Purity of exponential sums on $\mathbb{A}^n$

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Abstract

We give a purity result for two kinds of exponential sums of the type $\sum_{x\in k^n}\psi(f(x))$, where k is a finite field of characteristic p and $\psi : k \to \mathbb{C}^\star$ is a non-trivial additive character. In the first case $f \in k[x_1, \ldots, x_n]$ is a polynomial of degree divisible by p whose highest degree homogeneous form defines a non-singular projective hypersurface, and in the second one f is a polynomial of degree prime to p whose highest degree homogeneous form defines a projective hypersurface with isolated singularities.

1. Introduction

Let k be a finite field of characteristic p and cardinality q, and let $f \in k[x_1, \ldots, x_n]$ be a polynomial of degree d. Pick a non-trivial additive character $\psi : k \to \mathbb{C}^\star$, and consider the sum $\sum_{x\in k^n}\psi(f(x))$. In [Del74] Deligne proved, as a corollary to his proof of the Riemann hypothesis for projective varieties over finite fields, the following estimate:

Theorem 1. ([Del74], Théorème 8.4) *Suppose that*

i) *The highest degree homogeneous form f_d of f defines a nonsingular hypersurface in $\mathbb{P}^{n-1}_{\bar{k}}$.*
ii) *d is prime to p.*

Then we have the estimate

$$\left|\sum_{x\in k^n}\psi(f(x))\right| \leqslant (d-1)^n \cdot q^{n/2}.$$

Moreover, he showed that the sum is *pure* of weight n and rank $(d-1)^n$. In particular, there are $(d-1)^n$ complex algebraic numbers $\alpha_1, \ldots, \alpha_{(d-1)^n}$, all pure of weight n (meaning that all their conjugates over $\mathbb{Q}$ have absolute value $q^{n/2}$) such that, for every integer $m \geqslant 1$, if k_m denotes the degree m extension of k in a fixed algebraic closure $\bar{k}$, we have

$$(-1)^n \sum_{x\in k_m^n}\psi(\mathrm{Trace}_{k_m/k}(f(x))) = \sum_{i=1}^{(d-1)^n}\alpha_i^m.$$

What can we say in the case where p divides d? By perversity arguments (cf. [KL85], [Kat93], [Kat04]) we know that the sum is pure for *almost* all $f \in k[x_1, \ldots, x_n]$. More precisely, if we add a sufficiently general linear form to f (one that is contained in a suitable Zariski dense open subset U of the dual affine space $\hat{\mathbb{A}}^n_k$ depending on ψ and q), the sum becomes pure of weight n. However, these results do not give us any information about the sum associated to a particular f. On the other hand, in [AS00b] Adolphson and Sperber show, using p-adic methods, that if f satisfies certain regularity hypotheses the L-function associated to the exponential sum (or its inverse) is

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a polynomial. In this article we will use these results to give a version of Theorem 1 for the case where p divides d.

Fix a prime $\ell \neq p$ and an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ so that we can speak about absolute values of elements of $\bar{\mathbb{Q}}_\ell$ and weights without ambiguity. From now on we will assume that such an isomorphism has been chosen, without making any further reference to it. Thus, for every $\alpha \in \bar{\mathbb{Q}}_\ell$, $|\alpha|$ will always mean $|\iota(\alpha)|$. We will also use this isomorphism to identify the sets of $\mathbb{C}^\star$-valued characters and of $\bar{\mathbb{Q}}_\ell^\star$-valued characters of any finite group. Consider the lisse Artin-Schreier $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1_k$ associated to the non-trivial additive character $\psi : k \to \mathbb{C}^\star$ (cf. [Del77], 1.7). For every finite extension k'/k and every $t \in \mathbb{A}^1(k') = k'$, the trace of the geometric Frobenius element in $\mathrm{Gal}(\bar{k}/k')$ acting on the stalk of $\mathcal{L}_\psi$ at a geometric point $\bar{t}$ over t is $\psi(\mathrm{Trace}_{k'/k}(t))$. In particular, since ψ takes its values among the roots of unity, $\mathcal{L}_\psi$ is pure of weight 0.

Let $\mathcal{L}_{\psi(f)}$ denote the pull-back $f^\star\mathcal{L}_\psi$ on $\mathbb{A}^n_k$. The cohomology groups with compact support $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ are endowed with an action of the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ and, in particular, of the geometric Frobenius element $F \in \mathrm{Gal}(\bar{k}/k)$. By the Grothendieck trace formula we have

$$\sum_{x\in k^n} \psi(f(x)) = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Trace}(F|\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})).$$

Our first result is the following

THEOREM 2. *Let d be divisible by p. Write $f = f_d + f_{d'} + f'$, where f_d is the degree d homogeneous component of f, d' is the degree of $f - f_d$ and $f_{d'}$ is the degree d' homogeneous component of f. Suppose that*

a) *$d'/d > p/(p + (p-1)^2)$ and d' is prime to p.*
b) *The equation $f_d = 0$ defines a non-singular hypersurface in $\mathbb{P}^{n-1}_{\bar{k}}$.*
c) *The hypersurface defined in $\mathbb{P}^{n-1}_{\bar{k}}$ by $f_{d'} = 0$ does not contain any of the common zeroes of $\frac{\partial f_d}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_n}$ in $\mathbb{P}^{n-1}_{\bar{k}}$.*

Then

1. *$\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = 0$ for $i \neq n$.*
2. *$\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ has dimension $(d'(d-1)^n + (-1)^n(d-d'))/d$ and is pure of weight n.*
3. *We have the estimate*

$$\left|\sum_{x\in k^n} \psi(f(x))\right| \leqslant \frac{d'(d-1)^n + (-1)^n(d-d')}{d} \cdot q^{n/2}.$$

For $d' = d-1$ (the generic case) the inequality in (a) holds as long as $d \geqslant 3$, and we get

COROLLARY 3. *Assume $d \geqslant 3$ is divisible by p. Let $f = f_d + f_{d-1} + f'$ be as above. Suppose that*

a) *The equation $f_d = 0$ defines a non-singular hypersurface in $\mathbb{P}^{n-1}_{\bar{k}}$.*
b) *The equation $f_{d-1} = 0$ defines a hypersurface in $\mathbb{P}^{n-1}_{\bar{k}}$ which does not contain any of the common zeroes of $\frac{\partial f_d}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_n}$ in $\mathbb{P}^{n-1}_{\bar{k}}$.*

Then

1. *$\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = 0$ for $i \neq n$.*
2. *$\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ has dimension $((d-1)^{n+1} - (-1)^{n+1})/d$ and is pure of weight n.*

3. *We have the estimate*

$$\left|\sum_{x\in k^n}\psi(f(x))\right| \leqslant \frac{(d-1)^{n+1}-(-1)^{n+1}}{d}\cdot q^{n/2}.$$

As usual, (3) is a consequence of the vanishing of the cohomology together with Deligne's theorem on weights (cf. [Del80], Corollaire 3.3.4).

The second result deals with another kind of sum studied by Adolphson and Sperber in [AS00b] and is a generalization of ([Gar98], Theorem 0.4). Let $f \in k[x_1,\dots,x_n]$ be a polynomial of degree d, which we will now assume to be prime to p. We will show

Theorem 4. *Write $f = f_d + f_{d'} + f'$ as in Theorem 2. Suppose that*

a) *$d'/d > p/(p+(p-1)^2)$ and d' is prime to p.*
b) *The hypersurface defined by $f_d = 0$ in $\mathbb{P}^{n-1}_{\bar{k}}$ has at worst weighted homogeneous isolated singularities of total degrees $d_1,\dots,d_s$ prime to p (cf. [AS00b], Section 2 or [Gar98], 0.3 for the definitions).*
c) *The hypersurface defined by $f_{d'} = 0$ in $\mathbb{P}^{n-1}_{\bar{k}}$ does not contain any of these singularities.*

Let $\mu_1,\dots,\mu_s$ be the Milnor numbers corresponding to the singularities of $f_d = 0$. Then

1. $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = 0$ *for $i \neq n$.*
2. $\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ *has dimension $(d-1)^n - (d-d')\sum_{i=1}^s \mu_i$ and is pure of weight n.*
3. *We have the estimate*

$$\left|\sum_{x\in k^n}\psi(f(x))\right| \leqslant ((d-1)^n-(d-d')\sum_{i=1}^{s}\mu_i)\cdot q^{n/2}.$$

2. A cohomological vanishing result

In this section we will begin the proof of Theorem 2. We will first use the method of pencils to show the vanishing of $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ for $i > n+1$. This requires studying the fibers of the map f, so the first thing we need to do is find a suitable compactification of f. Unfortunately, the compactification defined in [Kat99] by embedding $\mathbb{A}^n$ as a dense open subset of the subscheme of $\mathbb{P}^n \times \mathbb{A}^1$ given by the vanishing of $F - \lambda X_0^d$ no longer works in this case. The reason is that we are compactifying a map of degree divisible by p, and this may introduce some wild ramification at infinity in the higher direct images of the constant sheaf with respect to the compactified map.

Therefore, instead of directly compactifying f, the idea is to first write f as the composition of a closed embedding of $\mathbb{A}^n$ in $\mathbb{A}^n \times \mathbb{A}^1$ (given by the graph of f) followed by the projection, and then compactify the projection restricted to the image of $\mathbb{A}^n$. Since we are compactifying a map of degree 1, we do not run into any problems caused by wild ramification. However, one disadvantage of this compactification is that the fiber at infinity will always have a singular point, so we will only be able to deduce the vanishing of the cohomology groups for $i > n+1$.

Proposition 5. *Suppose that the equation $f_d = 0$ defines a non-singular hypersurface in $\mathbb{P}^{n-1}_{\bar{k}}$. Then $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = 0$ for $i > n+1$.*

Proof. Define Z to be the hypersurface in $\mathbb{P}^{n+1}_k$ (where we take coordinates $X_0,\dots,X_n,T$) defined by the vanishing of $F - TX_0^{d-1}$, where F is the homogenization of f with respect to the variable X_0 (i.e. $F(X_0,\dots,X_n) = X_0^d \cdot f(X_1/X_0,\dots,X_n/X_0)$). The affine space $\mathbb{A}^n_k$ is naturally an open subscheme of Z (just by embedding it in $\mathbb{A}^{n+1}_k$ using the graph of f, and then identifying $\mathbb{A}^{n+1}_k$ with $\mathbb{P}^{n+1}_k$ minus the hyperplane $X_0 = 0$).

Next, we define the incidence variety $\tilde{Z}$ as a divisor of $Z \times \mathbb{P}^1_k$, given (with coordinates $X_0, \ldots, X_n, T$ for the first factor and λ_0, λ_1 for the second one) by the zero locus of $\lambda_0 T - \lambda_1 X_0$. Thus

$$\tilde{Z}(\bar{k}) = \{((x_0, \ldots, x_n, t), (\lambda_0, \lambda_1)) \in Z(\bar{k}) \times \mathbb{P}^1(\bar{k}) : \lambda_0 t = \lambda_1 x_0\}.$$

Let $\tilde{f} : \tilde{Z} \to \mathbb{P}^1_k$ be the restriction to $\tilde{Z}$ of the canonical projection $\pi_2 : Z \times \mathbb{P}^1_k \to \mathbb{P}^1_k$. It is a proper map, being the composite of a closed immersion and a proper projection (since Z is projective).

The open subset $\mathbb{A}^n_k \hookrightarrow Z$ can be embedded as an open subscheme of $\tilde{Z}$ in the obvious way. Namely, we identify the point $x \in \mathbb{A}^n(\bar{k})$ with $(x, f(x)) \in \tilde{Z}(\bar{k})$. In this way we get a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^n_k & \longrightarrow & \tilde{Z} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ \mathbb{A}^1_k & \longrightarrow & \mathbb{P}^1_k \end{array}$$

where the horizontal arrows are open embeddings. The image of $\mathbb{A}^n_k$ in $\tilde{Z}$ can be described as the set of $(x, \lambda) \in \tilde{Z}$ such that $x \notin Z \cap \{X_0 = 0\}$.

Before going any further we need to show that $\tilde{f}$ is a flat map.

Lemma 6. *The map $\tilde{f} : \tilde{Z} \to \mathbb{P}^1_k$ is flat.*

Proof. By ([Har77], Proposition III.9.9) it suffices to show that all geometric fibers of $\tilde{f}$ have the same Hilbert polynomial. The fiber over a finite point $\lambda \in \mathbb{A}^1(\bar{k})$ is easily seen to be the complete intersection of the degree d hypersurface $F - \lambda X_0^d = 0$ and the hyperplane $T - \lambda X_0 = 0$. Similarly, the fiber over infinity is the complete intersection of the hypersurface $F = 0$ and the hyperplane $X_0 = 0$. Since the Hilbert polynomial of a complete intersection only depends on its multidegree, we conclude that it is the same for all geometric fibers of $\tilde{f}$. □

We extend by zero the sheaf $\mathcal{L}_\psi$ to the whole $\mathbb{P}^1_k$, and take its pull-back by $\tilde{f}$ to $\tilde{Z}$, which we will also denote by $\mathcal{L}_{\psi(f)}$. This is compatible with the previous notation, since its restriction to $\mathbb{A}^n_k$ is just the pull-back of $\mathcal{L}_\psi$ by f.

Lemma 7. *There is a quasi-isomorphism*

$$\mathrm{R}\Gamma_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}).$$

Proof. To simplify the notation, we will identify each homogeneous form with the projective hypersurface defined by its vanishing. It is clear that $\tilde{Z}_1 := (Z \cap T \cap X_0) \times \mathbb{P}^1_k$ is contained in $\tilde{Z}$ as a closed subscheme. Let $\tilde{Z}_0$ be its complement. The restriction of $\tilde{f}$ to $\tilde{Z}_1$ is just the second projection. From the decomposition

$$\tilde{Z}_0 \overset{j}{\hookrightarrow} \tilde{Z} \overset{i}{\hookleftarrow} \tilde{Z}_1$$

we get an exact sequence of sheaves

$$0 \to j_! j^\star \mathcal{L}_{\psi(f)} \to \mathcal{L}_{\psi(f)} \to i_\star i^\star \mathcal{L}_{\psi(f)} \to 0$$

from which we get a distinguished triangle in $\mathcal{D}^b(\bar{\mathbb{Q}}_\ell - \text{vector spaces})$

$$\mathrm{R}\Gamma_c(\tilde{Z}_0 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \to \mathrm{R}\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \to \mathrm{R}\Gamma_c(\tilde{Z}_1 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \to$$

Now in $\tilde{Z}_1 \cong (Z \cap T \cap X_0) \times \mathbb{P}^1_k$ the sheaf $\mathcal{L}_{\psi(f)}$ is just the external tensor product $\bar{\mathbb{Q}}_\ell \boxtimes \mathcal{L}_\psi$. Therefore by the Künneth formula we have

$$\mathrm{R}\Gamma_c(\tilde{Z}_1 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) = \mathrm{R}\Gamma_c((Z \cap T \cap X_0) \otimes \bar{k}, \bar{\mathbb{Q}}_\ell) \otimes \mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathcal{L}_\psi) = 0$$

since $\mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathcal{L}_\psi) = \mathrm{R}\Gamma_c(\mathbb{A}^1_{\bar{k}}, \mathcal{L}_\psi) = 0$ (cf. [Del77], Théorème 2.7*). Hence we get a quasi-isomorphism

$$\mathrm{R}\Gamma_c(\tilde{Z}_0 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}).$$

The image of the open immersion $h : \mathbb{A}^n_{\bar{k}} \hookrightarrow \tilde{Z}_0$ is the set of $(x, \lambda) \in \tilde{Z}$ such that $x \notin Z \cap X_0$. Its complement in $\tilde{Z}_0$ is the set of $(x, \lambda) \in \tilde{Z}$ such that $x \in Z \cap X_0$ and $x \notin Z \cap T$, so it maps to the point at infinity under $\tilde{f}$. Since the stalk of $\mathcal{L}_\psi$ at infinity is zero, we have an equality $h_! h^\star \mathcal{L}_{\psi(f)} = \mathcal{L}_{\psi(f)}$, and therefore a quasi-isomorphism

$$\mathrm{R}\Gamma_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\tilde{Z}_0 \otimes \bar{k}, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}).$$

□

We will also denote by $\tilde{f} : \tilde{Z} \otimes \bar{k} \to \mathbb{P}^1_{\bar{k}}$ the map deduced from $\tilde{f} : \tilde{Z} \to \mathbb{P}^1_k$ by extension of scalars to $\bar{k}$. Since $\tilde{f}$ is proper, we have (by composition of derived functors)

$$\mathrm{R}\Gamma_c(\tilde{Z} \otimes \bar{k}, \mathcal{L}_{\psi(f)}) = \mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}\tilde{f}_\star \mathcal{L}_{\psi(f)}).$$

On the other hand, by the projection formula we have

$$\mathrm{R}\tilde{f}_\star \mathcal{L}_{\psi(f)} = \mathrm{R}\tilde{f}_\star(\bar{\mathbb{Q}}_\ell \otimes \tilde{f}^\star \mathcal{L}_\psi) = \mathrm{R}\tilde{f}_\star \bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\psi$$

so Proposition 5 is equivalent to

Proposition 8. *Under the previous hypotheses the cohomology group $\mathrm{H}^i_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}\tilde{f}_\star \bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\psi)$ vanishes for $i > n+1$.*

Therefore we will prove Proposition 8 instead.

Proposition 9. *The sheaves $\mathrm{R}^i \tilde{f}_\star \bar{\mathbb{Q}}_\ell$ on $\mathbb{P}^1_{\bar{k}}$ are lisse for $i \geqslant n+1$. For $i = n$ it is the extension of a lisse sheaf by a punctual sheaf.*

Proof. The fiber of $\tilde{f}$ at a point $\lambda \in \mathbb{A}^1(\bar{k})$ is defined in $\mathbb{P}^{n+1}_{\bar{k}}$ (with the usual coordinates $X_0, \dots, X_n, T$) by the homogeneous ideal $(F - TX_0^{d-1}, T - \lambda X_0) = (F - \lambda X_0^d, T - \lambda X_0)$. Its intersection with the hyperplane $X_0 = 0$ is then defined by the ideal (F, X_0, T), and is therefore isomorphic to the hypersurface defined in $\mathbb{P}^{n-1}_{\bar{k}}$ by $f_d = 0$, which is non-singular by hypothesis. Therefore, the fiber itself has at worst isolated singularities. On the other hand, the fiber at $\lambda = \infty$ is defined in $\mathbb{P}^{n+1}_{\bar{k}}$ by the ideal (F, X_0). This is the projective cone over the hypersurface defined in $\mathbb{P}^{n-1}_{\bar{k}}$ by $f_d = 0$, so it has only one singular point (the vertex).

By ([SGA7I], Exposé I, Cor. 4.3) we deduce that for every $\lambda \in \mathbb{P}^1(\bar{k})$ the I_λ-invariant specialization map $(\mathrm{R}^i \tilde{f}_\star \bar{\mathbb{Q}}_\ell)_\lambda \to (\mathrm{R}^i \tilde{f}_\star \bar{\mathbb{Q}}_\ell)_{\bar{\eta}}$ (where $\bar{\eta}$ is a geometric generic point of $\mathbb{P}^1_{\bar{k}}$ and I_λ the inertia group at λ) is an isomorphism for $i > n$ and surjective for $i = n$. As a consequence, $\mathrm{R}^i \tilde{f}_\star \bar{\mathbb{Q}}_\ell$ is lisse at λ for $i > n$. For $i = n$ we have an exact sequence (cf. [Kat99], Theorem 13)

$$0 \to (\text{punctual sheaf}) \to \mathrm{R}^n \tilde{f}_\star \bar{\mathbb{Q}}_\ell \to j_\star j^\star \mathrm{R}^n \tilde{f}_\star \bar{\mathbb{Q}}_\ell \to 0$$

where j is the inclusion of an open subset of $\mathbb{P}^1_{\bar{k}}$ on which $\mathrm{R}^n \tilde{f}_\star \bar{\mathbb{Q}}_\ell$ is lisse. But since the specialization map $(\mathrm{R}^n \tilde{f}_\star \bar{\mathbb{Q}}_\ell)_\lambda \to (\mathrm{R}^n \tilde{f}_\star \bar{\mathbb{Q}}_\ell)_{\bar{\eta}}$ is surjective and I_λ-equivariant, the action of I_λ on $(\mathrm{R}^n \tilde{f}_\star \bar{\mathbb{Q}}_\ell)_{\bar{\eta}}$ is trivial. As a consequence, the sheaf $j_\star j^\star \mathrm{R}^n \tilde{f}_\star \bar{\mathbb{Q}}_\ell$ is lisse at λ. □

Proposition 10. *The cohomology group $\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}^b \tilde{f}_\star \bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\psi)$ vanishes for:*

i) *$a > 2$, all b*
ii) *$b > n$, all a*
iii) *$b = n$, $a > 0$.*

Proof. Part (1) is clear for cohomological dimension reasons. For $b > n$, the sheaf $\mathrm{R}^b\tilde{f}_\star\bar{\mathbb{Q}}_\ell$ is lisse on $\mathbb{P}^1_{\bar{k}}$ by Proposition 9. Since $\mathbb{P}^1_{\bar{k}}$ is simply connected, it must be constant. Then, if $\bar\eta$ is a geometric generic point of $\mathbb{P}^1_{\bar{k}}$, we get

$$\mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}^b\tilde{f}_\star\bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\psi) = (\mathrm{R}^b\tilde{f}_\star\bar{\mathbb{Q}}_\ell)_{\bar\eta} \otimes \mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathcal{L}_\psi) = 0$$

since $\mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathcal{L}_\psi) = 0$. This proves (2).

To prove (3), let $j : V \hookrightarrow \mathbb{P}^1_{\bar{k}}$ be as in Proposition 9, where V is a dense open set on which $\mathrm{R}^n\tilde{f}_\star\bar{\mathbb{Q}}_\ell$ is lisse, and let $\mathcal{H} = j_\star j^\star \mathrm{R}^n\tilde{f}_\star\bar{\mathbb{Q}}_\ell$. Then $\mathcal{H}$ is lisse on $\mathbb{P}^1_{\bar{k}}$ by Proposition 9, so exactly as above we get $\mathrm{R}\Gamma_c(\mathbb{P}^1_{\bar{k}}, \mathcal{H}\otimes\mathcal{L}_\psi) = 0$. From the exact sequence

$$0 \to \mathcal{I}\ (=\text{punctual sheaf}) \to \mathrm{R}^n\tilde{f}_\star\bar{\mathbb{Q}}_\ell \to \mathcal{H} \to 0$$

we get, after tensoring with $\mathcal{L}_\psi$,

$$0 \to \mathcal{I}\otimes\mathcal{L}_\psi \to \mathrm{R}^n\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi \to \mathcal{H}\otimes\mathcal{L}_\psi \to 0.$$

Now $\mathcal{I}\otimes\mathcal{L}_\psi$ is punctual, so $\mathrm{H}^i_c(\mathbb{P}^1_{\bar{k}}, \mathcal{I}\otimes\mathcal{L}_\psi) = 0$ for $i > 0$. From the long exact sequence of cohomology associated to the exact sequence above we get isomorphisms

$$\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}^n\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi) \xrightarrow{\sim} \mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, \mathcal{H}\otimes\mathcal{L}_\psi) = 0$$

for $a > 0$. This proves (3). □

We can now complete the proof of Proposition 8. We have a spectral sequence

$$\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}^b\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi) \Rightarrow \mathrm{H}^{a+b}_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi).$$

Suppose $a + b > n + 1$. Then either

- $a > 2$, so $\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}^b\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi) = 0$ by part (1) of Proposition 10,
- $b > n$, so $\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}^b\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi) = 0$ by part (2) of Proposition 10 or
- $a = 2$ and $b = n$, so $\mathrm{H}^a_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}^b\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi) = 0$ by part (3) of Proposition 10.

Therefore, the spectral sequence implies that $\mathrm{H}^i_c(\mathbb{P}^1_{\bar{k}}, \mathrm{R}\tilde{f}_\star\bar{\mathbb{Q}}_\ell\otimes\mathcal{L}_\psi)$ vanishes for $i > n+1$. □

3. A sum of Milnor numbers computation

Consider the L-function associated to the sheaf $\mathcal{L}_{\psi(f)}$ on $\mathbb{A}^n_k$:

$$L(T, \mathcal{L}_{\psi(f)}) = \exp\sum_{m=1}^\infty \frac{S_m}{m}T^m$$

where

$$S_m = \sum_{x\in k_m^n} \psi(\mathrm{Trace}_{k_m/k}(f(x)))$$

and k_m is the extension of degree m of k in $\bar{k}$. By the Grothendieck trace formula, we have

$$L(T, \mathcal{L}_{\psi(f)}) = \prod_{i=0}^{2n} \det(1 - T\cdot F|\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}))^{(-1)^{i+1}}$$

where $F \in \mathrm{Gal}(\bar{k}/k)$ is the geometric Frobenius element.

The following result of Adolphson and Sperber ([AS00b], Theorem 1.11 and Proposition 6.5) gives an important restriction on the shape of this L-function:

THEOREM 11. *Write $f = f_d + f_{d'} + f'$, where f_d is the degree d homogeneous component of f, d' is the degree of $f - f_d$ and $f_{d'}$ is the degree d' homogeneous component of f. Suppose that $d'/d > p/(p+(p-1)^2)$ and d' is prime to p. Suppose also that $\frac{\partial f_d}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_n}$ have a finite number of common zeroes in $\mathbb{P}^{n-1}_{\bar{k}}$ (which is automatic if the hypersurface $f_d = 0$ in $\mathbb{P}^{n-1}_{\bar{k}}$ is non-singular) and the hypersurface defined in $\mathbb{P}^{n-1}_{\bar{k}}$ by $f_{d'} = 0$ does not contain any of them. Then $L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n - (d-d')\sum_{i=1}^{s}\mu_i$, where the sum is taken over the set $\{P_1, \ldots, P_s\}$ of common zeroes of $\frac{\partial f_d}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_n}$ in $\mathbb{P}^{n-1}_{\bar{k}}$ and μ_i denotes the corresponding Milnor number*

$$\mu_i = \dim_{\bar{k}} \mathcal{O}_{S,P_i}.$$

Here S is the zero-dimensional subscheme of $\mathbb{P}^{n-1}_{\bar{k}}$ defined by the ideal $(\frac{\partial f_d}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_n})$, and $\mathcal{O}_{S,P_i}$ its local ring at P_i, which is a finite $\bar{k}$-algebra.

We will now compute this sum of Milnor numbers explicitly in the following more general setting

LEMMA 12. *Let $F_1, \ldots, F_n \in \bar{k}[x_1, \ldots, x_n]$ be (possibly zero) homogeneous polynomials of degree $d-1$. Suppose that*

i) *$F_1, \ldots, F_n$ have a finite number of common zeroes in $\mathbb{P}^{n-1}_{\bar{k}}$.*

ii) *We have the relation*

$$\sum_{i=1}^{n} x_i \cdot F_i = 0.$$

Let $\{P_1, \ldots, P_s\}$ be the set of common zeroes of $F_1, \ldots, F_n$ in $\mathbb{P}^{n-1}_{\bar{k}}$, and for every $i = 1, \ldots, s$ let μ_i be the corresponding Milnor number. Then we have

$$\sum_{i=1}^{s} \mu_i = \frac{(d-1)^n - (-1)^n}{d}.$$

Proof. By induction on n, we first prove it for $n = 2$. In this case, both F_1 and F_2 must be non-zero (otherwise, by (2) they would both be zero, and (1) would not hold). The relation $x_1F_1 + x_2F_2 = 0$ implies that x_1 divides F_2 and x_2 divides F_1. Let $F_1 = x_2G_1$ and $F_2 = x_1G_2$. Then $x_1x_2(G_1+G_2) = 0$, so $G_2 = -G_1$. Therefore the subscheme defined by F_1 and F_2 is the one defined by G_1, which is a polynomial of degree $d-2$. The common zeroes of F_1 and F_2 are then in one-to-one correspondence with the distinct linear factors of G_1, and the Milnor numbers are the corresponding multiplicities. Thus in this case we get $\sum_{i=1}^{s}\mu_i = d-2 = ((d-1)^2-1)/d$.

We assume now that the lemma is true for $n-1 \geqslant 2$, and prove it for n. Choose $(\alpha_1, \ldots, \alpha_{n-1}) \in \bar{k}^{n-1}$ such that none of the points $P_1, \ldots, P_s$ is contained in the hyperplane $x_n - \sum_{j=1}^{n-1}\alpha_j x_j = 0$. We construct the polynomials $F'_1, \ldots, F'_n$ given by

$$F'_i(x_1, \ldots, x_{n-1}, x_n) = F_i(x_1, \ldots, x_{n-1}, x_n + \textstyle\sum_{j=1}^{n-1}\alpha_j x_j) +$$
$$+\alpha_i F_n(x_1, \ldots, x_{n-1}, x_n + \textstyle\sum_{j=1}^{n-1}\alpha_j x_j) \text{ for } i = 1, \ldots, n-1$$

$$F'_n(x_1, \ldots, x_{n-1}, x_n) = F_n(x_1, \ldots, x_{n-1}, x_n + \textstyle\sum_{j=1}^{n-1}\alpha_j x_j)$$

Then the schemes S defined by the ideal $(F_1, \ldots, F_n)$ and S_1 defined by $(F'_1, \ldots, F'_n)$ correspond to each other via the automorphism φ of $\mathbb{P}^{n-1}_{\bar{k}}$ given by $\varphi(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, x_n + \sum_{j=1}^{n-1}\alpha_j x_j)$. In particular the sums of the Milnor numbers at the points of S and S_1 are the same.

Moreover, we have

$$\begin{gathered}\sum_{i=1}^{n} x_i \cdot F_i'(x_1,\ldots,x_{n-1},x_n) = \\ = \sum_{i=1}^{n-1} x_i \cdot (F_i(x_1,\ldots,x_{n-1},x_n + \sum_{j=1}^{n-1}\alpha_j x_j) + \\ +\alpha_i F_n(x_1,\ldots,x_{n-1},x_n + \sum_{j=1}^{n-1}\alpha_j x_j)) + \\ +x_n \cdot F_n(x_1,\ldots,x_{n-1},x_n + \sum_{j=1}^{n-1}\alpha_j x_j) = \\ = \sum_{i=1}^{n-1} x_i \cdot F_i(x_1,\ldots,x_{n-1},x_n + \sum_{j=1}^{n-1}\alpha_j x_j) + \\ +(x_n + \sum_{i=1}^{n-1}\alpha_i x_i) \cdot F_n(x_1,\ldots,x_{n-1},x_n + \sum_{j=1}^{n-1}\alpha_j x_j) = 0.\end{gathered}$$

If $P = (x_1,\ldots,x_n)$ is a common zero of $F_1',\ldots,F_n'$, then $\varphi(P) = (x_1,\ldots,x_{n-1},x_n + \sum_{j=1}^{n-1}\alpha_j x_j)$ is a common zero of $F_1,\ldots,F_n$ so, by the choice of the α_i, $\varphi(P)$ is not contained in the hyperplane $x_n - \sum_{j=1}^{n-1}\alpha_j x_j = 0$. Hence P is not contained in the hyperplane $x_n = 0$. Therefore we can assume, and we will, that none of the common zeroes of $F_1,\ldots,F_n$ is contained in the hyperplane $x_n = 0$.

Under this assumption, we claim that $F_1,\ldots,F_{n-1}$ form a regular sequence in $\bar{k}[x_1,\ldots,x_n]$ (compare [AS00b], Lemma 5.1). Otherwise, the subscheme defined by them in $\mathbb{P}^{n-1}_{\bar{k}}$ would have an irreducible component Y of dimension at least 1. From (2) we deduce that Y is contained in the hypersurface $x_n F_n = 0$. Being irreducible, it must be contained either in $x_n = 0$ or in $F_n = 0$. Furthermore, since it has dimension $\geqslant 1$, its intersections with both $x_n = 0$ and $F_n = 0$ are non-empty. So in either case, the intersection of $F_1,\ldots,F_{n-1},F_n$ and $x_n = 0$ would be non-empty, in contradiction with the assumption made above.

Denote by S_1 the subscheme of $\mathbb{P}^{n-1}_{\bar{k}}$ defined by $(F_1,\ldots,F_{n-1})$. The support of S_1 is the disjoint union of the points $P_1,\ldots,P_s$, which are contained in $F_n = 0$, and the points $P_{s+1},\ldots,P_{s+r}$, which are contained in $x_n = 0$. Let $\nu_1,\ldots,\nu_{s+r}$ be the corresponding Milnor numbers (i.e. $\nu_i = \dim_{\bar{k}} \mathcal{O}_{S_1,P_i}$). Since $F_1,\ldots,F_{n-1}$ form a regular sequence of polynomials of degree $d-1$, S_1 is a zero-dimensional complete intersection of degree $(d-1)^{n-1}$, therefore

$$\sum_{i=1}^{s+r} \nu_i = \dim_{\bar{k}} \Gamma(S_1, \mathcal{O}_{S_1}) = (d-1)^{n-1}.$$

For every $i = 1,\ldots,s$, x_n is invertible in the local ring $\mathcal{O}_{\mathbb{P}^{n-1},P_i}$. So from (2) we deduce that F_n is contained in the ideal generated by $F_1,\ldots,F_{n-1}$ in this local ring. Therefore

$$\begin{gathered}\mathcal{O}_{S,P_i} = \mathcal{O}_{\mathbb{P}^{n-1},P_i}/(F_1,\ldots,F_{n-1},F_n) = \\ = \mathcal{O}_{\mathbb{P}^{n-1},P_i}/(F_1,\ldots,F_{n-1}) = \mathcal{O}_{S_1,P_i}\end{gathered}$$

and in particular $\nu_i = \mu_i$.

On the other hand, for $i = 1,\ldots,r$, F_n is invertible in the local ring $\mathcal{O}_{\mathbb{P}^{n-1},P_{s+i}}$, so x_n is contained in the ideal generated by $F_1,\ldots,F_{n-1}$ in this local ring. Let $G_j = F_j(x_1,\ldots,x_{n-1},0)$, S_2 the subscheme of $\mathbb{P}^{n-2}_{\bar{k}}$ (which we identify with the hyperplane $x_n = 0$ in $\mathbb{P}^{n-1}_{\bar{k}}$) defined by $(G_1,\ldots,G_{n-1})$. The points $Q_1,\ldots,Q_r$ of S_2 are in one-to-one correspondence with $P_{s+1},\ldots,P_{s+r}$ via the inclusion $\mathbb{P}^{n-2}(\bar{k}) \hookrightarrow \mathbb{P}^{n-1}(\bar{k})$, and

$$\begin{gathered}\mathcal{O}_{S_2,Q_i} = \mathcal{O}_{\mathbb{P}^{n-2},Q_i}/(G_1,\ldots,G_{n-1}) = \\ = \mathcal{O}_{\mathbb{P}^{n-1},P_{s+i}}/(F_1,\ldots,F_{n-1},x_n) = \\ = \mathcal{O}_{\mathbb{P}^{n-1},P_{s+i}}/(F_1,\ldots,F_{n-1}) = \mathcal{O}_{S_1,P_{s+i}},\end{gathered}$$

so the Milnor numbers are the same.

Now $G_1, \ldots, G_{n-1}$ fall under the hypotheses of the lemma, so we can apply the induction hypothesis and deduce that $\sum_{i=s+1}^{s+r} \nu_i = ((d-1)^{n-1} - (-1)^{n-1})/d$. Therefore

$$\sum_{i=1}^{s} \mu_i = \sum_{i=1}^{s} \nu_i = \sum_{i=1}^{s+r} \nu_i - \sum_{i=s+1}^{s+r} \nu_i =$$

$$= (d-1)^{n-1} - ((d-1)^{n-1} - (-1)^{n-1})/d = ((d-1)^n - (-1)^n)/d.$$

□

Thus, under the hypotheses of Theorem 11, $L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}}$ is a polynomial of degree $(d-1)^n - (d-d')((d-1)^n - (-1)^n)/d = (d'(d-1)^n + (-1)^n(d-d'))/d$.

4. End of the proof of Theorem 2

Part (3) of the theorem is a direct consequence of the previous two parts via the trace formula and Deligne's theorem. So it suffices to prove (1) and (2). Fix a positive integer $d' < d$ prime to p such that $d'/d > p/(p+(p-1)^2)$. Denote by $\mathcal{P}_{d,d'}$ the affine space of all polynomials in $k[x_1, \ldots, x_n]$ of degree $\leqslant d$ whose homogeneous component of degree i is zero for all $d' < i < d$. Let $\pi_1 : \mathcal{P}_{d,d'} \times \mathbb{A}^n_k \to \mathcal{P}_{d,d'}$ be the projection and $ev : \mathcal{P}_{d,d'} \times \mathbb{A}^n_k \to \mathbb{A}^1_k$ the evaluation map. Let $K \in \mathcal{D}^b_c(\mathcal{P}_{d,d'}, \bar{\mathbb{Q}}_\ell)$ be the object $\mathrm{R}\pi_{1!} ev^\star \mathcal{L}_\psi[n + \dim \mathcal{P}_{d,d'}]$.

Lemma 13. *The object K is perverse and pure of weight $n + \dim \mathcal{P}_{d,d'}$.*

Proof. For $d' = d-1$ (i.e. when $\mathcal{P}_{d,d'}$ is the affine space of all polynomials of degree $\leqslant d$) this is ([Kat04], Part (1) of Theorem 3.1.2). We will see that the same proof works in general.

There is a natural finite map $\tau : \mathbb{A}^n_k \to \hat{\mathcal{P}}_{d,d'}$. Namely, for every $t \in \mathbb{A}^n(\bar{k})$, $\tau(t) \in \hat{\mathcal{P}}_{d,d'}(\bar{k})$ is the evaluation map at t, $ev(-,t) : \mathcal{P}_{d,d'}(\bar{k}) \to \bar{k}$. Since $\bar{\mathbb{Q}}_\ell[n]$ is perverse and pure of weight n on $\mathbb{A}^n_{\bar{k}}$, so is $\tau_\star \bar{\mathbb{Q}}_\ell[n]$ on $\hat{\mathcal{P}}_{d,d'}$. Its Fourier transform $T_\psi(\tau_\star \bar{\mathbb{Q}}_\ell[n]) \in \mathcal{D}^b_c(\mathcal{P}_{d,d'}, \bar{\mathbb{Q}}_\ell)$ with respect to ψ is K (by the very definition of K). Therefore K is perverse and pure of weight $n + \dim \mathcal{P}_{d,d'}$ (cf. [KL85], Section 2 or [KW01], Section III.8 for the definition and main properties of the Fourier transform). □

Notice that for every finite extension k'/k and every $f \in \mathcal{P}_{d,d'}(k')$, the trace of the geometric Frobenius element in $\mathrm{Gal}(\bar{k}/k')$ acting on the stalk of K at a geometric point over f is the sum

$$(-1)^{n+\dim \mathcal{P}_{d,d'}} \sum_{x \in k'^n} \psi(\mathrm{Trace}_{k'/k} f(x)).$$

Let $U \subset \mathcal{P}_{d,d'}$ be the maximal dense open set on which K has lisse cohomology sheaves. Then $\mathcal{H}^i(K)_{|U} = 0$ for $i \neq -\dim \mathcal{P}_{d,d'}$ and $\mathcal{F} := \mathcal{H}^{-\dim \mathcal{P}_{d,d'}}(K) = \mathrm{R}^n \pi_{1!} ev^\star \mathcal{L}_\psi$ is lisse and pure of weight n on U. Thus, for different finite extensions k'/k and polynomials $f \in U(k')$, the exponential sums $\sum_{x \in k'^n} \psi(\mathrm{Trace}_{k'/k} f(x))$ are pure of weight n and the same rank as $\mathcal{F}$.

Let $V \subset \mathcal{P}_{d,d'}$ (resp. $W \subset \mathcal{P}_{d,d'}$) be the dense open set of all polynomials f such that f_d defines a non-singular hypersurface on $\mathbb{P}^{n-1}_{\bar{k}}$ (resp. the dense open set of all f such that $\frac{\partial f_d}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_n}$ have a finite number of common zeroes in $\mathbb{P}^{n-1}_{\bar{k}}$ and the hypersurface $f_{d'} = 0$ does not contain any of them). We know that

i) For every $f \in V(k)$, we have $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = 0$ for $i \neq n, n+1$. For $i > n+1$, this is Proposition 5. For $i < n$ it is just Poincaré duality, since $\mathbb{A}^n_{\bar{k}}$ is smooth and $\mathcal{L}_{\psi(f)}$ is lisse.

ii) For every $f \in W(k)$, the L-function

$$L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}} = \prod_{i=0}^{2n} \det(1 - T \cdot F | \mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}))^{(-1)^{n+i}}$$

is a polynomial of degree $N := (d'(d-1)^n + (-1)^n(d-d'))/d$ (cf. Section 3).

Recall that a constructible $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on a smooth connected scheme S is said to be of *perverse origin* if there is a perverse sheaf $L \in \mathcal{D}^b_c(S, \bar{\mathbb{Q}}_\ell)$ such that $\mathcal{G} = \mathcal{H}^{-\dim S}(L)$ (cf. [Kat03], Section 1). In that case, we have the following (cf. [Kat03], Proposition 12):

Theorem 14. *The integer valued function defined by $s \mapsto$ rank $\mathcal{G}_{\bar{s}}$ on S (where $\bar{s}$ is a geometric point over s) is lower semicontinuous. In other words, the rank of $\mathcal{G}$ does not increase under specialization. In particular, the dimension of the stalk of $\mathcal{G}$ at any geometric point of S can never exceed the generic rank of $\mathcal{G}$. Moreover, the largest open set on which $\mathcal{G}$ is lisse is precisely the set where the rank of $\mathcal{G}$ is maximal (equal to the generic rank).*

Notice that on U the degree of the L-function is just the rank of $\mathcal{F}$. Therefore, on $U \cap W$, $\mathcal{F}$ is lisse of rank N. In particular, the generic rank of $\mathcal{F}$ is N. Since $\mathcal{F}$ is of perverse origin, from Theorem 14 we deduce that for every $f \in \mathcal{P}_{d,d'}(k)$ the cohomology group $\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ (which is the stalk of $\mathcal{F}$ at a geometric point over f) has dimension at most N.

Now let $f \in V \cap W(k)$. From (1) we have

$$L(T, \mathcal{L}_{\psi(f)})^{(-1)^{n+1}} = \frac{\det(1 - T \cdot F|\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}))}{\det(1 - T \cdot F|\mathrm{H}^{n+1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}))}.$$

On the other hand, from (2) we know that this is a polynomial of degree N. Therefore

$$\dim \mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) - \dim \mathrm{H}^{n+1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = N.$$

Since $\dim \mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) \leqslant N$ and $\dim \mathrm{H}^{n+1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ can not be negative, we conclude that $\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ has dimension N and the group $\mathrm{H}^{n+1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ vanishes.

From Theorem 14 we deduce that $\mathcal{F}$ is lisse on $V \cap W$, since it has maximal rank there. Furthermore, it is pure of weight n, because $K_{|V\cap W} = \mathcal{F}_{|V \cap W}[\dim \mathcal{P}_{d,d'}]$ is pure of weight $n + \dim \mathcal{P}_{d,d'}$. This completes the proof of Theorem 2.

Remarks 15. When $p = d = 2$ and n is even, the sum $\sum_{x \in k^n} \psi(f(x))$ is known to be pure of weight n and rank 1 if the hypersurface defined by $f_2 = 0$ is non-singular (cf. [AS00a], Section 6).

Remarks 16. We will see now that, for the rank formula in Theorem 2 to hold, the restriction $d'/d > p/(p + (p-1)^2)$ (or at least some milder lower bound for d') is essential. More precisely, let $d = p^a d_0$, where d_0 is prime to p. We claim that the formula is not true for $d' < d_0$. Let $\mathcal{P}_{d,-1}$ be the affine space of homogeneous polynomials of degree d. Let A (resp. B) be the generic rank of $\mathrm{R}^n\pi_{1!}ev^\star\mathcal{L}_\psi$ on $\mathcal{P}_{d,-1}$ (resp. $\mathcal{P}_{d,d'}$). By ([Kat04], Theorem 3.6.5) we know

$$A = \frac{(d-1)^n + (-1)^n(d-1)}{d} + \frac{d_0 - 1}{d}((d-1)^n - (-1)^n).$$

On the other hand, since $\mathcal{P}_{d,-1} \subset \mathcal{P}_{d,d'}$ and $\mathrm{R}^n\pi_{1!}ev^\star\mathcal{L}_\psi$ is of perverse origin, we have the inequality $A \leqslant B$. But it is easy to see that the inequality $A \leqslant (d'(d-1)^n + (-1)^n(d-d'))/d$ is equivalent to $d_0 \leqslant d'$. Therefore if $d' < d_0$ we can not have $B = (d'(d-1)^n + (-1)^n(d-d'))/d$.

5. Proof of Theorem 4

We will use a similar procedure to prove the second result, therefore we will first show

Proposition 17. *Suppose that d is prime to p and the hypersurface defined in $\mathbb{P}^{n-1}_{\bar{k}}$ by the equation $f_d = 0$ has at worst isolated singularities. Then $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = 0$ for $i > n+1$.*

Proof. This is already proven, although not explicitly stated, in [Kat99], Theorem 16. Let $\tilde{X}$ be the incidence variety defined in $\mathbb{P}^n_k \times \mathbb{A}^1_k$ (with coordinates $X_0, X_1, \ldots, X_n$ for the first factor and λ for the second one) by the vanishing of $F - \lambda X_0^d$, where F is again the homogenization of f with respect to the variable X_0. Let $\tilde{f} : \tilde{X} \to \mathbb{A}^1_k$ be the restriction of the second projection. The affine space $\mathbb{A}^n_k$ can be naturally embedded as a dense open subset of $\tilde{X}$ and, as in Lemma 7, there is a quasi-isomorphism

$$\mathrm{R}\Gamma_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\tilde{X} \otimes \bar{k}, \mathcal{L}_{\psi(f)})$$

where we also denote by $\mathcal{L}_{\psi(f)}$ the pull-back of $\mathcal{L}_\psi$ to $\tilde{X}$ by $\tilde{f}$. The proof of [Kat99], Theorem 16, applied to $X = \mathbb{P}^n_k$, $L = X_0$ and $H = F$ (hence $\delta = 0$, $\varepsilon = -1$) shows that

$$\mathrm{H}^a_c(\mathbb{A}^1_{\bar{k}}, \mathrm{R}^b \tilde{f}_\star \bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\psi) = 0$$

for $a + b \geqslant n + 2$. In particular, the spectral sequence

$$\mathrm{H}^a_c(\mathbb{A}^1_{\bar{k}}, \mathrm{R}^b \tilde{f}_\star \bar{\mathbb{Q}}_\ell \otimes \mathcal{L}_\psi) \Rightarrow \mathrm{H}^{a+b}_c(\tilde{X} \otimes \bar{k}, \mathcal{L}_{\psi(f)})$$

implies that $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) \cong \mathrm{H}^i_c(\tilde{X} \otimes \bar{k}, \mathcal{L}_{\psi(f)}) = 0$ for $i > n + 1$. ☐

We will think of the homogeneous form f_d and the integer d' as being fixed, and the degree $\leqslant d'$ part of f, which we will call g, as being variable. Let $\mathcal{P}_{d'}$ be the affine space of all polynomials of degree $\leqslant d'$. Let $\pi_1 : \mathcal{P}_{d'} \times \mathbb{A}^n_k \to \mathcal{P}_{d'}$ be the projection and $ev_{f_d} : \mathcal{P}_{d'} \times \mathbb{A}^n_k \to \mathbb{A}^1_k$ the map $(g, x) \mapsto f_d(x) + g(x)$. Let $K \in \mathcal{D}^b_c(\mathcal{P}_{d'}, \bar{\mathbb{Q}}_\ell)$ be the object $\mathrm{R}\pi_{1!} ev^\star_{f_d} \mathcal{L}_\psi[n + \dim \mathcal{P}_{d'}]$. Exactly as in Lemma 13 one shows

LEMMA 18. *The object K is perverse and pure of weight $n + \dim \mathcal{P}_{d'}$.*

For every finite extension k'/k and every $g \in \mathcal{P}_{d'}(k')$, the trace of the geometric Frobenius element of $\mathrm{Gal}(\bar{k}/k')$ acting on the stalk of K at a geometric point over g is the sum

$$(-1)^{n+\dim \mathcal{P}_{d'}} \sum_{x \in k'^n} \psi(\mathrm{Trace}_{k'/k}(f_d(x) + g(x))).$$

Now let $V \subset \mathcal{P}_{d'}$ be the open set of all polynomials g whose homogeneous component of degree d' is non-zero and the hypersurface it defines in $\mathbb{P}^{n-1}_{\bar{k}}$ does not contain any of the singularities of $f_d = 0$. For every $g \in V(k)$ we have $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)}) = 0$ for $i \neq n, n+1$, by Proposition 17 and Poincaré duality. On the other hand, by ([AS00b], Theorem 1.10 and Proposition 6.5) we know that

$$L(T, \mathcal{L}_{\psi(f_d+g)})^{(-1)^{n+1}} = \frac{\det(1 - T \cdot F | \mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)}))}{\det(1 - T \cdot F | \mathrm{H}^{n+1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)}))}$$

is a polynomial of degree $N' := (d-1)^n - (d - d') \sum_{i=1}^s \mu_i$.

Let $U \subset \mathcal{P}_{d'}$ be a dense open subset where K has lisse cohomology sheaves. Then $\mathcal{H}^i(K)_{|U} = 0$ for $i \neq -\dim \mathcal{P}_{d'}$ and $\mathcal{F} := \mathcal{H}^{-\dim \mathcal{P}_{d'}}(K) = \mathrm{R}^n \pi_{1!} ev^\star_{f_d} \mathcal{L}_\psi$ is lisse of rank N' and pure of weight n on U. Being of perverse origin, by Theorem 14 this implies that for any $g \in \mathcal{P}_{d'}(k)$ the cohomology group $\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)})$ has dimension at most N'. Moreover, if $g \in V(k)$, since

$$\dim \mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)}) - \dim \mathrm{H}^{n+1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)}) = N',$$

we conclude that $\mathrm{H}^{n+1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)}) = 0$ and $\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_d+g)})$ has dimension N'. In particular, $\mathcal{F}_{|V}$ has constant rank N', so it is lisse by Theorem 14. Therefore, since $K_{|V} = \mathcal{F}_{|V}[\dim \mathcal{P}_{d'}]$ and K is pure of weight $n + \dim \mathcal{P}_{d'}$, the sheaf $\mathcal{F}$ must be pure of weight n on V. This concludes the proof of Theorem 4.

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