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COMPLETE CHARACTERIZATIONS OF KADEC-KLEE PROPERTIES IN ORLICZ SPACES

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Abstract. We study the connections between the Kadec-Klee property for local convergence in measure H_ℓ, the Kadec-Klee property for global convergence in measure H_g and the Δ_2-condition for Orlicz function spaces L^φ equipped with either the Luxemburg norm $\|\cdot\|_\varphi$ or the Orlicz norm $\|\cdot\|_\varphi^0$. Nominally, we prove that for $(L^\varphi, \|\cdot\|_\varphi)$ the conditions: φ satisfies an appropriate Δ_2-condition and $L^\varphi \in H_\ell$, $L^\varphi \in H_g$ are equivalent, although $L^\varphi \in H_g$ is not equivalent to $E^\varphi \in H_g$. In contrast, we also prove that, in the case of a non-atomic infinite measure space, properties H_ℓ and H_g for $(L^\varphi, \|\cdot\|_\varphi^0)$ do not coincide. More precisely, we prove that if φ vanishes only at zero, then both these properties coincide and they are equivalent to $\varphi \in \Delta_2$. However, if φ vanishes outside zero, then $(L^\varphi, \|\cdot\|_\varphi^0) \in H_g$ if and only if $\varphi \in \Delta_2(\infty)$. Since in the last case $(L^\varphi, \|\ \|_\varphi^0)$ is not order continuous, properties H_ℓ and H_g differ. Analogous results are also proved for the subspace E^φ of L^φ. It is also worth mentioning that the criteria for $E^\varphi \in H_\ell$ as well as for $E^\varphi \in H_g$ were not previously known. It follows from the criteria that the appropriate regularity Δ_2-condition for φ is necessary for $E^\varphi \in H_\ell$, $E_0^\varphi \in H_\ell$, $E^\varphi \in H_g$ and $E_0^\varphi \in H_g$ although these spaces are order continuous for any φ.

1. Introduction

If $(E, \|\cdot\|_E)$ is a normed linear space, then E is said to have the Kadec-Klee property ($E \in H$) if sequential weak convergence on the unit sphere coincides with norm convergence. It is well known that the classical L_p-spaces, $1 < p < \infty$,

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have the Kadec-Klee property (see [23], [24]). Although the space $L_1(0,1)$ fails to have the Kadec-Klee property, Riesz showed that each sequence $\{x_n\}$ on the unit sphere of an L_p-space, $1 \leq p < \infty$, convergent almost everywhere to x from the unit sphere of L_p, is also norm-convergent.

Throughout this paper (Ω, Σ, μ) denotes a σ-finite complete measure space. Let E be a Banach function lattice over on (Ω, Σ, μ) (see [14]). The positive cone E^+ of E is defined by $E^+ = \{x \in E : x \geq 0\}$. E is said to have the Kadec-Klee property for global convergence in measure ($E \in H_g$), if for all $\{x_n\}$ and x in the unit sphere of E whenever $x_n \to x$ globally in measure on Ω, then $\|x_n - x\| \to 0$. E is said to have the Kadec-Klee property for local convergence in measure (i.e. convergence in measure on subsets of finite measure) ($E \in H_l$ for short), if for all $\{x_n\}$ and x in the unit sphere of E whenever $x_n \to x$ locally in measure on Ω, then $\|x_n - x\| \to 0$.

These properties were investigated in [4] and [20] for symmetric spaces defined on any interval $[0, \alpha), 0 < \alpha \leq \infty$, and on the interval $[0,1)$, respectively.

In this paper we study the connections between the Kadec-Klee property for local convergence in measure, the Kadec-Klee property for global convergence in measure and the Δ_2-condition in Orlicz function spaces and their subspaces of order continuous elements equipped with either the Luxemburg norm or the Orlicz norm.

We start by fixing some notations. In the following $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{N}$ will stand for the sets of real numbers, nonnegative numbers and positive integers, respectively. By $\varphi : \mathbb{R} \to [0, \infty]$ we denote an Orlicz function, i.e., φ is convex, even, left continuous on the whole of $\mathbb{R}^+$, $\varphi(0) = 0$ and φ is not identically equal to zero. For any Orlicz function φ we let

$$a_\varphi := \sup\{u \geq 0 : \varphi(u) = 0\}$$

and

$$c_\varphi := \sup\{u > 0 : \varphi(u) < \infty\}.$$

We shall say that an Orlicz function φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$ (at infinity) [at zero] if there are positive constants K and u_0 with $0 < \varphi(u_0) < \infty$ such that $\varphi(2u) \leq K\varphi(u)$ holds for all $u \in \mathbb{R}$ (for every $|u| \geq u_0$) [for every $|u| \leq u_0$]. Obviously, φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$ if and only if it satisfies the Δ_2-condition at zero and at infinity. We denote these conditions by $\varphi \in \Delta_2$ ($\varphi \in \Delta_2(\infty)$), [$\varphi \in \Delta_2(0)$], respectively.

For any Orlicz function φ the statement "*φ-satisfies the suitable Δ_2-condition*", will mean that:

φ satisfies the Δ_2-condition for all u if μ is nonatomic and infinite.

φ satisfies the Δ_2-condition at infinity if μ is nonatomic and finite.

φ satisfies the Δ_2-condition at 0 if μ is the counting measure.

In the following, $L^0(\mu)$ will stand for the space of all (equivalence classes of) Σ-measurable real functions defined on Ω. For a given Orlicz function φ we define on $L^0(\mu)$ a convex functional (called a pseudomodular, see [21]) by

$$I_\varphi(x) = \int_\Omega \varphi(x(t))d\mu.$$

The Orlicz space $L^\varphi(\mu)$ is defined to be the set of all $x \in L^0(\mu)$ such that $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$ depending on x. We endow $L^\varphi(\mu)$ with the Luxemburg norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \le 1\}$$

and with the Orlicz norm

$$\|x\|_\varphi^0 = \sup\Big\{\int_\Omega |x(t)y(t)|d\mu :\ y \in L^{\varphi^*}(\mu), I_{\varphi^*}(y) \le 1\Big\},$$

where the function φ^* is defined by the formula

$$\varphi^*(u) = \sup\{|u|v - \varphi(v) : v \ge 0\}$$

and called complementary to φ in the sense of Young.

It is well known that if φ is finitely valued and satisfies the condition

$$\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty,$$

then the following Amemiya formula for the Orlicz norm is true (see [22])

$$\|x\|_\varphi^0 = \inf\Big\{\frac{1}{k}\big(1 + \int_\Omega \varphi(k\,x(t))d\mu\big) : k > 0\Big\}.$$

Moreover, for any $x \in L^\varphi(\mu) \setminus \{0\}$ there is a positive number k^* at which the infimum is attained, that is

$$\|x\|_\varphi^0 = \frac{1}{k^*}\Big(1 + \int_\Omega \varphi(k^* x(t))d\mu\Big).$$

In [11] it is proved that the Amemiya formula for the Orlicz norm is true for any Orlicz function and in [12] it is proved that Orlicz spaces generated by Orlicz functions satisfying the Δ_2-condition have the Kadec-Klee property for local convergence in measure.

In the sequel we will need some results concerning Banach lattices with order continuous norms. Recall that a Banach lattice E is said to be *order continuous* (OC for short), if $x_n \downarrow 0$ implies $\|x_n\| \to 0$ (see [17]).

For the definition of a symmetric space E we refer to [15] (cf. also [1]). Let us only recall that for $x \in E$, we denote by x^* the nonincreasing rearrangement of x (see section 3).

The subspace E^φ of L^φ is defined as the space of all order continuous elements in L^φ, where an element $x \in L^\varphi$ is said to be order continuous if $\|x_n\|_\varphi \to 0$ whenever $0 \le x_n \le |x|$ for any $n \in \mathbb{N}$ and $x_n \to 0$ μ-a.e. in Ω. It is well known that if μ is nonatomic, then $E^\varphi \neq \{0\}$ if and only if φ is finitely valued and that $E^\varphi = L^\varphi$ if and only if $\varphi \in \Delta_2$ (see [3], [19], [21] or [22]). It is also known that in the case of any nonatomic σ-finite measure, $x \in E^\varphi$ if and only if $I_\varphi(\lambda x) < \infty$ for any $\lambda > 0$ (see [5]).

Recall that a Banach lattice $(E, \le)$ is called strictly monotone ($E \in STM$ for short) if for any x, $y \in E$ with $0 \le y \le x$ and $y \neq x$ we have $\|y\| < \|x\|$. E is called upper (respectively lower) locally uniformly monotone ($E \in ULU$ M, respectively $E \in LLUM$) if for any $x \in E$ and $\{x_n\} \subset E$, the conditions $0 \le x \le x_n$ (respectively $0 \le x_n \le x$) and $\|x_n\| \to \|x\|$ imply $\|x_n - x\| \to 0$ (see [2], [4], [10], [16] and [20]).

The following lemma will be useful in what follows. An easy proof may be found in [14].

Lemma 1.1. *Let E be a Banach function lattice over a σ-finite measure space. If $x_n \to x$ in E, then there exist $y \in E^+$, $\{x_{n_k}\} \subset \{x_n\}$ and $\varepsilon_{n_k} \subset \mathbb{R}^+$ with $\varepsilon_{n_k} \downarrow 0$ such that $|x_{n_k} - x| \le \varepsilon_{n_k} y$.*

We will also use the following remarkable result from [4].

Theorem 1.1. *If E is a separable symmetric space on the Lebesgue measure space $([0,\alpha), m)$, where $0 < \alpha \le \infty$, then the following are equivalent:*

(i) *E is strictly monotone and E has the property H_g.*
(ii) *E is upper locally uniformly monotone.*
(iii) *For any $x \in E$ and $\{x_n\} \subset E$ such that $0 \le x^* \le x_n^*$, for $n \in \mathbb{N}$, and $\|x_n\| \to \|x\|$ we have $\|x_n^* - x^*\| \to 0$.*

2. Luxemburg Norm

We start with the following general result.

Proposition 2.1. *If E is not an order continuous Banach function lattice, then $E \notin H_\ell$ and $E \notin LLUM$.*

Proof. If E is not order continuous, it is well known (see [17]) that there exists a sequence $\{x_n\}$ in E^+ with $\|x_n\| = 1$ and $\mathrm{supp} x_n \cap \mathrm{supp} x_m = \emptyset$ (which implies that $x_n \to 0$ μ-a.e.) and a function $x \in E^+$ such that $x_n \le x$ for any $n \in \mathbb{N}$.

Define

$$y = \sum_{n=1}^{\infty} x_n \quad \text{and} \quad y_n = y - x_n.$$

If we can show that $y_n \to y$ weakly, or equivalently $x_n \to 0$ weakly, the implication $\|y_n\| \to \|y\|$ can be deduced because we also have $0 \le y_n \le y$.

Let E^* denote the dual space of E. For any nonnegative $f \in E^*$ and for all $k \in N$ we have

$$\sum_{n=1}^{k} f(x_n) = f\Big(\sum_{n=1}^{k} x_n\Big) \le x^*(x),$$

whence it follows that $\sum_{n=1}^{\infty} f(x_n)$ converges, and so $f(x_n) \to 0$ as $n \to \infty$. Since every $f \in E^*$ can be written as a difference of two nonnegative functionals, we have shown that $x_n \to 0$ weakly. Therefore $\|y_n\| \to \|y\|$.

We also have that $y_n \to y$ μ-a.e.. However

$$\|y - y_n\| = \|x_n\| = 1$$

for any $n \in \mathbb{N}$, which means that $E \notin H_\ell$. The same proof gives $E \notin LLUM$. □

Corollary 2.1. *Let φ be an Orlicz function. If φ does not satisfy the suitable Δ_2-condition, then $L^\varphi \notin H_\ell$.*

Proof. The proof follows from Proposition 2.1 and the fact that the space $L^\varphi(\mu)$ is an order continuous Banach function lattice if and only if φ satisfies the suitable Δ_2-condition (see [5], [6], [13] and [25]). □

If μ is a finite measure, the Kadec-Klee properties for local and global convergence in measure are equivalent. So, in most of the results in this paper we will restrict ourselves to studying the case of an infinite measure.

Proposition 2.2. *Let (Ω, Σ, μ) be a nonatomic and infinite measure space and φ be an Orlicz function with $a_\varphi > 0$ and $c_\varphi = \infty$. If L^φ is endowed with the Luxemburg norm, then $L^\varphi \notin H_g$.*

Proof. Consider a sequence $\{A_n\}$ of measurable sets such that

$$\mu(A_n) = 2^{-n}.$$

Let $A = \bigcup A_n$ and define

$$x = a_\varphi \chi_{\Omega \setminus A} \quad \text{and} \quad x_n = a_\varphi \chi_{\Omega \setminus A} + b_n \chi_{A_n},$$

where $b_n > 0$ and $\varphi(b_n)\mu(A_n) = 1$. Such a sequence $\{b_n\}$ exists by the assumption that $c_\varphi = \infty$.

We first note that $x_n - x = b_n\chi_{A_n}$. Therefore, $x_n \to x$ globally in measure. Now we are going to show that

$$\|x\|_\varphi = \|x_n\|_\varphi = 1.$$

We have

$$I_\varphi(x) \le I_\varphi(x_n) = \varphi(a_\varphi)\mu(\Omega \setminus A) + \varphi(b_n)\mu(A_n) = 1,$$

whence (see [22])

$$\|x\|_\varphi \le \|x_n\|_\varphi = 1. \tag{2.1}$$

On the other hand for all $\lambda > 1$,

$$I_\varphi(\lambda x) = \varphi(\lambda a_\varphi)\mu(\Omega \setminus A) = \infty$$

because $\mu(\Omega \setminus A) = \infty$. So $\|\lambda x\|_\varphi \ge 1$ for all $\lambda > 1$, which implies $\|x\|_\varphi \ge 1$. Hence, by using (2.1), we obtain

$$\|x\|_\varphi = \|x_n\|_\varphi = 1.$$

In order to finish the proof, observe that

$$I_\varphi(x_n - x) = \varphi(b_n)\mu(A_n) = 1,$$

which implies that $\|x_n - x\|_\varphi = 1$ for all $n \in \mathbb{N}$. □

Proposition 2.3. *Let φ be an Orlicz function with $a_\varphi = 0$ and $c_\varphi = \infty$ and E^φ be endowed with the Luxemburg norm. Assume φ does not satisfy the Δ_2-condition at 0. Then $E^\varphi \notin H_g$ whenever (Ω, Σ, μ) is a nonatomic and infinite measure space.*

Proof. Since $\varphi \notin \Delta_2(0)$, there exists a sequence $\{u_n\}$ of positive real numbers with $u_n \to 0$ and

$$\varphi(2u_n) > 2^n\varphi(u_n)$$

for all $n \in \mathbb{N}$.

Let $x \in E^\varphi$, $x \ge 0$ and $\|x\|_\varphi = 1$. We claim that there exists a sequence $\{A_n\}$ in Σ such that $\mu(A_n) = \infty$ and $I_\varphi(2x\chi_{A_n}) \le 2^{-n}$ for all $n \in \mathbb{N}$. Indeed, by σ-finiteness of the measure space, there exists a sequence $\{C_n\}$ in Σ such that $C_n \uparrow$, $0 < \mu(C_n) < \infty$ for every $n \in \mathbb{N}$ and $\bigcup_n C_n = \Omega$. The Lebesgue dominated convergence theorem yields $I_\varphi(2x\chi_{\Omega\setminus C_n}) \to 0$ as $n \to \infty$.

Since $\mu(\Omega \setminus C_n) = \infty$ for any $n \in \mathbb{N}$, the claim is proved for $\{A_n\}$ being a subsequence of the sequence $\{\Omega \setminus C_n\}$.

Let $B_n \subset A_n$ be for any $n \in \mathbb{N}$ such that $x\chi_{B_n} \in L^\infty$ and $\varphi(u_n)\mu(B_n) = 2^{-n}$. Define

$$x_n = x + \frac{u_n}{2}\chi_{B_n} = x\chi_{\Omega\setminus B_n} + (x + \frac{u_n}{2})\chi_{B_n}.$$

Then $x_n \in E^\varphi$ for all $n \in \mathbb{N}$. Since $x_n \geq x \geq 0$, we have $\|x_n\|_\varphi \geq \|x\|_\varphi = 1$. On the other hand

$$\begin{aligned} I_\varphi(x_n) &= I_\varphi(x\chi_{\Omega\setminus B_n}) + I_\varphi\left(\frac{2x+u_n}{2}\chi_{B_n}\right) \\ &\leq I_\varphi(x\chi_{\Omega\setminus B_n} + \frac{1}{2}\{I_\varphi(2x\chi_{B_n}) + I_\varphi(u_n\chi_{B_n})\} \\ &\leq 1 + \frac{1}{2}\{2^{-n} + 2^{-n}\} = 1 + 2^{-n}, \end{aligned}$$

whence $1 \leq \|x_n\|_\varphi \leq 1+2^{-n}$, i.e. $\|x_n\|_\varphi \to \|x\|_\varphi = 1$. Since $x_n - x = \frac{1}{2}u_n\chi_{B_n}$ and $u_n \to 0$, we conclude that $x_n \to x$ globally in measure. However, $I_\varphi(4(x_n - x)) = \varphi(2u_n)\mu(B_n) > 2^n\varphi(u_n)\mu(B_n) = 1$, whence $\|x_n - x\|_\varphi \geq \frac{1}{4}$ for all $n \in \mathbb{N}$. This yields $E^\varphi \notin H_g$. □

Proposition 2.4. *If (Ω, Σ, μ) is a nonatomic measure space and φ is an Orlicz function with $c_\varphi < \infty$, then L^φ equipped with the Luxemburg norm fails to have property H_g.*

Proof. Choose a sequence $\{A_n\}$ of measurable and pairwise disjoint sets such that $\mu(A_n) > 0$ for any $n \in \mathbb{N}$ and $\sum_{n=1}^\infty \varphi(b_n)\mu(A_n) \leq 1$, where $0 < b_n \uparrow c_\varphi$ as $n \to \infty$. Define

$$x_n = \sum_{k\neq n} b_k\chi_{A_k} \quad \text{and} \quad x = \sum_{k=1}^\infty b_k\chi_{A_k}.$$

Then we have

$$I_\varphi(x_n) \leq I_\varphi(x) \leq 1.$$

On the other hand for any $\lambda > 1$, we have

$$I_\varphi(\lambda x) \geq I_\varphi(\lambda x_n) = \infty.$$

Therefore, $\|x\|_\varphi = 1$ and $\|x_n\|_\varphi = 1$ for all $n \in \mathbb{N}$.

Since $\mu(A_n) \to 0$ as $n \to \infty$, we have that $x_n \to x$ globally in measure. However, for any $\lambda > 1$, $I_\varphi(\lambda(x - x_n)) = \infty$ for n large enough. This implies that $\|x - x_n\|_\varphi \geq \frac{1}{\lambda}$ for n large enough, and consequently $L^\varphi \notin H_g$. □

Proposition 2.5. *Let φ be an Orlicz function, (Ω, Σ, μ) be a nonatomic measure space and E^φ be endowed with the Luxemburg norm. Assume that φ does not satisfy the Δ_2-condition at ∞ and $c_\varphi = \infty$. Then $E^\varphi \notin H_g$.*

Proof. If we assume that $\varphi \notin \Delta_2(\infty)$, then for all $c > 0$ and $n \in \mathbb{N}$ there exists $u_{n,c} \geq n$ such that

$$\varphi(2u_{n,c}) > c\varphi(u_{n,c}).$$

Taking $c = 2^{n+1}$ gives the existence of $u_n \geq n$ such that

$$\varphi(2u_n) > 2^{n+1}\varphi(u_n) \text{ for all } n \in \mathbb{N}.$$

Take any $x \in E^\varphi$ such that $x \geq 0$ and $\|x\|_\varphi = 1$. In the same way as in the proof of Proposition 2.3 one can find a sequence $\{A_n\} \subset \Sigma$ and a subsequence $\{v_n\}$ of $\{u_n\}$ such that $x\chi_{A_n} \in L^\infty$, $I_\varphi(x\chi_{A_n}) < 2^{-n}$ and $\varphi(v_n)\mu(A_n) = 2^{-n}$. Defining $x_n = x + \frac{v_n}{2}\chi_{A_n}$, we have

$$\begin{aligned} 1 = I_\varphi(x) &\leq I_\varphi(x_n) = I_\varphi(x\chi_{\Omega\setminus A_n}) + I_\varphi(\frac{1}{2}(x\chi_{A_n} + v_n\chi_{A_n})) \\ &\leq 1 + \frac{1}{2}\{I_\varphi(x\chi_{A_n}) + \varphi_n(v_n)\mu(A_n)\} = 1 + 2^{-n}, \end{aligned}$$

whence $1 \leq \|x\|_\varphi \leq \|x_n\|_\varphi \leq 1 + 2^{-n}$ for any $n \in \mathbb{N}$. Moreover, from the equality $x_n - x = \frac{v_n}{2}\chi_{A_n}$ and the fact that $\mu(A_n) \to 0$ as $n \to \infty$ it follows that $x_n \to x$ globally in measure. However, $x_n \in E^\varphi$ and $I_\varphi(4(x_n - x)) = \varphi(2v_n)\mu(A_n) > 2^n\varphi(v_n)\mu(A_n) = 1$, whence $\|x_n - x\|_\varphi \geq \frac{1}{4}$ for all $n \in \mathbb{N}$. This means that $E^\varphi \notin H_g$. □

Proposition 2.6. *Let (Ω, Σ, μ) be a nonatomic and infinite measure space and φ be an Orlicz function with $a_\varphi > 0$ and $c_\varphi = \infty$. Then $E^\varphi \notin H_\ell$.*

Proof. Devide Ω into $A \cup B$, where $\mu(A) = \mu(B) = \infty$ and $A \cap B = \emptyset$. Let $A = \cup_{n=1}^\infty A_n$, where A_n are pairwise disjoint and $\mu(A_n) \geq 1$ for any $n \in \mathbb{N}$. Take $a_0 \geq 2a_\varphi$ and $B_0 \in \Sigma \cap B$ such that $\varphi(a_0)\mu(B_0) = 1$. Define

$$x = a_0\chi_{B_0} \text{ and } x_n = x + a_\varphi\chi_{A_n}.$$

Then $I_\varphi(x) = I_\varphi(x_n) = 1$, whence $\|x\| = \|x_n\| = 1$ for any $n \in \mathbb{N}$. Since the sets A_n are pairwise disjoint, we have $x_n \to x$ μ-a.e.. However, $I_\varphi(2(x_n - x)) = \varphi(2a_\varphi)\mu(A_n) \geq \varphi(2a_\varphi)$, whence $\|x_n - x\|_\varphi \geq (1/2)\min(1, \varphi(2a_\varphi)) > 0$. Since $x \in E^\varphi$ and $x_n \in E^\varphi$ for each $n \in \mathbb{N}$, the proof is finished. □

Proposition 2.7. *Let (Ω, Σ, μ) be a nonatomic and infinite measure space and φ be an Orlicz function with $a_\varphi > 0$ and $\varphi \in \Delta_2(\infty)$. Then $E^\varphi \in H_g$.*

Proof. Assume that $x \in S(E^\varphi)$, $\{x_n\} \subset S(E^\varphi)$ and $x_n \to x$ globally in measure. We have $I_\varphi(x) = I_\varphi(x_n) = 1$ for each $n \in \mathbb{N}$. First we will prove that

$$I_\varphi(x_n\chi_A) \to I_\varphi(x\chi_A) \text{ for any } A \in \Sigma. \tag{2.2}$$

By the σ-finiteness of μ and the fact that $x_n \to x$ globally in measure we know that $\{x_n\}$ contains a subsequence convergent to x μ-a.e.. Assume without loss of generality that $x_n \to x$ μ-a.e.. Since $\varphi \in \Delta_2(\infty)$ and consequently φ is continuous, we have $\|\varphi\circ x_n\|_{L^1}=I_\varphi(x_n)=I_\varphi(x)=\|\varphi\circ x\|_{L^1}$ and $\varphi\circ x_n\to\varphi\circ x$ μ-a.e.. Since $L^1\in H_\ell$, we get $\|\varphi\circ x_n-\varphi\circ x\|_{L^1}\to 0$. Hence for any $A\in\Sigma$, we get $\|\varphi\circ x_n\chi_A-\varphi\circ x\chi_A\|_{L^1}\to 0$, whence $I_\varphi(x_n\chi_A)\to I_\varphi(x\chi_A)$. Now we are ready to prove that $\|x_n-x\|_\varphi\to 0$. Since $E\in H_g$ if and only if $E^+\in H_g$ for any order continuous Banach function lattice (see Proposition 1 in [12], where it was proved that $E\in H_\ell$ if and only if $E^+\in H_\ell$ whenever E is order continuous, and observe that the proof works also for H_g in place of H_ℓ), we may assume in the remaining part of the proof that $x_n\geq 0$ and $x\geq 0$. We need to show that $I_\varphi(\lambda(x_n-x))\to 0$ for any $\lambda>0$. Choose any $\lambda>0$ and define for any $\mathbb{N}$

$$A_n=\{t\in\Omega:|x_n(t)-x(t)|>a_\varphi/\lambda\}.$$

We know that $\mu(A_n)\to 0$ as $n\to\infty$, so passing to a subsequence if necessary, we may assume without loss of generality that $\mu(\cup_{n=1}^\infty A_n)<\infty$.

Let $A=\cup_{n=1}^\infty A_n$ and $A'=\Omega\setminus A$. Then $\varphi(\lambda|x_n(t)-x(t)|)=0$ for any $n\in\mathbb{N}$ and $t\in A'$. Consequently, $I_\varphi(\lambda(x_n-x))\chi_{A'})=0$ for any $n\in\mathbb{N}$. To finish the proof we only need to show that $I_\varphi(\lambda(x_n-x))\chi_A)\to 0$. Let us prove first that $I_\varphi((x_n-x)\chi_A)\to 0$. By the superadditivity of φ on $\mathbb{R}^+$ and the fact that $x_n\geq 0$ and $x\geq 0$ μ-a.e., we have

$$\varphi\circ((x_n-x)\chi_A)\leq|\varphi\circ(x_n\chi_A)-\varphi\circ(x_n\chi_A)|\,. \tag{2.3}$$

By condition (2.2) and the fact that $L^1\in H_\ell$, we have

$$\|\varphi\circ(x_n\chi_A)-\varphi\circ(x_n\chi_A)\|_{L^1}\to 0.$$

So, inequality (2.3) gives $\|\varphi\circ((x_n-x)\chi_A)\|_{L^1}=I_\varphi(x_n-x)\chi_A)\to 0$. Given $\lambda>0$, we may assume passing to a subsequence if necessary that $\varphi(\lambda(x_n-x)\chi_A)\to 0$ $\mu-a.e.$. Moreover, by inequality (2.3), Lemma 1.1 and the assumption that $\varphi\in\Delta_2(\infty)$, it follows that this sequence has an integrable majorant. Consequently, the Lebesgue dominated convergence theorem yields $I_\varphi(\lambda(x_n-x))\to 0$. This finishes the proof. □

We remark that the only reason that Proposition 2.7 is not true for L^φ instead of E^φ is that if φ does not satisfy suitable Δ_2-condition, then for $x\in L^\varphi$ it can happen that $\|x\|_\varphi=1$ and $I_\varphi(x)<1$.

The previous results can be summarized in the following theorem.

Theorem 2.1. *Let (Ω,Σ,μ) be a nonatomic measure space, φ be an arbitrary Orlicz function, and $(L^\varphi,\|\cdot\|_\varphi)$ be the Orlicz space endowed with the Luxemburg norm. The following statements are equivalent:*

1. $\varphi\in\Delta_2$.

2. $L^\varphi(\mu) \in H_\ell$.
3. $L^\varphi(\mu) \in H_g$.

Assuming additionally that $c_\varphi = \infty$, we have:

4. $E^\varphi \in H_\ell$ *if and only if* $L^\varphi \in H_\ell$.
5. $E^\varphi \in H_g$ *if and only if, either*
   (i) $a_\varphi = 0$ *and* $\varphi \in \Delta_2$, *or*
   (ii) $a_\varphi > 0$ *and* $\varphi \in \Delta_2(\infty)$.

PROOF. It is known (see [5] and [12]) that $L^\varphi \in H_\ell$ if and only if $\varphi \in \Delta_2$. The implication $2 \Rightarrow 3$ is obvious. By Corollary 2.1 and Propositions 2.2-2.5 we get the equivalence $2 \iff 3$. Statement 4 follows by Propositions 2.3, 2.4 and 2.6 and by the first part of the theorem. Finally, statement 5 follows from Propositions 2.2, 2.3, 2.5 and 2.7 and the first part of the theorem. □

Remark 2.1. *If $c_\varphi = \infty$ and $a_\varphi = 0$, statement* 4 *of Theorem* 2.1 *can also be deduced in a different way, by observing that under the assumptions, E^φ is the STM (see* [16]*). Consequently, by Theorem* 1.1*, $E^\varphi \in$ ULUM and, by Theorem* 2, 3 *in* [10]*, $\varphi \in \Delta_2$.*

Example 2.1. *Consider the Orlicz function $\varphi(u) = \max(0, |u| - 1)$ and assume that (Ω, Σ, μ) is a nonatomic measure space. Then $L^1 + L^\infty = L^\varphi$ and*

$$\|x\|_\varphi = \inf\{\max(\|u\|_1, \|v\|_\infty) : u \in L^1, v \in L^\infty \text{ and } u + v = x\}.$$

If μ is finite, then $L^\varphi = E^\varphi$ and $L^\varphi \in H_\ell$ since $\varphi \in \Delta_2(\infty)$. If μ is infinite, then $L^\varphi \notin H_g$ (see Proposition 2.2*) but $E^\varphi \in H_g$ (see Proposition* 2.7*). Recall that L^φ consists of those $x \in L^0$ that $\mu(\{t \in \Omega : |x(t)| > \lambda\}) < \infty$ for some $\lambda > 0$ and E^φ consists of those $x \in L^0$ that $\mu(\{t \in \Omega : |x(t)| > \lambda\}) < \infty$ for any $\lambda > 0$ (see* [9]*).*

3. Orlicz Norm

As usual, $L^1 := L^1(\mu)$ and $L^\infty := L^\infty(\mu)$ denote the Lebesgue spaces of μ-integrable functions and μ-essentially bounded functions, respectively. These spaces are equipped with the standard norms. The spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$ play an important role in the interpolation theory of symmetric spaces (see [1] and [15]). Usually these spaces are equipped with the following norms:

$$\|x\|_{L^1 \cap L^\infty} = \max\{\|x\|_1, \|x\|_\infty\}$$

and

$$\|x\|_{L^1 + L^\infty} = \inf\{\|u\|_1 + \|v\|_\infty : x = u + v, u \in L^1, v \in L^\infty\}.$$

It is well known that for the Orlicz functions $\psi(u) = |u|$ for $|u| \leq 1$, $\psi(u) = \infty$ for $|u| > 1$ and $\varphi(u) = \max(0, |u| - 1)$, we have $L^1 \cap L^\infty = L^\psi$ and $L^1 + L^\infty = L^\varphi$, with the equality of norms if L^ψ is equipped with the Luxemburg norm and L^φ is equipped with the Orlicz norm (see [7], [8] and [9] for details). Note that the functions ψ and φ are mutually complementary in the sense of Young and, moreover, the spaces $(L^1 \cap L^\infty, \|x\|_{L^1 \cap L^\infty}), (L^1 + L^\infty, \|x\|_{L^1 + L^\infty})$ and $(L^\psi, \|.\|_\psi), (L^\varphi, \|.\|_\varphi^0)$ form two couples of mutually dual spaces in the sense of Köthe. Hence

$$\|x\|_\varphi^0 = \|x\|_{L^1 + L^\infty}$$

holds for all $x \in L^\varphi$. In addition, the Amemiya formula for the norm in $L^1 + L^\infty$ is proved in [9].

For any $x \in L^0(\mu)$ the decreasing rearrangement of x is the function x^* defined for any $t > 0$ by

$$x^*(t) = \inf\{\lambda > 0 : d_x(\lambda) < t\},$$

where d_x is the distribution function defined by

$$d_x(\lambda) = \mu(\{\omega \in \Omega : |x(\omega)| > \lambda\}).$$

For our purpose, it is worthwhile to note (see for example [17]) that

$$(x + y)^*(s + t) \leq x^*(s) + y^*(t)$$

for any $s, t > 0$ and that for all $x \in L^1 + L^\infty$ we have

$$\|x\|_{L^1 + L^\infty} = \int_0^1 x^*(t)dt.$$

From Proposition 2.2 we know that $(L^\varphi, \|\cdot\|_\varphi)$ does not have the Kadec-Klee property for global convergence in measure if $a_\varphi > 0$ and $c_\varphi = \infty$. However, this fact is not true when the Orlicz norm is considered, because by Proposition 1.2 in [4], it follows that $L^1 + L^\infty \in H_g$. We will present here a simple alternative proof of this fact.

Assume that $\{x_n\} \subset L^1 + L^\infty$, $x \in L^1 + L^\infty$, $x_n \to x$ globally in measure and $\|x_n\|_{L^1 + L^\infty} = \|x\|_{L^1 + L^\infty} = 1$ for all $n \in \mathbb{N}$. Since $x_n \to x$ globally in measure, $x_n^* \to x^*$ a.e., and thus

$$x_n^* \chi_{(0,1)} \to x^* \chi_{(0,1)} \text{ a.e.}$$

(see [15]). Bearing in mind that $L^1 \in H_\ell$ and $\|x_n^* \chi_{(0,1)}\|_{L^1} = \|x^* \chi_{(0,1)}\|_{L^1} = 1$, we deduce that

$$\int_0^1 |x_n^*(s) - x^*(s)|ds \to 0.$$

By Lemma 1.1 there exists a subsequence $(x^*_{n_k})$ of (x^*_n) and $y \geq 0$, $y \in L^1(0,1)$ such that $|x^*_{n_k}(t) - x^*(t)| \leq y(t)$ a.e. in $(0,1)$.

By the assumption that $x_n - x \to 0$ globally in measure, it follows that $(x_n - x)^* \to 0$ a.e.. Moreover,

$$(x_{n_k} - x)^*(t) \leq x^*_{n_k}(t/2) + x^*(t/2) \leq 2x^*(t/2) + y(t/2).$$

Since $2x^*(t/2) + y(t/2) \in L^1(0,1)$, by applying the Lebesgue dominated convergence theorem, we obtain

$$\int_0^1 (x_{n_k} - x)^*(t)\, dt \to 0,$$

which is equivalent to

$$\|x_{n_k} - x\|_{L^1 + L^\infty} \to 0.$$

Thus, since for each subsequence of $\{x_n - x\}$ we can extract a subsequence which converges to 0 strongly in $L^1 + L^\infty$, the proof is finished.

Proposition 3.1. *If φ is an Orlicz function with $c_\varphi = \infty$ not satisfying the Δ_2-condition at ∞ and (Ω, Σ, μ) is a nonatomic measure space, then $(E^\varphi, \|\cdot\|^0_\varphi) \notin H_g$.*

Proof. If μ is finite this is obvious, because we have

$$H_g \Leftrightarrow H_\ell \Rightarrow OC \Rightarrow \Delta_2(\infty).$$

So, assume that μ is nonatomic and infinite and that $\varphi \notin \Delta_2(\infty)$. Then there exists a sequence $\{u_n\}$ of positive real numbers such that $u_n \uparrow \infty$ and

$$\varphi(2u_n) > 2^n \varphi(u_n).$$

Take any nonnegative $x \in E^\varphi$ with $\|x\|^0_\varphi = 1$. Since $\varphi \notin \Delta_2(\infty)$, we have $\varphi(t)/t \to \infty$ as $t \to \infty$, and so, in the Amemiya formula for the Orlicz norm $\|\cdot\|^0_\varphi$, the infimum is attained at some $k > 0$, that is, $\|x\|^0_\varphi = \frac{1}{k}(1 + I_\varphi(kx))$. Note that, since $\|x\|^0_\varphi = 1$, necessarily $k \geq 1$.

Further (see the proof of Proposition 2.3) there exists a sequence $\{A_n\}$ in Σ with $\mu(A_n) = \infty$ for any $n \in \mathbb{N}$ such that

$$I_\varphi(2kx\chi_{A_n}) \leq 2^{-n}.$$

Let for any $n \in \mathbb{N}$, $B_n \subset A_n$ be such that

$$\varphi(u_n)\mu(B_n) = 2^{-n}$$

and define

$$x_n = x + \frac{u_n}{2k}\chi_{B_n} = x\chi_{\Omega \setminus B_n} + (x + \frac{u_n}{2k})\chi_{B_n}.$$

Since $x_n \geq x \geq 0$, we have $\|x_n\|_\varphi^0 \geq \|x\|_\varphi^0 = 1$. On the other hand

$$\begin{aligned}
\|x_n\|_\varphi^0 &= \inf_{\rho>0} \frac{1}{\rho}(1 + I_\varphi(\rho x_n)) \\
&\leq \frac{1}{k}(1 + I_\varphi(kx_n)) \\
&= \frac{1}{k}(1 + I_\varphi(kx\chi_{\Omega\setminus B_n})) + I_\varphi(kx\chi_{B_n} + \frac{u_n}{2}\chi_{B_n}) \\
&\leq 1 + \frac{1}{2}(I_\varphi(2kx\chi_{B_n}) + I_\varphi(u_n\chi_{B_n})) \\
&\leq 1 + \frac{1}{2}(\frac{1}{2^n} + \frac{1}{2^n}) \to 1.
\end{aligned}$$

As a consequence, we obtain $\|x_n\|_\varphi^0 \to \|x\|_\varphi^0$ and, since $\mu(B_n) \to 0$, $x_n \to x$ globally in measure.

In order to finish the proof, we show that $\|x_n - x\|_\varphi^0 \geq \frac{1}{4k}$. We have

$$I_\varphi(4k(x_n - x)) = I_\varphi(2u_n\chi_{B_n}) = \varphi(2u_n)\mu(B_n)) > 1$$

for all $n \in \mathbb{N}$. Hence $\|x_n - x\|_\varphi \geq 1/4k$, and so the proof is finished, by observing that $\|x_n - x\|_\varphi^0 \geq \|x_n - x\|_\varphi$, and $x_n \in E^\varphi$ for all $n \in \mathbb{N}$. □

Proposition 3.2. *Let (Ω, Σ, μ) be a nonatomic measure space and φ be an Orlicz function such that $a_\varphi = 0$, $c_\varphi = \infty$, and $\lim_{t\to\infty} \varphi(t)/t = \infty$. Then $E_0^\varphi \notin H_g$ whenever $\varphi \notin \Delta_2$.*

Proof. First we will show that $E_0^\varphi \notin H_g$ if $\varphi \notin \Delta_2(\infty)$ (it does not matter if $\mu(\Omega) = \infty$ or $\mu(\Omega) < \infty$ in this case). Take any $x \in S(E_0^\varphi)$ such that $\mu(\Omega \setminus \operatorname{supp} x) > 0$). The assumption $\varphi \notin \Delta_2(\infty)$ implies that there exists a sequence $\{u_n\}$ of positive numbers such that $\varphi(2u_n) > 2^n\varphi(u_n)$ for each $n \in \mathbb{N}$ and $u_n \to \infty$. Passing to a subsequence of $\{u_n\}$ if necessary we may assume that $\varphi(u_n)\mu(B_n) = 2^{-n}$ for a sequence $\{B_n\}$ in $\Sigma \cap (\Omega \setminus \operatorname{supp} x)$. Defining $x_n = x + u_n\mu(B_n)$, we easily see that $x_n \in E_0^\varphi$ for any $n \in \mathbb{N}$, $x_n \to x$ globally in measure and $1 \leq \|x_n\|_\varphi^0 \leq 1 + 2^{-n}$. However

$$\|x_n - x\|_\varphi^0 \geq \frac{1}{2}\min\{1, I_\varphi(2(x_n - x))\} = \frac{1}{2}\min\{1, \varphi(2u_n)\mu(B_n)\} > 1/2,$$

which means that $E_0^\varphi \notin H_g$.

Assume now that $\mu(\Omega) = \infty$ and $\varphi \notin \Delta_2(0)$. Then there is a decreasing sequence $\{u_n\}$ of positive numbers with $u_n \to 0$ such that $\varphi(2u_n) > 2^n\varphi(u_n)$ for each $n \in \mathbb{N}$. Take any nonnegative $x \in E_0^\varphi$ with $\|x\|_\varphi^0 = 1$. We know (see the proof of Proposition 2.3) that there is a sequence $\{A_n\} \subset \Sigma$ such that $\mu(A_n) = \infty$

and

$$I_\varphi(2kx\chi_{A_n}) \le 2^{-n}$$

for each $n \in \mathbb{N}$, where $k \ge 1$ satisfies $k^{-1}(1 + I_\varphi(kx)) = \|x\|_\varphi^0 = 1$.

Let $B_n \subset A_n$ be such that $\varphi(u_n)\mu(B_n) = 2^{-n}$. Defining $x_n = x + u_n\mu(B_n)$, we can prove in the same way as in the proof of Proposition 3.1 that $1 \le \|x_n\|_\varphi^0 \le 1 + 2^{-n}$ for each $n \in \mathbb{N}$ and $x_n \to x$ globally in measure, but $\|x_n - x\|_\varphi^0 \ge 1/4k$ for all $n \in \mathbb{N}$. Consequently, $E_0^\varphi \notin H_g$. □

Proposition 3.3. *Assume φ is an Orlicz function with $c_\varphi < \infty$ and (Ω, Σ, μ) is a nonatomic measure space. Then $L_0^\varphi \notin H_g$.*

Proof. Let $\{\lambda_k\}$ be a sequence of positive numbers with $\lambda_k < c_{varphi}$ for any $n \in \mathbb{N}$ and $\lambda_k \uparrow c_\varphi$ as $k \to \infty$ and $\{A_n\} \subset \Sigma$ be a sequence of pairwise disjoint sets of finite positive measure such that $\varphi(\lambda_k)\mu(A_n) \le 2^{-k}$. Define $x = \sum_{k=1}^\infty \lambda_k c_\varphi \chi_{A_k}$ and $x_n = \sum_{k \ne n} \lambda_k c_\varphi \chi_{A_k}$. Then $0 \le x_n \le x$ and $x_n \to x$ a.e. in Ω. Since L_0^φ has the Fatou property, we get $\|x_n\|_\varphi^0 \to \|x\|_\varphi^0$. Moreover,

$$I_\varphi(x_n - x) = \varphi(\lambda_n c_\varphi)\mu(A_n) \le 2^{-n}$$

and

$$I_\varphi(\lambda(x_n - x)) = \varphi(\lambda\lambda_n c_\varphi \mu(A_n) = \infty$$

for any $\lambda > 1$ and n large enough. Therefore

$$\|x_n - x\|_\varphi^0 \ge \|x_n - x\|_\varphi = 1$$

for $n \in \mathbb{N}$ large enough. Since $x_n \to x$ globally in measure, $L_0^\varphi \notin H_g$. □

The results of this section are summarized as follows.

Theorem 3.1. *Suppose that (Ω, Σ, μ) is a nonatomic measure space and φ is an Orlicz function with $\lim_{t\to\infty} \varphi(t)/t = \infty$, $a_\varphi = 0$ and $c_\varphi = \infty$. Let L_0^φ and E_0^φ be the spaces L^φ and E^φ equipped with the 0rlicz norm. Then the following conditions are equivalent:*

(1) $L_0^\varphi \in H_\ell$.
(2) $L_0^\varphi \in H_g$.
(3) $E_0^\varphi \in H_g$.
(4) $E_0^\varphi \in H_\ell$.
(5) $\varphi \in \Delta_2$.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious. Let us prove that $3 \iff 4$. It is obvious that $4 \Rightarrow 3$. If $E_0^\varphi \in H_g$, then by Propositions 3.1 and 3.2, we get $\varphi \in \Delta_2$. Consequently (see [3], [5] and [12]), $L_0^\varphi \in H_\ell$ and so $E_0^\varphi \in H_\ell$, too.

Therefore, the equivalence 3 $\iff$ 4 and the implication 5 $\Rightarrow$ 1 are proved. By Propositions 2.3 and 2.5, we get 4 $\Rightarrow$ 5. $\square$

Remark 3.1. *The equivalence of conditions* 1, 2 *and* 5 *in Theorem* 3.1 *holds for* L^φ *generated by an Orlicz function* φ *with* $a_\varphi = 0$ *without the assumption that* $c_\varphi = \infty$ *because, by Proposition* 3.3, *the condition* $c_\varphi = \infty$ *is necessary for* $L_0^\varphi \in H_g$.

Remark 3.2. *If, in addition, we assume that the measure* μ *is separable and* $c_\varphi = \infty$, *then the space* E_0^φ *is separable and* $E_0^\varphi \neq \{0\}$. *Applying Theorem* 1.1, *we can recapture some implications of Theorem* 3.1 *in a different way. Namely, if* $a_\varphi = 0$, *then* $E_0^\varphi \in STM$. *We also know that* $\varphi \in \Delta_2$ *is necessary for* $E_0^\varphi \in ULUM$ (*see* [10]).

Theorem 3.2. *Assume* φ *is an Orlicz function such that* $a_\varphi > 0$ *and* $\varphi(t)/t \to \infty$ *as* $t \to \infty$ *and* (Ω, Σ, μ) *is a nonatomic measure space. Then* $L_0^\varphi \in H_g$ *if and only if* $\varphi \in \Delta_2(\infty)$. *If we assume additionally that* $c_\varphi = \infty$, *then* $E_0^\varphi \in H_g$ *if and only if* $\varphi \in \Delta_2(\infty)$.

Proof. The necessity of $\varphi \in \Delta_2(\infty)$ for $L_0^\varphi \in H_g$ follows by Propositions 3.1 and 3.3 and by Remark 3.4. When $c_\varphi = \infty$, the necessity of $\varphi \in \Delta_2(\infty)$ for $E_0^\varphi \in H_g$ follows by Propositions 3.1 and 3.2.

We present a proof of sufficiency of the respective conditions for $L_0^\varphi \in H_g$ only. The proof for E_0^φ in place of L_0^φ is the same. Let $x \in S(L_0^\varphi)$ and $\{x_n\}$ be a sequence in $S(L_0^\varphi)$ such that $x_n \to x$ globally in measure. By the assumption that $\lim_{t\to\infty} \varphi(t)/t = \infty$, there are $k \geq 1$ and $k_n \geq 1$ for $n \in \mathbb{N}$ such that

$$\|x\|_\varphi^0 = \frac{1}{k}(1 + I_\varphi(kx)) \text{ and } \|x_n\|_\varphi^0 = \frac{1}{k_n}(1 + I_\varphi(k_n x_n)).$$

We need to prove that $I_\varphi(\lambda(x_n - x)) \to 0$ for any $\lambda > 0$. Choose an arbitrary $\lambda > 0$ and define

$$A_n = \{t \in \Omega : |x_n(t) - x(t)| \leq a_\varphi/\lambda\}.$$

The assumption that $x_n \to x$ globally in measure yields $\mu(A'_n) \to 0$ as $n \to \infty$, where for any $A \in \Sigma$, $A' := \Omega \setminus A$. So, one can find a subsequence $\{A'_{n_k}\}$ of $\{A'_n\}$ such that $\mu(A'_{n_k}) < 2^{-k}$. Defining $A = \bigcup_{k=1}^\infty A'_{n_k}$ we have $\mu(A) \leq 1$. Note that $A' = \bigcap_{k=1}^\infty A_{n_k}$ and $|x_{n_k}(t) - x(t)| \leq a_\varphi/\lambda$ for all $k \in N$ whenever $t \in A'$. Consequently, $I_\varphi(\lambda(x_{n_k} - x)\chi_{A'}) = 0$ for all $k \in N$. In order to prove that

$I_\varphi(\lambda(x_{n_k} - x)) \to 0$, we need to show that $I_\varphi(\lambda(x_{n_k} - x)\chi_A) \to 0$. Let k and k_n be as above. We first prove that the sequence $\{k_n\}$ is bounded. Define

$$C_\varepsilon = \{t \in \Omega : |x(t)| > \varepsilon\}$$

for each $\varepsilon > 0$. Clearly, it is possible to choose an $\varepsilon_0 > 0$ such that $a := \mu(C_{\varepsilon_0}) > 0$. Since $x_n \to x$ globally in measure, there exists $m \in N$ such that

$$\mu(\{t \in C_{\varepsilon_0} : |x_n(t) - x(t)| > \varepsilon_0/2\}) < a/2 \text{ for all } n \geq m.$$

Let $D_n = \{t \in C_{\varepsilon_0} : |x_n(t) - x(t)| \leq \varepsilon_0/2\}$. Then we have $\mu(D_n) > a/2$ for all $n \geq m$, and so

$$|\ |x_n(t)| - |x(t)|\ | \leq \varepsilon_0/2$$

for all $t \in D_n$ and $n \geq m$. Consequently, whenever $n \geq m$, we have

$$|x_n(t)| \geq \varepsilon_0/2 \quad \text{for all } t \in D_n.$$

Assuming that $\ell := \sup_n k_n = \infty$ one can find a subsequence $\{k_{n_j}\}$ of $\{k_n\}$ such that $k_{n_j} \to \infty$ as $j \to \infty$. Hence we get,

$$\begin{aligned} 1 = \|x_{n_j}\|_\varphi^0 &= \frac{1}{k_{n_j}}(1 + I_\varphi(k_{n_j} x_{n_j}) \geq \frac{1}{k_{n_j}} I_\varphi(k_{n_j} x_{n_j}) \\ &\geq \frac{1}{k_{n_j}} \varphi(k_{n_j}\varepsilon/2)\mu(D_{n_j}) \to \infty, \end{aligned}$$

a contradiction, showing that $\sup_n k_n < \infty$. So, one can find a subsequence of $\{k_n\}$ convergent to a positive number ℓ. Assume without loss of generality that $k_n \to \ell$ as $n \to \infty$. Since the measure space is σ-finite and $x_n \to x$ globally in measure, we can assume without loss of generality that $x_n \to x$ μ-a.e. in Ω. Consequently, $\varphi \circ k_n x_n \to \varphi \circ lx$ μ-a.e. in Ω. By the Fatou Lemma, we get

$$I_\varphi(\ell x) \leq \liminf_{n\to\infty} I_\varphi(k_n x_n),$$

whence

$$1 = \|x\|_\varphi^0 \leq \frac{1}{\ell}(1 + I_\varphi(\ell x)) \leq \liminf_{n\to\infty} \frac{1}{k_n}(1 + I_\varphi(k_n x_n)) = 1 \tag{3.1}$$

for all $n \in \mathbb{N}$. This implies the equality $I_\varphi(\ell x) = \ell - 1$. Moreover, $I_\varphi(k_n x_n) = k_n - 1 \to \ell - 1$ as $n \to \infty$, whence we get by (3.1) that $I_\varphi(k_n x_n) \to I_\varphi(\ell x)$ or equivalently, $\|\varphi \circ k_n x_n\|_{L^1} \to \|\varphi \circ \ell x\|_{L^1}$. We get $\|\varphi \circ k_n x_n - \varphi \circ \ell x\|_{L^1} \to 0$ as $n \to \infty$ by $L^1 \in H_\ell$. Consequently, $\|(\varphi \circ k_n x_n - \varphi \circ \ell x)\chi_D\|_{L^1} \to 0$ as $n \to \infty$ for any $D \in \Sigma$. This yields that

$$I_\varphi(k_n x_n \chi_D) \to I_\varphi(\ell x \chi_D), \tag{3.2}$$

for any $D \in \Sigma$ as $n \to \infty$. Recall that we want to show that $I_\varphi(\lambda(x_n - x)\chi_A) \to 0$ as $n \to \infty$. Taking into account that $k_n \to \ell$, $\varphi \in \Delta_2(\infty)$ and $\mu(A) < \infty$, we conclude from (3.2) that $I_\varphi(x_n\chi_A) \to I_\varphi(x\chi_A)$. Since $x_n\chi_A \to x\chi_A$ in measure and $L^1 \in H_\ell$, we get $\|\varphi \circ x_n\chi_A - \varphi \circ x\chi_A\|_{L^1} \to 0$ as $n \to \infty$. Consequently, in view of Lemma 1.1, we may assume by passing to subsequence if necessary that the sequence $\{|\varphi \circ x_n\chi_A - \varphi \circ x\chi_A|\}$ has a majorant $z \in L^1$. Since

$$\begin{aligned} \varphi \circ \frac{x_n - x}{2}\chi_A &\leq \frac{1}{2}\big\{\varphi \circ x_n\chi_A + \varphi \circ x\chi_A\big\} \\ &\leq \frac{1}{2}\big\{|\varphi \circ x_n\chi_A + \varphi \circ x\chi_A| + 2\varphi \circ x\chi_A\big\} \\ &\leq \frac{1}{2}z + \varphi \circ x \end{aligned}$$

and $z/2 + \varphi \circ x \in L^1$, the Lebesgue dominated convergence theorem yields $I_\varphi(x_n - x)\chi_A/2) \to 0$ as $n \to \infty$. Further $\mu(A) < \infty$, $x_n \to x$ in measure and $\varphi \in \Delta_2(\infty)$ implies that $I_\varphi(\lambda(x_n - x)\chi_A) \to 0$ for any $\lambda > 0$. In consequence, we get $\|x_n - x\|_\varphi^0 \to 0$ as $n \to \infty$, which finishes the proof. □

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics 129, Boston 1988.

[2] B. Bru and H. Heinich, *Monotonies des espaces d'Orlicz*, C. R. Acad. Sci. Paris **301** (1985), 893–894.

[3] S. Chen, *Geometry of Orlicz Spaces*, Dissertationes Math. **356** (1996).

[4] V. I. Chillin, P. G. Dodds, A. A. Sedaev and F. A. Sukochev, *Characterization of Kadec-Klee properties in symmetric spaces of measurable functions*, Trans. Amer. Math. Soc. **348** (1996), 4895–4918.

[5] P. Foralewski and H. Hudzik, *Some basic properties of generalized Calderón-Lozanovskiĭ spaces*, Collectanea Math. **48** (1997), 523-538.

[6] H. Hudzik, *On some equivalent conditions in Musielak-Orlicz spaces*, Comment. Math. Prace Mat. **24** (1984), 57–64.

[7] H. Hudzik, *Intersections and algebraic sums of Musielak-Orlicz spaces*, Portugaliae Math. **40** (1993), 287–296.

[8] H. Hudzik, *On smallest and largest Orlicz spaces*, Math. Nachr. **141** (1989), 109–115.

[9] H. Hudzik, *On the distance from the subspace of order continuous elements in $L^1 + L^\infty$*, Functiones et Approximatio **25** (1997), 157–163.

[10] H. Hudzik and W. Kurc, *Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices*, J. Approx. Theory **95** (1998), 353–368.

[11] H. Hudzik and L. Maligranda, *Amemiya norm equals Orlicz norm in general*, Indag. Math. **11** (2000), 573–585.

[12] H. Hudzik and M. Mastyło, *Strongly extreme points in Köthe-Bochner spaces*, Rocky Mountain J. Math. 3, Vol. **23** (1993), 899–909.

[13] A. Kamińska, *Flat Orlicz-Musielak sequence spaces*, Bull. Acad. Polo. Sci. Math. **30** (1982), 347–352.
[14] L. V. Kantarovich and G. P. Akilov, *Functional Analysis*, 2nd edition, Moscow 1978 (in Russian).
[15] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscov 1978 (in Russian).
[16] W. Kurc, *Strictly and uniformly monotone Musielak-Orlicz spaces and applications in best approximation*, J. Approx. Theory **69** (1992), 173–187.
[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Springer-Verlag, Berlin (1979).
[18] W. A. J. Luxemburg, *Banach Function Spaces*, Thesis, Delft 1955.
[19] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Math. 5, Univ. Estadual de Campinas, Campinas SP, Brasil, 1989.
[20] A. Medzhitov and P. Sukochev, *The property (H) in Orlicz spaces*, Bull. Polish. Acad. Sci. Math. **40** (1992), 5–11.
[21] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. **1034**, Springer Verlag 1983.
[22] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Pure and Applied Mathematics, Marcel Dekker. New-York, 1991
[23] F. Riesz, *Sur la convergence en moyenne I*, Acta Sci. Math. **4** 1928/29, 58–64.
[24] F. Riesz, *Sur la convergence en moyenne II*, Acta Sci. Math. **4** 1928/29, 182–185.
[25] B. Turett, *Rotundity of Orlicz spaces*, Indag. Math. **38** (1976), 462–469.

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