J}$ -coprimary there exists  $m \geq 0$  such that  $I_{\rho, J}^m$  annihilates  $M$ . Consider a set of  $A$ -homogeneous elements  $m_1, \dots, m_k \in M$  generating  $M$  as  $R$ -module. This leads to a natural  $A$ -graded surjection  $\bigoplus_{i=1}^k R/I_{\rho, J}^m(-\text{deg}(m_i)) \twoheadrightarrow M$ . In particular, there is a surjective morphism of  $D$ -modules

$$\bigoplus_{i=1}^k \mathcal{H}_0(E - \beta, R/I_{\rho, J}^m(-\text{deg}(m_i))) \twoheadrightarrow \mathcal{H}_0(E - \beta, M)$$

inducing the inclusion:

$$\text{Ch}^L(\mathcal{H}_0(E - \beta, M)) \subseteq \mathcal{V}(\text{in}_L(I_{\rho, J}^m), Ax\xi) = \mathcal{V}(\text{in}_L(I_\rho), A_J x_J \xi_J, \xi_j : j \notin J).$$

Here  $(x, \xi)$  stands for the coordinates in the cotangent space  $T^*\mathbb{C}^n$ ,  $x\xi = (x_1 \xi_1, \dots, x_n \xi_n)$  and  $\mathcal{V}$  is the zero set in  $T^*\mathbb{C}^n$  of the corresponding ideal.

The equality  $\text{Ch}^L(M_A(I_{\rho, J}, 0)) = \mathcal{V}(\text{in}_L(I_\rho), A_J x_J \xi_J, \xi_j : j \notin J)$  follows from [SW08, (3.2.2) and Corollary 4.17]. Thus,

$$(4.2) \quad \text{Ch}^L(\mathcal{H}_0(E - \beta, M)) \subseteq \text{Ch}^L(M_A(I_{\rho, J}, 0))$$

Let us now prove the equality

$$(4.3) \quad \text{Ch}^L(\mathcal{H}_0(E - \beta, M)) = \text{Ch}^L(M_A(I_{\rho, J}, 0))$$

by induction on the length  $r$  of a toral filtration  $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$  of  $M$ .

If  $r = 1$  we have that  $M \simeq R/I_{\rho, J}(\gamma)$  for some  $\gamma \in \mathbb{Z}^d$  and  $-\beta \in \text{qdeg}(M)$  means that  $-\beta + \gamma \in \text{qdeg}(R/I_{\rho, J}) = \mathbb{C}A_J$ . Thus,  $\mathcal{H}_0(E - \beta, M) \simeq M_A(I_{\rho, J}, \beta - \gamma)$  and we have (4.3) because the  $L$ -characteristic variety of  $M_A(I_{\rho, J}, \beta')$  is independent of  $\beta' \in -\text{qdeg}(R/I_{\rho, J})$  by the results in [SW08].

Assume by induction that we have (4.3) for all toral  $I_{\rho, J}$ -coprimary modules  $M$  with a toral filtration of length  $r$  such that  $-\beta \in \text{qdeg}(M)$ .

Let  $M$  be a  $I_{\rho, J}$ -coprimary toral module with toral filtration of length  $r + 1$ , i.e.  $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{r+1} = M$ . From the exact sequence

$$0 \longrightarrow M_r \longrightarrow M \longrightarrow M/M_r \longrightarrow 0$$

we obtain the long exact sequence of Euler–Koszul homology

$$\cdots \longrightarrow \mathcal{H}_1(E - \beta, M/M_r) \longrightarrow \mathcal{H}_0(E - \beta, M_r) \longrightarrow \mathcal{H}_0(E - \beta, M) \longrightarrow \mathcal{H}_0(E - \beta, M/M_r) \longrightarrow 0.$$

Now, we need to distinguish two cases.

Assume first that  $-\beta \notin \text{qdeg}(M/M_r)$ . Thus,  $\mathcal{H}_j(E - \beta, M/M_r) = 0$  for all  $j$  by [DMM10, Theorem 4.5] and we have that  $\mathcal{H}_0(E - \beta, M_r) \simeq \mathcal{H}_0(E - \beta, M)$  so they both have the same  $L$ -characteristic variety. Notice that the fact that  $-\beta \in \text{qdeg}M \setminus \text{qdeg}(M/M_r)$  along with Lemma 3.3 guarantees that there exists some  $i \leq r$  such that  $M_i/M_{i-1} \simeq R/I_{\rho, J}(\gamma_i)$ . This implies that  $M_r$  is also  $I_{\rho, J}$ -coprimary and we can apply the induction hypothesis.

Assume now that  $-\beta \in \text{qdeg}(M/M_r)$ . In this case we still have that the  $L$ -characteristic variety of  $\mathcal{H}_0(E - \beta, M/M_r)$  is contained in the  $L$ -characteristic variety of  $\mathcal{H}_0(E - \beta, M)$ . If  $M/M_r \simeq R/I_{\rho, J}(\gamma)$  we have that  $\text{Ch}^L(M_A(I_{\rho, J}, 0)) \subseteq \text{Ch}^L(\mathcal{H}_0(E - \beta, M))$  and using (4.2) we get (4.3).

We are left with the case when  $-\beta \in \text{qdeg}(M/M_r)$  and  $M/M_r \simeq R/I_{\rho', J'}(\gamma)$  with  $I_{\rho, J} \subsetneq I_{\rho', J'}$ . This implies that  $M_r$  is also  $I_{\rho, J}$ -coprimary. Moreover, it is clear that  $-\beta \in \text{qdeg}(M_r)$  by using Lemma 3.3. Thus, we have by induction hypothesis that the  $L$ -characteristic variety of  $\mathcal{H}_0(E - \beta, M_r)$  is the  $L$ -characteristic variety of  $M_A(I_{\rho, J}, 0)$ .

Assume to the contrary that there exists an irreducible component  $C$  of the  $L$ -characteristic variety of  $M_A(I_{\rho, J}, 0)$  that is not contained in the  $L$ -characteristic variety of  $\mathcal{H}_0(E - \beta, M)$ . This implies that  $C$  is not contained in  $\text{Ch}^L(\mathcal{H}_0(E - \beta, M/M_r))$ , i.e. the multiplicity  $\mu_{A,0}^{L,C}(M/M_r, \beta)$  of  $C$  in the  $L$ -characteristic cycle of  $\mathcal{H}_0(E - \beta, M/M_r)$  is zero (see [SW08, Definition 4.7]). As a consequence, the multiplicity  $\mu_{A,i}^{L,C}(M/M_r, \beta)$  of  $C$  in the  $L$ -characteristic cycle of  $\mathcal{H}_i(E - \beta, M/M_r)$  is zero for all  $i \geq 0$  because we can use an adapted version of [SW08, Theorems 4.11 and 4.16] since  $M/M_r$  is a module of the form  $R/(I_{A, J'} + \mathfrak{m}_{J'}) (\gamma)$  after rescaling the variables via  $\rho$ . Now, using the long exact sequence of Euler–Koszul homology and the additivity of the  $L$ -characteristic cycle we conclude that  $\mu_{A,i}^{L,C}(M, \beta) = \mu_{A,i}^{L,C}(M_r, \beta)$  for all  $i \geq 0$ . In particular we have that  $\mu_{A,0}^{L,C}(M, \beta) > 0$  and thus  $C$  is contained in the  $L$ -characteristic variety of  $\mathcal{H}_0(E - \beta, M)$ . We conclude that the  $L$ -characteristic variety of  $M_A(I_{\rho, J}, 0)$  is contained in the  $L$ -characteristic variety of  $\mathcal{H}_0(E - \beta, M)$  and this finishes the induction proof.  $\square$

The following result is well known. We include a proof for the sake of completeness.

**Lemma 4.2.** *Let  $I_1, \dots, I_r$  be a sequence of ideals in  $R$  and  $\omega \in \mathbb{R}^n$  a weight vector. Then*

$$(4.4) \quad \bigcap_{j=1}^r \sqrt{\text{in}_\omega(I_j)} = \sqrt{\text{in}_\omega(\bigcap_j I_j)}$$

*Proof.* The inclusion  $\text{in}_\omega(\bigcap_j I_j) \subseteq \bigcap_{j=1}^r \text{in}_\omega(I_j)$  is obvious and then we can take radicals.

Let us see that  $\bigcap_{j=1}^r \text{in}_\omega(I_j) \subseteq \sqrt{\text{in}_\omega(\bigcap_j I_j)}$ . Let us consider an  $\omega$ -homogeneous element  $f \in \bigcap_{j=1}^r \text{in}_\omega(I_j)$ ; then for all  $j = 1, \dots, r$  there exists  $g_j \in I_j$  such that  $f = \text{in}_\omega(g_j)$ . Thus we have  $\prod_j g_j \in \bigcap_j I_j$  and  $f^r = \prod_j \text{in}_\omega(g_j) = \text{in}_\omega(\prod_j g_j) \in \text{in}_\omega(\bigcap_j I_j)$ . In particular,  $f \in \sqrt{\text{in}_\omega(\bigcap_j I_j)}$ . This finishes the proof as the involved ideals are  $\omega$ -homogeneous.  $\square$

The following result is a direct consequence of [DMM10, Theorem 6.8] and Proposition 4.1 when  $-\beta \notin \mathcal{Z}_{\text{primary}}(I)$ . However, we want to prove it when  $-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$  that is a weaker condition.

**Theorem 4.3.** *Let  $I$  be a  $A$ -graded binomial ideal and consider a binomial primary decomposition  $I = \bigcap_{\rho, J} C_{\rho, J}$ . If  $M_A(I, \beta)$  is holonomic (equivalently,  $-\beta$  lies outside the Andean arrangement) then the  $L$ -characteristic variety of  $M_A(I, \beta)$  coincide with the union of the  $L$ -characteristic varieties of  $M_A(I_{\rho, J}, 0)$  for all associated toral primes  $I_{\rho, J}$  of  $I$  such that  $-\beta \in \text{qdeg}(R/C_{\rho, J})$ . In particular, the slopes of  $M_A(I, \beta)$  along a coordinate subspace in  $\mathbb{C}^n$  coincide with the union of the set of slopes of  $M_A(I_{\rho, J}, 0)$  along the same coordinate subspace for  $I_{\rho, J}$  varying between all the associated toral primes of  $I$  such that  $-\beta \in \text{qdeg}(R/C_{\rho, J})$ .*

*Proof.* By Proposition 3.2, we have that  $M_A(I, \beta)$  is isomorphic to  $M_A(I_\beta, \beta)$ . We also have that

$$(4.5) \quad \bigcup_{-\beta \in \text{qdeg}(R/C_{\rho, J})} \text{Ch}^L(M_A(C_{\rho, J}, \beta)) \subseteq \text{Ch}^L(M_A(I_\beta, \beta)) \subseteq \mathcal{V}(\text{in}_L(I_\beta), Ax\xi)$$

On the other hand, by Lemma 4.2 we have that  $\mathcal{V}(\text{in}_L(I_\beta)) = \cup \mathcal{V}(\text{in}_L(C_{\rho, J})) = \mathcal{V}(\text{in}_L(I_{\rho, J}))$ . Here  $\mathcal{V}$  is the zero set of the corresponding ideal. The result in the statement follows from the last inclusion, the inclusions (4.5) and Proposition 4.1.  $\square$

**Remark 4.4.** Notice that Theorem 4.3 implies that the map from  $\mathbb{C}^d \setminus \mathcal{Z}_{\text{Andean}}(I)$  to *Sets* sending  $\beta$  to the set of slopes of  $M_A(I, \beta)$  along any fixed coordinate subspace is upper-semi-continuous in  $\beta$ . The Examples 4.5 and 4.6 illustrate Theorem 4.3. [ES96, Theorem 4.1] has been very useful in order to construct binomial  $A$ -graded ideals  $I$  starting from some toral primes that we wanted to be associated primes of  $I$ .

**Example 4.5.** The binomial ideal

$$I = \langle \partial_1 \partial_3, \partial_1 \partial_4, \partial_2 \partial_3, \partial_2 \partial_4, \partial_3 \partial_4, \partial_1^4 \partial_2^3 - \partial_1 \partial_5, \partial_1^3 \partial_2^4 - \partial_2 \partial_5, \partial_3^4 - \partial_3 \partial_5, \partial_4^4 - \partial_4 \partial_5^2 \rangle$$

is  $A$ -graded for the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$$

Its primary components are the toral primes  $I_i := I_{\rho_i, J_i}$ ,  $i = 1, 2, 3, 4$ , where  $J_1 = \{3, 5\}$ ,  $J_2 = \{4, 5\}$ ,  $J_3 = \{1, 2, 5\}$ ,  $J_4 = \{5\}$  and  $\rho_i : \ker_{\mathbb{Z}} A_{J_i} \rightarrow \mathbb{C}^*$  is the trivial character for  $i = 1, 2, 3, 4$ . Notice that  $\text{qdeg}(R/I_i) = \mathbb{C}A_{J_i} = \mathbb{C}\binom{1}{1}$  for  $i = 1, 2, 4$  and  $\text{qdeg}R/I_3 = \mathbb{C}^2$ . Using Theorem 4.3, Remark 3.4 and the results in [SW08] we have the following:

If  $\beta \in \mathbb{C}^2 \setminus \mathbb{C}\binom{1}{1}$  then  $M_A(I, \beta) \simeq M_A(I_3, \beta)$  has a unique slope  $s = 6$  along the hyperplane  $\{x_5 = 0\}$  and it is regular along the other coordinate hyperplanes.

If  $\beta \in \mathbb{C}\binom{1}{1}$  then  $M_A(I, \beta)$  has the slopes  $s_1 = 3/2$ ,  $s_2 = 3$  and  $s_3 = 6$  along  $\{x_5 = 0\}$ .

**Example 4.6.** The binomial ideal  $I = \langle \partial_4 \partial_5, \partial_3 \partial_5, \partial_2 \partial_5, \partial_1 \partial_5, \partial_3 \partial_4 \partial_6, \partial_2 \partial_4 \partial_6, \partial_1 \partial_3 \partial_4, \partial_1 \partial_2 \partial_4, \partial_5^3 - \partial_6 \partial_5, \partial_2 \partial_3^2 \partial_4 - \partial_3 \partial_4^2, \partial_1^2 \partial_4^3 - \partial_4 \partial_6, \partial_1^2 \partial_2^2 \partial_3^3 - \partial_3 \partial_6, \partial_1^2 \partial_2^3 \partial_3^2 - \partial_2 \partial_6 \rangle$  is  $A$ -graded for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Using Macaulay2 we get a primary decomposition of  $I$  where the primary components are the toral primes  $I_i = I_{\rho_i, J_i}$ ,  $i = 1, \dots, 6$ , where  $J_1 = \{1, 2, 3, 6\}$ ,  $J_2 = \{2, 3, 4\}$ ,  $J_3 = \{2, 4\}$ ,  $J_4 = \{5, 6\}$ ,  $J_5 = \{1, 4, 6\}$ ,  $J_6 = \{1, 6\}$  and  $\rho_i : \ker_{\mathbb{Z}} A_{J_i} \rightarrow \mathbb{C}^*$  is the trivial character for  $i = 1, \dots, 6$ .

We have  $I_1 = \langle \partial_1^2 \partial_2^2 \partial_3^2 - \partial_6, \partial_4, \partial_5 \rangle$ ,  $I_2 = \langle \partial_2 \partial_3 - \partial_4, \partial_1, \partial_5, \partial_6 \rangle$ ,  $I_3 = \langle \partial_1, \partial_3, \partial_5, \partial_6 \rangle$ ,  $I_4 = \langle \partial_5^2 - \partial_6, \partial_1, \partial_2, \partial_3, \partial_4 \rangle$ ,  $I_5 = \langle \partial_1^2 \partial_4^2 - \partial_6, \partial_2, \partial_3, \partial_5 \rangle$  and  $I_6 = \langle \partial_2, \partial_3, \partial_4, \partial_5 \rangle$ .

We know that  $\text{qdeg} R/I_i = \mathbb{C}A_{J_i}$  for  $i = 1, \dots, 6$ . In particular,  $R/I_1$  is the unique component with Krull dimension  $d = 3$ .

There are four components with Krull dimension  $d - 1 = 2$ , namely  $R/I_2, R/I_3$  have quasidegrees set  $\mathbb{C}A_{J_2} = \mathbb{C}A_{J_3} = \{y_1 = 0\} \subseteq \mathbb{C}^3$  and  $R/I_5, R/I_6$  have quasidegrees set  $\mathbb{C}A_{J_5} = \mathbb{C}A_{J_6} = \{y_2 = y_3\} \subseteq \mathbb{C}^3$ . There is one component  $R/I_4$  with Krull dimension one and quasidegrees set equal to the line  $\mathbb{C}A_{J_4} = \{y_1 = y_2 = y_3\} \subseteq \mathbb{C}^3$ .

Thus, in order to study the behavior of  $M_A(I, \beta)$  when varying  $\beta \in \mathbb{C}^3$  it will be useful to stratify the space of parameters  $\mathbb{C}^3$  by the strata  $\Lambda_1 = \mathbb{C}^3 \setminus \{y_1(y_2 - y_3) = 0\}$ ,  $\Lambda_2 = \{y_1 = 0\} \setminus \{y_2 - y_3 = 0\}$ ,  $\Lambda_3 = \{y_2 - y_3 = 0\} \setminus \{y_1 = 0\}$ ,  $\Lambda_4 = \{y_1 = y_2 = y_3\} \setminus \{0\}$ ,  $\Lambda_5 = \overline{\Lambda_2} \cap \overline{\Lambda_3} \setminus \{0\} = \{y_1 = 0 = y_2 - y_3\} \setminus \{0\}$  and  $\Lambda_6 = \{0\}$ .

Let us compute the slopes of  $M_A(I, \beta)$  along coordinate hyperplanes according with Theorem 4.3, Remark 3.4 and the results in [SW08]. Recall that  $a_1, \dots, a_6$  stand for the columns of the matrix  $A$ . We have the following situations:

- 1) If  $-\beta \in \Lambda_1$  then  $R/I_1$  is the unique component whose quasidegrees set contains  $-\beta$ . Thus,  $M_A(I, \beta) \simeq M_A(I_1, \beta)$  has a unique slope  $s = 6$  along the hyperplane  $\{x_6 = 0\}$  because  $a_6/s = [1/3, 1/3, 1/3]^t$  belongs to the plane passing through  $a_1, a_2, a_3$ .
- 2) If  $-\beta \in \Lambda_2$ , then  $-\beta \in \text{qdeg}(R/I_i)$  if and only if  $i \in \{1, 2, 3\}$  so  $M_A(I, \beta)$  has the slope  $s = 6$  along  $\{x_6 = 0\}$  arising from  $I_1$  and the slope  $s = 2$  along  $\{x_4 = 0\}$  arising from  $I_2$  (since  $a_4/2$  lie in the line passing through  $a_2, a_3$ ).
- 3) If  $-\beta \in \Lambda_3$ , then  $-\beta \in \text{qdeg} R/I_i$  if and only if  $i \in \{1, 5, 6\}$ .  $M_A(I, \beta)$  has the slopes  $s = 4$  (arising from  $I_5$ ) and  $s = 6$  (arising from  $I_1$ ) along  $\{x_6 = 0\}$ .
- 4) If  $-\beta \in \Lambda_4$ , then  $-\beta \in \text{qdeg} R/I_i$  if and only if  $i \in \{1, 4, 5, 6\}$ .  $M_A(I, \beta)$  has the slopes  $s = 2$  (arising from  $I_4$ ),  $s = 4$  (arising from  $I_5$ ) and  $s = 6$  (arising from  $I_1$ ) along  $\{x_6 = 0\}$ .
- 5) If  $-\beta \in \Lambda_5 = \cap_{i \neq 4} \text{qdeg} R/I_i \setminus \text{qdeg} R/I_4$  then  $M_A(I, \beta)$  has the slopes  $s = 4$  (arising from  $I_5$ ) and  $s = 6$  (arising from  $I_1$ ) along  $\{x_6 = 0\}$  and the slope  $s = 2$  (arising from  $I_2$ ) along  $\{x_4 = 0\}$ .
- 6) If  $-\beta \in \Lambda_6$  (i.e.  $\beta = 0$ ) we have that  $-\beta$  is in the quasidegrees set of all the components  $R/I_i$ . Thus,  $M_A(I, \beta)$  has the slopes  $s = 2$  (arising from  $I_4$ ),  $s = 4$  (arising from  $I_5$ )

and  $s = 6$  (arising from  $I_1$ ) along  $\{x_6 = 0\}$  and the slope  $s = 2$  (arising from  $I_2$ ) along  $\{x_4 = 0\}$ .

In all the cases there are no more slopes along coordinate hyperplanes. Notice that when we move  $-\beta$  from one stratum  $\Lambda_i$  of dimension  $r$ ,  $1 \leq r \leq d = 3$ , to another stratum  $\Lambda_j \subseteq \overline{\Lambda_i}$  of dimension  $r - 1$  then  $M_A(I, \beta)$  can have new slopes along a hyperplane but no slope disappears.

**Remark 4.7.** By [DMM10, Lemma 7.2], all toral primes of a lattice-basis ideal  $I(B)$  have dimension exactly  $d$  and are minimal primes of  $I(B)$ . Thus, the  $L$ -characteristic varieties and the set of slopes of  $M_A(I(B), \beta)$  are independent of  $-\beta \notin \mathcal{Z}_{\text{Andean}}(I(B))$ .

To finish this Section we are going to compute the multiplicities of the  $L$ -characteristic cycle of a holonomic binomial  $D$ -module  $M_A(I, \beta)$  for  $\beta$  generic. Recall that the volume  $\text{vol}_\Lambda(B)$  of a matrix  $B$  with columns  $b_1, \dots, b_k \in \mathbb{Z}^d$  with respect to a lattice  $\Lambda \supseteq \mathbb{Z}B$  is nothing but the Euclidean volume of the convex hull of  $\{0\} \cup \{b_1, \dots, b_k\}$  normalized so that the unit simplex in the lattice  $\Lambda$  has volume one.

From now on we assume that  $\mathcal{Z}_{\text{Andean}}(I) \neq \mathbb{C}^d$  and that  $\beta \in \mathbb{C}^d$  is generic. In particular we assume that all the quotients  $R/C_{\rho,J}$  whose quasidegrees set contain  $-\beta$  are toral and have Krull dimension  $d$ . The generic condition will also guarantee that  $\beta$  is not a rank-jumping parameter of any hypergeometric system  $\mathcal{H}_0(E - \beta, I_{\rho,J})$ .

Under this assumptions it is proved in [DMM10, Theorem 6.10] that the holonomic rank of  $M_A(I, \beta)$  equals

$$\text{rank}(M_A(I, \beta)) = \sum_{R/I_{\rho,J} \text{ toral } d\text{-dimensional}} \mu_{\rho,J} \text{vol}_{\mathbb{Z}A_J}(A_J)$$

We will use the same strategy in order to compute the multiplicities in the  $L$ -characteristic cycle  $\text{CCh}^L(M_A(I, \beta))$ . It is enough to compute the multiplicities in the  $L$ -characteristic cycle of  $M_A(C_{\rho,J}, \beta)$  for each  $d$ -dimensional toral component  $C_{\rho,J}$  of  $I$  and then apply [DMM10, Theorem 6.8].

In [SW08, Section 3.3] the authors give an index formula for the multiplicity  $\mu_{A,0}^{L,\tau}(\beta)$  of the component  $\overline{C_A^\tau}$  in the  $L$ -characteristic cycle  $\text{CCh}^L(M_A(\beta))$  of a hypergeometric  $D$ -module; see equality (4.1). They prove that these multiplicities are independent of  $\beta$  if  $\beta$  is generic (see [SW08, Theorem 4.28]). Let us denote by  $\mu_{A,0}^{L,\tau}$  this constant value.

If  $M$  is a finitely generated  $R$ -module, we denote by  $\mu_{A,0}^{L,\tau}(M, \beta)$  the multiplicity of the component  $\overline{C_A^\tau}$  in the  $L$ -characteristic cycle  $\text{CCh}^L(\mathcal{H}_0(E - \beta, M))$  (see [SW08, Definition 4.7]).

For  $J \subset \{1, \dots, n\}$  we denote  $A_J$  the submatrix whose columns are indexed by  $J$ ,  $D_J$  the Weyl algebra with variables  $\{x_j, \partial_j \mid j \in J\}$  and  $L_J$  the filtration on  $D_J$  induced by the weights  $(u_j, v_j)$  for  $j \in J$ . In particular, we can define the multiplicity  $\mu_{A_J}^{L_J,\tau}$  for any face  $\tau$  of the  $(A_J, L_J)$ -umbrella  $\Phi_{A_J}^{L_J}$ .

**Theorem 4.8.** *Let  $R/C_{\rho,J}$  be a toral  $d$ -dimensional module and let  $\beta$  be generic. We have for all  $\tau \in \Phi_{A_J}^{L_J}$  and for any filtration  $L$  on  $D$  that*

$$\mu_{A,0}^{L,\tau}(R/C_{\rho,J}, \beta) = \mu_{\rho,J} \mu_{A_J}^{L_J,\tau}.$$

*Proof.* It follows the ideas of the last part of the proof of [DMM10, Theorem 6.10] (see also the proof of Theorem 3.7). We write  $M = R/C_{\rho,J}$  and consider a toral filtration  $M_0 = (0) \subseteq$

$M_1 \subseteq \cdots \subseteq M_r = M$  each successive quotient  $M_i/M_{i-1}$  being isomorphic to  $\frac{R}{I_{\rho_i, J_i}}(\gamma_i)$  for some  $\gamma_i \in \mathbb{Z}^d$ . The number of successive quotients of dimension  $d$  is the multiplicity  $\mu_{\rho, J}$  of the ideal  $I_{\rho, J}$  in  $C_{\rho, J}$ . From the assumption on  $\beta$  we can take  $-\beta$  outside the union of the quasidegree sets of  $\frac{R}{I_{\rho_i, J_i}}$  with Krull dimension  $< d$ . Then

$$\mathcal{H}_j(E - \beta, M_i/M_{i-1}) = \begin{cases} 0 & \text{if } I_{\rho_i, J_i} \neq I_{\rho, J} \\ \mathcal{H}_j(E - \beta + \gamma_i, R/I_{\rho, J})(\gamma_i) & \text{otherwise.} \end{cases}$$

Again using that  $\beta$  is generic, we have that  $\mathcal{H}_j(E - \beta, M_i/M_{i-1}) = 0$  for any  $i$  and any  $j \geq 1$ . The statement of the Theorem follows by applying decreasing induction on  $i$  and the additivity of  $\mu_{A, 0}^{L, \tau}$  with respect to the exact sequence

$$0 \longrightarrow \mathcal{H}_0(E - \beta, M_{i-1}) \longrightarrow \mathcal{H}_0(E - \beta, M_i) \longrightarrow \mathcal{H}_0(E - \beta, M_i/M_{i-1}) \longrightarrow 0.$$

We notice here that the multiplicity  $\mu_{A, 0}^{L, \tau}$  for  $\mathcal{H}_0(E - \beta + \gamma_i, R/I_{\rho, J})$  equals  $\mu_{A, J}^{L, \tau}$  for the hypergeometric  $D_J$ -module  $M_{A, J}(\beta - \gamma_i)$  because  $\beta$  is generic.  $\square$

## 5. ON THE GEVREY SOLUTIONS AND THE IRREGULARITY OF BINOMIAL $D$ -MODULES

Let us denote by  $Y_i$  the hyperplane  $x_i = 0$  in  $\mathbb{C}^n$ . Again by [DMM10, Theorem 6.8], in order to study the Gevrey solutions and the irregularity of a holonomic binomial  $D$ -module  $M_A(I, \beta)$  for generic parameters  $\beta \in \mathbb{C}^d$  it is enough to study each binomial  $D$ -module  $M_A(C_{\rho, J}, \beta)$  arising from a  $d$ -dimensional toral primary component  $R/C_{\rho, J}$ . For any real number  $s$  with  $s \geq 1$ , we consider, the irregularity complex of order  $s$ ,  $\text{Irr}_{Y_i}^{(s)}(M_A(C_{\rho, J}, \beta))$  (see [Meb90, Definition 6.3.1]). Since  $M_A(C_{\rho, J}, \beta)$  is holonomic, by a result of Z. Mebkhout [Meb90, Theorem 6.3.3] this complex is a perverse sheaf and then for  $p \in Y_i$  generic it is concentrated in degree 0. For  $r \in \mathbb{R}$  with  $r \geq 1$  we denote by  $L_r$  the filtration on  $D$  induced by  $L_r = F + (r - 1)V$  and we will write simply  $\Phi_A^r$  instead of  $\Phi_A^{L_r}$  and  $\mu_{A, 0}^{r, \tau}$  instead of  $\mu_{A, 0}^{L_r, \tau}$ .

**Theorem 5.1.** *Let  $R/C_{\rho, J}$  be a toral  $d$ -dimensional module,  $\beta$  generic,  $p \in Y_i$  generic,  $i = 1, \dots, n$  and  $s$  a real number with  $s \geq 1$ . We have that*

$$\dim_{\mathbb{C}} \mathcal{H}^0(\text{Irr}_{Y_i}^s(M_A(C_{\rho, J}, \beta)))_p = \mu_{\rho, J} \sum_{i \notin \tau \in \Phi_{A, J}^s \setminus \Phi_{A, J}^1} \text{vol}_{\mathbb{Z}A, J}(A_{\tau})$$

*Proof.* We follow the argument of the proof of Theorem 7.5 in [Fer10]. We apply results of Y. Laurent and Z. Mebkhout [LM99, Lemme 1.1.2 and Section 2.3] to get

$$\dim_{\mathbb{C}} \mathcal{H}^0(\text{Irr}_{Y_i}^s(M_A(C_{\rho, J}, \beta)))_p = \mu_{A, 0}^{s+\epsilon, \emptyset} - \mu_{A, 0}^{1+\epsilon, \emptyset} + \mu_{A, 0}^{1+\epsilon, \{i\}} - \mu_{A, 0}^{s+\epsilon, \{i\}}.$$

To finish the proof we apply Theorem 4.8 and Theorem 7.5 [Fer10].  $\square$

**Remark 5.2.** Notice that the above formula for  $\dim_{\mathbb{C}} \mathcal{H}^0(\text{Irr}_{Y_i}^s(M_A(C_{\rho, J}, \beta)))_p = 0$  yields zero if  $i \notin J$  since in that case the induced filtration  $(L_s)_J$  (denoted just by  $s$  by abuse of notation) is constant and so  $\Phi_{A, J}^s \setminus \Phi_{A, J}^1 = \emptyset$ .

Let us see how to compute Gevrey solutions of a binomial  $D$ -module  $M_A(I, \beta)$ . By (3.3) in [DMM<sub>2</sub>10] the  $I_{\rho, J}$ -primary component  $C_{\rho, J}$  of an irredundant primary decomposition of any

$A$ -graded binomial ideal  $I$  (for some minimal associated prime  $I_{\rho,J} = I_{\rho} + \mathfrak{m}_J$  of  $I$ ) contains  $I_{\rho}$ . Thus,

$$(5.1) \quad I_{\rho} + \mathfrak{m}_J^r \subseteq C_{\rho,J} \subseteq \sqrt{C_{\rho,J}} = I_{\rho,J} = I_{\rho} + \mathfrak{m}_J$$

for sufficiently large integer  $r$ . In fact, it is not hard to check that  $C_{\rho,J} = I_{\rho} + B_{\rho,J}$  for some binomial ideal  $B_{\rho,J} \subseteq R$  such that  $\mathfrak{m}_J^r \subseteq B_{\rho,J} \subseteq \mathfrak{m}_J$ . Let us fix such an ideal  $B_{\rho,J}$ .

For any monomial ideal  $\mathfrak{n} \subseteq C_{\rho,J}$  such that  $\sqrt{\mathfrak{n}} = \mathfrak{m}_J$  we have that

$$H_A(I_{\rho} + \mathfrak{n}, \beta) \subseteq H_A(C_{\rho,J}, \beta) \subseteq H_A(I_{\rho,J}, \beta).$$

Let us fix such an ideal  $\mathfrak{n}$ . In particular, any formal solution of  $M_A(I_{\rho,J}, \beta)$  is a solution of  $M_A(C_{\rho,J}, \beta)$  and any solution of  $M_A(C_{\rho,J}, \beta)$  is a solution of  $M_A(I_{\rho} + \mathfrak{n}, \beta)$ .

Let us assume that  $C_{\rho,J}$  is toral (i.e.  $R/I_{\rho,J}$  has Krull dimension equal to  $\text{rank } A_J$ ). We will also assume that  $\text{rank } A_J = \text{rank } A$  in order to ensure that  $\text{qdeg}(R/C_{\rho,J}) = \mathbb{C}^d$ .

On the one hand, both the solutions of  $M_A(I_{\rho,J}, \beta)$  and the solutions of  $M_A(I_{\rho} + \mathfrak{n}, \beta)$  can be described explicitly if the parameter vector  $\beta$  is generic enough. More precisely, a formal solution of the hypergeometric system  $M_A(I_{\rho,J}, \beta)$  with very generic  $\beta$  is known to be of the form

$$\phi_v = \sum_{u \in \ker A_J \cap \mathbb{Z}^J} \rho(u) \frac{(v)_{u_-}}{(v+u)_{u_+}} x_J^{v+u}$$

where  $v \in \mathbb{C}^J$  such that  $A_J v = \beta$  and  $(v)_w = \prod_{j \in J} \prod_{0 \leq i \leq w_j - 1} (v_j - i)$  is the Pochhammer symbol (see [GZK89, SST00]). Here,  $v$  needs to verify additional conditions in order to ensure that  $\phi_v$  is a formal series along a coordinate subspace or a holomorphic solution.

The vectors  $v$  you need to consider to describe a basis of the space of Gevrey solutions of a given order along a coordinate subspace of  $\mathbb{C}^n$  for the binomial  $D$ -module  $M_A(I_{\rho,J}, \beta)$  are the same that are described in [Fer10] for the hypergeometric system  $M_{A_J}(\beta)$ .

On the other hand, for  $\gamma$  in  $\mathbb{N}^{\bar{J}}$  let  $G_{\gamma}$  be either a basis of the space of holomorphic solutions near a non singular point or the space of Gevrey solutions of a given order along a coordinate hyperplane of  $\mathbb{C}^J$  for the system  $M_{A_J}(I_{\rho}, \beta - A_{\bar{J}}\gamma)$ , where  $\bar{J}$  denotes the complement of  $J$  in  $\{1, \dots, n\}$  and  $x_{\bar{J}}^{\gamma}$  runs in the set  $S_{\bar{J}}(\mathfrak{n})$  of monomials in  $\mathbb{C}[x_{\bar{J}}]$  annihilated by the monomial differential operators in  $\mathfrak{n}$ . Then a basis of the same class of solutions for the system  $M_A(I_{\rho} + \mathfrak{n}, \beta)$  is given by

$$\mathcal{B} = \{x_{\bar{J}}^{\gamma} \varphi : x^{\gamma} \in S_{\bar{J}}(\mathfrak{n}), \varphi \in G_{\gamma}\}$$

We conclude that any holomorphic or formal solution of  $M_A(C_{\rho,J}, \beta)$  can be written as a linear combination of the series in  $\mathcal{B}$ . The coefficients in a linear combination of elements in  $\mathcal{B}$  that provide a solution of  $M_A(C_{\rho,J}, \beta)$  can be computed if we force a general linear combination to be annihilated by the binomial operators in a set of generators of  $B_{\rho,J}$  that are not in  $\mathfrak{n}$ .

Thus, the main problem in order to compute formal or analytic solutions of  $M_A(C_{\rho,J}, \beta)$  is that the ideal  $B_{\rho,J}$  is not a monomial ideal in general and that a minimal set of generators may involve some variables  $x_j$  for  $j \in J$ . Let us illustrate this situation with the following example.

**Example 5.3.** Let us write  $x = x_1, y = x_2, z = x_3, t = x_4$  and consider the binomial ideal  $C_{\rho,J} = I_{\rho} + B_{\rho,J} \subseteq \mathbb{C}[\partial_x, \partial_y, \partial_z, \partial_t]$  where  $J = \{1, 2\}$ ,  $\rho : \ker(A_J) \cap \mathbb{Z}^2 \rightarrow \mathbb{C}^*$  is the trivial character,  $A$  is the row matrix  $(2, 3, 2, 2)$ ,  $I_{\rho} = \langle \partial_x^3 - \partial_y^2 \rangle$  and  $B_{\rho,J} = \langle \partial_z^2 - \partial_x \partial_t, \partial_t^2 \rangle$ .

Notice that  $C_{\rho,J}$  is  $A$ -graded for the row matrix  $A = (2 \ 3 \ 2 \ 2)$  and that  $C_{\rho,J}$  is toral and primary.

Since  $C_{\rho,J}$  is primary and its radical ideal is  $I_{\rho} + \mathfrak{m}_J = \langle \partial_x^3 - \partial_y^2, \partial_z, \partial_t \rangle$ , we have that  $M_A(C_{\rho,J}, \beta)$  is an irregular binomial  $D$ -module for all parameters  $\beta \in \mathbb{C}$  (see Theorem 3.7) and that it has only one slope  $s = 3/2$  along its singular locus  $\{y = 0\}$ .

We are going to compute the Gevrey solutions of  $M_A(C_{\rho,J}, \beta)$  corresponding to this slope.

By the previous argument and using that  $\mathfrak{n} = \langle \partial_z^4, \partial_t^2 \rangle \subseteq B_{\rho,J}$  we obtain that any Gevrey solution of  $M_A(C_{\rho,J}, \beta)$  along  $\{y = 0\}$  can be written as

$$f = \sum_{\gamma,k} \lambda_{\gamma,k} z^{\gamma_z} t^{\gamma_t} \phi_k(\beta - 2\gamma_z - 2\gamma_t)$$

where  $\lambda_{\gamma,k} \in \mathbb{C}$ ,  $\gamma = (\gamma_z, \gamma_t)$ ,  $\gamma_z \in \{0, 1, 2, 3\}$ ,  $\gamma_t, k \in \{0, 1\}$  and

$$\phi_k(\beta - 2\gamma_z - 2\gamma_t) = \sum_{m \geq 0} \frac{((\beta - 3k)/2 - \gamma_z - \gamma_t)_{3m}}{(k + 2m)_{2m}} x^{(\beta - 3k)/2 - \gamma_z - \gamma_t - 3m} y^{k+2m}$$

is a Gevrey series of index  $s = 3/2$  along  $y = 0$  at any point  $p \in \{y = 0\} \cap \{x \neq 0\}$  if  $(\beta - 3k)/2 - \gamma_z - \gamma_t \notin \mathbb{N}$ .

We just need to force the condition  $\partial_x \partial_t(f) = \partial_z^2(f)$  in order to obtain the values of  $\lambda_{\gamma,k}$  such that  $f$  is a solution of  $M_A(C_{\rho,J}, \beta)$ .

In this example, we obtain the conditions  $\lambda_{(2,1),k} = \lambda_{(3,1),k} = 0$  for  $k = 0, 1$  and

$$\lambda_{(\gamma_z+2,0),1} = \frac{((\beta - 3k)/2 - \gamma_z)}{(a + 1)(a + 2)} \lambda_{(\gamma_z,1),k}$$

for  $k, \gamma_z = 0, 1$ .

In particular we get an explicit basis of the space of Gevrey solutions of  $M_A(C_{\rho,J}, \beta)$  along  $y = 0$  with index equal to the slope  $s = 3/2$  and we have that the dimension of this space is 8. Notice that  $8 = 4 \cdot 2$  is the expected dimension (see Theorem 5.1) since  $\mu_{\rho,J} = 4$  and the dimension of the corresponding space for  $M_A(I_{\rho,J}, \beta)$  is 2 (see [FC11, FC08]).

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