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ON ARBITRARY SETS AND *ZFC*

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Abstract. Set theory deals with the most fundamental existence questions in mathematics–questions which affect other areas of mathematics, from the real numbers to structures of all kinds, but which are posed as dealing with the existence of sets. Especially noteworthy are principles establishing the existence of some infinite sets, the so-called "arbitrary sets." This paper is devoted to an analysis of the motivating goal of studying arbitrary sets, usually referred to under the labels of *quasi-combinatorialism* or *combinatorial maximality*. After explaining what is meant by definability and by "arbitrariness," a first historical part discusses the strong motives why set theory was conceived as a theory of arbitrary sets, emphasizing connections with analysis and particularly with the continuum of real numbers. Judged from this perspective, the axiom of choice stands out as a most central and natural set-theoretic principle (in the sense of quasi-combinatorialism). A second part starts by considering the potential mismatch between the formal systems of mathematics and their motivating conceptions, and proceeds to offer an elementary discussion of how far the Zermelo–Fraenkel system goes in laying out principles that capture the idea of "arbitrary sets". We argue that the theory is rather poor in this respect.

The concept of set, the axiom system *ZFC* and its alleged intuitive underpinnings, the universe V of all sets—all are topics on which mathematicians and philosophers offer many, contrasting opinions. Talk of the concept of set as intuitive or commonsense has been widespread since the emergence of set theory around 1870. However, the view that the concept of set builds upon everyday notions is quite problematic: if we understand 'collection' in anything close to its everyday sense, it is impossible to iterate the process of formation of collections in the way needed for set theory, so as to build 'collections' like $\{\{a\}, \{a, b\}\}$. The intuitive image of a collection, whether isolated or in combination with other intuitive notions, does not offer a

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solid basis for building the concept of a set.[1] This task requires crucial assumptions—beginning with the objecthood or elementhood of sets—that cannot be justified by starting from everyday notions. Only through such assumptions can one elaborate a concept of set worthy of that name.

Yet this is not to say that set theory was allowed many degrees of freedom in its formative period. In this respect I disagree with the following sentences of Kanamori:

> unlike the emergence of mathematics from market-place arithmetic and Greek geometry, sets and transfinite numbers were neither laden nor buttressed with substantial antecedence. Like strangers in a strange land stalwarts developed a familiarity with them guided hand in hand by their axiomatic scaffolding. (Kanamori [1996, p. 12].)

Such a depiction seems suitable for the metatheoretic period that set theory as a field lived from about 1950, an era of metamathematical results that explored the landscape of large cardinals, the technique of forcing, and the world of models of *ZFC*, but *not* for the more properly theoretical and axiomatic period that antedated 1940. It was in this period, 1904 to 1940, that the core of an understanding was gained of set theory, its axiomatic underpinnings, the universe V, and even 'small' large cardinals[2]—deficient understanding, maybe, but not so much definitive progress has been made since.[3] And I would like to emphasize that such understanding indeed had strong roots and substantial antecedence: basically it was a matter of *understanding and clarifying the concepts of number and function*.

Zermelo himself reflected the situation clearly when he stated that set theory is "the branch of mathematics whose task is to investigate mathematically the fundamental notions 'number', 'order', and 'function', taking them in their pristine, simple form" (Zermelo [1908, p. 200]; quite obviously, he wrote those words with the problem of well-ordering in mind). But as soon as we mention number and function as understood by 1900, ideas of order, topology, and structure are by necessity implicated: total and partial orders, rings and fields, topological completion, etc. As I emphasized in

[1] The concept is highly theoretical in nature, as I have argued elsewhere, e.g., in the Epilogue to Ferreirós [2007].

[2] Interestingly, these elements basically agree with the limited amount of set theory that Tait (2005, chap. 6) is able to develop "from below" in a way that largely follows Cantor and Zermelo. As Martin (1998, p. 229) notes, this does not mean the evidence for *ZFC* and weak large cardinals is purely "intrinsic" and of a different kind from the "extrinsic" evidence for other assumptions. Already the evidence for an axiom like Infinity is largely extrinsic (Maddy 1988 and 1997).

[3] By saying this I do not mean to deny, in any way, the enormous development of several different research programs in set theory in recent decades; see above all Kanamori & Foreman [2010].

Ferreirós [2007], the origins of set theory were linked with a vast territory of the mathematics of 1900.

The main purpose of this paper will be to offer some reflections on (i) the origins of the guiding principle of admitting arbitrary subsets, and (ii) the way in which it is captured inside the formal axiomatic system *ZF* with Choice, *ZFC*. In the first part I shall argue that the Axiom of Choice (AC) is an absolutely natural assumption, lest set theory deviate strongly from the work of Dedekind, Cantor, Zermelo, Hausdorff, *and* from the classical understanding of the real numbers. Moreover, I shall argue that AC comes closest to capturing (a bit of) the idea of arbitrary, non-definable sets—also called "quasi-combinatorialism" or "combinatorial maximality." The second part is devoted to arguing that, all things considered, *ZFC* is a poor system compared with its motivating goal of studying "all possible" arbitrary subsets (which suggests deep reasons for the failure of this system and its extensions to settle the truth or falsity of CH).

§**1. Quasi-combinatorialism.** From the beginning, the idea of admitting *arbitrary subsets* was crucial motivation for the founding fathers—and here I refer particularly to Cantor, Dedekind, Hilbert and Zermelo, as the situation with Frege and Russell may have been different. When we say that there exists, in set theory, the set of all subsets of N, this set $\wp(N)$ is assumed to have as elements *all possible subsets of N, whether definable or not*, be they finite or infinite. Surprisingly, Cantor and Dedekind never made that idea sufficiently explicit in their published attempts to formulate systematic treatments of set theory—Dedekind [1888] dealing with finite and denumerable sets, and Cantor's *Beiträge* [1895 & 1897] dealing with sets of the first two transfinite cardinalities.[4] However, the principle is clearly required by their contributions to set theory and the foundations of mathematics, and I believe it fair to say that it is distinctly characteristic of their approaches. Until 1900 it remained something like a crucial but *implicit* guiding principle, and it also deviated from the views of many of their contemporaries, including very important contributors to set theory such as Borel, Baire, and Lebesgue.

1.1. The Dedekind–Cantor approach has been described as a *quasi-combinatorial* conception, for reasons that we now review.[5] A contrast has to be made with constructivistic conceptions, but also with definabilist views.

[4]Dedekind was careful to indicate that the restriction to countable sets was done for practical purposes, "der Deutlichkeit und Einfachheit halber" (see 1888, p. 387, footnote). Analogously, Cantor's plan was to deal with arbitrary cardinals in a third part of his *Beiträge* (see Ferreirós [2007], 291*ff* and Epilogue).

[5]See Maddy [1997] for a good philosophical analysis of the crucial role of quasi-combinatorialism (or "combinatorial maximality," as she tends to say) in set theory.

The basic distinction (which is less clear than desirable) is between *definable* sets and *arbitrary* sets.[6] The concept of definable set may seem clear enough: consider the set of all even natural numbers $\{2, 4, 6, 8, \dots\}$, defined in the obvious way, or the set $\{2, 3, 5, 7, 11, \dots\}$ of prime numbers, definable as the collection of all numbers n such that m/n implies $m = n$ or $m = 1$. Even the set $\{3, 14, 159, 2653, \dots\}$, formed by grouping ciphers in the decimal expansion of π, can be defined in terms of properties of the natural numbers, finitarily in a language for the structure $\langle N, 0, +, \cdot\rangle$—despite the fact that π is a transcendental number.[7]

The matter, of course, is entirely different with *arbitrary* (non-definable) subsets of N. Our previous example using the decimal expansion of π may suffice to convince the reader that it is impossible to offer a single specific example of an "arbitrary set" of natural numbers. Concrete examples of infinite sets of naturals that we can offer are *ipso facto* definable sets of naturals (and this includes not only the sets studied in descriptive set theory, but also, e.g., the case of $0^{\#}$ and other sharps, see section 6 below). The more one reflects on this matter, the more obvious it becomes; eventually one may come to think that the idea of a "concrete" example of an "arbitrary" anything is an oxymoron. However, the point is not emphasized often, but rather is commonly obscured, and so it seems important both to underscore it and to fully absorb it.

Consider the light it throws on the postulation of powersets: although anything we may be able to specify is definable sets, we still postulate the existence of a *totality* of (definable *and* non-definable) sets of natural numbers, the elements of $\wp(N)$. The requirement this imposes is totally different from postulating a totality of natural numbers, N, in which case there is no problem with exhibiting concrete instances.

1.2. To avoid paradoxes such as Richard's paradox,[8] the notion of definability must be understood as relative to some specified formal system. Thus definable sets will correspond to sentences in some (finitary or recursively specifiable) formal language. One may consider expanding the language, or using several formal languages together, but this does not alter the essential

[6] Although it is customary to use "arbitrary" for sets in general, whether definable or not, here I shall tend to understand that word to mean non-definable in the context of a specified theory. I believe the context suffices to dispel ambiguity, but on occasion I will use "non-definable."

[7] To be sure, the definition is prolix and involved. One can use, for instance, Leibniz's formula $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$ as a basis for an explicit definition of the sequence $3, 14, 159, \dots$ in the language of arithmetic. I thank Juan Arias de Reyna for offering me important clarifications on this point.

[8] The paradox is discussed in good treatises on logic and set theory; a detailed treatment can be found in Fraenkel, Bar-Hillel, and Levy [1973]; see Richard's original paper in Heijenoort [1967], pp. 142–144. In this connection other important paradoxes are König's (of the least real r which is not in the sequence of finitely definable real numbers), and Berry's.

point: the system(s) under consideration must be fully specifiable, which amounts to the requirement of strict formality, i.e., recursive specification.

Since definable sets correspond to sentences in a finitary or recursive formal language, given a set C of any cardinality and a corresponding formal language, there exist only countably many definable subsets of C. Cantor's Theorem guarantees that there are non-definable subsets of C. In the clear case of the natural numbers, even if we allow (countably) many different formal systems, there exist only countably many definable subsets of N. Indeed Cantorian set theory guarantees that there are continuum-many arbitrary subsets of N. Intuitively, then, a randomly chosen subset of N will be entirely arbitrary, non-definable.

The importance of Cantor's diagonal procedure is, precisely, that it constitutes a method for transcending any given sequence of definable subsets of N (analogously for other sets). In and of itself, however, Cantor's diagonal method does not lead to arbitrary sets. In fact, if a countable sequence of definable sets of natural numbers is given explicitly, so that we can compute whether n belongs to the n^{th} set, the diagonal procedure yields a computation of the truth value of $n \in B$ (where B refers to the new set defined by Cantor's method).[9] This was the feature exploited in Richard's Paradox, which in turn is solved by restricting definability to specified formal languages.

A more liberal notion is that of *definability with parameters*, which amounts to the same as relativized definability. The idea is the following. Even though most real numbers are not definable from the natural number system, we can assume that the set R is given and consider sets of real numbers definable in a language for the structure $\langle R, 0, 1, +, \cdot \rangle$ with arbitrary parameters in R. Notice that this requires us to adopt a form of methodological platonism; we introduce an assumption that some system we cannot fully specify is *given*.[10] Just like before, since the language must be formally specified, it follows that only countably many subsets of R are definable, and (by Cantor's Theorem) that there are many more non-definable subsets.

When employing definability with parameters the domain of the parameters must be fixed from the start. Richard's Paradox exploited an ambiguity here, for if we regard that domain as generated along the way, a contradiction seems unavoidable. The contrast between the static conception of set theory and generative or constructivistic conceptions of mathematical objects was

[9]Compare the case with algebraic numbers: one can *define* in the strict sense a transcendental number by using Cantor's diagonal argument applied to an explicit enumeration of the set of algebraic numbers. The idea is to combine theorem 1 of Cantor [1874] with the diagonal procedure in Cantor [1892]. See, e.g., R. Gray, *Georg Cantor and Transcendental numbers*, ***The American Mathematical Monthly***, vol. 101 (1994), pp. 819–832.

[10]I rely here on the distinction between methodological platonism and (philosophical or ontological) realism. This is a common point among philosophers of set theory, see for instance Shapiro [1997], p. 38.

not sufficiently clear around 1900. Sometimes this contrast is obscured, even today, by careless expositors of set theory.

1.3. The reason for calling the classical standpoint of Dedekind and Cantor quasi-combinatorial is the following. In traditional combinatorics we are given a finite number of elements, and all their possible combinations are then regarded as given. When the number of elements is very large, say $10^{10^{10}}$, it may be unfeasible to actually produce all of their combinations, but if we disregard limitations due to time and speed of operation, they may be conceived as reachable—and this is the traditional standpoint. What about combinations of infinitely many elements? Here we have two possible answers, which characterize the standpoints of set theory vs. constructive approaches to mathematics.[11] Constructivists will insist that the infinitary case is essentially different from the finite one, so that both cannot be put on a par.

Set theory and modern mathematics, following the lead of Cantor and Dedekind, insist on considering the infinitary case on a par with the finitary. Hence the name "quasi-combinatorial" to emphasize the fact that a strong analogy is drawn with the finitary, combinatorial case. That name was coined and the conception explained by Bernays [1935] in words that deserve quotation:

> But analysis is not content with this modest variety of platonism [to take the collection of all natural numbers as given]; it reflects it to a stronger degree with respect to the following notions: set of numbers, sequence of numbers, and function. It abstracts from the possibility of giving definitions of sets, sequences, and functions. These notions are used in a "quasi-combinatorial" sense, by which I mean: in the sense of an analogy of the infinite to the finite.
>
> Consider, for example, the different functions which assign to each member of the finite series $1, 2, \ldots, n$ a number of the same series. There are n^n functions of this sort, and each of them is obtained by n independent determinations. Passing to the infinite case, we imagine functions engendered by an infinity of independent determinations which assign to each integer an integer, and we reason about the totality of these functions.
>
> In the same way, one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded. We add to this the idea of the totality of these sets. Sequences of real numbers and sets of real numbers are envisaged in an analogous manner. From this point of view, constructive definitions of specific functions, sequences,

[11]The issue of intuitionism is too specific and delicate for us to consider here. Thus when I speak of constructivism I shall mean restrictions of classical mathematics and not intuitionism (especially not the introduction of free choice sequences).

> and sets are only ways to pick out an object which exists independently of, and prior to, the construction.
>
> The axiom of choice is an immediate application of the quasi-combinatorial concepts in question. . . . (Bernays [1935, pp. 259–260].)

Bernays' neat explanation of the intuitive idea of set would have been impossible without the contrast with constructivistic approaches to mathematics.[12] In this and some other ways, constructivistic critics have contributed centrally to laying out the foundations of modern mathematics.

Notice also how Bernays begins with an explanation for functions and immediately moves on to sets. The essential equivalence between 'set' and 'function' in a modern set-theoretic setting was obvious for members of his generation. It was made clear in the 1920s with the full reduction of functions to sets in *ZFC*, but also with the fact that von Neumann's original system employed *argument* and *function* (what he called I-objects and II-objects) as primitive notions (not set-membership); see Heijenoort [1967, p. 399]. That essential equivalence is a crucial realization with respect to sets. Once it is understood, I believe, the temptation of thinking naively that sets are a matter of common sense vanishes.

1.4. I said above that the situation with contemporaries like Frege, Peano, and the young Russell may have been different because one may interpret that, for them, sets are and can only be definable. Frege started with concepts,[13] a set being the extension of a concept, and thus for a set to be given, a concept must be available. Under which conditions a concept is "available" was an unresolved matter. At that time they hoped to develop a symbolic calculus which could completely mirror the realm of arithmetic truths, and implicitly this included the hope that the realm of definable subsets of N might reach *all* sets of natural numbers.

Concepts may be available in at least two ways, as abstractly given, or by means of a linguistic specification; since by 1890 and even as late as 1910 notions were still unclear and in flux, Frege and others (including the French analysts Borel, Baire, Lebesgue) could hope that both avenues for "availability" might coincide. After the limiting results concerning formal systems, we know that Frege and the young Russell would have needed some kind of "quasi-combinatorial concepts," which is almost a contradiction in terms (since one must abstract from the possibility of giving definitions, i.e.,

[12]Consider, e.g., the difference between a strict understanding of functional *law vs.* the modern idea of *mapping* (see section 2.1 below), a difference that seems to have gone unnoticed in the early period of development of set theory, and was brought to the fore by the criticisms of constructivistically inclined mathematicians.

[13]The full story is, of course, that Frege regarded concepts as particular cases of function (with co-domain the truth values V,F); but for our present purposes we can simplify in exactly the same way he did in *Grundlagen der Arithmetik*.

conceptual determinations, of such "concepts"). But the need for that assumption, and its implications, were still very far from clear as of 1900. This explains why Frege and contemporaries could still make their definabilist tendencies seem compatible with adherence to classical analysis.[14]

If one wishes to avoid the metaphysics of concept-platonism and the unclarities remaining in the early views, a concept should be seen as the counterpart, the 'phantom' or 'shadow' of a linguistic expression of a certain kind, e.g., *twin primes* or *perfect Polish spaces*.[15] Modern logical theory recommends to identify the vague notion of a "concept" with the crisp concept[16] of *open sentence in a given formal language*. Let us, for the sake of brevity and precision, call this a *formal predicate*.

This makes it possible to mathematise and extensionalise the notion of a definable set, saying that a set C is definable over a structure M when and only when there is a formal predicate in one free variable (in the language $\mathsf{L_M}$) such that members of C satisfy the formal predicate. Using $\varphi(x)$ as a schematic representation of the formal predicate in language $\mathsf{L_M}$, this is nothing but the very familiar $C = \{x : \varphi(x)\}$. C is formed by all objects (in a certain domain) that fall under the formally specified concept, to use Frege's phraseology.

1.5. Naturally, the idea of a definable set is then relative to the formal language(s) under consideration; when we deal with definability with parameters, the domain of interpretation of the language(s) also becomes crucial. This creates the need for some care in the handling of set theories, since reliance on different formal systems might entail significant differences in the definable sets. Anyhow, with the move we have discussed one has obtained a clear, mechanisable characterisation of the definable sets.[17]

All the rest is arbitrary sets, non-definable sets, and as you know, $\wp(N)$ contains only denumerably many definable sets, continuum-many non-definable sets; similarly, $\wp(R)$ contains $2^{2^{\aleph_0}}$ non-definable sets, and so on.

[14] Frege and Russell avoided constructivism because they left the idea of a concept or a function—more precisely, the extent of the realm of concepts and functions—unspecified. A clear discussion of this point can be found in Weyl [1944]. Also relevant is the fact that Frege allowed impredicative methods, so that his requirement of definability did not exclude impredicative definitions.

[15] Although in elementary cases concepts are the linguistic expression of certain cognitive representations; consider, e.g., the notorious concept of *chair*.

[16] Pun is intended, obviously. The need for restriction to open sentences in a (first order) formal language was urged by Skolem and Weyl in the context of the 1920s, and it was then a very radical move; see Ferreirós [2007], chap. X, especially pp. 357*ff.* But I take this formalistic tendency to be the most defining trait of 20[th]-century logic, hence the unhistorical phraseology that I have employed above.

[17] A natural refinement of the dichotomy definable vs. arbitrary is the following trichotomy: predicatively definable sets, impredicatively definable sets, arbitrary sets. But just like before, the arbitrary sets are simply "all the rest;" maybe only the first two classes allow for precise specification. Gödel's constructible sets are somehow transversal: his constructible hierarchy L takes the first notion but combines it with the assumption of all transfinite ordinals.

The intuitive idea behind arbitrary sets is that of "an infinity of independent determinations," to use Bernays' words, which assign each integer to the set or not in the case of $\wp(N)$; or assign each real to the set or not in the case of $\wp(R)$, etc. Gödel presented this notion in the language of a "random set" of numbers, to emphasize that those independent determinations are to be regarded as random—not determined by formal predicates or the like.[18] Terms like 'random' or 'arbitrary' are inevitable if we aim to pinpoint the notion we are talking about. (Other options, such as talk of 'free choice,' are less satisfactory insofar as their anthropomorphic suggestions conflict with the static, platonistic orientation of set theory. In fact, they invite us to constructivism.)

We allow ourselves to reason about the totality of such arbitrary or random sets, considering them as given in the same sense that we regard N as given. But notice that one might wish to establish differences between $\wp(N)$ and $\wp(R)$, insofar as single elements of N are fully specifiable while those of R are not—that is, insofar as the reals raise definability issues that do not appear for the naturals. This means that the iterability of the powerset operation is not obvious. But the distinctive standpoint of set theory, in the Cantor–Dedekind tradition, is to disregard such distinctions.

The crucial question is, do we have mathematical control of that "rest," the arbitrary subsets? And, how much of the idea is taken care of by *ZFC*? What set-theoretic axioms play a role in making more precise the notion of an arbitrary set? But, before going into those issues, let us consider the historical origins in more detail.

Part 1. Arbitrary sets behind *ZFC*

§2. Functions, real numbers, and arbitrary sets. Two main lines in the development of analysis converged in promoting the admission of arbitrary sets, namely, ideas concerning decimal expansions of real numbers and the (Dirichlet–Riemann) notion of so-called *arbitrary* functions. The second line was more consciously explicit in the period 1870–1930, and its importance is well known.

2.1. The attempt to make precise Dirichlet's approach to functions was one of the crucial driving forces in the emergence of set theory. Gustav Lejeune–Dirichlet presented in 1837 a "purely conceptual" idea of function as any (many-to-one) correspondence of numerical values—regardless of whether the correlation can be determined by a formula or not. This was part of his attempt to build mathematics upon the basis of general concepts, not formulas or analytic expressions. He referred to such as "arbitrary" functions, and in the 19th century it was customary to speak of "Dirichlet's concept" of arbitrary function. Notice that Dirichlet's approach, because

[18] Feferman [1990], p. 259, footnote 14.

it relegated explicit analytic formulas to a secondary role, forced mathematicians into specific consideration of the domain and co-domain of their functional correspondences, the arbitrary functions.

Thirty years later, the study of discontinuous functions started in earnest, due to Riemann's new definition of the integral, and this again induced mathematicians to set-theoretic considerations. Such studies, including attempts at extending Dirichlet's work on Fourier series, led mathematicians like Hankel, du Bois–Reymond, and Cantor to analyse properties of the sets of points at which a given function is discontinuous (or where the series representation fails, in the case of Cantor), the so-called sets of uniqueness.

The study concerned definable subsets of R, now in the sense of descriptive set theory (which in essence amounts to definability with parameters in R), but research was guided by the idea that arbitrary subsets had to be admitted. An arbitrary infinite point-set $P \subseteq R$ was assumed to be given, and one proceeded to its study, e.g., through properties of the set of its limit points. The infinite point-set is entirely arbitrary, but one concentrates on certain classes of such sets according to specified properties (e.g., that after countably many iterations the operation *derived set* leads to an empty set). Cantor and others introduced complex examples of point-sets in the course of their studies, most famously Cantor's ternary set. More generally, one studies sets obtained from open (closed) sets by complementation, countable union, projection, iterated countably many times, which leads into the Borel sets, analytic sets, projective sets.[19] It is well known that the main questions in this field are settled by Projective Determinacy and large cardinal axioms (see Maddy [1997]).[20]

Dedekind, too, was strongly motivated by the ideas of Dirichlet and Riemann on arbitrary functions. This led him to introduce arbitrary mappings, as an explicit concept in set theory, in his work of 1888. (They were already present in his algebraic and number-theoretic research since 1871, together with the notions of homomorphism and isomorphism.) But the intended arbitrariness was still quite obscure, and as late as 1888 he spoke of the mapping φ of a set S as being determined by "a law" (Dedekind [1888, p. 799]). From our standpoint (sect. 1.2) the notion of an "arbitrary law" makes no sense, or else it is just a reduplication of the set-theoretic problem at the level of intensional objects (as would be the case with "quasi-combinatorial concepts", see 1.2 above). But it is abundantly clear that Dedekind was a partisan of Dirichlet's concept of arbitrary function.[21]

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[19] For details about the history of descriptive set theory, see Cooke [1993], Kanamori [1995].

[20] The difference between definable-in-real-parameters and non-definable-in-real-parameters marks the distinction between descriptive set theory and general topology of R, and the main questions of this latter domain remain open (I owe this observation to a referee). When this distinction became clear to mathematicians is an interesting historical question that deserves closer study.

[21] Hence the only reasonable interpretation of Dedekind's 'definition' is that by referring to "a law" he aimed (i) to capture the idea that a mapping must be fully specified for each

In connection with section 1, one should notice the strict analogy between functions given by formulas vs. functions in general, and definable sets vs. arbitrary sets. This is a crucial point, but I must refrain from entering into more historical detail in the interest of space.[22]

2.2. Renewed study of the concept of real number was motivated by the beginnings of studies of point-sets, just mentioned (Cantor, du Bois–Reymond), and more generally by the contemporaneous rigorisation of analysis (Dedekind, Weierstrass, etc.). The very influential approach of arithmetisation, and associated ideas about pure mathematics, established a new methodological framework that led to significant novelties in the treatment of the real numbers.[23]

The resulting drive to develop the real number system out of the set Q of rational numbers led Cantor and Dedekind to introduce in practice (though not clearly enough at the level of theory) two core assumptions of set theory: the axioms of Infinity and Powerset; this was accompanied by the assumption that *arbitrary* subsets of Q had to be admitted. That step was taken in 1872 by both Dedekind and Cantor, *but only implicitly*; as a matter of fact, their reasonings posited simultaneously infinite sets, their power domains, and quasi-combinatorial subsets.[24]

Retrospectively, the roots of these assumptions can be found in traditional views about decimal expansions. After the introduction of decimal expansions, eventually it became clear that all the different lengths within an interval of unit length can be associated with *all possible* decimal expansions after the colon, i.e., all possible assignments of ciphers to the finite ordinal numbers.

I have expressed myself with some care, writing 'associated' intentionally, to avoid the modern idea of an *identification* of real numbers with infinite decimal expansions. From a methodological standpoint, this is an important distinction. Notice the following: it is one thing to admit that, given a point on a line, an infinite decimal expansion (possibly non-periodic) is determined

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element of the domain, $\varphi(s)$ univocally determined for each s; and at the same time (ii) to abstract from the possibility of giving an actual definition of φ by means of a formula. In fact, this reading emerges simply from applying to mappings, *mutatis mutandi*, what he had just emphasized for sets: "In what manner this determination is brought about, and whether we know a way of deciding upon it, is a matter of indifference for all that follows" (Dedekind [1888], p. 797, footnote).

[22] See, e.g., classical expositions of the history of set theory by Jourdain [1906–1914] or Cavaillès [1938], more recent ones by Dauben or Moore [1982], my Ferreirós [2007], or philosophically oriented books like Maddy [1997], Lavine [1994].

[23] See the chapters by Ferreirós (p. 235) and Schappacher & Petri (p. 343) in Goldstein *et al.* [2007], and also Ferreirós [2007], chap. IV.

[24] Jané [2005a] has written that the positing of power domains was, historically and conceptually, the *first act* of set theory. (The terminology of "first" and "second act" comes from Brouwer.)

or can be produced—and quite another thing, to define the real numbers in the interval $(0, 1)$ as *all the possible* infinite decimal expansions

$$0, c_1 c_2 c_3 \ldots c_i \ldots,$$

where the c_i are ciphers in decimal notation. The first principle was traditional, admitted since (at least) about 1600,[25] but the second approach is typically modern. Within the first approach, questions of existence in mathematics can still be conceived in agreement with Euclid's *Elements* (points are given by explicit diagrammatic constructions), and decimal expansions can be regarded as numerical approximations to such entities.

The second viewpoint is detached from such constructive and/or intuitive geometric underpinnings, necessitating the introduction of an infinitary standpoint and abstract principles of existence. This modern approach stems from about 1850–70; it emerged within the context of the new methodologies of pure mathematics, modern analysis and algebra, and arithmetisation.

Let us now employ the Dirichlet notion of arbitrary functions to make more explicit and clear this modern conception of the connection between real numbers and decimal expansions. The real numbers in a unit interval can be understood in terms of sequences of the ten ciphers, and thus in terms of *all* arbitrary functions

$$f\colon\ N \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.^{26}$$

This standpoint suggests that the secret of the continuum will be found in arbitrary functions, in the arbitrary sequences of ciphers, in the idea of *all possible* assignments.

Of course, instead of the ten digits of decimal notation one can just as well employ any finite number of digits—in particular, ternary or binary expansions. And if instead of $f\colon\ N \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ we consider the binary notation,

$$f\colon\ N \to \{0, 1\},$$

it becomes obvious that one can study the continuum via two-valued functions on N. These functions in turn can be seen to code subsets of N. Thus the "secret" of the continuum is nothing but the secret of the powerset $\wp(N)$.

[25] Historically, decimal expansions were used as early as the 10^{th} century by al-Uqlidisi (*Kitab al-fusul fi al-hisab al-Hindi*, part IV), and their use continued in the coming centuries, e.g., with the Persian mathematician al-Kashi about 1400 (see A. S. Saidan, *The arithmetic of al-Uqlidisi*, Dordrecht, Reidel, 1978). It is well known that Simon Stevin became influential when he employed them around 1600. A different question is, when we may assume mathematicians to have understood the relation between infinitary combinations of the ten ciphers, and the continuum.

[26] Obviously, one has to introduce a convention in order to exclude the possibility that two different functions correspond to the same number (corresponding to what happens with $0.1000\ldots = 0.0999\ldots$). From a more abstract point of view, one may avoid this distinction, leaving behind the topology of R, which leads to the fundamental Baire space of descriptive set theory.

Cantor saw clearly at least part of this assertion,[27] and understood fully clearly how to generalize it to any set S whatsoever simply by considering $f\colon S \to \{0, 1\}$ (Cantor [1892]). The task was, then, to study "all possible" subsets of N, or in general of S, however big the cardinality of S may be; that is to say, the question is *to understand arbitrary subsets*.

A further historical annotation is in order. To rethink the real numbers by means of Dirichletian arbitrary functions is not a rational reconstruction of my own imagination. And not merely because the corresponding realisation must have been lurking in the background, in particular among experts in arithmetic methods such as Cantor and Weierstrass, well acquainted with continued fractions, power series, and of course digital expansions (decimal, binary, ternary). A well-known paper by Cantor [1892] and a manuscript of Dedekind are witnesses that the founding fathers of set theory reconceived the real number system exactly along these lines. Dedekind's short piece, dated 1891, is noteworthy because it introduces Baire space (in the sense of descriptive set theory, not the more usual one of topology) eight years before René Baire's dissertation.[28] Working directly upon the structure $\langle N, 1, \sigma \rangle$ of natural numbers defined in Dedekind [1888], he considers the "continuous" set of all mappings $\varphi\colon N \to N$. That constitutes a substantial move of liberation from traditional ideas about the number system and its so-called "genetic construction," towards a more purely set-theoretic approach to the continuum.

2.3. Defining the real numbers on the basis of the ordered field Q, Cantor relied on the totality of sequences of rational numbers, while Dedekind relied on the totality of cuts on the rational numbers. In the first case one needs to admit arbitrary (non-definable) sequences, in the second one needs arbitrary (non-definable) cuts. But this postulation remained fuzzy in their work of 1872, a matter of hypothesizing that "all possible" sequences of rationals or subsets of Q be given, perhaps as elements of a new set.

It surely is relevant that neither Dedekind nor Cantor went on to formulate explicitly the Powerset principle, at least in print. The idea was left somewhat vague, in the form of a practically crucial but implicit guiding principle. Cantor would make the principle explicit in a letter to Hilbert (formulated as "The multiplicity of *all the subsets* of an available set is an available set," in an obscure context however), but only to express doubts in the next letter.[29] At

[27] His considerations were complicated by doubts about the acceptability of $\wp(S)$ given S (see below).

[28] See 'Stetiges System aller Abbildungen der natürlichen Zahlenreihe N in sich selbst,' in Dedekind [1932], vol. 2, pp. 371–72. However, the piece is very short, one page only, and merely introduces a total order of the set of all mappings $\varphi\colon N \to N$; it does not enter into its promised topic, to investigate the "continuous" structure of that set.

[29] See Meschkowski and Nilson [1991], pp. 396–398 and Ferreirós [2007], pp. 447–448. Especially noteworthy is the letter of 12.10.1898, where Cantor expressed a negative opinion about the Powerset principle (he calls it "illusory").

any rate, their theories of the real numbers presupposed full powersets, and the axiom's necessity for understanding the continuum as a point-set offers the strongest case in its favour. There are other points in the works of Cantor and Dedekind that are related to Powerset or at least to arbitrary subsets: Cantor's studies of point-sets proceeded on the assumption of arbitrary sets of reals, and the same applies to Dedekind's work on ideals (infinite sets of complex integers) where he routinely assumed all possible ideals on any ring of integers to be given.[30]

As one can see, ideas about arbitrary functions, arbitrary real numbers, and arbitrary sets were a crucial background to the emergence of set theory—from Dedekind and Cantor, to Lebesgue, Zermelo and beyond. Under the influence of such ideas, the motivating principle of *accepting totalities of arbitrary subsets* was put in place. This crucial principle became the main bone of contention for critics of set theory in the decade after 1900, despite the fact that it was not made explicit by either Dedekind or Cantor. Only with the Axiom of Choice did the principle surface more clearly.

§**3. The principle of Choice.** From the standpoint of a principled acceptance of arbitrary subsets, it is obvious that one should accept choice sets. Just like one regards a set of integers as the "result" of infinitely many "independent acts" deciding for each number whether it should be included or excluded, one views a choice set for a family F of nonvoid sets as the result of transfinitely many "independent acts" assigning to each $y \in F$ an element $x \in y$. To use Gödel's terminology of "random sets," it is clear that among the random subsets of F there are those that satisfy the condition just enunciated. And set theory is guided by the idea that all such random sets are to be regarded as given.

Perhaps the most characteristic trait of quasi-combinatorialism is a negative trait: one *disregards completely* issues of explicit definition (through formal predicates in any given system) as irrelevant to the "givenness" of infinite sets. From this point of view, explicit definitions of specific functions, sequences, or sets are only "ways to pick out" an object which exists independently of, and prior to, the construction or definition. Wholly independently of the mathematician's choices, moves, or constructions, choice sets and choice functions are given. It is for these reasons that Bernays remarked that AC is "an immediate application" of the quasi-combinatorial viewpoint.

[30] Cantor's work on ordinals and well-ordering seems also worth mentioning here, for given any set Cantor regards as given all possible well-orderings on it. Finally, in the case of Dedekind, some reconstructions of his celebrated categoricity argument of 1888 assume that it relies on Powerset (see Reck and Awodey [2002]), although the matter is contentious (see, e.g., Parsons [2008]).

What caused trouble with the Axiom of Choice around 1905 is precisely the insistence on definability, the idea that infinite sets should be determined by a concept (section 1.2). It was because many mathematicians understood sets as concept-extensions that they found it unacceptable to postulate the "existence" of sets such as those guaranteed by Choice. (Similarly, many thought that functions ought to be given by explicit formulas—a view that was strongly promoted from Berlin, as early as 1870.[31]) To be more precise, many mathematicians of the period were inclined towards a constructivist notion of mathematical existence (for the real numbers) and showed definabilist preferences concerning sets of reals, all of which caused them to object to the Axiom of Choice.

All of this was particularly clear in the case that was of general concern around 1905: the existence of a well-ordering for the set R of real numbers. Zermelo's proof that such a well-ordering exists was no proof for mathematicians who thought of sets as determined by concepts. It was becoming increasingly clear then, as it is clear today, that such a well-ordering is *not definable*.

Notice that a concrete, definable well-ordering is what Hilbert asked for in Problem 1 of the famous 1900 list:

> It appears to me most desirable to obtain a direct proof of this remarkable statement of Cantor's, perhaps by actually giving an arrangement of [the real] numbers such that in every partial set a first number can be pointed out. (Hilbert [1900, p. 1104].)

In light of what was said above, Hilbert's statement of the problem is vague, since any precise formulation must refer to a formal system. In order to make it fully precise, the most natural formulation is to ask whether a well-ordering of R is *ZFC*-definable. This Hilbertian question was solved in the negative by Solomon Feferman using forcing methods to prove that "*it is consistent with ZF,* AC *and* GCH *that there is no set-theoretically definable well-ordering of the continuum*" (Feferman [1965, p. 342]).

Reflecting on Zermelo's proof of the Well-Ordering theorem, Émile Borel offered in 1905 some very perceptive remarks. Zermelo had shown the equivalence between two problems: the problem of well-ordering a certain set S, and the problem of defining a choice function on the powerset $\wp(S)$. In Borel's view, this second problem was in no way to be regarded as solved in general. He thus kept emphasizing the need for mathematicians to actually specify the infinitary objects or processes they aim to study. He insisted on the need to define infinite sets by concepts, while those who accepted AC were, more or less consciously, emphasizing their view that definability is no central matter, while arbitrary sets (*resp.* quasi-combinatorialism) are primary.

At this level, of course, the complexity of (often incompatible) viewpoints in the early development of set theory becomes utterly clear. While Cantor,

[31] See Ferreirós [2007], pp. 183–186, and also chaps. I and V.

Dedekind and Zermelo were the great champions of arbitrary sets, Peano had considered the Axiom of Choice and rejected the idea, Russell remained agnostic throughout the 1900s and in ***Principia Mathematica***, the French analysts Baire, Borel and Lebesgue rejected the principle they had previously relied on ... and so forth. (See Moore [1982] for a plethora of details.)

Some mathematicians went on to insist throughout their career that infinite sets can only be accepted provided that they are determined by an explicit definition. Hermann Weyl provides us with the example of a very consistent standpoint along these lines. He argued that, as *inexhaustibility* is a characteristic trait of infinite sets, a quasi-combinatorial viewpoint simply *cannot* be applied to them (Weyl [1918, pp. 13–15, 32–33]). It has been said that Weyl never proved a result that depended on the Axiom of Choice and, although I have not verified that this is so, it is clear that he did his best to ensure the independence of his mathematical work from an assumption he considered so questionable.

The reactions of Borel, Baire, Lebesgue, Weyl, and others to Zermelo's system, and their principled objections to AC, can be read conversely as so many arguments for the inadequacy of the image of sets as concept-extensions. Let me emphasize again that this is so as long as we refrain from postulating the existence of a Logician's Paradise in which all the required "quasi-combinatorial" concepts are already given (section 1.2).

One aspect requires further clarification. The effects of Choice as a principle of set existence are *relative to* assumptions concerning the background model. It is well known that under assumptions such as constructibility ($V = L$) the principle of Choice becomes a theorem; we shall come back to this below. Postulating the existence of sets provided by Choice becomes objectionable only because it is (tacitly or explicitly) assumed that the background model *does include* arbitrary sets. This is almost trivial in the context of R and its well-orderings, which we just discussed, since the 'classical' notion of real number is intimately entangled with the assumptions of actual infinity and of arbitrary subsets (section 2.3).

Let me put this point differently. I shall argue below that AC is the strongest embodiment of the ideal of quasi-combinatorialism inside *ZFC*, but even this embodiment does not capture the idea of arbitrary set directly, in some absolute sense—but only relatively and partially. After one further century of developing set theory, one may suspect that the ultimate reason why the founding fathers failed to make explicit their basic motivation of fully admitting arbitrary sets was simply that the idea is impossible to pin down completely—it cannot be made fully explicit.

§4. Interlude on constructibility. An important element in gauging the import of quasi-combinatorialism has been the study of the constructible universe.[32] This has helped make mathematically clearer what the quasi-combinatorial (or full) universe is meant to be. So let us briefly consider the axiom of Constructibility, usually presented by writing (in class notation) $V = L$, which we shall also have to take into account in connection with AC later.

Gödel's idea of constructibility arose as a characteristic mix of modern or 'classical' procedures and constructivistic restrictions. The main effect of the constructibility requirement is to abolish impredicative definition of sets, and to conceive sets as introduced step by step through explicit definition; but on the other hand, transfinite iteration of the process is allowed, with steps $\omega, \omega + 1, \ldots$ and in general a step α for each transfinite ordinal α. As Gödel emphasized, this is an entirely nonconstructive element, and it counterbalances the elimination of impredicative definitions (i.e., the universe is generated over the class of ordinals as impredicatively given). Thus the approach uses a transfinite iteration of predicative definability.

Gödel's initial hope may have been that, doing so, one might capture a lot of the impredicative and arbitrary sets. This at least could be the reason why he wrote, in the last paragraph of his 1938 abstract on the consistency of AC and GCH:

> The proposition [$V = L$] added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. (Feferman [1990, p. 27].)[33]

Later work suggests that it is not so, namely, that transfinite iteration of predicative definability captures little (or even nothing?) of arbitrary sets, as will become clearer in what follows. Moreover, the assumption $V = L$ seems to go against the motivating ideas behind set theory, specifically against combinatorial maximality. But this was not clear from the beginning.

The constructibility requirement can most simply be presented as a restriction on the von Neumann hierarchy: while the usual cumulative hierarchy is assumed to allow arbitrary sets when domain $V_{\alpha+1}$ is defined as comprising all the subsets of V_α,[34] the constructible hierarchy introduces $L_{\alpha+1}$ as the set of all predicatively defined subsets of L_α (with parameters in L_α). But while a strictly predicative approach would allow only finite levels, and thus would be insufficient to introduce many ordinals, Gödel's constructible hierarchy has levels α for all ordinals.

[32]The main reference is Devlin [1984]. See also Maddy [1997].

[33]Notice, by the way, Gödel's acknowledgement that the idea of an arbitrary infinite set may be "vague," which runs counter to the strong platonism of his post-war days. This contrast has been remarked by many authors, from Feferman and M. Davis to several others; see, e.g., Gödel's *Collected Works*, vol. III, pp. 36–44 and 156–163.

[34]The von Neumann hierarchy inherits the same weaknesses we shall discuss (section 6) apropos the Powerset axiom. Informal descriptions of the iterative conception are just that—informal thoughts.

The detailed work of Jack Silver, Robert Solovay, and others, leading to the codification of $0^{\#}$ (zero sharp), strongly supports the view that Constructibility is not compatible with the maximality that set theory is meant to incorporate. After a pathbreaking result of Dana Scott established that the existence of a measurable cardinal entails $V \neq L$, in the sense that V is strictly larger than L, some experts began to work on results about the structure of L that could shed more light on this matter. For instance, F. Rowbottom was guided by the insight that, under certain structural assumptions, every ordinal definable in L is merely countable, hence so must be $\wp(\omega)$, $\wp\,\wp(\omega)$, etc.! In 1964 he established that a combinatorial (partition) property of measurable cardinals already implies that there are only countably many reals in L. As a consequence, even the existence of a Ramsey cardinal implies that there are only countably many constructible sets of integers.[35]

This line of work soon found a definitive formulation thanks to what has been called the Silver–Solovay theory of indiscernibles for L. Scott's line of work led to the intrinsic characterization of measurable cardinals as critical points of elementary embeddings $j\colon V \to M$ of the universe V into some inner model M. Combining this with the concept of indiscernibility that had been developed in model theory by Ehrenfeucht and Mostowski, the path was found to the distillation of $0^{\#}$. If there exists a non-trivial elementary embedding of L into itself, then there is a closed unbounded proper class of ordinals that are indiscernible in L. $0^{\#}$ is defined as the set of natural numbers that codes the Gödel numbers of the true formulas about the indiscernibles in L.

Beyond coding the sentences true of indiscernibles in L, $0^{\#}$ is a "blueprint" with complete genetic information for the uniform generation of L, thus having a crucial role in the structural theory of the constructible universe. Work on the topic continued in the 1970s with R. Jensen's fine structure theory of the constructible universe, including his celebrated Covering Theorem in 1974, a deep structural statement about the proximity of V to L, "easily the most significant result of the 1970s in set theory."[36]

All of these results are conditional, since it might be the case that there are no measurable cardinals (indeed, Jensen's work was partly guided by the conviction that the postulation of measurables is inconsistent) or no

[35] For details on all these technical matters, see Kanamori [1994], section 8 and 9. This work is noteworthy for the plethora of historical and technical information it contains, and my exposition in this section merely follows Kanamori.

[36] Kanamori [1994], p. 111. The Covering Theorem establishes that, if $0^{\sharp}$ does not exist, then for every uncountable set X of ordinals there is a constructible Y such that $X \subset Y$ and Y has the same cardinality as X. Furthermore, L is then the core model (see Jensen [1995]). For further details on all these issues, see the newly published *Handbook of Set Theory*, Foreman & Kanamori (2010), volume 3 (e.g., the chapter by William J. Mitchell, 'Beginning Inner Model Theory').

non-trivial elementary embeddings of L into itself. One might thus wonder whether a set of integers with the properties of $0^{\#}$ does or does not exist. This, however, is not likely: at worst, $0^{\#}$ is extracted from assumptions that turn out to be inconsistent, in which case $0^{\#}$ would be the set of all Gödel numbers of sentences in the language under consideration (no trouble as to its definition and existence); and while it is in principle possible that there are no ordinals indiscernible in L, this runs counter to the whole development of set theory in the last half century,[37] and against the guiding notion of quasi-combinatorialism.

Despite not yielding absolute results—for, as we have seen, its results are conditional on large cardinal assumptions—that whole sophisticated body of theory has led most set theorists to the conviction that L is a *very thin subclass* of the class V of all quasi-combinatorial sets. There is interesting evidence that many uncountable cardinals in L are merely countable, and most experts think that $0^{\#}$ is just another set of integers, its peculiarity coming merely from its being defined metatheoretically. Thus, contrary to what Gödel may have thought in 1938, it turns out that a predicative restriction of *ZF* is not compensated by the trick of generating the universe over the class of ordinals as impredicatively given (transfinite induction over all ordinals).

We do not need to enter into further details concerning the ways in which Constructibility runs against the ideal of quasi-combinatorialism, since there is excellent work on this topic.[38] In any event, such results as we have just reviewed help make mathematically clearer what the quasi-combinatorial universe V is meant to be. But such negative, partial characterizations are not the same as a full, positive definition. And this brings us back to our main topic.

Part 2. Arbitrary sets in *ZFC*

In this second part I shall offer some arguments aimed at establishing the poverty of *ZFC* with respect to the guiding idea of arbitrary subsets. While the axiom system is a series of formulae in a fully specified formal language, quasi-combinatorialism has played the role of a guiding thought that promoted the adoption of certain methods, codified in the axioms. But there is no a priori warrant that the quasi-combinatorial ideal can be fully captured by formal statements.

[37]If $0^{\sharp}$ does not exist, then the hierarchy of large cardinals would stop around the level of weakly compact cardinals, and certainly below measurable cardinals and Ramsey cardinals, which would also be non-existent. *ZFC* + “$0^{\sharp}$ does not exist” is a consistent theory, but it is also widely believed that *ZFC* + “$0^{\sharp}$ exists” is consistent.

[38]Readers who wish to see an in-depth study, from a philosophical point of view, are referred to Maddy [1997].

Some axioms of the *ZFC* system can hardly contribute anything to it, as is clearly the case with Extensionality, Pairing, Regularity, Infinity, even Union.[39] We shall consider the remaining three axioms, namely the axiom of Separation, the axiom of Choice, and the axiom of Powersets. These are the postulates that have to do with the matter of which subsets of a given set S exist. (Separation is a consequence of Replacement but not vice versa, so it might seem necessary to discuss the latter; but the logical features of both axioms are similar with respect to any of the issues we shall be discussing, hence we can limit our discussion to Separation without loss.) In all of these cases, what we shall say in section 5 about the potential mismatch between conception (understanding, thought) and formal axioms is relevant.

The standpoint from which we shall revisit those three axioms is to consider (i) the extent to which the idea of arbitrary set is necessary to motivate them intuitively, and (ii) the requirement each one of them imposes on models of axiomatic set theory, or to put it differently, what these axioms enforce upon set-theoretic domains. Well known results in the metatheory of *ZFC*, beginning with Skolem's result that it admits countable models, already indicate that the axioms do not enforce very much. But I believe that an elementary discussion of the matter should be welcome, and this is what I shall offer.

Naturally, it will be necessary to keep in mind that an axiomatic requirement is modified in the presence of other axioms: if it is the case that one of the axioms does the job of providing arbitrary subsets, all of the other axioms will do so secondarily.

§5. Complementarity in mathematics, and powersets. In this section we introduce a short digression that is relevant to all that follows, although this cannot be the place to enter into a fully detailed discussion of the standpoint presented and my reasons for it.[40]

5.1. In the development of mathematics, we find noteworthy instances of two contrary tendencies. One has been the drift towards reducing mathematics to a purely symbolic system; notable examples are found in 20th century strict formalism,[41] but also in Lagrange, Peano, and so on. The

[39]It has been brought to my notice that Regularity is like Choice in that it postulates the existence of an object x that is not concretely specified (for any nonvoid set S there is an element x of S such that $x \cap S = \varnothing$). However, the axiom is not relevant for mainstream mathematics, and it does not affect the question of which subsets of a given set do exist, which is why I shall not deal with it here.

[40]An important part of my forthcoming book *Mathematical Knowledge and the Interplay of Practices* is devoted to such a discussion.

[41]A colorful quotation was provided by Wittgenstein in 1939: "The only meaning they have in mathematics is what the calculation gives them ... if you think you're seeing into unknown depths—that comes from a wrong imagery." (Wittgenstein [1976], p. 254). Whether Wittgenstein was a formalist during this intermediate period is a question I must leave to

other has been the attempt to reduce mathematics to a purely conceptual system; noteworthy instances can be found in the 19th century trend that was labeled "the conceptual approach" (e.g., Dedekind) but also in some 20th century proponents of category theory.[42] The impression one gets from studying these developments is that none of them has been successful (in their ultimate reductionistic goal; they have led to advances and partial successes).

For a long time, mathematicians and philosophers have entertained the hope that a symbolic system could be developed which would mirror perfectly the thought-processes in the mind of the mathematician. Leibniz's ideal of a *mathesis universalis* was inspired by this hope, and his views influenced many later authors, including Boole, Grassmann, Frege, and Hilbert. (In previous work, referring to efforts to submit logical inference to algorithmic mathematical treatment, I called this "the principle of the calculus" (Ferreirós [2001, sec. 3.2]).) Obviously such a goal has been of great consequence for the development of modern mathematical logic and foundational studies, and, obviously too, Gödel's incompleteness theorems established crucial limitations it faces. On the other hand, the 19th century "conceptual" trend reconceived all of mathematics by means of sets and structures, but the paradoxes of set theory forced it to find a safe haven in the highly formalized axiomatic systems of Zermelo and others. In general, mathematicians sometimes manage to avoid symbolic systems of a certain kind, but only to develop some new systems.

In my view, the failure of both radical tendencies is of the essence. The standpoint I adopt emphasizes the need to consider the meaning or thought that accompanies formulae and calculations. (This is no doubt shared by many other philosophers, but the question is how to proceed.) Mathematical symbolism cannot be mastered without immersion in a practice, and by learning the practice we learn to associate representations and meaning to the formulae. Normal (so-called informal) symbolic systems and theories cannot be made to stand alone outside of practice; and when systems and theories are formalized and made to stand alone, the phenomenon of nonstandard interpretations arises in a natural way.

Indeed, I defend the *complementarity of symbolic means and thought* in mathematics—each one joined by the other, none of them reducible to the other. For obvious reasons, it is more difficult to deny the role and importance of the symbolic component in mathematics, but substantial arguments can be given for a similar conclusion concerning the conceptual component.

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experts in the topic, but other authors (like H. B. Curry) offer straightforward examples of a formalistic standpoint.

[42]I have discussed the 19th century conceptual approach in Ferreirós [2007]. As for category theory, see, e.g., W. Lawvere & Stephen H. Schanuel, ***Conceptual Mathematics: A First Introduction to Categories*** (Cambridge University Press, 2005) and Krömer [2007].

For my purposes here, I shall be content with the modest claim that, in light of developments in mathematics and its foundations during the 20th century, such a standpoint deserves to be seriously considered as an option.

The complementarity of symbolic means and thought explains the distance between formulae, say formal axioms, and conceptual understanding. A really trivial example is the following:

$$2 + 2 = 5 \text{ is obviously correct,}$$

provided '5' is the cipher associated with number four. The Axiom of Powersets provides a non-trivial example (where the discussion of quasi-combinatorialism vs. definable subsets is relevant, see section 1).

5.2. Intuitively, by $\wp(\mathrm{C})$ we mean the set whose elements are all subsets of C—*really all*, we might say in a useless effort to emphasize the point. But the usual formal axiom in *ZFC* only ensures that, in a domain Δ that models the formal system, there is an object that bears the ε-relation to all objects in Δ that are subsets of C, i.e., that 'act' as subsets of C *in the domain*. By itself, the effect of postulating $\wp(\mathrm{S})$ is to "collect" into a set all of the subsets of S that are *given* in a domain.

This example is linked with the well-known phenomenon of *Skolem's paradox*, the fact that there are non-standard models of first-order *ZFC*, sometimes called 'non-intended' models.[43] By the Löwenheim–Skolem Theorem we get Skolem's paradox: first-order *ZFC* has denumerable models; in those models there is both ω and the power-set $\wp(\omega)$, but $\wp(\omega)$, being part of a countable model, is itself denumerable. As you know, there is no contradiction here, just a paradox: *in* the model there is no one-to-one correspondence between $\wp(\omega)$ and ω, but 'from the outside' (in a wider model) one may recognize the existence of such a correspondence.[44]

The paradoxical in Skolem's paradox comes from the distance between the formal axioms and our conceptual understanding. We may be inclined to say that, in a non-intended Skolem model, the formally-given $\wp(\mathrm{C})$ "cannot really have" all subsets. By doing so we manifest our impression (our thought) that the formal system does not capture what we meant. The complementarity phenomenon that I described above is thus emphasized.

Notice that recourse to second-order logic is of no help here. A second-order version of *ZFC* can only exclude non-intended models provided that

[43]The "intention" here is given by what we mean or think; from a strict formalist standpoint, talk of 'intended' models is nonsense.

[44]On Skolem's paradox see Jané [2001]. Of course the phenomenon is not particularly linked with powersets. Similar thoughts apply to almost all other formal systems, e.g., to the Dedekind–Peano axioms for arithmetic, or to Hilbert's axioms for the real number system. Consider for instance the Axiom of Induction in first-order *PA*, where instead of considering all arbitrary subsets of N (or perhaps all number-theoretic properties, but see section 1.2) we restrict our attention to well-formed formulae $\Phi(x)$ in the formal language. Similarly with the Completeness Axiom in Hilbert's system.

the second-order quantifiers are given a special meaning. This difficult problem is usually trivialized by saying that this special meaning is "the standard semantics" of second-order logic. The rhetorical effect is clear, but not so the theoretical rationale: if we make a second-order system stand alone, Henkin semantics is just as natural as the preferred one, or even more natural. The "standard" semantics is preferred just because it agrees with our preferred set-theoretic standpoint, that is to say, with quasi-combinatorialism. But the formal system *per se* is unable to do the job. And if our solution is to have recourse to meaning, we might as well read the intended meanings into the first-order axiom system.[45]

In the move of adopting a second-order version of *ZFC*, formal systems and thoughts become entangled in an unclear way. Strictly speaking, that move does not respect the rules of the game of formalization, leading us beyond formal logic. This may be admissible in normal mathematical work, where one presupposes a "standard" set theory anyway, but not in the context of studying the foundations of set theory.[46] Powerset in a first-order version is too schematic to do the work, and the "standard" second-order version begs the question, for it presupposes that the conception of arbitrary sets is definite, transparent.

It is the business of set theory to make explicit the principles of set existence, especially those principles having to do with existence of infinite sets. This includes of course the principles of existence for arbitrary subsets of infinite sets. To relegate some of these to an underlying logic and its supposedly "standard" semantics is merely to obscure that key goal of set theory.[47] Combinatorial maximality is a clear motivation for set theory, but if we secure it by brute force, that amounts to renouncing the goal of a full axiomatic analysis.

I guess belief in the primitive nature and the "logical transparency" of the idea of *all subsets* of any given domain is widely held, but it does not go beyond the general thought of combinatorial maximality. A second-order theory of sets does not make the nature of powersets clearer, but

[45]For further arguments concerning this controversial matter, see among others Shapiro [1991], Ferreirós [2001], Väänänen [2001], Jané [2005b]. While preparing the final version of this paper, a referee's comment led me to Weston [1976], a paper that criticizes Georg Kreisel's views about the second-order decidability of CH, and which offers a long, detailed presentation of exactly the same line of argument as I have just given (see especially the first half, sections I–IV; I cannot agree with several of the ideas presented in the rest).

[46]The illegitimate move is exactly what Shapiro [1991] avoids by renouncing foundationalism.

[47]From this standpoint, the idea that there is "a more primitive, logical notion of set" which is formalized in second-order logic and can be used as a basis for building up set theory (Tait [2005], p. 134) becomes hardly tenable. There exist other approaches to second-order logic which avoid the pitfalls of the one discussed above, such as those employed in proof theory, where comprehension principles explicitly regulate the existence of sets (see, e.g., Simpson [1999]).

merely assumes the question has already been solved. By taking this idea for granted, and treating the special "standard" meaning of second-order quantifiers as a logical primitive, we may simply be obscuring the "nebulous character," the "inherent vagueness" of the notions of arbitrary subset and powerset.[48] At any rate, it should be clear that the advantage of first-order set theory is that it forces the experts to confront explicitly the task of clarifying those notions.

To sum up, the axiom of Powersets postulates a somehow *maximal* set of subsets of any given S, with the maximality remaining fuzzy—or perhaps better, with it remaining an ideal horizon that might even be impossible to make fully concrete in mathematical terms. The requirement is clear mostly in the negative: if it were the case that a subset T of S can be shown to exist, then (by maximality of the powerset) T must be an element of $\wp(S)$. We want $\wp(\omega)$ to be combinatorially maximal, but in order to make this precise one should spell out what it is to mean, offering a full axiomatic analysis. Which sets must be given in the domain will depend on the remaining axioms, perhaps in combination with Powersets itself (a proviso that is made particularly necessary by the impredicativity of the *ZFC* system).

§6. On arbitrary sets in *ZFC*: Subsets by Separation. The axiom of Separation could also be called the axiom of Definable Subsets. A subset T of S is given by Separation *iff* there is a formal predicate $\varphi(x)$ in the language of *ZFC* that characterizes the elements of S belonging to T. Because of this strict correspondence between subsets given by Separation and formal predicates $\varphi(x)$, the axiom we are considering seems to fall short of the more arbitrary range of quasi-combinatorialism entirely.

Nevertheless, it is instructive to consider what a constructivist may find objectionable in the postulate. This of course is the impredicativity of Separation: the axiom can be used to specify a certain subset of S, say by reference to the powerset $\wp(S)$, that we may be unable to characterize otherwise. This method is unacceptable for a predicativist and from the viewpoint of stricter constructivist positions.

It is not clear to what extent the impredicativity of Separation may capture some of the quasi-combinatorial idea. The language of first-order logic, by virtue of its richness of expression and in particular its use of multiple quantifiers, can represent very subtle interrelations between elements of the domain. Consider a first-order sentence of the kind: $\forall x\, \exists y\, \Phi(x, y)$, which may express the existence of a unique y for every x in the domain.[49] In such cases, first-order logic is powerful enough to define a functional relation between elements of the domain. This and impredicativity may well make Separation capture a bit of the idea of arbitrary sets.[50]

[48] Quotations come from Weyl [1918], p. 15 and Feferman [1998], p. 73.

[49] I mean a sentence like: $\forall x\, \exists y\, [\Phi(x, y) \wedge \forall u\, \forall v\, (\Phi(x, u) \wedge \Phi(x, v) \rightarrow u = v)]$.

[50] I thank John Steel for pointing me to this possibility (personal communication).

However, given the way in which we have framed the notion of "definable set" in section 1, that much has been included in the *definable* portion. As we pointed out there (section 1.2, footnote), a natural refinement of the dichotomy definable vs. arbitrary is a trichotomy: predicatively definable sets, impredicatively definable sets, arbitrary sets. I must leave open the question whether this trichotomy is helpful for a more detailed analysis of the topic.

The arbitrary sets are simply "all the rest". That this rest is non-void is warranted by the fact that we have only denumerably many formal predicates in the languages under consideration—and this is independent of the impredicativity issue. Thus Separation can only provide us with denumerably many subsets of N, and Cantor's Theorem establishes the existence of subsets not given by Separation. Here the fact that Cantor's Theorem is an indirect result is essential: my claim is not contradicted by the fact that the 'diagonal set' employed in the proof is introduced by appeal to the axiom of Separation.

Notice that this use of Separation is dependent on the initial assumption that we are given an α-sequence of sets, with α some ordinal (just ω above). Notice also that we read into the proof the quasi-combinatorial idea, by assuming the possibility that elements of this α-sequence be arbitrary sets. In the practice of proving, the interaction between thought and formulae becomes particularly delicate—and this is not captured by purely formal systems.

§7. On arbitrary sets in *ZFC*: Subsets by Choice. Readers who have accepted the view that both Separation and Powerset fall short of incorporating the quasi-combinatorial ideal should be inclined to think that the axiom of Choice plays a central role in its incorporation. We have emphasized that trouble with AC arose, around 1905, from the idea that infinite sets should and can only be determined by a concept, i.e., from resistance to quasi-combinatorialism, and Bernays remarked that Choice is "an immediate application" of the quasi-combinatorial idea.

7.1. It has long been established that AC is indispensable in certain contexts. Well-Ordering is equivalent to AC in the axiom system *ZF*, and so are Zorn's Lemma, or within topology Tychonoff's Theorem. An early milestone in studies of axiomatic dependences involving AC was the work of Sierpinski, who devoted a lengthy paper in 1918 to spell out how deeply entrenched the axiom is in analysis.[51] It was indeed usual in analysis to extract, from a sequence of (countably many) nested domains converging to a point, a sequence of points; AC is just a generalization of this procedure. An interesting but little known fact is that AC did not originate with Zermelo himself, but was suggested to him by Erhard Schmidt, a specialist in

[51] For a recent discussion of this topic, see E. Schechter, *Handbook of Analysis and its Foundations*, San Diego, Academic Press, 1997.

functional analysis.[52] It is telling that the axiom originated in analysis and not pure set theory.

Indispensable uses of AC occur precisely in cases where infinitely many sets are assumed to be given, with possibly arbitrary sets among them, and a set of corresponding elements is needed. Russell's famous exemplification of the axiom with infinite sets of socks, vs. shoes, is aimed precisely at underscoring that AC is unnecessary when a formal predicate can be specified that "does the choosing". When the axiom is used indispensably, it offers the best currently available examples of quasi-combinatorialism at work (see section 3).[53]

The development of mathematical theories is usually opaque to mathematicians when things are in the making—at least when the theory is of a certain degree of complexity. It might have turned out that the adoption of AC suffices to capture axiomatically the full idea of combinatorial maximality (this would have been a lucky accident, but past examples of such things can probably be found). On reflection and with hindsight it appears that the situation is the opposite, and one wonders whether quasi-combinatorialism will ever be captured by a formal system.

7.2. Contrary to the analysis of AC I have just given, one might reply that, since the axiom is valid in the constructible universe, it captures nothing of the idea of arbitrary set.[54] However, the validity of AC under the assumption of Constructibility ($V = L$) does not constitute an argument against AC capturing some of combinatorial maximality. The fact that AC is true in the constructible universe is a product of the strong restriction imposed on set-formation by Constructibility (evidence for the thinness of L has been reviewed in section 4) and AC is true of L because *the whole* constructible universe can be well-ordered—as a result of its sets being ordinal-definable by predicative sentences. Constructibility is strictly stronger than the Axiom of Choice, in the axiom system *ZF*; we have just noticed that it implies Global Choice.

The thesis I want to defend is the following. AC is the axiom that comes closest to capturing the idea of arbitrary set, but it does not capture the full idea, and it does so merely relatively. It is only *relative to the implicit assumption* that some arbitrary sets are given in the universe that AC specifies

[52] Zermelo acknowledged this in the paper containing his first proof of Well-Ordering, which took the form of a letter to Hilbert: "I owe to Herr Erhard Schmidt the idea that, by invoking this principle [the "logical principle" of Choice], we can take an *arbitrary* [choice function] γ as a basis for well-ordering" (Van Heijenoort 1967, p. 141). Schmidt is remembered from the Gram–Schmidt orthonormalization process; he was a student of Hilbert, taking his Ph.D. in 1905 with a dissertation on integral equations and Hilbert space.

[53] Russell's agnosticism with regards to AC, or the "multiplicative axiom", is telling; see section 1.3 and Russell [1910] or [1919], chap. XII.

[54] This reply was actually offered to me by one member of the audience during a presentation of my ideas at the Berkeley Logic Colloquium in December 2006.

a condition which enforces quasi-combinatorialism to some extent. The reader will quickly concede that it is only relative to such an assumption that AC becomes questionable. Given a family of sets F, the axiom gives us a choice set C; for the axiom to be employed unavoidably, either some of the sets S ∈ F are entirely arbitrary, hence defining conditions cannot be had; or the set F itself collects the definable sets S in such an arbitrary way that no uniform condition picking up the elements of S ∈ F that belong to C can be had; or both at a time. Notice that the arbitrariness making the use of AC unavoidable is presupposed; it must affect either the family F or (some of) its elements S, or both.

What troubled critics of the axiom is that the choice set C would then have been "determined arbitrarily," which for them, since they were adopting some form of definabilism (not quasi-combinatorialism), meant that C has not been determined at all. Since they were mostly discussing the case of *R*, the matter seemed clear, since the postulation that *R* is given as a whole makes the above assumption explicit. What troubled the critics is the *conditional* arbitrariness introduced with the stipulation of AC, but notice that this is a relative form of quasi-combinatorialism.

Once more, the fact that AC is true of *L* hardly counts, for in contracting *V* to *L* we do eliminate that background assumption of arbitrariness. (Remember that the existence of a Ramsey cardinal implies that *L* has only countably many reals, and other similar results mentioned in section 4.) Let me also suggest that, from the beginning, the fact that AC is theorematic in *L* was evidence that the constructibility conjecture is too restrictive, for the simple reason that AC had been shown indispensable in the context of analysis. That is, it was indispensable under the assumption that the universe *does* include some non-definable sets, such as non-definable reals (arbitrary sets of natural numbers), sets of reals, and so on.

Under the assumption of quasi-combinatorialism, vague as it may be, the universe of sets is richer than *L* and contains non-definable sets. In such a richer universe, AC is used indispensably to work with arbitrary sets. But the point remains that this "greater richness" of the universe is not captured by the axioms, above and beyond the conditional requirement that AC makes explicit. We believe that Constructibility is not a natural axiom because it contradicts the quasi-combinatorial ideal, and thus we aim to work with richer universes, but we have been unable to formulate low-level axioms that further specify that richness.

§**8. Conclusion.** Forty years ago, Mostowski suggested that the "intuitive notion" of set is "too vague to allow us to decide whether that axiom of choice and the continuum hypothesis are true or false" (Mostowski [1967, p. 89]). In this spirit, many logicians are willing to abandon AC, working with assumptions that contradict it. I have argued, however, that the Axiom of Choice is a key ingredient of set theory, lest one abandon the focus on arbitrary sets (quasi-combinatorialism). It follows that the status of AC is quite different from that of the Continuum Hypothesis—that, of course, is the situation in practice, but I have argued that it should be that way for principled reasons. The notion of set is tight enough to provide a clear motivation for AC, and at the same time, it is perhaps too vague to settle CH. To eliminate AC in set theory means to go in a direction highly deviant from set theory as inaugurated by Dedekind, Cantor, Zermelo.[55]

Even if the platonism of set theory and modern mathematics turns out to be admissible conceptually and from a methodological point of view, as I am inclined to think, it is a different matter whether and to what extent we can hope to *specify and bring under axiomatic control* the postulation of totalities of arbitrary sets. The above discussion provides strong reasons to believe that *ZFC* is *too poor* an axiom system, vis-à-vis its motivation of laying out a quasi-combinatorial theory of sets which allows for "all possible" arbitrary subsets of any given set. And we have found reasons to suspect that the idea of arbitrary sets is impossible to pin down.

If so, we should expect this guiding thought to play the role of a "regulatory ideal," but we should not expect to obtain a fully closed theoretical system by its analysis. Some principles will turn out to be well grounded in the quasi-combinatorial ideal, a case in point being AC. But we should be prepared to find that there are questions we are not in a position to answer definitely. This may well be the case with Cantor's continuum problem, which, if so, would be asking us to specify matters beyond the limits of our conceptual possibilities. One has to insist that the continuum problem—which cardinal $\aleph_\alpha$ measures the size of $\wp(\omega)$ and hence R—boils down to a most basic question: is there an infinite subset of $\wp(\omega)$ which cannot be bijected with ω nor with $\wp(\omega)$ itself?

Mathematicians have been able to solve this question for an interesting class of parameter-definable subsets of $\wp(\omega)$ such as the Borel sets and the analytic sets (and even the whole hierarchy of projective sets under the assumption of Projective Determinacy) with the answer No, which of course is compatible with CH. But the problem is that, because of quasi-combinatorialism, we assume the existence of further subsets of $\wp(\omega)$. AC implies that there are sets of reals that *do not* have the perfect set property.

[55]As one might expect, there exists also the opposite tendency, from Hilbert [1925] to Hintikka [1999], to view AC as a logical principle. This is natural given the blurred boundaries between logic and mathematics, which induce a certain ambiguity of the term 'logic.' First-order logic already has the power to capture a weak notion of function, and if we strengthen this feature we can obtain forms of AC (Hintikka employs a rule of functional instantiation in IF-logic). However, because of the intimate link between AC and quasi-combinatorialism, in my opinion it is more appropriate to avoid making such moves (see also Ferreirós [2001], sec. 2.1 and 2.2).

Generally speaking, the strategy of descriptive set theory seems very good for enlarging the domain of sets for which CH has been established. It seems that the most one could expect from this approach is eventually to find a counterexample to Cantor's hypothesis. But it does not constitute an attempt to clarify and bring under mathematical control the postulation of arbitrary sets.

If *ZFC* falls short of specifying what is meant by combinatorial maximality, it is no wonder that it leaves unsettled the truth or falsehood of CH. This problem asks for exacting precision as to the reach or extent of the domain of sets introduced via quasi-combinatorialism, in the simplest case of $\wp(\omega)$. Cantor's hypothesis (or any of the alternatives) can then be an essential addition by helping make much more concrete and precise the vague assumption involved in the Powerset axiom. It should thus come as no surprise that CH or alternatives (like $2^{\aleph_0} = \aleph_2$ or, why not, $2^{\aleph_0} = \aleph_{\omega+1}$)[56] do significant work in further specifying the set-theoretic standpoint.

If the formal system *ZFC* did capture satisfactorily the idea of combinatorial maximality, one should expect it to settle the truth or falsehood of CH. I do not mean to say that a formal system capturing the quasi-combinatorial ideal should be complete.[57] What I am trying to point out here is quite independent of the phenomenon of formal incompleteness. It is perfectly conceivable that a deep analysis of the quasi-combinatorial ideal could lead to a formal theory that settles a question like CH, without violating incompleteness. I have in mind principles that would, so to speak, 'fix' that area of the theory which surrounds the proposition CH—an obvious candidate being CH itself as an axiom, but this would be a recourse to brute force. We desire something subtler, conceptually more penetrating; in particular we should aim to find principles that are strongly backed by the quasi-combinatorial ideal (just like AC). This kind of 'fixing' has happened in the past with many mathematical questions and results, and it is a natural kind of fact in mathematical experience.

On the other hand, considerations offered by the experts that lead to the suggestion that Cantor's problem is unsolvable should count as so many arguments for the impossibility of formally capturing the notion of arbitrary subsets, the idea of combinatorial maximality.[58] Which implies also, as we

[56] $2^{\aleph_0} = \aleph_2$ is a consequence of Martin's Maximum, and it is also favoured by Woodin [2001] on the basis of his research on Ω-logic and the "Ω-conjecture" of its completeness; it was favoured by Gödel in the past, and it is by other experts today, too, but by no means all of them. The matter remains controversial, especially because some of the crucial assumptions behind Woodin's highly technical and difficult work are introduced on grounds of simplicity and fertility, and are seemingly unrelated to the quasi-combinatorial ideal.

[57] That would give a trivial knock-down argument against ever being able to capture it via a formal system.

[58] Compare the views of a notable critic of the meaningfulness of the continuum problem: "The fact that it [CH] has not been settled by any remotely plausible assumption leads me,

have spelled out above, the impossibility to capture formally the idea of real number in its full intended generality. This of course should be striking for any mathematician, given the central role that R plays alongside N as a core object or structure of mathematics, pure and applied.

What can we suggest along more positive lines? The recommendation that suggests itself naturally, as a result of our discussion, is to try to move beyond AC in the analysis of arbitrary sets, along lines which (it seems, given the current state of the matter) ought to be orthogonal to the direction of Gödel's Program for large cardinals. Up to the present, large cardinal assumptions have not enabled the experts to specify the "thickness" of the set-theoretic universe, beyond the conflict of measurable cardinals with $V = L$, and so it seems natural to look for principles of a different kind. Perhaps one has to look at low levels of the cumulative hierarchy again. New principles entailing a richer structure for $\wp(N)$ are needed, or if you don't like this way of expression, we need new axioms—inspired in quasi-combinatorialism—that capture more precisely the extraordinarily rich structure of $\wp(N)$. We have seen that merely postulating that $\wp(N)$ is the set of "all" subsets of N is no precise specification, as it relies on what the rest of the axiom system specifies about this "all."

I suppose many people must have been looking for such principles, and failed. If the situation does not change, it will become natural to be sceptical about the postulation of powersets and related issues, such as the possibility of settling CH. It would even become natural to emphasize that R is a totality of a different kind than N. Of course, set theorists have tried to develop approaches leading to progress in determining the "thickness" of the universe (e.g., forcing axioms). I must leave it to the experts to determine how far such assumptions go, and whether they constitute answers to the kind of question I am proposing: formulate axioms inspired in quasi-combinatorialism that further specify the "richness" or "thickness" of the universe of sets.

If *ZFC* is poor as I claim, it seems quite noteworthy that, here too, a little bit goes a very long way (to borrow a favourite expression of Feferman's). For the poor expression of the quasi-combinatorial standpoint distilled in *ZFC* is still powerful enough to allow for an interpretation of almost all 20th-century mathematics.

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for one, to agree with Weyl that it is an inherently indefinite problem which will never be "solved"." (Feferman [1998], p. 73). One could add other voices, with opinions going one way or the other; see, e.g., Maddy [1988] and [1997].

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