OPTIMAL CONTROL FOR THE DEGENERATE ELLIPTIC LOGISTIC EQUATION

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Abstract

We consider the optimal control of the harvesting of the diffusive degenerate elliptic logistic equation. Under certain assumptions, we prove the existence and uniqueness of an optimal control. Moreover, the optimality system and a characterization of the optimal control are also derived. Sub-supersolution method, singular eigenvalue problem and differentiability with respect to the positive cone are the techniques used to get our results.

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Running head. Optimal control for degenerate logistic equation

1 Introduction

This work considers the optimal harvesting control of a species whose state is governed by the degenerate (nonlinear slow diffusion) elliptic logistic equation, i.e.,

$$\begin{cases} -\Delta w^m = (a-f)w - ew^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, $N \geq 1$; $m > 1$; a, f and e are bounded functions with some restrictions that will be detailed below.

Equ. (1.1) was introduced in populations dynamics by Gurtin and MacCamy in [5] describing the behaviour of a single species inhabiting in Ω and whose population density is $w(x)$. Since the population is subject to homogeneous Dirichlet boundary conditions, we are assuming that Ω is fully surrounded by inhospitable areas. In such model, the positive function $e(x)$ describes the limiting effects of crowding in the species and $a(x)$ represents the growth rate of the species. The function $f(x)$ denotes the distribution of control harvesting of the species. Since f will be considered non-negative, observe that f leads by reducting the growth rate. Finally, the operator $-\Delta$ measures the diffusion, i.e., the moving rate of the species from high density regions to low density areas. In this case, $m > 1$ (nonlinear slow diffusion) means that the diffusion is slower than in the linear case $m = 1$, which gives rise to more realistic biological results, see [5].

To study (1.1), we make the change of variables $w^m = u$ and obtain

$$\begin{cases} -\Delta u = (a-f)u^\alpha - eu^\beta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $\alpha = 1/m$ and $\beta = 2/m$. Under hypothesis $(H2)$ below, we prove that for each f, there exists a unique positive solution of (1.2), that it will be denoted by u_f. The optimal control criteria is to maximize the payoff functional

$$J(f) := \int_\Omega (\lambda u_f h(f) - k(f)),$$

where $h \in C^1(\mathbb{R}^+; \mathbb{R}^+)$, $k \in C^2(\mathbb{R}^+; \mathbb{R}^+)$ and $\lambda > 0$ will be considered as parameter. J represents the difference between economic revenue measured by $\int_\Omega \lambda u_f h(f)$ and the control cost measured by $\int_\Omega k(f)$. Here, λ describes the quotient between the price of the species and

the cost of the control.
The special case (quadratic functional)

$$h(t) = t \qquad \text{and} \qquad k(t) = t^2,$$

was introduced in dynamics population by Leung and Stojanovic in [10] (see also [3], [9] and references therein).

An optimal control is a function $f \in \mathcal{C}$, where $\mathcal{C}$ is a suitable subset of $L^\infty(\Omega)$, such that

$$J(f) = \sup_{g \in \mathcal{C}} J(g).$$

In the case $m = 1$, i.e., $\alpha = 1$ and $\beta = 2$, and $h(t) = t$ and $k(t) = t^2$, this problem has been studied in detail in [3], [10] and [11]. In fact, some results of this work have been motivated by [3]. In these papers, under certain assumptions in the coefficients of the problem, the authors obtained the existence and uniqueness of the optimal control, as well as a characterization of the optimal control by means the solution of the optimality system. To obtain the results, the authors used mainly the sub-supersolution method, the derivability of the maps $f \mapsto u_f$ and $f \mapsto J(f)$ and the expressions of their derivatives.
When $m > 1$, i.e. $\alpha < 1$, this derivability is rather difficult than in the case $m = 1$, because it involves linear elliptic and eigenvalue problems with unbounded potentials in a neighbourhood of $\partial\Omega$. These difficulties have been solvented by using results of singular eigenvalue problems from [2] and [6], and some classical ones of Krasnoselskii, see [7]. They let us deduce the Fréchet derivability from the Gâteaux derivability with respect to the positive cone. Moreover, the introduction of the functions h and k in the payoff functional leads us to establish the hypotheses to assure the existence and uniqueness of the optimal control.

An outline of this work is as follows: in Section 2 we introduce some notations and we give some results of the existence and uniqueness of the principal eigenvalue and of solution of a linear elliptic problems with unbounded potentials. In Section 3 we show the existence and uniqueness of positive solution of (1.2), collecting a result from [4]. Moreover, we study the derivability of the map $f \mapsto u_f$ giving an explicit expression of that. In Section 4, we show that for λ sufficiently small there exists a unique optimal control. In the last Section we characterize the optimal control. This characterization provides us the optimality system and certain regularity of the optimal control. It is well known that this regularity can suggest numeric methods to approximate the optimal control, which are not considered in this work.

2 Preliminaries

In this paper we use the following notation: Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $\gamma \in (0,2)$ fixed. For any $f \in L^\infty(\Omega)$ we denote

$$f_M := \operatorname{ess\,sup} f \qquad f_L := \operatorname{ess\,inf} f,$$

$$L^\infty_+(\Omega) := \{f \in L^\infty(\Omega) : f_L \geq 0\} \quad L^\infty_-(\Omega) := \{f \in L^\infty(\Omega) : f_M \leq 0\}.$$

Moreover, we denote by P the non-negative cone of $C^1_0(\overline{\Omega})$, whose interior is

$$int(P) := \{u \in C^1_0(\overline{\Omega}) : \ u > 0 \quad \text{in } \Omega, \ \partial u/\partial n < 0 \quad \text{on } \partial\Omega\}$$

where $C^1_0(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u = 0 \quad \text{on } \partial\Omega\}$ and n is the outward unit normal at $\partial\Omega$.
Finally, for any $\Omega' \subset \Omega$, $\sigma_1^{\Omega'}$ and $\varphi_1^{\Omega'}$ stand for the principal eigenvalue and the corresponding positive eigenfunction of the operator $-\Delta$ and homogeneous boundary Dirichlet condition with $\|\varphi_1^{\Omega'}\|_\infty = 1$. In particular, we write $\sigma_1 := \sigma_1^\Omega$ and $\varphi_1 := \varphi_1^\Omega$.
Assume

$$(H1) \qquad M \in L^\infty_{loc}(\Omega) \quad \text{verifying} \quad M(x) d_\Omega(x)^\gamma \in L^\infty(\Omega),$$

where $d_\Omega(x) := dist(x, \partial\Omega)$.

Given $\sigma \in \mathbb{R}$ and $f \in L^\infty(\Omega)$, we consider the following problems

$$\begin{cases} -\Delta u + M(x)u &= \sigma u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

$$\begin{cases} -\Delta u + M(x)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Remark 2.1 *Observe that we are not assuming that* $M \in L^\infty(\Omega)$ *and that a weak solution of* (2.2) *or an associated eigenfunction to the eigenvalue* σ *of* (2.1) *are well defined by the Hardy inequality, see for instance [8].*

The next result follows from [2] and [6]. We include it for the reader's convenience.

Theorem 2.2 *Assume that* M *satisfies* $(H1)$*. Then:*

a) *There exists a unique principal eigenvalue (i.e., a real eigenvalue with an associated eigenfunction in int(P)), which is simple and we denote it by* $\sigma_1(-\Delta + M)$*. Moreover, it satisfies*

$$\sigma_1(-\Delta + M) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^2 + \int_\Omega M(x)u^2}{\displaystyle\int_\Omega u^2} \right\}.$$

b) *(Strong Maximum Principle)* $\sigma_1(-\Delta + M) > 0$ *if and only if* $v \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$*, with* $p > N$ *such that* $v \neq 0$, $-\Delta v + M(x)v \geq 0$ *in* Ω, $v \geq 0$ *on* $\partial\Omega$*, then* $v \in int(P)$.

By the variational characterization of $\sigma_1(-\Delta + M)$, it follows:

Proposition 2.3 *a) (Monotonicity respect to the potential) Assume that* M_i, $i = 1, 2$ *satisfy* $(H1)$ *and* $M_1 \leq M_2$*. Then*

$$\sigma_1(-\Delta + M_1) \leq \sigma_1(-\Delta + M_2).$$

b) (Continuity respect to the potential) Assume that M_n, M, $n \in \mathbb{N}$ *satisfy* $(H1)$ *with*

$$\int_\Omega M_n\varphi^2 \to \int_\Omega M\varphi^2, \quad \text{as } n \to \infty \text{ and for all } \varphi \in H_0^1(\Omega). \tag{2.3}$$

Then,

$$\sigma_1(-\Delta + M_n) \to \sigma_1(-\Delta + M) \quad \text{as } n \to \infty.$$

The following estimate will play an important role in the next sections.

Lemma 2.4 *Assume that* M_n, M, $n \in \mathbb{N}$ *satisfy* $(H1)$, $\sigma_1(-\Delta+M) > 0$ *and* (2.3)*. Then, there exist a positive constant* $C_0 < 1$ *(independient of* n*) and* $n_0(C_0) \in \mathbb{N}$ *such that*

$$C_0 \int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla u|^2 + \int_\Omega M_n u^2 \quad \forall u \in H_0^1(\Omega),\ \forall n \geq n_0. \tag{2.4}$$

Proof: Since $\sigma_1(-\Delta + KM) \to \sigma_1(-\Delta + M) > 0$ as $K \downarrow 1$, there exists $K_0 > 1$ such that $\sigma_1(-\Delta + K_0 M) > 0$. Let C_0 be such that $K_0 = 1/(1 - C_0)$.

To prove (2.4) it is sufficient to show that $\sigma_1(-\Delta + K_0 M_n) \geq 0$ for $n \geq n_0$. But $\sigma_1(-\Delta + K_0 M_n) \to \sigma_1(-\Delta + K_0 M) > 0$. □

The following result shows that (2.2) possesses a unique solution.

Theorem 2.5 *Assume that M satisfies $(H1)$ and $\sigma_1(-\Delta+M)>0$. Then, there exists a unique solution $u\in C^{1,\kappa}(\overline{\Omega})$, for some $\kappa\in(0,1)$, of (2.2). Moreover, there exists a constant $K>0$ (independient of f) such that*

$$\|u\|_{C^{1,\kappa}(\overline{\Omega})}\leq K\|f\|_\infty. \tag{2.5}$$

Proof: For $v\in C_0^1(\overline{\Omega})$ we consider the problem

$$\begin{cases} -\Delta u &= -M(x)v \quad \text{in } \Omega,\\ \quad u &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

By Proposition 2.3 in [6], there exists a unique solution $u\in C^2(\Omega)\cap C^{1,\kappa}(\overline{\Omega})$, for some $\kappa\in(0,1)$, of (2.6) with

$$\|u\|_{C^{1,\kappa}(\overline{\Omega})}\leq K_1\|v\|_{C^1(\overline{\Omega})}.$$

Define $G_1: C_0^1(\overline{\Omega})\mapsto C_0^{1,\kappa}(\overline{\Omega})$, $v\mapsto G_1(v)$ the unique solution of (2.6). We have shown that G_1 is bounded.

For $h\in L^\infty(\Omega)$ we consider the problem

$$\begin{cases} -\Delta u &= h(x) \quad \text{in } \Omega,\\ \quad u &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

It is well known that fixed $h\in L^\infty(\Omega)$, there exists a unique solution $u\in W^{2,p}(\Omega)$ of (2.7) for all $p>1$, and

$$\|u\|_{C^{1,\kappa}(\overline{\Omega})}\leq K_1\|u\|_{W^{2,p}(\Omega)}\leq K_2\|h\|_\infty.$$

We can define the map $G_2: L^\infty(\Omega)\mapsto C_0^{1,\kappa}(\overline{\Omega})$, $h\mapsto G_2(h)$ the unique solution of (2.7). We have got that G_2 is bounded.

Now, if we define

$$H: C_0^1(\overline{\Omega})\mapsto C_0^1(\overline{\Omega}),\qquad H(u):=u-G_1(u),$$

denote by $i: C_0^{1,\kappa}(\overline{\Omega})\mapsto C_0^1(\overline{\Omega})$ the compact imbedding and we pose $G:=H\circ i: C_0^{1,\kappa}(\overline{\Omega})\mapsto C_0^{1,\kappa}(\overline{\Omega})$, then we can rewrite (2.2) as

$$G(u)=G_2(f)$$

being G a compact pertubation of the identity. Since $\sigma_1(-\Delta + M) > 0$, G is inyective. The Fredholm's Theorem provides us the existence and uniqueness of solution $u \in C_0^{1,\kappa}(\overline{\Omega})$ of (2.2) satisfying (2.5). □

The next result is an easy consequence of Theorem 2.2 b).

Lemma 2.6 *a) Assume that M satisfies $(H1)$ and $\sigma_1(-\Delta + M) > 0$. Consider $f_i \in L^\infty(\Omega)$, $i = 1, 2$ with $f_1 \leq f_2$ and let u_i, $i = 1, 2$ be the respective solutions of* (2.2)*. Then, $u_1 \leq u_2$.*

b) Assume that M_i, $i = 1, 2$ satisfy $(H1)$ and $M_1 \leq M_2$ with $\sigma_1(-\Delta + M_1) > 0$. Let u_i, $i = 1, 2$ be the respective solutions of (2.2)*. Then, $u_2 \leq u_1$.*

3 The degenerate logistic equation

Consider

$$\begin{cases} -\Delta u = bu^\alpha - eu^\beta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and assume that

$(H2)$ $\quad 0 < \alpha < 1 \leq \beta, \quad b \in L^\infty_+(\Omega) \backslash \{0\}, \quad e \in \mathcal{A},$

where

$$\mathcal{A} := \{f \in L^\infty(\Omega) : f_L > 0\}.$$

The next result has been proved in [4] when $b, e \in C^\nu(\overline{\Omega})$, $\nu \in (0, 1)$. The proof is also valid in this case.

Theorem 3.1 *Assume $(H2)$. The following assertions are true:*

a) There exists a unique strictly positive solution u_b of (3.1)*. Moreover, by elliptic regularity $u_b \in W^{2,p}(\Omega), p > 1$, and so $u_b \in C^{1,\kappa}(\overline{\Omega}) \cap int(P)$, with $0 < \kappa \leq 1 - N/p$.*

b) We have the following a priori bound,

$$\|u_b\|_\infty \leq \left(\frac{b_M}{e_L}\right)^{1/(\beta-\alpha)}. \tag{3.2}$$

c) If $b_L > 0$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, it holds

$$\varepsilon\varphi_1(x) \leq u_b(x) \quad c.p.d.\ x \in \Omega$$

where $\varepsilon_0 > 0$ satisfies

$$b_L - \sigma_1\varepsilon^{1-\alpha} - e_M\varepsilon^{\beta-\alpha} = 0.$$

d) If $b_L = 0$, since $b_M > 0$ there exists a ball $B := B(x_0, r)$ such that $b_{L,B} > 0$ in B, where $b_{L,B}$ is the essential infimum of b in B. Hence, $\varepsilon\varphi_1^B \leq u_b$ c.p.d. in B for all $\varepsilon \leq \varepsilon_1$ and where $\varepsilon_1 > 0$ satisfies

$$b_{L,B} - \sigma_1^B\varepsilon^{1-\alpha} - e_{M,B}\varepsilon^{\beta-\alpha} = 0.$$

Remark 3.2 *By (H2), (3.1) satisfies the strong maximum principle and then there exist two positive constants k_1, k_2 such that*

$$k_1 d_\Omega(x) \leq u_b(x) \leq k_2 d_\Omega(x) \quad \forall x \in \Omega. \tag{3.3}$$

The following result plays an important role along the work.

Theorem 3.3 *Assume (H2). Then, the map $b \in \mathcal{A} \subset L^\infty(\Omega) \mapsto u_b \in int(P) \subset C_0^1(\overline{\Omega})$ is increasing, continuous and C^1.*

For the proof of this result we use the following elementary lemma.

Lemma 3.4 *a) Let $\alpha \in (0,1]$ and $0 < t_1 < t_2$ be. Then*

$$\alpha t_2^{\alpha-1}(t_2 - t_1) \leq t_2^\alpha - t_1^\alpha \leq \alpha t_1^{\alpha-1}(t_2 - t_1).$$

b) Let $\beta \in [1, +\infty)$ and $0 \leq t_1 < t_2$ be. Then

$$\beta t_1^{\beta-1}(t_2 - t_1) \leq t_2^\beta - t_1^\beta \leq \beta t_2^{\beta-1}(t_2 - t_1).$$

Proof of Theorem 3.3: It follows easily that the map is increasing. For the continuity, let $b_n, b \in \mathcal{A}$ be such that $b_n \to b$ in L^∞, then $(b_n)_M \to b_M$. Hence, fixed $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\|u_{b_n}\|_\infty \leq \left(\frac{(b_n)_M}{e_L}\right)^{1/(\beta-\alpha)} \leq \left(\frac{b_M + \delta}{e_L}\right)^{1/(\beta-\alpha)} = C \quad \text{(independient of } n\text{)},$$

and so, the sequence $\{u_{b_n}\}$ is bounded in $W^{2,p}(\Omega), p > 1$. There exists a subsequence, relabeled by n, such that $u_{b_n} \to u$ in $C^{1,\kappa}(\overline{\Omega}), \kappa < 1 - N/p$. Moreover, u is a weak solution of (3.1). It remains to prove that $u = u_b$. By the uniqueness of positive solution of (3.1), it suffices to prove that $u > 0$. Since $b_M > 0$, there exist $x_0 \in \Omega$, $r_0 > 0$, such that $(b_n) \geq (b_n)_{L,B} > 0$ c.p.d. in $B = B(x_0, r_0)$, for $n \geq n_0$. By Theorem 3.1 d), we have that there exist $\varepsilon_n > 0$ such that $\varepsilon_n \varphi_1^B \leq u_{b_n}$ c.p.d. in B where ε_n is such that

$$(b_n)_{L,B} - \sigma_1^B \varepsilon_n^{1-\alpha} - e_{M,B} \varepsilon_n^{\beta-\alpha} = 0.$$

Since $(b_n)_{L,B} \to b_{L,B}$, it follows that $\varepsilon_n \to \varepsilon > 0$ where ε is such that

$$b_{L,B} - \sigma_1^B \varepsilon^{1-\alpha} - e_{M,B} \varepsilon^{\beta-\alpha} = 0,$$

and so $\varepsilon \varphi_1^B \leq u$ c.p.d. in B and then $u > 0$.

For the derivability we use the Implicit Function Theorem. Fixed $p > N$, we define the map $\mathcal{F} : \mathcal{A} \times \mathcal{U} \subset L^\infty(\Omega) \times C_0^1(\overline{\Omega}) \mapsto L^p(\Omega)$ where $\mathcal{U} := W^{2,p}(\Omega) \cap int(P)$, as

$$\mathcal{F}(b, u) := -\Delta u - b u^\alpha + e u^\beta.$$

$\mathcal{A}$ is an open set in $L^\infty(\Omega)$ and it is well known, see [1], that $\mathcal{U}$ is also open in $C_0^1(\overline{\Omega})$. It is clear that $\mathcal{F}(b_0, u_{b_0}) = 0$.

We show that $\mathcal{F}$ is C^1, for which it is sufficient to show it for the second component. We calculate the Gâteaux derivative respect to this, which will be denoted by $D_G\mathcal{F}$. Let $(b, u) \in \mathcal{A} \times \mathcal{U}$ and $\xi \in C_0^1(\overline{\Omega})$ be, then

$$D_G\mathcal{F}(b,u)\xi := \lim_{\varepsilon \to 0} \frac{\mathcal{F}(b, u + \varepsilon\xi) - \mathcal{F}(b,u)}{\varepsilon} = -\Delta\xi - b \lim_{\varepsilon \to 0} \frac{(u+\varepsilon\xi)^\alpha - u^\alpha}{\varepsilon} + e \lim_{\varepsilon \to 0} \frac{(u+\varepsilon\xi)^\beta - u^\beta}{\varepsilon}.$$

We claim that:

$$\frac{(u+\varepsilon\xi)^\beta - u^\beta}{\varepsilon} \to \beta u^{\beta-1}\xi \quad \text{and} \quad \frac{(u+\varepsilon\xi)^\alpha - u^\alpha}{\varepsilon} \to \alpha u^{\alpha-1}\xi \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0. \tag{3.4}$$

Assume $\varepsilon \downarrow 0$. Using Lemma 3.4, to prove (3.4) it is sufficient to show that

$$(u+\varepsilon\xi)^{\beta-1}\xi \to u^{\beta-1}\xi \qquad \text{and} \qquad (u+\varepsilon\xi)^{\alpha-1}\xi \to u^{\alpha-1}\xi \qquad \text{in } L^p(\Omega) \text{ as } \varepsilon \downarrow 0.$$

The first one is true because $\beta \geq 1$. For the second one, we have

$$\|[(u+\varepsilon\xi)^{\alpha-1} - u^{\alpha-1}]\xi\|_p = \|[(u+\varepsilon\xi)^\alpha - (u+\varepsilon\xi)u^{\alpha-1}]\left(\frac{\xi}{u+\varepsilon\xi}\right)\|_p. \tag{3.5}$$

Since $u \in int(P)$, there exist $\varepsilon_0 > 0$ and $k(\varepsilon)$ such that $u + \varepsilon\xi \in int(P)$ for $\varepsilon \leq \varepsilon_0$ and

$$k(\varepsilon) := \inf_{x \in \overline{\Omega}} \frac{u(x) + \varepsilon\xi(x)}{d_\Omega(x)} > 0.$$

Clearly $k_0 := \min\{k(0), k(\varepsilon_0)\}$ (independent of ε) verifies

$$k_0 d_\Omega(x) \leq u + \varepsilon\xi. \tag{3.6}$$

On the other hand, since $\xi \in C_0^1(\overline{\Omega})$, it follows that from the Mean Value Theorem that

$$|\xi(x)| \leq d_\Omega(x)\|\xi\|_{C^1(\overline{\Omega})} \quad \text{for } x \in \Omega. \tag{3.7}$$

Then, using (3.6) and (3.7), we get

$$\frac{|\xi|}{u+\varepsilon\xi} \leq \frac{|\xi|}{k_0 d_\Omega(x)} \leq C\|\xi\|_{C^1(\overline{\Omega})}. \tag{3.8}$$

Moreover, by (3.3) and (3.7)

$$(u+\varepsilon\xi)u^{\alpha-1} \leq (u+\varepsilon\xi)k_1^{\alpha-1}d_\Omega(x)^{\alpha-1} \leq K d_\Omega(x)^\alpha \in L^\infty(\Omega),$$

and so,

$$(u+\varepsilon\xi)^\alpha - (u+\varepsilon\xi)u^{\alpha-1} \in L^\infty(\Omega).$$

Therefore, from (3.5) and (3.8), it follows that

$$\|[(u+\varepsilon\xi)^{\alpha-1} - u^{\alpha-1}]\xi\|_p \leq C\|(u+\varepsilon\xi)^\alpha - (u+\varepsilon\xi)u^{\alpha-1}\|_p\|\xi\|_{C^1(\overline{\Omega})} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

This proves (3.4), and so that the linear and continuous map

$$D_G\mathcal{F}(b,u)\xi = -\Delta\xi - \alpha b u^{\alpha-1}\xi + \beta e u^{\beta-1}\xi, \quad \forall \xi \in C_0^1(\overline{\Omega})$$

is the Gâteaux derivative.

For the continuity of this map, we have to prove that if $(b_n, u_n) \to (b, u)$, then

$$\|D_G\mathcal{F}(b_n,u_n) - D_G\mathcal{F}(b,u)\|_{\mathcal{L}(C_0^1(\overline{\Omega});L^p(\Omega))} \to 0,$$

for which, thanks to $\beta \geq 1$, it is sufficient to show that

$$\sup_{\|\xi\|_{C^1(\overline{\Omega})}=1} \{\|(b_n u_n^{\alpha-1} - b u^{\alpha-1})\xi\|_p\} \to 0.$$

Firstly, observe that

$$(b_n u_n^{\alpha-1} - b u^{\alpha-1})\xi = (b_n - b)u_n^{\alpha-1}\xi + b(u_n^{\alpha-1} - u^{\alpha-1})\xi. \tag{3.9}$$

Since $u_n \in int(P)$, it is well-defined

$$0 < k_n := \inf_{x\in\overline{\Omega}} \frac{u_n(x)}{d_\Omega(x)}.$$

By the continuty of the infimum, it follows that

$$k_n \to k_0 := \inf_{x\in\overline{\Omega}} \frac{u(x)}{d_\Omega(x)} > 0$$

since $u \in int(P)$.

Hence, using (3.9) and (3.7), as $n \to \infty$ we get

$$\begin{aligned}
\|(b_n u_n^{\alpha-1} - bu^{\alpha-1})\xi\|_p &\le \|b_n - b\|_\infty \|u_n^{\alpha-1}\xi\|_\infty + \|b\|_\infty \|(u_n^{\alpha-1} - u^{\alpha-1})\xi\|_\infty \\
&\le \|b_n - b\|_\infty k_n^{\alpha-1} d_\Omega^\alpha \|\xi\|_{C^1(\overline{\Omega})} + \|b\|_\infty \|(u_n^\alpha - u_n u^{\alpha-1})\frac{\xi}{u_n}\|_\infty \\
&\le \|\xi\|_{C^1(\overline{\Omega})}(\|b_n - b\|_\infty k_n^{\alpha-1} d_\Omega^\alpha + \|b\|_\infty \|u_n^\alpha - u_n u^{\alpha-1}\|_\infty k_n^{-1}) \to 0.
\end{aligned}$$

Therefore, $\mathcal{F}$ is C^1 respect to the second component and the Gâteaux derivative coincides with the Fréchet one. Denote it by $D_2\mathcal{F}$.
Finally, we will prove that $D_2\mathcal{F}(b_0, u_{b_0})$ is non singular showing that

$$\sigma_1(-\Delta - \alpha b_0 u_{b_0}^{\alpha-1} + \beta e u_{b_0}^{\beta-1}) > 0. \tag{3.10}$$

Indeed, define

$$M_b := -\alpha b u_b^{\alpha-1} + \beta e u_b^{\beta-1}. \tag{3.11}$$

We will prove that M_{b_0} satisfies (H1) and $\sigma_1(-\Delta + M_{b_0}) > 0$. Observe that $M_{b_0} \in L^\infty_{loc}(\Omega)$ and that by (3.3), there exists $k_1 > 0$ such that $k_1 d_\Omega(x) \le u_{b_0}$. Then,

$$|M_{b_0}(x)| d_\Omega(x) = |-\alpha b_0 + \beta e(x) u_{b_0}^{\beta-\alpha}| u_{b_0}^{\alpha-1} d_\Omega(x) \le C u_{b_0}^{\alpha-1} d_\Omega(x) \le C k_1^{\alpha-1} d_\Omega(x)^{\alpha-1} d_\Omega(x),$$

i.e., $M_{b_0}(x) d_\Omega(x)$ is bounded.

On the other hand, since u_{b_0} is solution of (3.1) we get that $\sigma_1(-\Delta - b_0 u_{b_0}^{\alpha-1} + e u_{b_0}^{\beta-1}) = 0$, and so by (H2) and Proposition 2.3 it follows that

$$\sigma_1(-\Delta - \alpha b_0 u_{b_0}^{\alpha-1} + \beta e u_{b_0}^{\beta-1}) > \sigma_1(-\Delta - b_0 u_{b_0}^{\alpha-1} + e u_{b_0}^{\beta-1}) = 0.$$

This proves (3.10). Now, the Implicit Function Theorem assures that there exist two open neighbourhoods $\mathcal{N}$, of b_0 in $L^\infty(\Omega)$ and $\mathcal{M}$, of u_{b_0} in $C_0^1(\overline{\Omega})$, and a C^1 map $\Phi : \mathcal{N} \mapsto \mathcal{M}$ such that

a) $\Phi(b_0) = u_{b_0}$,

b) $\mathcal{F}(s,\Phi(s))=0$ for any $s\in\mathcal{N}$,

c) $\mathcal{F}(s,y)=0$ with $s\in\mathcal{N}$, $y\in\mathcal{M}$, then $y=\Phi(s)$.

Since for s near b, the equation possesses a unique solution, then $\Phi(s)=u_s$. Therefore, $b\mapsto u_b$ is C^1 and the proof is complete. □

Along this work, we need the Gâteaux derivative of the map $b\in L^\infty_+(\Omega)\backslash\{0\}\mapsto u_b\in int(P)$.

Lemma 3.5 *Let $b\in L^\infty_+(\Omega)\backslash\{0\}$, $g\in L^\infty_+(\Omega)$ or $g\in L^\infty_-(\Omega)$, and $\varepsilon\simeq 0$ be such that $b+\varepsilon g\in L^\infty_+(\Omega)\backslash\{0\}$. Then,*

$$\frac{u_{b+\varepsilon g}-u_b}{\varepsilon}\rightharpoonup \xi_{b,g}\quad \text{in } H^1_0(\Omega) \text{ as } \varepsilon\to 0,$$

where $\xi_{b,g}$ is the unique solution of

$$\begin{cases} -\Delta\xi+M_b(x)\xi &= gu_b^\alpha \quad \text{in } \Omega,\\ \xi &= 0 \qquad \text{on } \partial\Omega,\end{cases}\tag{3.12}$$

and M_b is defined in (3.11).

Remark 3.6 *Since M_b satisfies $(H1)$ and by* (3.10), *it follows from Theorem* 2.5 *the existence and uniqueness of $\xi_{b,g}\in C^1_0(\overline{\Omega})$.*

Proof: Let $g\in L^\infty_+(\Omega)$, $\varepsilon>0$ be and define

$$\xi_\varepsilon := \frac{u_{b+\varepsilon g}-u_b}{\varepsilon}.$$

It is easy to show that ξ_ε satisfies

$$\begin{cases} -\Delta\xi_\varepsilon+(-bA_\varepsilon+eB_\varepsilon)\xi_\varepsilon &= gu^\alpha_{b+\varepsilon g} \quad \text{in } \Omega,\\ \xi_\varepsilon &= 0 \qquad \text{on } \partial\Omega,\end{cases}$$

where

$$A_\varepsilon(x):=\frac{u^\alpha_{b+\varepsilon g}(x)-u^\alpha_b(x)}{u_{b+\varepsilon g}(x)-u_b(x)}\qquad B_\varepsilon(x):=\frac{u^\beta_{b+\varepsilon g}(x)-u^\beta_b(x)}{u_{b+\varepsilon g}(x)-u_b(x)}$$

Since $b+\varepsilon g>b$ and by the monotony of the map $b\mapsto u_b$, it follows that $A_\varepsilon, B_\varepsilon\in C^1(\Omega)$. In fact, $\alpha u^{\alpha-1}_{b+\varepsilon g}\le A_\varepsilon\le\alpha u_b^{\alpha-1}$ and $\beta u_b^{\beta-1}\le B_\varepsilon\le\beta u^{\beta-1}_{b+\varepsilon g}\in L^\infty(\Omega)$ and so,

$$B_\varepsilon\to\beta u_b^{\beta-1}\quad \text{as } \varepsilon\downarrow 0,$$

and $|A_\varepsilon| d_\Omega \in L^\infty(\Omega)$. So, $\sigma_1(-\Delta - bA_\varepsilon + eB_\varepsilon)$ is well defined. Moreover,

$$-bA_\varepsilon + eB_\varepsilon \geq -\alpha b u_b^{\alpha-1} + \beta e u_b^{\beta-1}, \tag{3.13}$$

and by a similar reasoning to used in (3.4), we have that as $\varepsilon \downarrow 0$

$$\int_\Omega A_\varepsilon \varphi^2 \to \alpha \int_\Omega u_b^{\alpha-1}\varphi^2, \quad \int_\Omega B_\varepsilon \varphi^2 \to \beta \int_\Omega u_b^{\beta-1}\varphi^2, \quad \forall \varphi \in H_0^1(\Omega).$$

Hence, by Proposition 2.3, we get

$$\sigma_1(-\Delta - bA_\varepsilon + eB_\varepsilon) \to \sigma_1(-\Delta - \alpha b u_b^{\alpha-1} + \beta e u_b^{\beta-1}) \quad \text{as } \varepsilon \downarrow 0.$$

and by (3.10) and (3.13),

$$\sigma_1(-\Delta - bA_\varepsilon + eB_\varepsilon) \geq \sigma_1(-\Delta - \alpha b u_b^{\alpha-1} + \beta e u_b^{\beta-1}) > 0.$$

Then, applying Lemma 2.4, there exists a constant C (independient of ε) such that

$$C\int_\Omega |\nabla \xi_\varepsilon|^2 \leq \int_\Omega |\nabla \xi_\varepsilon|^2 + \int_\Omega (-\alpha b u_b^{\alpha-1} + \beta e u_b^{\beta-1})\xi_\varepsilon^2 \leq \int_\Omega |\nabla \xi_\varepsilon|^2 + \int_\Omega (-bA_\varepsilon + eB_\varepsilon)\xi_\varepsilon^2 = \int_\Omega g u_{b+\varepsilon g}^\alpha \xi_\varepsilon$$

and so, using (3.2), we obtain

$$\|\xi_\varepsilon\|_{H_0^1(\Omega)} \leq C \quad \text{(independient of } \varepsilon).$$

Then, of each bounded sequence considered, there exists a weakly convergent sub-sequence. It is not hard to prove that the limit verifies (3.12), and by the uniqueness of solution it follows that $\xi_\varepsilon \rightharpoonup \xi_{b,g}$ in $H_0^1(\Omega)$.

In the case $g \in L_-^\infty(\Omega), \varepsilon > 0$, it holds $\alpha u_b^{\alpha-1} \leq A_\varepsilon \leq \alpha u_{b+\varepsilon g}^{\alpha-1}$ and $\beta u_b^{\beta-1} \geq B_\varepsilon \geq \beta u_{b+\varepsilon g}^{\beta-1}$, and and so, instead of (3.13), we have

$$-bA_\varepsilon + eB_\varepsilon \geq -\alpha b u_{b+\varepsilon g}^{\alpha-1} + \beta e u_{b+\varepsilon g}^{\beta-1}.$$

As $\varepsilon \downarrow 0$, we have

$$\int_\Omega (-\alpha b u_{b+\varepsilon g}^{\alpha-1} + \beta e u_{b+\varepsilon g}^{\beta-1})\varphi^2 \to \int_\Omega (-\alpha b u_b^{\alpha-1} + \beta e u_b^{\beta-1})\varphi^2 \quad \forall \varphi \in H_0^1(\Omega).$$

By Proposition 2.3, we get

$$\sigma_1(-\Delta - bA_\varepsilon + eB_\varepsilon) \geq \sigma_1(-\Delta - \alpha b u_{b+\varepsilon g}^{\alpha-1} + \beta e u_{b+\varepsilon g}^{\beta-1}) \to \sigma_1(-\Delta - \alpha b u_b^{\alpha-1} + \beta e u_b^{\beta-1}) > 0.$$

Again, applying Lemma 2.4 we obtain the result. □

4 Existence and uniqueness of optimal control

Consider a such that

$(H3)\qquad a \in \mathcal{A}.$

We define the set

$$\mathcal{C} := \{f \in L^\infty_+(\Omega) : \ f \le a\}.$$

When $f \in \mathcal{C}$, we have proved in the previous Section that there exists a unique positive solution of (3.1) with $b = a - f$, and it will denote by u_f (if $f = a$, then $u_f := 0$.)

For $\lambda > 0$ we consider the functional $J : \mathcal{C} \mapsto \mathbb{R}$

$$J(g) := \int_\Omega (\lambda h(g) u_g - k(g)),$$

where $h \in C^1(\mathbb{R}^+; \mathbb{R}^+)$, h' is Lipschitz continuous function and $h(s) = 0$ if and only if $s = 0$; $k \in C^2(\mathbb{R}^+; \mathbb{R}^+)$ is a convex function and there exists $C > 0$ such that $|k(s)| \le Cs^2$ and $k''(s) \ge k_0 > 0$. We assume:

$(H4)\qquad \lim_{t\to 0} \frac{k(t)}{h(t)} = 0,$

$(H5)\qquad \lim_{t\to +\infty} \frac{k(t)}{h(t)} = +\infty, \quad t \mapsto \frac{h(t)}{t}$ is non-increasing, $\quad t \mapsto \frac{k(t)}{t}$ is increasing.

In this Section we want to prove the existence and uniqueness of the optimal control under suitable assumptions. The following result gives us the existence of optimal control.

Theorem 4.1 *Assume $(H3) - (H4)$. There exists an optimal control, i.e., $f \in \mathcal{C}$ such that*

$$J(f) = \sup_{g\in\mathcal{C}} J(g).$$

Moreover, the benefit is positive, i.e., $\sup_{g\in\mathcal{C}} J(g) > 0$.

Proof: By (3.2), it follows that

$$\sup_{g\in\mathcal{C}} J(g) < +\infty,$$

and so, there exists a maximizing sequence $f_n \in \mathcal{C}$. Then, there exists a subsequence, relabeled by f_n, such that

$$f_n \rightharpoonup f \in \mathcal{C} \quad \text{in } L^2(\Omega) \text{ and } \quad u_{f_n} \to u_f \quad \text{in } H^1_0(\Omega),$$

and by the regularity of h,

$$h(f_n) \rightharpoonup h(f) \in \mathcal{C} \quad \text{in } L^2(\Omega),$$

and then,

$$\int_\Omega h(f_n)u_{f_n} \to \int_\Omega h(f)u_f.$$

By the hypothesis on k, the map $\Phi : L^2(\Omega) \mapsto \mathbb{R}$ defined by

$$\Phi(g) := \int_\Omega k(g)$$

is continuous (see Lemma 17.1 in [12], for instance) and convex, and so w.l.s.c. Then,

$$\int_\Omega k(f) \leq \underline{\lim}_{n\to\infty} \int_\Omega k(f_n).$$

Hence,

$$J(f) = \int_\Omega \lambda h(f)u_f - k(f) \geq \overline{\lim}_{n\to\infty} \int_\Omega \lambda h(f_n)u_{f_n} - k(f_n) = \sup_{g\in\mathcal{C}} \int_\Omega \lambda h(g)u_g - k(g).$$

Finally, we take $f = \varepsilon > 0$, then

$$J(\varepsilon) = h(\varepsilon) \int_\Omega \left(\lambda u_\varepsilon - \frac{k(\varepsilon)}{h(\varepsilon)}\right),$$

and so, since $u_\varepsilon \to u_0 > 0$ and by $(H4)$, it follows that $J(\varepsilon) > 0$ for ε sufficiently small. This completes the proof. □

The following result gives us a bound of the optimal control, and it will be used to prove its uniqueness.

Lemma 4.2 *Assume $(H3) - (H5)$. If $f \in \mathcal{C}$ is an optimal control, then*

$$f \leq T_\lambda$$

where

$$T_\lambda := \inf\{t \in \mathbb{R}^+ : \frac{k(t)}{h(t)} = \lambda\mathcal{K}\}, \quad \text{and} \quad \mathcal{K} := \left(\frac{a_M}{e_L}\right)^{1/(\beta-\alpha)}.$$

Remark 4.3 *a) By $(H4)$ and $(H5)$, it follows that $T_\lambda > 0$ and that $T_\lambda \to 0$ as $\lambda \downarrow 0$.*

b) Theorem 4.1 and Lemma 4.2 are generalizations of Theorem 2.1 in [3], which has been proved in the case $m = 1$, $h(t) = t$ and $k(t) = t^2$.

Proof: Let $f \in L^\infty_+(\Omega)$ be. By $(H5)$, there exists $t_0 > 0$ such that $k(t_0)/h(t_0) = \lambda\mathcal{K}$. We consider

$$g := \min\{f, t_0\},$$

and we will prove that $J(g) > J(f)$, whence the result follows.

By definition, $g \leq f$ and then $u_g \geq u_f$. If $x_0 \in \Omega$ is such that $f(x_0) = g(x_0)$ then

$$\lambda u_g(x_0)h(g(x_0)) - k(g(x_0)) \geq \lambda u_f(x_0)h(f(x_0)) - k(f(x_0)).$$

On the other hand, if $f(x_0) > g(x_0) = t_0 > 0$, then by (3.2)

$$\lambda u_g(x_0)h(g(x_0)) - k(g(x_0)) \leq \lambda\mathcal{K}h(t_0) - k(t_0) = 0,$$

and so by $(H5)$, we get

$$0 \geq \lambda u_g(x_0)\frac{h(g(x_0))}{g(x_0)} - \frac{k(g(x_0))}{g(x_0)} > \lambda u_f(x_0)\frac{h(f(x_0))}{f(x_0)} - \frac{k(f(x_0))}{f(x_0)}.$$

Then,

$$J(g) = \int_{\{f=g\}} \lambda h(g)u_g - k(g) + \int_{\{f>g\}} \lambda h(g)u_g - k(g) \geq \int_{\{f=g\}} \lambda h(f)u_f - k(f) +$$

$$+ \int_{\{f>g\}} (\lambda\frac{h(g)}{g}u_g - \frac{k(g)}{g})g > \int_{\{f=g\}} \lambda h(f)u_f - k(f) + \int_{\{f>g\}} \lambda h(f)u_f - k(f) = J(f).$$

□

For the uniqueness, we use the argument described in Section 6 in [3]. Firstly, we prove the next result.

Proposition 4.4 *Let* $J : \mathcal{D} := \{f \in L^\infty(\Omega) : (a - f) \in \mathcal{A}\} \subset L^\infty(\Omega) \mapsto \mathbb{R}$ *be. Then* J *is Fréchet continuously differentiable and*

$$J'(f)(g) = \int_\Omega (\lambda h'(f)u_f - \lambda u_f^\alpha P_f - k'(f))g, \quad \forall f \in \mathcal{D},\ \forall g \in L^\infty(\Omega), \tag{4.1}$$

where for any $f \in \mathcal{D}$, $P_f \in C^1_0(\overline{\Omega})$ *is the unique solution of*

$$\begin{cases} -\Delta P_f + M_f(x)P_f &= h(f) \quad \text{in } \Omega, \\ P_f &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

being $M_f := -\alpha(a - f)u_f^{\alpha-1} + \beta e u_f^{\beta-1}$.

To prove this result, we need some previous ones. For $f \in \mathcal{D}$ and $g \in L^\infty(\Omega)$, let $\xi_{f,g}$ be the unique solution of

$$\begin{cases} -\Delta\xi + M_f(x)\xi &= -g u_f^\alpha \quad \text{in } \Omega, \\ \xi &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Observe that (4.2) and (4.3) have a unique solution because $\sigma_1(-\Delta + M_f) > 0$ (see (3.10) and (3.11)) and Theorem 2.5.

Lemma 4.5 *The map $f \in \mathcal{D} \mapsto P_f \in C_0^1(\overline{\Omega})$ is continuous.*

Proof: Fixed $p > N$, we consider the map $\mathcal{G} : \mathcal{D} \times (C_0^1(\overline{\Omega}) \cap W^{2,p}(\Omega)) \mapsto L^p(\Omega)$ defined by

$$\mathcal{G}(f, P) = -\Delta P + M_f P - h(f).$$

Observe that $\mathcal{G}$ is continuous. Indeed, the continuity of the map $f \mapsto M_f P$ follows with a similar argument to the one used in the proof of Theorem 3.3 to show that the map $D_G\mathcal{F}$ is continuous. On the other hand, it is clear that $\mathcal{G}(f, P_f) = 0$. Given $\xi \in C_0^1(\overline{\Omega}) \cap W^{2,p}(\Omega)$ is easy to prove that $D_2\mathcal{G}(f, P_f)\xi = -\Delta\xi + M_f\xi$. Moreover, as in (3.10), $\sigma_1(-\Delta + M_f) > 0$ and so $D_2\mathcal{G}(f, P_f)$ is non singular. The Implicit Function Theorem completes the proof. □

The next result is due by Krasnoselskii, see [7], where we send for the definitions of the following concepts.

Lemma 4.6 *Let E be a Banach space ordered by a generating positive cone P, F a Banach space and $T : E \mapsto F$. Assume that the Gâteaux derivative of T with respect to P, denoted by $D_{G,P}T$, exists and it is continuous in a neighbourhood of $x_0 \in E$. Then, the Fréchet derivative coincides with the Gâteaux derivative and T is C^1 near x_0.*

Recall that P is generating if $E = P - P$. It is well known, see Proposition 1.7 in [1], that if $int(P) \neq \emptyset$, then P is generating.

Proof of Proposition 4.4: Firstly, we compute the Gâteaux derivative respect to the cone, denoted by $D_{G,P}J$. Let $g \in L_+^\infty(\Omega)$, $f \in \mathcal{D}$ and $\varepsilon > 0$ be such that $f + \varepsilon g \in \mathcal{D}$. Using Lemma 3.5 and (4.3)

$$D_{G,P}J(f)g := \lim_{\varepsilon \downarrow 0} \frac{J(f + \varepsilon g) - J(f)}{\varepsilon} = \int_\Omega \lambda\xi_{f,g} h(f) + \lambda h'(f) u_f g - k'(f) g.$$

By the equation that satisfy $\xi_{f,g}$ y P_f (see (4.3) and (4.2)), it follows that

$$\int_\Omega h(f)\xi_{f,g} + \int_\Omega g u_f^\alpha P_f = 0,$$

and so,

$$D_{G,P}J(f)(g) = \int_\Omega (\lambda h'(f)u_f - \lambda u_f^\alpha P_f - k'(f))g, \qquad \forall g \in L^\infty_+(\Omega).$$

Let $f_n \to f \in \mathcal{D}$ be in L^∞ and $g \in L^\infty(\Omega)$. Then, by Theorem 3.3 and Lemma 4.5 it follows

$$\sup_{\|g\|_\infty \le 1} |D_{G,P}J(f_n)(g) - D_{G,P}J(f)(g)| \le$$

$$\le \sup_{\|g\|_\infty \le 1} \int_\Omega |\lambda(h'(f_n)u_{f_n} - h'(f)u_f) - \lambda(u_{f_n}^\alpha P_{f_n} - u_f^\alpha P_f) - (k'(f_n) - k'(f))g| \to 0.$$

and so, $D_{G,P}J$ is continuous. Applying Lemma 4.6, the Gâteaux derivative coincides with the Fréchet derivative and that the map is C^1. □

The next result shows that some maps involved in (4.1) are Lipschitz continuous.

Lemma 4.7 *Assume* $(H3) - (H5)$. *There exists* $\Lambda > 0$ *such that for* $0 < \lambda < \Lambda$ *the maps* $f \in [0, T_\lambda] \mapsto u_f, P_f, u_f^\alpha P_f \in L^\infty(\Omega)$ *are Lipschitz continuous.*

Proof: Let $f, g \in [0, T_\lambda]$ be, by the monotony of the map $f \mapsto u_f$, it follows that

$$0 < u_{T_\lambda} \le u_f, u_g \le u_0$$

for λ such that $a - T_\lambda > 0$, that is $\lambda < \lambda_0$ for some λ_0 (see Remark 4.3 a)). To the end of the proof we take $\lambda < \lambda_0$. By the Mean Value Theorem,

$$\begin{gathered} u_f^\alpha - u_g^\alpha = \alpha\xi^{\alpha-1}(f,g)(u_f - u_g), \quad u_f^\beta - u_g^\beta = \beta\eta^{\beta-1}(f,g)(u_f - u_g) \quad \text{with} \\ 0 < u_{T_\lambda} \le \min\{u_f, u_g\} \le \xi(f,g), \eta(f,g) \le \max\{u_f, u_g\} \le u_0. \end{gathered} \tag{4.4}$$

Let $w := u_f - u_g$ be. Then, w satisfies

$$(-\Delta + N(f,g))w = (g - f)u_g^\alpha, \quad \text{in } \Omega, \quad w = 0 \qquad \text{on } \partial\Omega,$$

where $N(f,g) := -\alpha(a - f)\xi^{\alpha-1}(f,g) + \beta e\eta^{\beta-1}(f,g)$. Using $f \ge 0$ and (4.4), it follows that

$$N(f,g) \ge -\alpha a\xi^{\alpha-1}(f,g) + \beta e\eta^{\beta-1}(f,g) \ge -\alpha a u_{T_\lambda}^{\alpha-1} + e\beta u_{T_\lambda}^{\beta-1}.$$

It is not hard to show that as $\lambda \downarrow 0$

$$\int_\Omega (-\alpha a u_{T_\lambda}^{\alpha-1} + e\beta u_{T_\lambda}^{\beta-1})\varphi^2 \to \int_\Omega (-\alpha a u_0^{\alpha-1} + e\beta u_0^{\beta-1})\varphi^2 \quad \forall \varphi \in H_0^1(\Omega),$$

and so, by Proposition 2.3 we obtain that

$$\sigma_1(-\Delta + N(f,g)) \geq \sigma_1(-\Delta - \alpha a u_{T_\lambda}^{\alpha-1} + e\beta u_{T_\lambda}^{\beta-1}) \to \sigma_1(-\Delta - \alpha a u_0^{\alpha-1} + e\beta u_0^{\beta-1}) > 0$$

as $\lambda \downarrow 0$. Hence, there exists $\lambda_1 > 0$ such that

$$N(f,g) \geq -\alpha a u_{T_{\lambda_1}}^{\alpha-1} + e\beta u_{T_{\lambda_1}}^{\beta-1} \tag{4.5}$$

and

$$\sigma_1(-\Delta + N(f,g)) \geq \sigma_1(-\Delta - \alpha a u_{T_{\lambda_1}}^{\alpha-1} + e\beta u_{T_{\lambda_1}}^{\beta-1}) > 0. \tag{4.6}$$

Then, by (4.5), (4.6) and Lemma 2.6, we have that $w \leq \psi_1$ where ψ_1 is the unique solution of

$$\left\{ \begin{array}{rcll} -\Delta u + (-\alpha a u_{T_{\lambda_1}}^{\alpha-1} + e\beta u_{T_{\lambda_1}}^{\beta-1})u & = & (g-f)u_g^{\alpha} & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{4.7}$$

Interchanging f and g, we get that $-w \leq \psi_2$ where ψ_2 is the unique solution of (4.7) with second member $(f-g)u_f^{\alpha}$. Then, taking into account that u_f possesses a priori bound independient of f (see (3.2)) and Theorem 2.5, it follows that

$$\|u_f - u_g\|_\infty = \|w\|_\infty \leq \max\{\|\psi_1\|_\infty, \|\psi_2\|_\infty\} \leq \max\{\|\psi_1\|_{C^1(\overline{\Omega})}, \|\psi_2\|_{C^1(\overline{\Omega})}\} \leq C\|f-g\|_\infty. \tag{4.8}$$

This shows that the map $f \mapsto u_f$ is Lipschitz.

Before proving the Lipschitz character of the map $f \in [0, T_\lambda] \mapsto P_f$, we see that

$$P_f \leq \mathcal{P} \quad \text{in } \Omega, \tag{4.9}$$

where $\mathcal{P} \in C_0^1(\overline{\Omega})$, independient of f. Indeed, let $f \in [0, T_\lambda]$ be, then $M_f \geq -\alpha a u_{T_\lambda}^{\alpha-1} + \beta e u_{T_\lambda}^{\beta-1}$, and so, using again Lemma 2.6 b), $P_f \leq \mathcal{P}$ where $\mathcal{P}$ is the unique solution of

$$\left\{ \begin{array}{rcll} -\Delta u + (-\alpha a u_{T_{\lambda_1}}^{\alpha-1} + e\beta u_{T_{\lambda_1}}^{\beta-1})u & = & T & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

where $T := \max_{f \in [0,T_\lambda]} \max_{x \in \overline{\Omega}} h(f(x))$. This implies (4.9).

We will prove now that the map is Lipschitz. Let $f, g \in [0, T_\lambda]$ and $z := P_f - P_g$ be. Then z satisfies

$$-\Delta z + M_f z = T(f,g), \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega,$$

where

$$T(f,g) = h(f) - h(g) + P_g[\alpha(a-f)(u_f^{\alpha-1} - u_g^{\alpha-1}) - \beta e(u_f^{\beta-1} - u_g^{\beta-1})] + \alpha(g-f)P_g u_g^{\alpha-1}.$$

Applying again the Mean Value Theorem, we get

$$u_f^{\alpha-1} - u_g^{\alpha-1} = (\alpha-1)\xi^{\alpha-2}(f,g)(u_f - u_g), \quad u_f^{\beta-1} - u_g^{\beta-1} = (\beta-1)\eta^{\beta-2}(f,g)(u_f - u_g)$$

$$0 < u_{T_\lambda} \leq \min\{u_f, u_g\} \leq \xi(f,g), \eta(f,g) \leq \max\{u_f, u_g\} \leq u_0. \tag{4.10}$$

Hence,

$$T(f,g) = h(f) - h(g) + P_g[\alpha(\alpha-1)(a-f)\xi^{\alpha-2} - \beta(\beta-1)e\eta^{\beta-2}](u_f - u_g) + \alpha(g-f)P_g u_g^{\alpha-1}.$$

By a similar argument to the used in the proof of (4.8), we obtain

$$\|P_f - P_g\|_\infty = \|z\|_\infty \leq C\|T(f,g)\|_\infty. \tag{4.11}$$

Since $\mathcal{P} \in C_0^1(\overline{\Omega})$, and using (3.3), (3.7), (4.9) and (4.10), we obtain

$$\begin{aligned}
\|\alpha(f-g)P_g u_g^{\alpha-1}\|_\infty &\leq C\|f-g\|_\infty \|P_g u_{T_{\lambda_1}}^{\alpha-1}\|_\infty \\
&\leq C\|f-g\|_\infty k_1^{\alpha-1}\|\mathcal{P} d_\Omega^{\alpha-1}\|_\infty \\
&\leq C\|f-g\|_\infty \|d_\Omega^\alpha\|_\infty \|\mathcal{P}\|_{C^1(\overline{\Omega})} \\
&\leq C\|f-g\|_\infty \quad \text{with } C \text{ independient of } f \text{ and } g.
\end{aligned}$$

On the other hand, using (4.8), (4.9) and (4.10)

$$\begin{aligned}
\|\alpha(\alpha-1)(a-f)P_g\xi^{\alpha-2}(u_f - u_g)\|_\infty &\leq C\|\mathcal{P}\xi^{\alpha-2}(u_f - u_g)\|_\infty \\
&\leq C\|\mathcal{P}\xi^{\alpha-2}\max\{|\psi_1|, |\psi_2|\}\|_\infty \\
&\leq C\|\mathcal{P} d_\Omega^{\alpha-2}\max\{|\psi_1|, |\psi_2|\}\|_\infty \\
&\leq C\|\mathcal{P}\|_{C^1(\overline{\Omega})}\|d_\Omega^\alpha\|_\infty \max\{\|\psi_1\|_{C^1(\overline{\Omega})}, \|\psi_2\|_{C^1(\overline{\Omega})}\} \\
&\leq C\|f-g\|_\infty
\end{aligned}$$

with C independient of f and g. Analogously it can be treated the term $-e\beta(\beta-1)P_g\eta^{\beta-2}(u_f - u_g)$. Then, since h is Lipschitz in $[0, T_\lambda]$ and by (4.11), it follows that the map $f \mapsto P_f$ is Lipschitz.

Let $f, g \in [0, T_\lambda]$ be, we have

$$\|u_f^\alpha P_f - u_g^\alpha P_g\|_\infty \le \|(u_f^\alpha - u_g^\alpha)P_f\|_\infty + \|u_g^\alpha(P_f - P_g)\|_\infty.$$

By the Mean Value Theorem,

$$\|(u_f^\alpha - u_g^\alpha)P_f\|_\infty = \|\alpha\xi^{\alpha-1}P_f(u_f - u_g)\|_\infty \le C\|\varphi\|_{C^1(\overline{\Omega})}\|f-g\|_\infty \le C\|f-g\|_\infty.$$

It is sufficient to take $\Lambda := \min\{\lambda_0, \lambda_1\}$. This completes the proof. □

Theorem 4.8 *Assume* $(H3)-(H5)$*. Then, there exists* $\Lambda_0 > 0$ *such that if* $\lambda < \Lambda_0$ *there exists a unique optimal control.*

Proof: Let $f \in \mathcal{C}$ be an optimal control, then by Lemma 4.2

$$f \in I := [0, T_\lambda]_\infty.$$

We take $\lambda < \Lambda$ (the constant obtained in Lemma 4.7) and sufficiently small λ such that $I \subset \mathcal{C}$. In I, convex, the strictly concave character of J is equivalent to the monotony of J'. Hence, by (4.1), for $f, g \in I$, we have that

$$(J'(f) - J'(g))(f-g) = \int_\Omega [\lambda(h'(f)u_f - h'(g)u_g) + \lambda(u_g^\alpha P_g - u_f^\alpha P_f) - (k'(f) - k'(g))](f-g) \le$$
$$\le \int_\Omega (\lambda L - k_0)(f-g)^2 < 0,$$

taking $\lambda < k_0/L := \Lambda_1$, where L the Lipschitz constant of the maps h', $f \mapsto u_f$, $f \mapsto P_f$ and $f \mapsto u_f^\alpha P_f$ (see Lemma 4.7). □

5 Regularity of the optimal control and optimality system

In this section we consider the special case $h(t) = t$ and $k(t) = t^2$, which satisfy clearly $(H4)$ and $(H5)$. Moreover, in this case

$$T_\lambda = \lambda\mathcal{K}.$$

The following result provides us of a caracterization of an optimal control. It follows as Theorem 3.1 in [10], using now our Lemma 3.5.

Lemma 5.1 *Assume* $f \in \mathcal{C}$ *and* $(H3)$*. If* f *is an optimal control, then*

$$f = \frac{\lambda}{2}u_f(1 - u_f^{\alpha-1}P_f)^+.$$

The next result says us that the optimal control is a Hölder continuous function when λ is small and it lets us write the optimality system.

Proposition 5.2 *Assume $(H3)$. There exists Λ_1 such that if $\lambda \le \Lambda_1$, then $P_f \le u_f^{1-\alpha}$. So, if f is an optimal control, we have that*

$$f = \frac{\lambda}{2} u_f (1 - u_f^{\alpha-1} P_f). \tag{5.1}$$

Proof: Let f be an optimal control. For $\lambda < \lambda_0 := a_L/\mathcal{K}$, we have that $u_f \ge u_{\lambda\mathcal{K}} > 0$. As in Lemma 4.7, it follows the existence of λ_1 such that there exists a unique positive solution ψ of

$$\begin{cases} -\Delta\psi + (-a\alpha u_{\lambda_1\mathcal{K}}^{\alpha-1} + \beta e u_{\lambda_1\mathcal{K}}^{\beta-1})\psi = \mathcal{K} & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.6 and (3.2), it follows that

$$P_f \le \lambda\psi \quad \text{for } \lambda \le \lambda_1. \tag{5.2}$$

We define now

$$\lambda_2 := \inf_{x\in\overline{\Omega}} \frac{u_{\lambda_1\mathcal{K}}^{1-\alpha}}{\psi} \le \inf_{x\in\overline{\Omega}} \frac{u_f^{1-\alpha}}{\psi}.$$

Observe that $\lambda_2 > 0$. Indeed, since ψ and $u_{\lambda_1\mathcal{K}}$ are positive functions, it follows the existence of a constant $k > 0$ such that

$$\frac{u_{\lambda_1\mathcal{K}}^{1-\alpha}}{\psi} > k d_\Omega^{-\alpha} > 0.$$

Taking $\Lambda_1 := \min\{\lambda_0, \lambda_1, \lambda_2\}$ and taking into account (5.2) and the definition of λ_2, it follows $P_f \le u_f^{1-\alpha}$, and as a consequence of Lemma 5.1, we obtain (5.1). □

The following result is an easy consequence of the previous result and it provides us with the optimality system.

Corollary 5.3 *Assume $(H3)$ and $\lambda \le \Lambda_1$. Then any optimal control f may be expressed as in (5.1), where the pair $(u_f, P_f) := (u, P)$ satisfies*

$$\begin{cases} -\Delta u = u^\alpha(a - \frac{\lambda}{2}u + \frac{\lambda}{2}u^\alpha P - e u^{\beta-\alpha}) & \text{in } \Omega, \\ -\Delta P + (-\alpha a u^{\alpha-1} + \beta e u^{\beta-1})P = \frac{\lambda}{2}(u - u^\alpha P(1+\alpha) + \alpha u^{2\alpha-1}P^2) & \text{in } \Omega, \\ u = P = 0 & \text{on } \partial\Omega, \end{cases}$$

and $u > 0$.

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References

[1] Amann H (1976) Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Review 18:620-709.

[2] Bertsch M, Rostamian R (1985) The principle of linearized stability for a class of degenerate diffusion equations. J. Diff. Eqns. 57:373-405.

[3] Cañada A, Gámez JL, Montero JA (1998) Study of an optimal control problem for diffusive nonlinear elliptic equations of logistic type. SIAM J. Control Optim. 36:1171-1189.

[4] Delgado M, Suárez A (2000) On the existence of dead cores for degenerate Lotka-Volterra models. Proc. Royal Society of Edin. 130 A:743-766.

[5] Gurtin ME, MacCamy RC (1977) On the diffusion of biological populations. Math. Biosci. 33:35-49.

[6] Hernández J, Mancebo F, Vega de Prada JM On the linearization of some singular nonlinear elliptic problem and applications, to appear in Ann. Inst. H. Poincare Anal. Non-Linearie.

[7] Krasnoselskii MA (1964) Positive solutions of operator equations. Noordhoff, Groningen.

[8] Kufner A (1980) Weighted Sobolev Spaces. Text zur Mathematik, 31, Teubner, Leipzig.

[9] Leung AW (1995) Optimal harvesting-coefficient control of steady-state prey-predator diffusive Volterra-Lotka systems. Appl. Math. Optim. 31:219-241.

[10] Leung AW, Stojanovic S (1993) Optimal control for elliptic Volterra-Lotka type equations. J. Math. Anal. Appl. 173:603-619.

[11] Montero JA (2000) A uniqueness result for an optimal control problem on a diffusive elliptic Volterra-Lotka type Equation. J. Math. Anal. Appl. 243:13-31.

[12] Kavian O (1993) Introduction à la théorie des points critiques et applications aux problèmes elliptiques. Springer-Verlag, Paris.