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Recent results on stabilization of PDEs by noise

Tomás Caraballo

Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. Correos 1160, 41080–Sevilla, Spain.

caraball@us.es

Dedicated to Ana María Heredia[1] with my utmost Gratitude and Affection

Abstract

This paper is intended to be a brief review on some recent results on the stabilization effect produced by noise in phenomena modelled by partial differential equations. We emphasyse the different effects that distinct interpretations of the noise may cause on the same system, and we focus on two classical and canonical interpretations (Itô versus Stratonovich). Finally, we comment on some open problems.

Key words: *exponential stability stabilization Ito's noise Stratonovich's noise stochastic PDE.*

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1 Introduction: Why is Stratonovich noise more significant than Itô noise?

The use of stochastic partial differential equations in physics, chemistry, biology, economics, engineering, etc. is widespread. The addition of random elements (noise) is based on the assumption that such equations are a better model of reality than their deterministic counterparts. Depending on the situation, one can find arguments justifying either of the canonical choices of noise (Itô or Stratonovich). We will not discuss this is detail here, but will emphasyse that these different types of noise can

[1]Quiero dedicar este trabajo a Doña Ana María Heredia, como muestra de mi gratitud y cariño. Tuve la oportunidad de disfrutar de sus enseñanzas de verdadera MAESTRA DE ESCUELA durante los dos últimos años de mis estudios de primaria, y desde entonces tengo la suerte de poder contar con su amistad y aprecio. Ella es una de las personas “culpables” de que hoy pueda estar dedicándole este modesto homenaje.

produce solutions with very different long-time behaviours (see Caraballo & Langa [11]).

A fundamental question is the following: assuming that the real world is actually non-deterministic, are deterministic models good approximations? If the answer is affirmative, then the use of such models would be justified. Otherwise, then in some situations the addition of noise could produce dramatic changes in the behaviour. Here, we analyse the long-time behaviour of the solutions and investigate the potential stabilization (or destabilization) effect of the addition of noise.

In the finite-dimensional context, there is a wide literature about such problems; see Arnold [4], Arnold et al. [6], Mao [34], Scheutzow [38], etc. Many results on the stabilization and destabilization produced by both Itô and Stratonovich noise have been obtained, and these have also been applied to construct feedback stabilisers (an important tool in control problems). Although in each particular situation one or other choice of the noise may be more appropriate, it is stabilization by Stratonovich noise that might be more significant. To explain this in more detail, let us consider the linear n-dimensional ordinary differential equation

$$\dot{x} = Ax, \tag{1}$$

where A could have unstable directions, and the following stochastic versions of this equation, corresponding to the two different interpretations of the stochastic integral

$$\mathrm{d}x = Ax\,\mathrm{d}t + \sigma x \circ \mathrm{d}W(t) \qquad \text{(Stratonovich)} \tag{2}$$

$$\mathrm{d}y = Ay\,\mathrm{d}t + \sigma y\,\mathrm{d}W(t), \qquad \text{(Itô)} \tag{3}$$

where $W(t)$ is a standard Wiener process on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see e.g. Arnold [2] for the constructions and properties of both types of differential equations). The initial value problems for (2) and (3) corresponding to the data $x(0) = x_0$, $y(0) = y_0$, can be solved explicitly, and their solutions are given by

$$x(t) = \mathrm{e}^{\sigma W(t)} \exp(tA)x_0 \quad \text{and} \quad y(t) = \mathrm{e}^{-\frac{\sigma^2}{2}t + \sigma W(t)} \exp(tA)y_0.$$

Taking into account the properties of the Wiener process (see again Arnold [2]), it follows that, for σ large enough, the zero solution is exponentially stable for the Itô equation (with probability one), while the same is not true for the Stratonovich equation. This seems to imply that Itô noise has a more profound stabilising effect than Stratonovich noise.

However, this argument is somewhat misleading. Indeed, (1) can be obviously stabilised by using a simple deterministic feedback control, i.e., the new equation

$$\dot{x} = Ax + \lambda x \tag{4}$$

becomes exponentially stable provided $\lambda < 0$ and $|\lambda|$ is large enough. Similarly, it could be stabilised with the periodic control $\lambda(t)x$,

$$\dot{x} = Ax + \lambda(t)x,$$

where $\lambda(t) = \lambda_0 + \sin t$, with $\lambda_0 < 0$ and $|\lambda_0|$ large enough. The function $\sin t$ is, in some sense, a mean-zero function. The same is true with faster mean-zero periodic fluctuations: stabilization takes place because of the systematic dissipative term $\lambda_0 x$ in the equation, while the mean-zero property means that the other term has no influence on the asymptotic behaviour. We can write such an equation in the form

$$\dot{x} = Ax + \lambda_0 x + x\dot{W}_\varepsilon(t)$$

where $\dot{W}_\varepsilon(t)$ denotes the zero-mean periodic term and may be considered as a physically realistic approximation of the ideal white noise $\dot{W}(t)$. It is therefore not surprising that the same result is true in the limit when the regular mean-zero fluctuations tend to a mean-zero white noise, and the systematic term $\lambda_0 x$ is still present. Now, it is well known that in such a limit (more precisely in all cases where there are rigorous results concerning the Wong-Zakai [40] approximation of a stochastic equation by a random equation with regularised noise), the correct stochastic interpretation for the equation is the Stratonovich one (see, e.g., Sussmann [39] for a more detailed analysis):

$$\mathrm{d}x = Ax\,\mathrm{d}t + \lambda_0 x\,\mathrm{d}t + \sigma x \circ \mathrm{d}W(t),$$

where $\sigma > 0$ describes the intensity of the noise. While it should be intuitively clear that this equation is exponentially stable when $\lambda_0 < 0$ is sufficiently small, a rigorous proof follows from Itô's formula, since as we previously mentioned an explicit form for the solution is

$$x(t) = \mathrm{e}^{\lambda_0 t + \sigma W(t)} \exp(At)x(0).$$

Notice that this property is independent of σ. However, the previous Stratonovich equation can be rewritten in its equivalent Itô form:

$$\mathrm{d}x = Ax\,\mathrm{d}t + \lambda_0 x\,\mathrm{d}t + \sigma x\,\mathrm{d}W(t) + \frac{\sigma^2}{2}x\,\mathrm{d}t.$$

If we choose a white noise with intensity such that $\frac{\sigma^2}{2} = -\lambda_0$, we arrive at the Itô equation

$$\mathrm{d}x = Ax\,\mathrm{d}t + \sigma x\,\mathrm{d}W(t). \tag{5}$$

As a consequence of this elementary analysis, it is clear that this equation is exponentially stable with probability one for σ large enough.

In general, what this example means is that Itô equations with multiplicative noise correspond to the limit of deterministic equations with a mean-zero fluctuating control plus a stabilising systematic control. So, the fact that an Itô equation such as (5) is exponentially stable, even if the equation (1) is not, should not be much more surprising than the fact that (4) with sufficiently small $\lambda < 0$ is exponentially stable; it is only that the mathematics required for the proof are more elaborate.

There is a non-trivial literature on stabilization by *Stratonovich* noise, with both mathematical and engineering contributions (see [4],[6] and the references therein). Since such a noise behaves like a periodic zero-mean feedback control, its stabilising effect is unexpected and very intriguing. In the finite-dimensional case, Arnold and his collaborators have proved that the linear differential system (1) can be stabilised by the addition of a collection of multiplicative noisy terms,

$$\mathrm{d}x = Ax\,\mathrm{d}t + \sum_{i=1}^{d} B_i x \circ \mathrm{d}W_i(t), \tag{6}$$

where the W_i are mutually independent Wiener processes and the B_i are suitable skew-symmetric matrices, if and only if

$$\mathrm{tr}\ A < 0. \tag{7}$$

Note that the form of the noise is more complex than just a single multiplicative term of the form $\sigma x \circ \mathrm{d}W_t$.

The corresponding problem for linear partial differential equations has remained open for a long time, perhaps because one avenue would be to follow a similar approach but in the infinite-dimensional case, e.g. by proving a version of the celebrated Oseledec Multiplicative Ergodic Theorem for infinite-dimensional spaces. Fortunately, in [19] a very simple argument is successfully used to show that one can obtain a stabilization result for a linear PDE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au$$

with a finite sum of Stratonovich terms as in (6).

However, to the best of our knowledge, the problem of stabilization of nonlinear PDEs by Stratonovich noise is still an unsolved problem in general, while it has already been solved, in various interesting applications, by using Itô's noise (see [9] ,[11],[16] ,[14],[17] ,[30],[32] amongst many others).

On the other hand, it is worth mentioning that the analysis carried out in this field is only concerned with the stabilization of the trivial solution. Nevertheless, going deeper in the investigation of the nonlinear models, we can find in the stochastic models some special solutions called *stationary* but which are not stationary in the deterministic sense. These solutions sometimes become random attractors for some systems, so their existence and properties are very important. Some preliminary results have been obtained in [10].

The aim of this paper is to review on the recent results obtained in this field, to outline the basic techniques used in this area, and to comment on some open questions. To be more precise, we consider a linear evolution equation on a separable Hilbert space H given by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \tag{8}$$

where A is a linear (unbounded) operator, i.e., $A : D(A) \subset H \mapsto H$, and the stochastically perturbed evolution equation

$$\mathrm{d}u = Au\,\mathrm{d}t + \sum_{i=1}^{N} B_i u \circ \mathrm{d}W_i, \tag{9}$$

where the $B_i : D(B_i) \subset H \mapsto H$ are linear operators and the W_i are mutually independent Wiener process on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In Section 2 we first prove that the stability of (8) and (9) are equivalent if the operators B_i and A are mutually commuting and satisfy other additional assumptions (we call this situation "fully commuting"). It is remarkable that, if the noise is considered in the Itô sense, it may produce stabilization or even destabilization under the same assumptions (see Caraballo & Langa [11] for a detailed analysis). Then, we prove that (9) becomes exponentially stable with probability one (w.p.1) for suitable operators B_i if and only if the trace of A is negative, an infinite-dimensional analogue of the results of Arnold et al. [6]. In Section 3 we consider the nonlinear framework and split our analysis into two cases. First, we show how the theory of stabilization has been widely developed by considering the noise in the Itô sense, having been applied to several interesting examples arising in applications. Next, we consider the stabilization problem for nonlinear PDEs by using Stratonovich noise and we show how to prove this fact in a canonical example as the Chafee-Infante equation in one spatial dimension. Also, for this example, it can be shown a "super-stabilization" effect produced by a rich enough additive noise. We end the paper with some additional comments and stating some open problems.

2 Stabilization and destabilization of a linear PDE

In this section we first establish some results concerning the exponential stability of the null solution to a linear stochastic PDE. Our main purposes are, on the one hand, to point out the different effects that the interpretation of the noise may produce in the final results, and on the other, to characterise the stabilization of linear PDEs by Stratonovich noise.

To start with, we can consider the problem in the Itô formulation, so that we can apply a result due to Da Prato & Zabczyk [25] which ensures the equivalence of the stochastic PDE to a nonautonomous deterministic equation depending on a random parameter, i.e. a random PDE. Then, we will transform our Stratonovich model to an equivalent[2] Itô model and will apply this result.

Let us consider the Cauchy problem

$$\begin{cases} \mathrm{d}u = Au\,\mathrm{d}t + \sum\limits_{k=1}^{d} B_k u\,\mathrm{d}W_k, \\ u(0) = u_0 \in H, \end{cases} \tag{10}$$

where $A : D(A) \subset H \to H$, $B_k : D(B_k) \subset H \to H$, $k = 1, 2, \cdots, d$ are generators of C_0-semigroups $S_A(t)$ and S_k respectively, and the $W_1, \cdots, W_d$ are independent real Wiener processes on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We need the following additional assumptions:

- (A1) The operators $B_1, \cdots, B_d$ generate mutually commuting C_0-groups S_k.
- (A2) $D(B_k^2) \supset D(A)$ for $k = 1, \cdots, d$ and $\bigcap_{k=1}^d D((B_k^*)^2)$ is dense in H, where B_k^* denotes the adjoint operator of B_k.
- (A3) $C = A - \frac{1}{2}\sum_{k=1}^d B_k^2$ generates a C_0-semigroup S_C.

Given a fixed realisation of our Wiener processes $W_k(t,\omega)$, $\omega \in \Omega$, in order to solve (10) we define

$$U_\omega(t) = \prod_{k=1}^{d} S_k(W_k(t,\omega)) \quad \text{and} \quad v(t) = U_\omega^{-1}(t)u(t), \qquad t \geq 0, \tag{11}$$

and we consider the auxiliary system

$$\begin{cases} \dot{v}(t) = U_\omega^{-1}(t) C U_\omega(t) v(t) \\ v(0) = u_0, \end{cases} \tag{12}$$

which is a deterministic Cauchy problem depending on the parameter ω. The following result, along with the definition of a strong solution, can be found in Da Prato & Zabczyk [25]:

[2] This equivalence has been proved by Kunita [28] for suitable partial differential operators. We implicitly assume that we are considering this case. It is undoubtedly an important task to develop a general theory of stochastic PDEs in the Stratonovich sense.

Proposition 1 *Let assumptions* (A1)–(A3) *be satisfied. Then, if u is a strong solution to* (10)*, the process $v(t,\omega)$ defined by* (11) *satisfies* (12)*. Conversely, if v is a predictable process whose trajectories are continuously differentiable and satisfy* (12) *$\mathbb{P}$-a.s., then the process $u(t,\omega) = U_\omega(t)v(t,\omega)$ takes values in $D(C)$ $\mathbb{P}$-a.s and for almost all t, and is a strong solution to* (10)*.*

Remark 1 One can also find in Da Prato & Zabczyk [25] a sufficient condition ensuring the solvability of (12) which could be useful in applications (see [25, pp. 177–179] for more details).

Now, we consider the Stratonovich version of the problem:

$$\begin{cases} \mathrm{d}u = Au\,\mathrm{d}t + \sum\limits_{k=1}^{d} B_k u \circ \mathrm{d}W_k, \\ u(0) = u_0 \in H. \end{cases} \tag{13}$$

To ensure existence of solutions to this problem, we can consider its equivalent Itô version

$$\begin{cases} \mathrm{d}u = (A + \frac{1}{2}\sum\limits_{k=1}^{d} B_k^2)u\,\mathrm{d}t + \sum\limits_{k=1}^{d} B_k u\,\mathrm{d}W_k, \\ u(0) = u_0 \in H. \end{cases} \tag{14}$$

If we now assume (A1)–(A2) and, instead of (A3), the following

- (A3') $C = A + \frac{1}{2}\sum_{k=1}^{d} B_k^2$, generates a C_0-semigroup S_C,

then, thanks to Proposition 1, problem (13) can be equivalently rewritten as

$$\begin{cases} \dot{v}(t) = U_\omega^{-1}(t)AU_\omega(t)v(t) \\ v(0) = u_0. \end{cases} \tag{15}$$

We can now proceed with our stability analysis.

2.1 Itô vs Stratonovich in the fully commuting case

Now we establish a result which characterises the asymptotic stability of the Stratonovich model (13) under the assumption that all the operators involved in the equation mutually commute (we call this the "fully commuting" case) and weshow how, under the same assumptions, the Itô equation (10) may exhibit very different asymptotic behaviour.

The next result can be found in [19] (see also [11]).

Theorem 2 *In addition to assumptions* (A1)–(A2) *and* (A3')*, suppose that A commutes with each $S_k(t)$. Then the strongly continuous semigroup $S_A(t)$ generated by A is exponentially stable, i.e. there exist $M_0, \gamma > 0$ such that $|S_A(t)| \leq M_0 e^{-\gamma t}$ for all $t > 0$, if and only if there exist $\alpha, C > 0$ and $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 0$ such that for any $\omega \notin \Omega_0$ there exists $T(\omega) > 0$ such that the following holds for the solution of* (13)*:*

$$|u(t)| \leq C|u_0|e^{-\alpha t} \quad \textit{for} \;\; t \geq T(\omega)\ \mathbb{P} - a.s.$$

Proof. (Sketch) Denote by $u(t) = u(t,\omega;0,u_0)$ the solution of (13) for $u_0 \in D(A)$, and by $v(t) = v(t,\omega;0,u_0)$ the corresponding solution to (15), i.e. $v(t) = U_\omega^{-1}(t)u(t)$. Owing to the commutativity of the operator A and the operators S_k, the problem (15) can be written as

$$\begin{cases} \dot{v}(t) = Av(t), \\ v(0) = u_0 \in D(A), \end{cases} \tag{16}$$

whose solution is given by $v(t) = S_A(t)u_0$, so we have an explicit expression for our solution $u(t)$:

$$u(t) = u(t,\omega;0,u_0) = U_\omega(t)S_A(t)u_0. \tag{17}$$

Taking into account now the properties of strongly continuous semigroups and the Wiener processes (especially that $\lim_{t\to+\infty} |W_k(t,\omega)|/t = 0$, for all $k = 1,\cdots,d$), it is not difficult to conclude the proof. □

Consequently, in this fully commuting case the stability properties of the deterministic problem (8) and the stochastic (13) are equivalent, so we can ensure the adequacy of the deterministic model to the stochastic real phenomenon. However, if we interpret the noise in the sense of Itô, we may have very different results. It may happen that (8) is stable and (10) remains stable (persistence of stability from the deterministic to the stochastic model), or (8) is unstable and (10) becomes stable (stabilization produced by the noise), or (8) is stable and (10) becomes unstable (destabilization), etc. Let us now illustrate these facts.

2.1.1 Persistence of stability and stabilization by Itô noise

To illustrate these features, we will analyse the following example. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ ($d \le 3$) with C^∞-boundary, and consider the reaction-diffusion equation

$$\begin{cases} \mathrm{d}u(t,x) = (\Delta u(t,x) + \alpha u(t,x))\,\mathrm{d}t + \gamma u(t,x)\,\mathrm{d}W(t), \\ u(t,x) = 0, \quad t > 0, \quad x \in \partial\mathcal{O}, \\ u(0,x) = u_0(x), \quad x \in \mathcal{O}, \end{cases} \tag{18}$$

where, as usual, Δ denotes the Laplacian operator and $W(t)$ is a scalar Wiener process. To set this problem in our framework, we take $H = L^2(\mathcal{O})$, $A = \Delta + \alpha I$ and $B = \gamma I$. It then holds that $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. Let $\lambda_1 > 0$ denote the first eigenvalue of $-\Delta$. Then, as a consequence of Theorem 3 in Subsection 2.2 (see also Kwiecinska [29]), it is easy to check that the null solution of (18) is exponentially stable $\mathbb{P}$-a.s. if the parameters in the equation satisfy

$$2(\alpha - \lambda_1) - \gamma^2 < 0.$$

Let us now discuss what this condition means.

First, notice that when $\alpha < \lambda_1$, the deterministic equation (i.e. Eq. (18) with $\gamma = 0$) is exponentially stable. Then, for any $\gamma \in \mathbb{R}$ (in other words, no matter how large or small the intensity of the noise might be), the stochastic equation (18) remains exponentially stable $\mathbb{P}$-a.s. So, the persistence of stability takes place in the presence of noise.

However, if $\alpha > \lambda_1$, then the deterministic equation is not stable (see, for instance, Example 2.1.2 below for a more detailed analysis in a case of one spatial dimension). But, if we choose γ large enough so that $2(\alpha - \lambda_1) - \gamma^2 < 0$, then the stochastic equation becomes exponentially stable $\mathbb{P}$-a.s. Consequently, a large intensity of the noise has produced a stabilization effect on the system.

2.1.2 Destabilization produced by Itô noise

Consider the deterministic heat equation but now in one spatial dimension:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + \alpha u(t,x), \quad t>0, \quad 0<x<\pi, \\ u(t,0)=u(t,\pi)=0, \quad t>0, \\ u(0,x)=u_0(x), \quad x\in[0,\pi]. \end{cases} \tag{19}$$

Set again $H = L^2([0,\pi])$ and let $A = \frac{\partial^2}{\partial x^2} + \alpha$. It then follows that $D(A) = H_0^1([0,\pi]) \cap H^2([0,\pi])$. Notice that this system can be explicitly solved and its solution is given by

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-(n^2-\alpha)t} \sin nx,$$

where $u_0(x) = \sum_{n=1}^{\infty} a_n \sin nx$. Hence, exponential stability holds if and only if $\alpha < n^2$ for all $n \in \mathbb{N}$, i.e. if and only if $\alpha < 1$.

Consider now the problem

$$\begin{cases} \mathrm{d}u(t,x) = Au(t,x)\,\mathrm{d}t + Bu(t,x)\,\mathrm{d}W(t), \\ u(0,x)=u_0(x), \end{cases} \tag{20}$$

where B is defined by $Bu(x) = \delta \frac{\partial u(x)}{\partial x}$, for any $u \in H_0^1([0,\pi])$, $\delta \in \mathbb{R}$.

We will show that, if we choose δ such that

$$\frac{\delta^2}{2} < 1 \quad \text{and} \quad \frac{\delta^2}{2} - 1 + \alpha \ge 0,$$

then the stochastic problem becomes unstable.

Indeed, denoting by $C = A - \frac{1}{2}B^2$, the stability of problem (20) is equivalent to the stability of

$$\begin{cases} \mathrm{d}u(t,x) = Cu(t,x)\,\mathrm{d}t + Bu(t,x) \circ \mathrm{d}W(t) \\ u(0,x)=u_0(x). \end{cases} \tag{21}$$

But, due to the commutativity property of the operators involved in the equation, Theorem 2 ensures that the stability of (21) is equivalent to the stability of the deterministic problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = (1 - \dfrac{\delta^2}{2}) \dfrac{\partial^2 u(t,x)}{\partial x^2} + \alpha u(t,x), \\ u(t,0)=u(t,\pi)=0, t>0, \\ u(0,x)=u_0(x), x\in[0,\pi]. \end{cases}$$

This is exponentially stable if and only if $\alpha < 1 - \frac{\delta^2}{2}$. Since our constants satisfy the opposite inequality, we have that the noise has destabilised the deterministic exponentially stable system.

As a conclusion in this fully commuting case, it is thus evident that we should be very careful with the interpretation given to the noise since, depending on that, the behaviours of the deterministic and stochastic models may be completely different. More precisely, the Stratonovich noise does not modify the stability properties of the deterministic model, while Itô noise can produce very different effects.

2.2 Stabilization of a linear PDE in the non-fully commuting case

Notice that under our fully commuting assumptions in the previous subsection, the deterministic problem is exponentially stable if and only if the stochastically perturbed equation has the same property. However, an immediate question arises. What happens if no commutativity holds between A and some B_k? In this case, one can find in [19] some sufficient conditions ensuring the persistence of exponential stability from the deterministic to the stochastic model.

2.2.1 Straightforward stabilization produced by a simple multiplicative Itô noise

First, we point out that a simple multiplicative noise in the sense of Itô can always stabilise the deterministic linear partial differential equation (8) in a lot of cases. So, if we are interested in finding appropriate types of Itô noise to produce stabilization, we do not need to worry too much about looking for a very complicate expression of the noise. Just a term like

$$\sigma u \dot{W}(t)$$

can produce that effect. However, this stabilization can be produced for more general terms and, moreover, we can determine in some cases even the decay rate of the solutions (exponential, sub- or super-exponential, etc).

We now include a result which is a particular situation of a much more general nonlinear theorem (see Section 3 for more details).

First, recall that a linear operator A generates a strongly continuous semigroup $S_A(t)$ satisfying $|S_A(t)| \leq \mathrm{e}^{\alpha t}$, $\alpha \in \mathbb{R}$, if and only if $(Au, u) \leq \alpha |u|^2$, for all $u \in D(A)$.

Theorem 3 *Assume that A generates a strongly continuous semigroup $S_A(t)$ satisfying $|S_A(t)| \leq \mathrm{e}^{\alpha t}$, $\alpha \in \mathbb{R}$, and $B : D(B) \subset H \mapsto H$ is a linear (bounded or unbounded) operator with $D(A) \subset D(B)$. Suppose that the two following hypotheses hold:*

i) There exists $\beta \in \mathbb{R}$ such that

$$(Au, u) + \frac{1}{2}|Bu|^2 \leq \beta |u|^2, \forall u \in D(A) \tag{22}$$

(which is immediately fulfilled for $\beta = \alpha + \frac{1}{2}\|B\|^2$, if B is bounded).

ii) There exists $b, \widetilde{b} \in \mathbb{R}$, with $0 \leq b \leq \widetilde{b}$, such that

$$b|u|^2 \leq (u, Bu) \leq \widetilde{b}|u|^2 \quad \forall u \in D(B). \tag{23}$$

Then, for every $u_0 \in D(A), u_0 \neq 0$, the solution $u(t) = u(t, \omega; 0, u_0)$ to the problem

$$\begin{cases} \mathrm{d}u = Au\,\mathrm{d}t + Bu\,\mathrm{d}W, \\ u(0) = u_0 \in H, \end{cases}$$

satisfies

$$\limsup_{t\to+\infty} \frac{1}{t} \log |u(t; u_0)|^2 \leq -(b^2 - \beta), \quad \mathbb{P} - a.s.$$

Notice that, in the particular case in which B is defined by $Bu = bu$ with $b \in \mathbb{R}$, then $\beta = \alpha + \frac{1}{2}b^2$ and therefore $b^2 - \beta = \frac{1}{2}b^2 - \alpha$, which is positive when $|b|$ is large enough (i.e. large intensity of the noise produces stabilization on the null solution).

2.2.2 Not so straightforward stabilization produced by Stratonovich noise

However, to obtain the same effect using Stratonovich noise turns out to be a completely different and much more difficult problem as commented in the Introduction. But, surprisingly, a very simple trick (discovered long time after the results in the finite-dimensional case were obtained) will allow to prove that the negative trace assumption (7) is a necessary and sufficient condition for the stabilization of a linear PDE by using a suitable Stratonovich noise (see [19] for a detailed exposition on this problem). Instead of establishing and proving this stabilization result, we will motivate the problem with an example on which the proof of the theorem is based.

Consider the following one-dimensional heat equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + 2u(t,x), \quad t > 0, \quad 0 < x < \pi, \\ u(t,0) = u(t,\pi) = 0, t > 0, \\ u(0,x) = u_0(x), \quad x \in [0,\pi]. \end{cases} \tag{24}$$

This problem can be formulated in our framework by setting $H = L^2([0,\pi])$ and $A = \frac{\partial^2}{\partial x^2} + 2I$. It follows that $D(A) = H_0^1([0,\pi]) \cap H^2([0,\pi])$. Recall that this problem can be solved explicitly as in Section 2.1.2, yielding

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-(n^2-2)t} \sin nx,$$

where $u_0(x) = \sum_{n=1}^{\infty} a_n \sin nx$. Hence, it is clear that the zero solution of our problem (24) is not stable. But we will choose appropriate operators $B_k : H \to H,\ k = 1, 2, ...d$, such that

$$\begin{cases} \mathrm{d}u(t,x) = Au(t,x)\,\mathrm{d}t + \sum\limits_{k=1}^{d} B_k u(t,x) \circ \mathrm{d}W_k(t) \\ u(0,x) = u_0(x), \end{cases} \tag{25}$$

becomes exponentially stable with probability one. It is worth pointing out that the operators B_k cannot commute with A.

Notice that A possesses a sequence of eigenvalues given by $\lambda_n = 2 - n^2, n \geq 1$, with associated eigenfunctions $e_n = \sqrt{\frac{2}{\pi}} \sin nx$, which form an orthonormal basis of the Hilbert space H. This means that any $u \in H$ can be represented in the form

$$u = \sum_{k \geq 1} (u, e_k) e_k = \sum_{k \geq 1} u_k e_k.$$

Now we define $B : H \mapsto H$ as $Be_1 = -\sigma e_2$, $Be_2 = \sigma e_1$ and $Be_n = 0$ for any $n \geq 3$, which is a linear operator (and does not commute with A). Then, using the Fourier representation for the solution $u(t)$ to (25), our problem can be re-written as

$$\begin{cases} \sum_{k \geq 1} \mathrm{d}u_k(t)\, e_k = \sum_{k \geq 1} \lambda_k u_k(t) e_k \,\mathrm{d}t + (\sigma u_2(t) e_1 - \sigma u_1(t) e_2) \circ \mathrm{d}W(t) \\ u(0) = u_0 = \sum_{k \geq 1} u_{0k} e_k. \end{cases} \tag{26}$$

Identifying the coefficients, we get two coupled problems.The first one is a 2-dimensional stochastic ordinary differential system, and the second one is an infinite-dimensional system which is exponentially stable since $\lambda_n < 0$ for all $n \geq 3$):

$$\left\{\begin{array}{l} \begin{pmatrix} \mathrm{d}u_1(t) \\ \mathrm{d}u_2(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \circ \mathrm{d}W(t) \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_{01} \\ u_{02} \end{pmatrix}, \end{array}\right. \tag{27}$$

and

$$\left\{\begin{array}{l} \sum_{k\geq 3} \mathrm{d}u_k(t) e_k = \sum_{k\geq 1} \lambda_k u_k(t) e_k \, \mathrm{d}t, \\ \sum_{k\geq 3} u_k(0) e_k = \sum_{k\geq 3} u_{0k} e_k. \end{array}\right. \tag{28}$$

Now, since the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a basis for the linear space of skew symmetric 2×2 matrices, the results in Arnold *et al.* [6] prove that the leading Lyapunov exponent of solutions to (27) tends to $(\lambda_1 + \lambda_2)/2 = -1/2$ as the parameter σ grows to $+\infty$. As it easily follows that the leading Lyapunov exponent for the solutions to (28) is $\lambda_3 = -7$, we can ensure that the top Lyapunov exponent for the solutions of (26) is negative.

Thus, the main idea for the stabilization is to decompose the problem into two new problems: a finite-dimensional one which can be stabilised by using previously available methods from the finite dimensional framework, and another infinite-dimensional system which is already exponentially stable. This idea can be extended in a general way to solve the stabilization problem for a class of deterministic PDEs which appears very frequently in applications.

Consider the infinite-dimensional linear system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \tag{29}$$

where $A : D(A) \subset H \mapsto H$ is a linear operator which has a sequence of eigenvalues λ_j with associated eigenfunctions e_j. We assume that these eigenfunctions form an orthonormal basis of the separable Hilbert space H, and that the eigenvalues λ_j are bounded above (but not necessarily below), so that they can be ordered in the form $\lambda_1 \geq \lambda_2 \geq \ldots$. We denote by $|\cdot|$ the norm in H and by $(\cdot,\cdot)$ its associated scalar product.

Now, we can establish our main stabilization result:

Theorem 4 *(Caraballo & Robinson [19]) Assume that the trace of A is negative, i.e.*

$$\sum_{j=1}^{\infty} \lambda_j < 0. \tag{30}$$

Then there exist linear operators $B_k : H \mapsto H$, $k = 1, 2, \ldots, d$, such that, for

$$\mathrm{d}u = Au \, \mathrm{d}t + \sum_{j=1}^{d} B_k u \circ \mathrm{d}W_k, \tag{31}$$

the zero solution is exponentially stable $\mathbb{P}$-a.s. The operators B_k are such that for some $N > 0$, the $N \times N$ matrices $D_1, ..., D_k$ defined as

$$D_k = \begin{pmatrix} (B_k e_1, e_1) & (B_k e_2, e_1) & \cdots & (B_k e_N, e_1) \\ (B_k e_1, e_2) & (B_k e_2, e_2) & \cdots & (B_k e_N, e_2) \\ \vdots & \vdots & \ddots & \vdots \\ (B_k e_1, e_N) & (B_k e_2, e_N) & \cdots & (B_k e_N, e_N) \end{pmatrix}$$

are skew-symmetric.

Conversely, if there exist linear operators $B_k : H \mapsto H$, $k = 1, 2, ..., d$, with the above properties, for which the zero solution of (31) is exponentially stable with probability one then the trace of A is negative.

3 Stabilization of nonlinear PDEs

The objectives of this section are the following. First, we will show that there exists a well developed theory concerning the stabilization of nonlinear PDEs by Itô noise with applications to several interesting examples. On the other hand, since not much is known about the same topic but involving Stratonovich noise, we will analyse a particular example (which, somehow, can be considered as canonical) in which the previous Theorem 4, jointly with some order preserving properties will allow to prove stabilization for the Chafee-Infante equation by using Stratonovich noise. A more complete study for more general nonlinear equations is to be done.

3.1 Some representative results concerning the stabilization of nonlinear PDEs by Itô noise

First, we introduce the framework where our analysis is going to be carried out.

Let H be a real, separable Hilbert space and V a real, reflexive and separable Banach space such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V',$$

where the injections are continuous and dense. In particular, we also assume that both V and V' are uniformly convex.

We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V', respectively; by $\langle \cdot, \cdot \rangle$ the duality product between V, V', and by $(\cdot, \cdot)$ the scalar product in H. Let a_1 be the constant of the injection $V \hookrightarrow H$, i.e.

$$a_1 |u|^2 \leq \|u\|^2 \quad \forall u \in V.$$

Consider the following problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = F(t, u), \\ u(0) = u_0 \in H, \end{cases} \tag{32}$$

where $F(t, \cdot) : V \mapsto V'$, $t \in \mathbb{R}_+$, is a family of (nonlinear) operators satisfying $F(t, 0) = 0$ and the following hypothesis:

There exist a continuous function $\nu(\cdot)$ and a real number $\nu_0 \in \mathbb{R}$ such that

$$2\langle u, F(t, u) \rangle \leq \nu(t)|u|^2 \quad \forall u \in V, \tag{33}$$

where

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \nu(s)\, ds \leq \nu_0. \tag{34}$$

Assume that for each $u_0 \in H$ there exists a unique strong solution $u = u(t; u_0)$ to (32), with $u(t; u_0) \in L^2(0, T; V) \cap C^0([0, T]; H)$. Observe that, when $F(t, \cdot)$ satisfies a coercivity condition of the type

$$2\langle u, F(t, u) \rangle \leq -\varepsilon \|u\|^p + \alpha |u|^2, \;\; \forall u \in V, \; \varepsilon > 0, \; \alpha \in \mathbb{R}, \; p > 1$$

and a monotonicity hypothesis, there exists a unique strong solution $u = u(t; u_0)$ to (32) in $L^p(0, T; V) \cap C^0([0, T]; H)$ (see, for instance, Lions [33]).

Note that this coercivity assumption obviously implies (33).

Now, we will see that (32) can be stabilised by using a stochastic perturbation of the kind $g(t, u(t))\mathrm{d}W(t)$. Here, $W(t)$ is (for simplicity) a standard real Wiener process defined on a certain complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ and $g(t, \cdot) : H \to H$ satisfies $g(t, 0) = 0$ and the following condition

$$|g(t, u) - g(t, v)|^2 \leq \lambda(t)|u - v|^2 \quad \forall t \in \mathbb{R}_+, \ \forall u, v \in H, \tag{35}$$

where $\lambda(\cdot)$ is a nonnegative continuous function such that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \lambda(s)\, ds \leq \lambda_0 \in \mathbb{R}_+. \tag{36}$$

Indeed, let us consider the following perturbed problem

$$\begin{cases} \mathrm{d}u(t) = F(t, u(t))\ \mathrm{d}t + g(t, u(t))\ \mathrm{d}W(t), \quad t > 0, \\ u(0) = u_0 \in H. \end{cases} \tag{37}$$

We suppose that for each $u_0 \in H$ there exists a unique strong solution to (37) in $I^p(0, T; V) \cap L^2(\Omega; C^0([0, T]; H))$ for all $T > 0$ and certain $p > 1$,where $I^p(0, T; V)$ denotes the space of all V-valued measurable processes $u = u(t)$ satisfying $E \int_0^T \|u(t)\|^p\, dt < +\infty$ (see for instance Pardoux [36] for conditions under which there exists a unique solution for each $u_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$).

Assume that $V : \mathbb{R}_+ \times H \to \mathbb{R}_+$ is a $C^{1,2}$-positive functional such that, for any $u \in V$ and $t \in \mathbb{R}_+$, $V'_u(t, u) \in V$. We define the operators L and Q as follows: for each $u \in V$, $t \in \mathbb{R}_+$,

$$LV(t, u) = V'_t(t, u) + < V'_u(t, u), F(t, u) > +\frac{1}{2}\big(V''_{u,u}(t, u)g(t, u), g(t, u)\big)$$

and

$$QV(t, u) = \big(V'_u(t, u), g(t, u)\big)^2.$$

Theorem 5 *(Caraballo et al. [16]) Assume that the solution of* (37) *satisfies that* $|u(t)| \neq 0$ *for all* $t \geq 0$ $\mathbb{P}$*-a.s. provided* $|u_0| \neq 0$ $\mathbb{P}-a.s.$ *Let* $V \in C^2(H; \mathbb{R}_+)$ *and let* ψ_1 *and* ψ_2 *be two real-valued continuous functions on* $\mathbb{R}_+$*, with* $\psi_2 \geq 0$*. Assume that there exist* $p > 0$, $\gamma \geq 0$ *and* $\theta \in \mathbb{R}$ *such that*

(a). $\quad |u|^p \leq V(u) \quad \forall u \in V;$

(b). $\quad LV\big(t, u\big) \leq \psi_1(t)V(u), \quad \forall u \in V \quad \forall t \in \mathbb{R}_+;$

(c). $\quad QV\big(t, u\big) \geq \psi_2(t)V^2(u), \quad \forall u \in V \quad \forall t \in \mathbb{R}_+;$

(d). $\quad \limsup_{t\to\infty} \frac{\int_0^t \psi_1(s)ds}{t} \leq \theta, \qquad \liminf_{t\to\infty} \frac{\int_0^t \psi_2(s)ds}{t} \geq 2\gamma.$

Then the unique strong solution of (37) *satisfies*

$$\limsup_{t\to\infty} \frac{\log\big|u(t, u_0)\big|}{t} \leq -\frac{\gamma - \theta}{p} \qquad \mathbb{P}-a.s.,$$

whenever $u_0 \in H$ *is an* $\mathcal{F}_0$*-measurable random vector such that* $|u_0| \neq 0$ *a.s. In particular, if* $\gamma > \theta$*, the solution is* $\mathbb{P}$*-a.s. exponentially stable.*

Proof. (Sketch) The proof relies on Itô's formula, the exponential martingale inequality and the Borel-Cantelli lemma. To be more precise, let us fix $u_0 \in H$ such that $|u_0| \neq 0$ $\mathbb{P}-$a.s. By Itô's formula,

$$\begin{aligned} \log V(u(t,u_0)) &\leq \log V(u(0)) + M(t) \\ &\quad + \int_0^t \Big(\frac{LV(s,u(s))}{V(u(s))} - \frac{1}{2}\frac{QV(s,u(s))}{V^2(u(s))}\Big) ds, \end{aligned} \tag{38}$$

where $M(t) = \int_0^t \frac{1}{V(u(s))}\big(V_x'(u(s)), g(s,u(s))\big) dw(s)$.

From the exponential martingale inequality, we can deduce that

$$\mathbb{P}\{\,\omega : \sup_{0\leq t\leq w} \Big[M(t) - \int_0^t \frac{u}{2}\frac{1}{V^2(u(s))} QV(s,u(s))ds\Big] > v\,\} \leq e^{-uv}$$

for any positive u, v and w. Assigning $\epsilon > 0$ arbitrarily, taking

$$u = \alpha, \quad v = 2\alpha^{-1}\log k, \quad w = k\epsilon \qquad (k \geq 1)$$

where $0 < \alpha < 1$ and applying the well-known Borel-Cantelli lemma, we see that there exists an integer $k_0(\epsilon,\omega) > 0$ for almost all $\omega \in \Omega$ such that

$$M(t) \leq 2\alpha^{-1}\log k + \frac{\alpha}{2}\int_0^t \frac{QV(s,u(s))}{V^2(u(s))} ds$$

for all $0 \leq t \leq k\epsilon$, $k \geq k_0(\epsilon,\omega)$. Replacing this in (38) and using conditions (b) and (c), we deduce that there exists a positive random integer $k_1(\epsilon)$ such that

$$\log V(u(t)) \leq \log V(u(0)) + 2\alpha^{-1}\log k + \int_0^t \psi_1(s)ds - \frac{1}{2}(1-\alpha)\int_0^t \psi_2(s)ds$$

$\mathbb{P}$-a.s. for all $(k-1)\epsilon \leq t \leq k\epsilon$ and $k \geq \max(k_0(\epsilon,\omega) \vee k_1(\epsilon)$. Now, assumption (d) implies that

$$\begin{aligned} \frac{\log|u(t)|}{t} &\leq \frac{\log V(u(t))}{pt} \\ &\leq \frac{1}{pt}\Big(\log V(u(0)) + 2\alpha^{-1}\log k + \big(\theta+\epsilon\big)t - \frac{1}{2}(1-\alpha)\big(2\gamma+\epsilon\big)t\Big). \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\log|u(t)|}{t} \leq \frac{1}{p}\Big[\big(\theta+\epsilon\big) - (1-\alpha)\big(\gamma + \frac{\epsilon}{2}\big)\Big] \qquad a.s.$$

Letting $\alpha \to 0$ and $\epsilon \to 0$, we obtain:

$$\limsup_{t\to\infty} \frac{\log\big|u(t,u_0)\big|}{t} \leq -\frac{\gamma-\theta}{p} \qquad a.s.$$

□

As a direct consequence of Theorem 5, by using the function $V(t,u) = |u|^2$, we can prove the following:

Theorem 6 *Assume that the solution of* (37) *satisfies* $|u(t,u_0)| \neq 0$ *for all* $t \geq 0$ $\mathbb{P}$*-a.s. provided* $|u_0| \neq 0$ $\mathbb{P}$*-a.s. In addition to hypotheses* (33) – (36)*, assume that*

$$(g(t,u),u)^2 \geq \rho(t)|u|^4 \quad \forall u \in H, \tag{39}$$

where $\rho(\cdot)$ *is a nonnegative continuous function such that*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \rho(s)\,ds \geq \rho_0, \quad \rho_0 \in \mathbb{R}_+. \tag{40}$$

Then, if $u = u(t,u_0)$ *denotes the solution to* (37)*, it follows that*

$$\limsup_{t\to\infty} \frac{1}{t}\log|u(t,u_0)|^2 \leq -(2\rho_0 - \nu_0 - \lambda_0) \quad P-a.s. \tag{41}$$

for any $u_0 \in H$*. In particular, if* $2\rho_0 > \nu_0+\lambda_0$*, the equation* (37) *is* $\mathbb{P}$*-a.s. exponentially stable.*

3.1.1 Stability properties of a general nonlinear example

Now, we are going to apply Theorem 6 to analyse the pathwise stability of a nonlinear stochastic partial differential equation. Consider the following problem previously studied, among others, by Pardoux [36], Caraballo and Liu [15]:

$$\begin{cases} \mathrm{d}u(t) = A(t,u(t))\,\mathrm{d}t + B(t,u(t))\,\mathrm{d}W(t), & t > 0, \\ u(0) = u_0 \in H. \end{cases} \tag{42}$$

where $A(t,\cdot) : V \mapsto V'$ is a family of nonlinear operators defined for almost every t (t-a.e. for short), satisfying $A(t,0) = 0$ for $t \in \mathbb{R}_+$; $B(t,\cdot) : V \mapsto H$, satisfies

(b.1) $B(t,0) = 0$;

(b.2) There exists $k > 0$ such that

$$|B(t,y) - B(t,x)| \leq k\|y - x\|, \quad \forall x,y \in V,\ t\text{-a.e.}$$

In [15], the following result is proved:

Theorem 7 *In addition to* (b.1)–(b.2)*, assume the following coercivity condition: there exist* $\alpha > 0$*,* $p > 1$ *and* $\lambda \in \mathbb{R}$ *such that, for almost all* $t \in \mathbb{R}_+$ *and for all* $x \in V$*, one has*

$$2\langle x, A(t,x)\rangle + |B(t,x)|^2 \leq -\alpha\|x\|^p + \lambda|x|^2. \tag{43}$$

Then there exists $r > 0$ *such that*

$$E|u(t;u_0)|^2 \leq E|u_0|^2 e^{-rt} \quad \forall t \geq 0,$$

if at least one of the following hypotheses holds:

(i) $\lambda < 0$*;*

(ii) $\lambda\beta^2 - \alpha < 0$ *and* $p = 2$*.*

Furthermore, under the same assumptions the solution is $\mathbb{P}$*-a.s. exponentially stable. That is, there exist positive constants* ξ*,* η *and a subset* $\Omega_0 \subset \Omega$ *with* $\mathbb{P}(\Omega_0) = 0$ *such that, for each* $\omega \notin \Omega_0$*, there exists a positive random number* $T(\omega)$ *satisfying*

$$|u(t,\omega;u_0)|^2 \leq \eta|u_0|^2 e^{-\xi t}, \quad \forall t \geq T(\omega).$$

Observe that, in many applications, conditions (i) and (ii) mean that the term containing B must be small enough with respect to A. For example, let $\mathcal{O}$ be an open, bounded subset in $\mathbb{R}^N$ with regular boundary and let $2 \le p < +\infty$. Consider the Sobolev spaces $V = W_0^{1,p}(\mathcal{O})$, $H = L^2(\mathcal{O})$ with their usual inner products, and the operator $A\colon V \mapsto V'$ defined as

$$\langle v, Au\rangle = -\sum_{i=1}^{N} \int_{\mathcal{O}} \left|\frac{\partial u(x)}{\partial x_i}\right|^{p-2} \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}\, dx + \int_{\mathcal{O}} au(x)v(x)\, dx \;\; \forall u, v \in V,$$

where $a \in \mathbb{R}$. We also introduce B, with $B(t,u) \equiv bu$, where $b \in \mathbb{R}$. Finally, let $W(t)$ be a standard real Wiener process.

Then,

$$2\langle x, A(t,x)\rangle + |B(t,x)|^2 = -2\|x\|^p + 2a|x|^2 + b^2|x|^2 \;\; \forall x \in V, \tag{44}$$

so (43) holds with equality for $\alpha = 2$ and $\lambda = 2a + b^2$. Now, condition (i) requires $2a + b^2 < 0$, so $a < 0$ and $b^2 < -2a$. On the other hand, (ii) holds whenever $(2a+b^2)a_1^{-1} - 2 < 0$, that is, $b^2 < 2a_1 - 2a$. Therefore, Theorem 7 guarantees the exponential stability of paths $\mathbb{P}$-a.s. only for these values of a and b, which means that the deterministic system $\mathrm{d}u(t) = A(t,u(t))\,\mathrm{d}t$ is exponentially stable and the random perturbation is small enough. However, Theorem 6 ensures exponential stability for sufficiently large perturbations although the deterministic system is unstable. Note that, in this case, it is not difficult to see that

$$2\langle x, A(t,x)\rangle = -2\|x\|^p + 2a|x|^2 \le \begin{cases} 2a|x|^2, & \text{if } p > 2, \\ (2a - 2a_1)|x|^2, & \text{if } p = 2, \end{cases} \tag{45}$$

so that

$$\nu(t) = \nu_0 = \begin{cases} 2a, & \text{if } p > 2, \\ 2a - 2a_1, & \text{if } p = 2, \end{cases}$$

$$\lambda_0 = \rho_0 = b^2$$

and thus Theorem 6 yields

$$\limsup_{t\to\infty} \frac{1}{t} \log|u(t;u_0)|^2 \le \begin{cases} -(b^2 - 2a), & \text{if } p > 2, \\ -(b^2 - 2a + 2a_1), & \text{if } p = 2. \end{cases}$$

Consequently, we get pathwise exponential stability $\mathbb{P}$-a.s. if

$$b^2 > \begin{cases} 2a, & \text{if } p > 2, \\ 2a - 2a_1, & \text{if } p = 2. \end{cases}$$

In general, we have the following result:

Theorem 8 *Assume* (b.1)–(b.2), (43) *and that there exists a nonnegative continuous function $b = b(t)$ such that*

$$(B(t,x), x)^2 \ge b(t)|x|^4 \quad \forall x \in V, \tag{46}$$

with

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t b(s)\, ds \ge b_0 \in \mathbb{R}_+. \tag{47}$$

Then, $\mathbb{P}$-a.s. it follows that

$$\limsup_{t\to+\infty} \frac{1}{t} \log|u(t;u_0)|^2 \le \begin{cases} -(2b_0 - \lambda) & \text{if } p > 1, \\ -(2b_0 - \lambda + \alpha a_1) & \text{if } p = 2, \end{cases} \tag{48}$$

for any $u_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$ such that $|u_0| \neq 0$, $\mathbb{P}$-a.s.

Remark 2 Observe that if $\lambda < 0$ then (42) is pathwise exponentially stable $\mathbb{P}$-a.s. for all $p > 1$ and all $b_0 \in \mathbb{R}_+$; while when $\lambda > 0$, (42) is stable if $2b_0 > \lambda$ (for $p \neq 2$) or $2b_0 > \lambda - \alpha a_1$ (for $p = 2$). Now, taking into account our previous theorems, we can summarize the analysis for the preceding example:

- *Case 1: The nonlinear problem, i.e., $p > 2$.* Observe that in this case, the problem is exponentially stable for all $b \in \mathbb{R}$ when $a \leq 0$. However, if $a > 0$ Theorem 7 gives stability provided $b^2 > 2a$. Note that we do not know what happens if $a > 0$ and $b^2 \leq 2a$.
- *Case 2: The linear problem, i.e., $p = 2$.* As in the preceding case, when $a \leq 0$ the system is $\mathbb{P}$-a.s. exponentially stable for all $b \in \mathbb{R}$. But if $a > 0$ we need to check (ii), which requires $b^2 < 2a_1 - 2a$, or it should hold $b^2 > 2a - 2a_1$. So, if $a \leq a_1$ exponential stability $\mathbb{P}$-a.s. follows for all $b \in \mathbb{R}$. But, when $a > a_1$, we only can ensure stability for $b^2 > 2a - 2a_1$ and we do not know what happens for $b^2 \leq 2a - 2a_{1.}$.

In conclusion, our results guarantee exponential stability for a wide range of values of a and b. Of course, this also means that, given the deterministic system $\mathrm{d}x(t) = A(t, x(t))\,\mathrm{d}t$, if a stochastic perturbation of the type $bx(t)\,\mathrm{d}W(t)$ appears, the perturbed system becomes exponentially stable when the parameter of the noise satisfies the conditions above. But when this does not happen, that is, when we do not know whether the system is stable or not, what could we say? Is it possible to add another stochastic term in order to stabilise the stochastic PDE? The answer is positive and some results on this direction can be found, for instance, in [16].

Remark 3 It is remarkable that Theorem 7 is a particular case of a more general result which ensures stabilization with general decay rate (super- or sub-exponential). Also, these results can be used to construct stabilisers of PDEs (see [9] for more details on these topics).

Remark 4 The technique used to prove our previous theorems can be adapted to study interesting examples arising in applications. Two important cases are the 2D Navier-Stokes equations (see [14]) and the 3D Lagrangian averaged Navier-Stokes α-model, also called Camassa-Holm equation (see [18, 17]).

3.2 A first stabilization result of a nonlinear PDEs by Stratonovich noise

As far as we know, there are no general results concerning the stabilization of nonlinear PDEs by Stratonovich noise. The problem seems to be very difficult and challenging. The unique work in this direction involves a canonical model whose dynamics is very well known in the deterministic case. This is the Chafee-Infante equation

$$\frac{\partial u}{\partial t} = \Delta u + \beta u - u^3, \quad \text{for} \quad x \in D \quad \text{and} \quad u|_{\partial D} = 0, \tag{49}$$

where D is a smooth bounded domain in $\mathbb{R}^m$.

In [8] it is proved that the *nonlinear* equation (49) can be stabilised by adding a collection of noisy terms similar to the linear case in Section 2:

$$\mathrm{d}u = [\Delta u + \beta u - u^3]\,\mathrm{d}t + \sum_{i=1}^{d} B_i u \circ \mathrm{d}W_i. \tag{50}$$

Essentially it is shown that solutions of (50) can be bounded using appropriate *positive* solutions of the linear equation

$$\mathrm{d}u = [\Delta u + \beta u]\,\mathrm{d}t + \sum_{i=1}^{d} B_i u \circ \mathrm{d}W_i. \tag{51}$$

Since (51) can be stabilised via a suitable choice of $\{B_i\}$, so can (50). The proof makes essential and continual use of the order-preserving properties of (50).

To set this problem in a suitable context, we choose $H = L^2(D)$; denote by $-\mathbb{A}$ the linear operator in H associated to the Laplacian. We then take $A = \mathbb{A} + \beta I$, which clearly satisfies the conditions of Theorem 4, and let N be the smallest integer such that $\sum_{j=1}^{N}(\beta - \lambda_j) < 0$. It follows that there exist linear operators $B_k : H \to H$ such that the zero solution of

$$\mathrm{d}u = [-\mathbb{A}u + \beta u]\,\mathrm{d}t + \sum_{k=1}^{d} B_k u \circ \mathrm{d}W_k(t) \tag{52}$$

is exponentially stable $\mathbb{P}$-a.s.

This fact can be used to deduce the stabilization of the nonlinear equation via the addition of the same noisy terms. The next result can be found in [8]:

Theorem 9 *There exist bounded linear operators $B_k : H \mapsto H$ and independent real Wiener processes W_k , $k = 1, \ldots d$, such that the zero solution of*

$$\mathrm{d}u = (-\mathbb{A}u + \beta u - u^3)\,\mathrm{d}t + \sum_{j=1}^{d} B_k u \circ \mathrm{d}W_k(t) \tag{53}$$

is exponentially stable $\mathbb{P}$-a.s.

This simple, but illustrative, example may help to solve the stabilization problem for more general nonlinear PDEs appearing in applications. To the best of our knowledge, this is still an open problem.

4 "Super"-stabilization to a non-trivial stationary solution (random fixed point)

In the previous sections, we have exhibited a collection of results and techniques to stabilise the null solution of some classes of partial differential equations by using either Stratonovich or Itô noisy terms. However, there exists a very important effect of the noise not matter the interpretation we could give that can be regarded as a "super"-stabilization effect, since somehow the instabilities of the problem "dissapear" when the noise is added to the equation. It is worth mentioning that, in many situations, the noise may appear in an additive way, so that the Itô and Stratonovich interpretations coincide. In [8] it is shown that the addition of an additive noise that is rich enough may produce such a "super-stabilization" effect on the system. In this case, the dynamics of the model will be conduced to a stationary process (random fixed point). We first need some preliminaries.

4.1 Random dynamical systems and random attractors

In the interest of brevity we only state the definitions here: for more background on random dynamical systems, see Arnold [4].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\vartheta_t : \Omega \to \Omega,\ t \in \mathbb{R}\}$ be a family of measure preserving transformations such that $(t, \omega) \mapsto \vartheta_t\omega$ is measurable, $\vartheta_0 = \mathrm{id}$, and $\vartheta_{t+s} = \vartheta_t\vartheta_s$ for all $s, t \in \mathbb{R}$. The flow ϑ_t together with the corresponding probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ is called a *(measurable) dynamical system.*

A continuous *random dynamical system* (RDS) on a Polish space (X, d) with Borel σ-algebra $\mathcal{B}$ over ϑ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable map

$$\begin{gathered}\varphi : \mathbb{R}^+ \times \Omega \times X \to X\\ (t, \omega, x) \mapsto \varphi(t, \omega)x\end{gathered}$$

such that $\mathbb{P}$-a.s.

i) $\varphi(0, \omega) = \mathrm{id}$ on X

ii) $\varphi(t + s, \omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s, \omega)$ for all $t, s \in \mathbb{R}^+$ (cocycle property)

iii) $\varphi(t, \omega) : X \to X$ is continuous.

A *random attractor* for an RDS φ is a random set $\omega \mapsto \mathcal{A}(\omega)$ such that

(i) $\mathcal{A}$ is a random compact set, that is, $\mathbb{P}$-a. s., $\mathcal{A}(\omega)$ is compact, and for all $x \in X$ the map $\omega \mapsto \mathrm{dist}\big(x, \mathcal{A}(\omega)\big)$ is measurable with respect to $\mathcal{F}$.

(ii) $\mathbb{P}$-a. s. $\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\vartheta_t\omega)$ for all $t \geq 0$, and

(iii) for every $D \subset H$ bounded, $\mathbb{P}$-a. s.,

$$\lim_{t\to\infty} \mathrm{dist}\big(\varphi(t, \vartheta_{-t}\omega)D, \mathcal{A}(\omega)\big) = 0.$$

To set our equation in the framework of random dynamical systems, we let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space generating the two-sided Wiener process $W(t)$, and define a shift ϑ_t on Ω by

$$W(t, \vartheta_s\omega) = W(t + s, \omega) - W(s, \omega),$$

the additional subtracted term ensuring that $W_\cdot(\vartheta_s\omega)$ is still a Brownian motion.

From Pardoux [36] we deduce that for any initial condition $u_0 \in L^2(D)$ and $T > 0$, there exists a unique strong solution

$$u(t; u_0) \in L^2(\Omega \times (0, T); H_0^1(D)) \cap L^4(\Omega \times (0, T) \times D) \cap L^2(\Omega; C(0, T; L^2(D)),$$

which generates a random dynamical system φ on the phase space $L^2(D)$ by setting $\varphi(t, \omega)u_0 = u(t; u_0)$.

In [12], it is proved that

$$\mathrm{d}u = [\Delta u + \beta u - u^3]\,\mathrm{d}t + \sigma u \circ \mathrm{d}W \tag{54}$$

has a random attractor. We then showed that the Hausdorff dimension of the attractor is bounded by d when

$$\beta < \frac{1}{d}\sum_{j=1}^{d} \lambda_j,$$

where λ_j are the eigenvalues of the Laplacian arranged in increasing order. Recently, Langa & Robinson proved in [31] the same upper bound for the upper box-counting (fractal) dimension of the attractor. Since $\lambda_n \sim n^{2/m}$ this implies that $d(\mathcal{A}(\omega)) \leq c\beta^{m/2}$.

In [13] we show that, provided $m \leq 5$, the unstable manifold near the origin has dimension at least d when $\beta > \lambda_d$. This leads to a lower bound on the dimension of the same order as the upper bound, and hence together show that the dimension of the random attractor is of the same order as its deterministic counterpart, namely

$$d(\mathcal{A}(\omega)) \sim \beta^{m/2}.$$

In this sense, the addition of a single multiplicative Stratonovich noise has no effect on the asymptotic complexity of the dynamics.

4.2 Collapse of the random attractor produced by additive noise

In this final section we show that the addition of a sufficiently rich additive white noise will reduce the random attractor of the equation to a single (random) point.

Such behaviour was originally demonstrated for the one-dimensional ordinary differential equation

$$\mathrm{d}x = [\alpha x - x^3]\,\mathrm{d}t + \epsilon\,\mathrm{d}W, \qquad \text{with} \qquad \alpha > 0$$

by Crauel & Flandoli [22], and has recently been shown by Robinson & Tearne [37] for a general gradient ODE of the form

$$\mathrm{d}x = -\nabla V(x) + \epsilon\,\mathrm{d}W$$

where $x \in \mathbb{R}^m$, W_t is an m-dimensional Brownian motion, and ϵ is sufficiently small (note that this is not in general an order-preserving system).

In [8] we prove a similar result for the equation

$$\mathrm{d}u = [\Delta u + \beta u - u^3]\,\mathrm{d}t + \sqrt{C}\,\mathrm{d}W, \qquad x \in D = [0, L], \tag{55}$$

where W, $t \in \mathbb{R}$, is a two-sided Q-cylindrical Wiener process on $H = L^2(D)$ (see Da Prato & Zabczyk [25] for the description and properties of the cylindrical Wiener process) and C is a bounded linear operator with bounded inverse on H. Here, we restrict ourselves to a one-dimensional domain.

The argument used in this case could be generalised to treat more abstract problems (cf. Chueshov & Scheutzow [20]) but the underlying idea is simple: Results of Arnold & Chueshov [5] on the structure of random attractors in order-preserving systems guarantee the existence of two random fixed points $\underline{a}$ and $\overline{a}$ that are contained in the attractor and are such that

$$\underline{a}(\omega) \leq u \leq \overline{a}(\omega), \quad \forall u \in \mathcal{A}(\omega).$$

Corresponding to these random fixed points there are invariant measures $\delta_{\underline{a}(\omega)}$ and $\delta_{\overline{a}(\omega)}$. Since the noise in (55) is sufficiently rich to guarantee that the equation has a *unique* invariant measure (e.g. Da Prato, Debussche, & Goldys [24]), it follows that the laws of $\underline{a}(\omega)$ and $\overline{a}(\omega)$ must coincide. It is only a small step from this, using the fact that $\underline{a}(\omega) \leq \overline{a}(\omega)$, to the deduction that $\underline{a}(\omega) = \overline{a}(\omega) = a(\omega)$, and hence that $\mathcal{A}(\omega) = \{a(\omega)\}$, i.e. the attractor is a single point.

Let us now recall the formal existence and uniqueness results for (55), and let us establish that the random attractor is a point.

We take $(\Omega, \mathcal{F}, \mathbb{P})$ to be the probability space that generates the Q-cylindrical Wiener process $W(t)$, and define a shift ϑ_t on Ω by $W(t, \vartheta_s\omega) = W(t+s, \omega) - W(s, \omega)$.

Under these assumptions (see Da Prato & Zabczyk [25]), for each $u_0 \in L^2(D)$ and $T > 0$ there exists a unique solution $u(t; u_0)$ for (55), with

$$u(t; u_0) \in L^2\big(\Omega \times (0,T); H_0^1(D)\big) \cap L^4\big(\Omega \times (0,T) \times D\big) \cap L^2\big(\Omega; C(0,T; L^2(D))\big).$$

It follows that the solutions of (55) generate a random dynamical system on $L^2(D)$ if we define

$$\varphi(t, \omega)u_0 = u(t; \omega, u_0),$$

where $u(t; \omega, u_0)$ is the solution of (55) with noise ω and initial condition $u(0) = u_0$.

Then, we have the following result (see [8] for the proof):

Theorem 10 *The random attractor for* (55) *consists of a single point, i.e. there exists a random variable* $a : \Omega \mapsto H$ *with*

$$\varphi(t, \omega)a(\omega) = a(\vartheta_t\omega) \qquad \text{for every } t \geq 0, \qquad \mathbb{P} - a.s.$$

such that $\mathcal{A}(\omega) = \{a(\omega)\}$.

The above result would extend to m-dimensional domains if the existence of a unique invariant measure could be guaranteed in this case (cf. Hairer [26]).

5 Final remarks and some open problems

Needless to say that what we have included in the previous section does not cover all the aspects involving the asymptotic behaviour of stochastic partial differential equations. We have only emphasised some facts related to the stabilising effect that can be produced by the appearance of noise in deterministic PDEs. Therefore, many other topics could have been considered. For instance, we have only mentioned the effects produced by the noise in the structure of the global attractor in a very particular example, so it is very interesting to analyse these potential effects that different classes of noise can produce on the global attractors for other interesting models from applications.

Going deeper in this direction, there are many situations in which the uniqueness of solution is not known or cannot be guaranteed, or even that the evolution of the system can be better modelled by a differential inclusion. To our knowledge, there are no results on the stabilization effect of the noise in multivalued dynamical systems.

Sometimes, the consideration of delay terms in the equations of some models is fully justified. The problem of stabilization of delay (ordinary or partial) differential equations is therefore an important task. In the finite dimensional context, there are only a few results on the stabilization by the Itô noise when the delay is small enough (see Appleby and Mao [1]), but nothing is known for systems with arbitrary delay (finite or infinite) neither by using Itô nor Stratonovich noise.

Another problem that may be even closer to reality is related to the effect produced by the noise when it acts only on (part of) the boundary of the domain and not in the forcing term in the equation. For instance, if we are considering an oceanic model, the stochastic disturbances may appear on the ocean surface and not in the equations driving the system. Some preliminary results are to appear in the future (see [7]).

Of course, we could list some more interesting and challenging situations but we content ourselves with the previous ones since this area is still in its infancy, and too much work is to be done in the future. We hope we can contribute to solve some of these problems.

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