3rd International Conference on Mechanical Models in Structural Engineering
CMMoST 2015
24-26 June 2015 Seville SPAIN

On the Insurmountable Size of Truss-like Structures

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ABSTRACT

Galileo postulated the existence of an insurmountable size for stone columns bearing a useful load as the size for which the structure is only able to resist its self-weight. Herein a method for the determination of the unsurmountable size for truss-like structures is shown, given the form of these structures and the ratio between the allowable stress and the specific weight of the material (the material structural scope). Three types of bars are considered: straight bars, with solid and hollow rectangular cross-section, and catenary bars with circular cross-section —a limit and theoretical case for estimating a meaningful upper bound of the structural scope—. An approximate rule to estimate the structural efficiency —here named GA rule— is shown, and is compared with numerical solutions using the proposed method.

Keywords: structural design, insurmountable size, structural scope, trusses, self-weight.

1. THE GALILEO PROBLEM AND THE AROCA RULE

In a first approximation, we can represent the physical cost of a structure by its self-weight, as many cost during the manufacturing, but not all, are approximately proportional to the self-weight of the structure: CO_2 emissions, mineral resources consumption, etc. For a given structural problem, we define the structural efficiency as the ratio between the useful load and the whole load (i.e., the useful load plus the self-weight) required to solve that problem in a particular structure.

Galileo [1] postulated the existence of insurmountable sizes for structures, as well as the relationship between the size of a structure and its ability to resist a useful load: let us imagine a cylindrical stone column, at the limit of its resistance only as a result of its self-weight; we name *structural scope* of the column to the height ($\mathcal{L}$) of this column, which cannot resist any additional load, hence being null its efficiency. A useful column with the same base must have therefore a height L smaller than $\mathcal{L}$. This new column can resist an additional useful load (Q), the value of which is at most the weight difference between the two columns. According to the previous definition of efficiency:

$$r \le \frac{\mathcal{L} - L}{\mathcal{L}} = 1 - \frac{L}{\mathcal{L}} = 1 - t \qquad (1)$$

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We define *structural* (or *relative*) *size* (t) as the ratio between the height L of a column and its scope $\mathcal{L}$. This structural size t can have values between 0 and 1. Note that Eq.(1), that we name Galileo's rule, is exact in the case of linear pieces with no tangential stress, but it is not proved that it would be a general rule. We define cost (k) as the inverse of efficiency, hence always higher than unity. Then, the self-weight of the column is:

$$P = (k-1)Q \qquad \text{with } k = 1/r \tag{2}$$

Let us define the *material structural scope* ($\mathcal{A}$), a characteristic length, as the ratio between the admissible stress and its specific weight [2]. This amount is the only information that we will need to obtain the structural scope of the structure itself, besides its geometry and the useful load distribution to be supported. In the case of cylindrical columns, the scope of the columns is the scope of the material: let f be the admissible stress of the material, and let A be the area of the cross-section, equating the strength capacity of the column base to the weight of the column of maximum height, we obtain $f\ A = \mathcal{L}\ A\ \rho$ (where ρ is the specific weight of the material). Then:

$$\mathcal{L} = \frac{f}{\rho} = \mathcal{A} \tag{3}$$

Maxwell [3] found the way to compare structural costs for a given structural problem. We name [4] *Maxwell problem* to the problem that consist of defining a structure capable of supporting a equilibrated set of external forces defined both in position and magnitude, and we name *Maxwell structure* to the structure that resolves a Maxwell problem and is built out of elements that work uniaxially, in tension or in compression. Maxwell showed that for two of those structures solving the same problem, the difference in cost is proportional to the difference in the *stress volume* $\mathcal{V}$ of each, defined as:

$$\mathcal{V} = \sum |e|\,\ell \tag{4}$$

where e is the value of the internal force in each element of the structure and ℓ its length.

Later Michell [5] showed that the self-weight of a structure is minimal if its stress volume $\mathcal{V}$ is also minimal. Also he found a necessary criterion so that a Maxwell structure would be an absolute minimum, from which are derived some optimal layouts for some specific problems. We have defined [6] Michell's number (ν) to the dimensionless ratio between the stress volume of a structure and the product of the total useful load times the size of the problem (the height in the case of columns, the span for beams, etc.):

$$\nu = \frac{\mathcal{V}}{QL} \tag{5}$$

Then a structural form is an absolute minimum when its Michell number is lesser or equal to any other structure solving the same Maxwell problem.

The contributions of Maxwell and Michell on the measure of structural efficiency do not take into account the self-weight, as it can be seen in the definition of Maxwell problem. Ricardo Aroca [2], [4], [6], [7], [8] joined the theories of Galileo and the theories of Maxwell and Michell: given a Maxwell problem and a structure with stress volume $\mathcal{V}$ that resolves it, the volume and the self-weight of such structure are:

$$V = \frac{\mathcal{V}}{\sigma} \qquad \mathcal{V} = \nu QL \qquad P = V\rho = \frac{\mathcal{V}}{\mathcal{A}} = \nu\frac{QL}{\mathcal{A}} \tag{6}$$

But previous expressions are only accurate for the null size, since they do not take into account that the self-weight must also be equilibrated. So for structures with $t > 0$:

$$\mathcal{V} \neq \nu QL \qquad P \neq \nu\frac{QL}{\mathcal{A}} \tag{7}$$

Usually the distribution of self-weight will be different to the useful load, but in many cases of interest (for example, in bending structures) both distributions can be represented by distributions of similar forces. We can then estimate the scope of the structural form as the size $L = \mathcal{L}$ for which the structure just resists its self-weight, without possibility of adding any additional load. It results then the following *approximate* expression, substituting the useful load Q by the self-weight P in the last side of last expression into Eq. (6):

$$P \approx \nu\frac{P\mathcal{L}}{\mathcal{A}} \qquad \Rightarrow \qquad \mathcal{L} \approx \frac{\mathcal{A}}{\nu} \tag{8}$$

These expressions are Aroca's rule. Substituting the last expression in Eq. (1), we have the following rule for the efficiency (which we name GA rule, honoring both Galileo and Aroca):

$$r \approx 1 - \nu\frac{L}{\mathcal{A}} \tag{9}$$

This estimate of the efficiency will be exact for linear structures whose distribution of self-weight is isomorphic to useful load distribution and whose equilibrium requires no tangential stress. In any other case it will be only an approximation whose usefulness must be proved.

2. THE GALILEO PROBLEM FOR TRUSS-LIKE STRUCTURES

Here we propose a similar approach to that described above, now for the case of trusses with useful load arranged as forces applied at the nodes, to obtain the structural scope of these structures, subjected to the following limitations: (i) bars are of constant cross-section; (ii) identical tension-compression patterns on bars due to both useful load and self-weight (i.e., equal sign of the internal force in each bar in these two independent load conditions); (iii) the material has the same absolute value of allowable stress in tension than in compression; and (iv) buckling of compressed bars is not taken into consideration. The aim is to obtain a first estimate of the structural scope for a given structural form, and the structural efficiency.

The method proposed here aim to include the cost of transmitting the distributed self-weight along the length of each bar to its extremes, because in large structure this cost will generally be important in relative terms. Rozvany [9] proposed an optimization method that include self-weight but that require bars with variable cross-section —excluded in our approach, limitation (i)—, in fact adopting an exponential function. Such form for bars (close to the so named 'constant maximum stress design') require a stress tensor that does not fulfil the differential equations of equilibrium [17], so the solutions obtained cannot be considered feasible solutions for the problem. Other authors [10], [11], [12], circumvent the problem of having to consider any bending effects including only half of the bar weight in each of its nodes, but then smart algorithms, as Simulated Annealing, will choose solutions with very large length, as the bending is free of cost [10]. The difficulties of tackling with self-weight in truss-like structures disappear in continuous structures using for example FEM [13], [14]. The selected limitations are justified for several reasons. (i) has a practical meaning. With (ii), we avoid special cases whereby the self-weight of a structure can be equilibrated by the external loading, as pointed out by Bendsoe [15]. With (iii) and (iv), we keep the model simple, but (iii) it is not difficult to overcome and (iv) maybe be suppressed in future research applying new results on this subject [16] .

2.1. Equilibrium equations with useful loads and self-weight

Let be $\mathbf{N_Q}$ the internal forces in a Maxwell structure under the action of the useful load $\mathbf{Q}$ (hereafter bold capital denote vector or arrays). Suppose that we have solved the problem of designing with bars that include their self-weight. Such bars represent an additional load due to its self-weight that we can introduce using statically equivalent forces at their ends, $\mathbf{P}$. The local equilibrium of the self-weight at interior points of the bar depends on the type of bar: for straight beams, on its bending; for cables or bars without bending stiffness, on the curvature of the bar itself. The Maxwell structure will have to develop additional internal forces for these new loads, $\mathbf{N_P}$. We can consider that the design problem is solved if it results that for each bar an axil internal force in the χ direction defined by its ends is developed, $N_\chi = N_Q + N_P$, and the bar is dimensioned to strictly resist the resulting stresses. The above (ii) limitation can now be expressed saying that $\operatorname{sgn}(N_P) = \operatorname{sgn}(N_Q)$, and hence $\operatorname{sgn}(N_\chi) = \operatorname{sgn}(N_Q)$.

The equilibrium equations are the same for both set of loads:

$$\mathbf{Q} = \mathbf{HN_Q} \qquad \mathbf{P} = \mathbf{HN_P} \qquad \mathbf{Q} + \mathbf{P} = \mathbf{HN}_\chi \tag{10}$$

Let ω_i be the ratio between the equivalent weight due to self-weight in the vertex i of the bar and the internal force, i.e., $\omega_i = P_i / N_\chi$. Then:

$$\mathbf{P} = \mathbf{\Omega}_L \mathbf{N}_\chi \tag{11}$$

In this expression $\mathbf{\Omega}_L$ is the matrix of coefficients ω_{ij} of each bar j for each component i of $\mathbf{P}$, depending on L, the size of the structure. Of course, each column of $\mathbf{\Omega}_L$ has only two non-null components. Therefore,

$$\mathbf{Q} + \mathbf{\Omega}_L \mathbf{N}_\chi = \mathbf{H}\mathbf{N}_\chi \qquad \mathbf{Q} = (\mathbf{H} - \mathbf{\Omega}_L)\mathbf{N}_\chi \tag{12}$$

Note that these equations are nonlinear, as $\mathbf{\Omega}_L$ depends on the size L of the structure and on the sign pattern of $\mathbf{N}_\mathbf{Q}$, which is given.

When $\mathbf{Q} \rightarrow \mathbf{0}$, i.e., when the structure cannot resist more than its self-weight (and its size L is then equal to the scope $\mathcal{L}$ of its form), results:

$$(\mathbf{H} - \mathbf{\Omega}_\mathcal{L})\mathbf{N}_\chi = \mathbf{0} \tag{13}$$

And the value of $\mathcal{L}$ is determined as the lowest value of L for which $(\mathbf{H} - \mathbf{\Omega}_L)$ is singular, excluding $L = 0$.

2.2. Beams (straight bars)

Let ℓ be the length of the bar, and β the angle formed by the bar and the horizontal, see Fig. 1. The expressions for the internal forces along the bar (tension is positive, compression is negative) are:

$$N(s) = N_\chi + \rho A \sin\beta \left(s - \tfrac{1}{2}\ell\right); \qquad M(s) = \tfrac{1}{2}\rho A \cos\beta \cdot s\,(\ell - s) \tag{14}$$

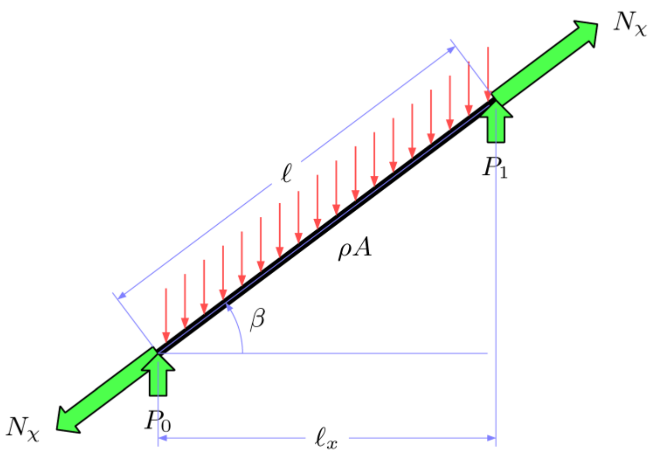


Figure 1. *Beam.*

We will use the following design conditions: the bars have defined the depth h and the radius of gyration i as fractions of the length ℓ of the bar: $h = k_1 \cdot \ell \qquad i = k_2 \cdot h = k_3 \cdot \ell$

The maximum normal stresses depending on s (position on the χ axis of the bar) are:

$$\sigma(s) = \frac{N(s)}{A} \pm \frac{M(s)}{W} \tag{15}$$

$$\sigma(s) = \frac{N(s)}{A} \pm \frac{hM(s)}{2Ai^2} = \frac{N(s)}{A} \pm \frac{k_1 M(s)}{2Ak_3^2 \ell} \tag{16}$$

$$A\sigma(s) = N(s) \pm \frac{2k_4 M(s)}{\ell}; \quad with\ k_4 = \frac{1}{4}\frac{k_1}{k_3^2} = \frac{1}{4k_1 k_2^2} \tag{17}$$

$$A\sigma(s) = N_\chi + \rho As\left(\left(1 - \frac{\ell}{2s}\right)\sin\beta \pm k_4\left(1 - \frac{s}{\ell}\right)\cos\ \beta\right) \tag{18}$$

The stress is maximum for $s = k_{5i}\ell$, with

$$k_{5i} = \frac{k_4 cos\ \beta \pm \sin\beta}{2k_4 \cos\beta} \tag{19}$$

$k_{5T} = \frac{1}{2}\left(1 + \frac{1}{k_4}\tan\beta\right)$, in the situation with lower compression or with higher tension (20)

$k_{5C} = \frac{1}{2}\left(1 - \frac{1}{k_4}\tan\beta\right)$, in the situation with lower tension or with higher compression (21)

which means that for β such that $\tan\beta \geq k_4$, the stress is maximum at the ends of the bar.

Taking $k_{6T} = \min\{1, k_{5T}\}$:

$$A\sigma_T = N_\chi + \rho A\ k_{6T}\left(\sin\beta\ \left(1 - \frac{1}{2k_{6T}}\right) + k_4 \cos\beta(1 - k_{6T})\right)\ \ell \tag{22}$$

By grouping the parameters corresponding to the bar, with the following definition of a new constant k_i:

$$k_i = k_{6T}\left(\sin\beta\ \left(1 - \frac{1}{2k_{6T}}\right) + k_4 \cos\beta(1 - k_{6T})\right) \tag{23}$$

we can write:

$$A\sigma_T = N_\chi + \rho A k_i \ell \tag{24}$$

and similarly:

$$A\sigma_C = N_\chi - \rho A k_i \ell \tag{25}$$

Being f the allowable stress of the material; making $\sigma_T = -\sigma_C = f$ to select the appropriate constant cross-sectional area A, it results, depending on the sign of N_χ:

$$\frac{A}{N_\chi} = \pm\frac{1}{f-\rho k_i \ell} \tag{26}$$

The coefficients ω_i of the matrix $\mathbf{\Omega}$ are:

$$\omega_i = \pm\rho\frac{\ell}{2}\frac{A}{N_\chi} = \mathrm{sgn}(N_Q)\frac{1}{2\left(\frac{\mathcal{A}}{\ell}-k_i\right)} \tag{27}$$

As the Eq. (18) only considers the component σ_χ of the stress tensor with the model based on the hypothesis of Navier only appropriate for very slender beams, for which the effects of Saint Venant's principle can be neglected, the expression in Eq. (27) is a good approximation for the self-weight of very slender beams. We keep us on this simple model in this initial work for the sake of simplicity. More accurate models will be used in future research.

2.2.1. Bars with rectangular section

With rectangular section, $k_2 = 1/\sqrt{12}$, being the width b the *free* design parameter. The cross-sectional area is $= h \cdot b = k_1 b \cdot \ell$. Using this value of k_2 to calculate k_i, from Eq.(27) we can calculate the values of ω_i for given values of k_1.

2.2.2. Bars with hollow rectangular section

The radius of gyration of a hollow rectangular section with depth h, width b and thickness t is:

$$i = \sqrt{\frac{12ht^3 - 6h^2t^2 - 6bht^2 + 3bh^2t - 8t^4 + 4bt^3 + h^3t}{-24t^2 + 12ht + 6bt}} \tag{28}$$

Eliminating terms with powers of t, we obtain the value of i when $t \to 0$:

$$i = \sqrt{\frac{3bh^2 + h^3}{12(h+b)}} \tag{29}$$

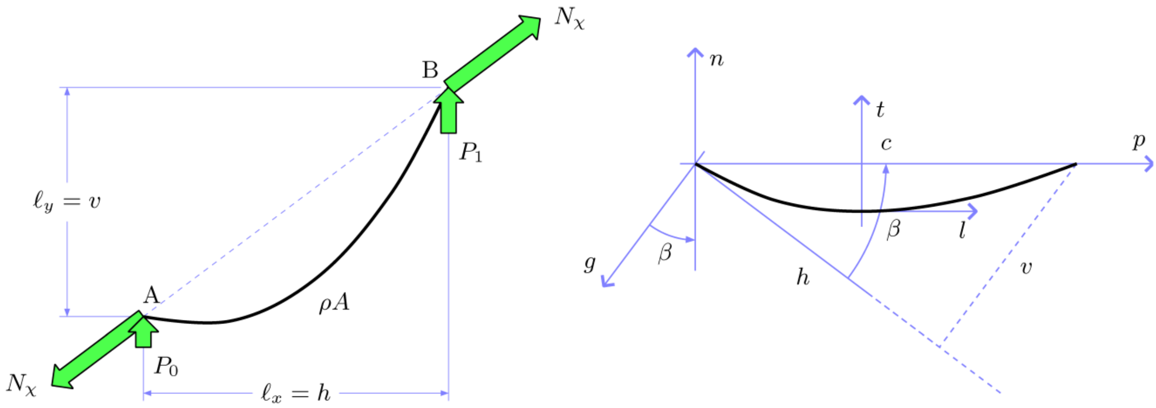
Taking for this case as an additional design decision that the width b is proportional to the depth ($b = k_b h$), resulting the value of k_2:

$$k_2 = \frac{1}{\sqrt{12}}\sqrt{\frac{3k_b + 1}{k_b + 1}} \tag{30}$$

being the thickness t the free design parameter. Now for given values of k_1 and k_b , we can take the area of the section as $A = 2(h+b)t = 2(1+k_b)k_1 t \cdot \ell$, and using this value of k_2 we can calculate the values of ω_i.

2.3. Catenary bars of constant cross-section

Among all the alternatives for design with constant cross-section and self-weight, the catenary arc, see Figure 2(a), is the better known in respect to efficiency, because there is no tangential stress involved [17].



(a) xy axes (b) Local axes

Figure 2. *Catenary arc.*

Of course this is a theoretical solution, very difficult to build in practice. But as the optimal solutions from Michell's theory –in fact, funicular structures and hence intrinsic instable ones– their study lead to theoretical limits that no other solution with constant cross-section can exceed.

Let p, n be axes such that axis p follows direction χ, see Figure 2(b). The gravity axis is g, so β is the angle formed by horizontal axis and p. The chord of the catenary arc is c, its height is v, and its base is h ($h = c\cos\beta$; $v = c\sin\beta$).

The basic equation result from the equilibrium of a differential arc $\mathrm{d}s$: the variation of the internal force plus the weight must be null:

$$\mathrm{d}\,\vec{N} + \rho A(-\sin\beta, -\cos\beta)\mathrm{d}s = \vec{0} \tag{31}$$

and it can be integrated as:

$$\vec{N} = \rho A(s\sin\beta + K_1, s\cos\beta + K_2) \tag{32}$$

Let s_0 be the point of the curve with parallel tangent to p axis:

$$\vec{N}(s_0) = N(s_0)\cdot(1{,}0) = \rho A(s_0\sin\beta + K_1, s_0\cos\beta + K_2) \tag{33}$$

Furthermore, $N(s_0)$ is the oblique component of the internal force, O hereafter, i.e., the component in the chord direction for the oblique p, g axes, i.e. N_χ. Hence:

$$\vec{N}(s) = (N_p, N_n) = \rho A((s - s_0)\sin\beta + k, (s - s_0)\cos\beta), \quad \text{with } k = \frac{O}{\rho A} \tag{34}$$

Now, changing to the l, t axes of the figure, $s_0 = 0$, $l(0) = 0$, y $t(0) = 0$:

$$\vec{N}(s) = (N_l, N_t) = \rho A(s\sin\beta + k, s\cos\beta), \quad \text{with } k = \frac{O}{\rho A} \tag{35}$$

Note the definition of the s coordinate: s grows with the concavity on the left, i.e., in the growing direction of $\mathrm{d}t/\mathrm{d}l$. In the figure, s grows to the right and the concavity points up. As a result, c will be negative for a compression arch, accordingly with the definition of h and v as function of c and β above.

Defining $\tan\alpha$ as the slope of the tangent to the curve,

$$\tan\alpha = \frac{N_t}{N_l} \qquad \cos\alpha = \frac{N_l}{\sqrt{N_l^2 + N_t^2}} \qquad \sin\alpha = \frac{N_t}{\sqrt{N_l^2 + N_t^2}} \tag{36}$$

and the parametric equations for this curve can be obtained by integration :

$$l(s) = \int \cos\alpha \,\mathrm{d}s \qquad t(s) = \int \sin\alpha \,\mathrm{d}s \tag{37}$$

Defining

$$\phi_1(s) = k\sinh^{-1}\left(\frac{s + k\sin\beta}{k\cos\beta}\right) \qquad \phi_2(s) = \sqrt{k^2 + s^2 + 2ks\sin\beta)} \tag{38}$$

The parametric equations are

$$l(s) = \big(\phi_1(s) - \phi_1(0)\big)\cos^2\beta + \big(\phi_2(s) - \phi_2(0)\big)\sin\beta \tag{39}$$
$$t(s) = -\big(\phi_1(s) - \phi_1(0)\big)\sin\beta\cos\beta + \big(\phi_2(s) - \phi_2(0)\big)\cos\beta \tag{40}$$

Note that the limit of $l(s)$ for $\pm\pi/2$ is $\pm(s - |k|) + k$, and the limit of $t(s)$ is null (a vertical line).

The modulus of the axial force is:

$$|\vec{N}(s)| = \rho A\sqrt{k^2 + s^2 + 2ks\sin\beta} = \left|\frac{O}{k}\right|\phi_2(s) \tag{41}$$

The coordinates s_1 and s_2 are determined by the fact that the ends of the arc must be the points (l_1, t_1) and (l_2, t_2) with the conditions $l_2 - l_1 = c$ and $t_1 = t_2$, i.e.:

$$l(s_1) = l_1, \quad t(s_1) = t_1; \qquad l(s_2) = l_1 + c, \quad t(s_2) = t_2; \tag{42}$$

Accordingly, defining $\Phi_i(s) = \phi_i(s) - \phi_i(0)$:

$$(\Phi_1(s_2) - \Phi_1(s_1)) \cos^2 \beta - (\Phi_2(s_2) - \Phi_2(s_1)) \sin \beta = c \quad (43)$$
$$\{(\Phi_1(s_2) - \Phi_1(s_1)) \sin \beta + (\Phi_2(s_2) - \Phi_2(s_1))\} \cos \beta = 0 \quad (44)$$

The failure criterion is the additional equation for determining k. For example, if $k > 0$ y $\beta > 0$, the maximum slope with the horizontal line is at s_2 and $N(s_2) = fA$ for strictly choosing the value of the constant cross-sectional area A:

$$\sqrt{k^2 + s_2^2 + 2ks_2 \sin \beta} = \mathcal{A} \quad (45)$$

Hence, in general

$$\sqrt{k^2 + s_i^2 + 2ks_i \sin \beta} = \mathcal{A} \qquad \text{with } i = 1 \text{ or } 2 \quad (46)$$

$\{k, s_1, s_2\}$ is the solution of {(43), (44), (46)}. In this way, k, s_1, s_2 can be determined numerically for given values of $|c|$, β, $\mathcal{A}$ and $\mathrm{sgn}(N_Q)$.

The weights on the Maxwell structure are determined with the ω coefficients:

$$\omega_{p=0} = \frac{P_{p=0}}{N_\chi} = -\frac{s_1}{k} \qquad \omega_{p=c} = \frac{P_{p=c}}{N_\chi} = \frac{s_2}{k} \quad (47)$$

3. AN EXAMPLE

We will use the example from Figure 3 as a study case.

The equilibrium equations (10) at nodes $1, 2, 3$ with point loads (H_i, V_i) and the equations (11) relating coefficients ω with weights P and oblique internal forces O are:

$$\begin{bmatrix} H_1 \\ V_1 \\ H_2 \\ V_2 \\ H_3 \end{bmatrix} = \overbrace{\begin{bmatrix} K_1 & -K_1 & -1 & 0 & 0 \\ -K_2 & -K_2 & 0 & 0 & 0 \\ 0 & K_1 & 0 & 1 & -1 \\ 0 & K_2 & 0 & 0 & 0 \\ -K_1 & 0 & 0 & -1 & 0 \end{bmatrix}}^{\mathbf{H}} \begin{bmatrix} O_j \\ O_p \\ O_{cs} \\ O_t \\ O_{ci} \end{bmatrix} \quad \text{with } K_1 = a/\sqrt{a^2 + h^2}\,; \quad K_2 = h/\sqrt{a^2 + h^2}$$

$$\begin{bmatrix} 0 \\ P_1 \\ 0 \\ P_2 \\ 0 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \omega_{j1} & \omega_{p1} & \omega_{cs1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_{p2} & 0 & \omega_{t2} & \omega_{ci2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}^{\mathbf{\Omega}} \begin{bmatrix} O_j \\ O_p \\ O_{cs} \\ O_t \\ O_{ci} \end{bmatrix} \quad (48)$$

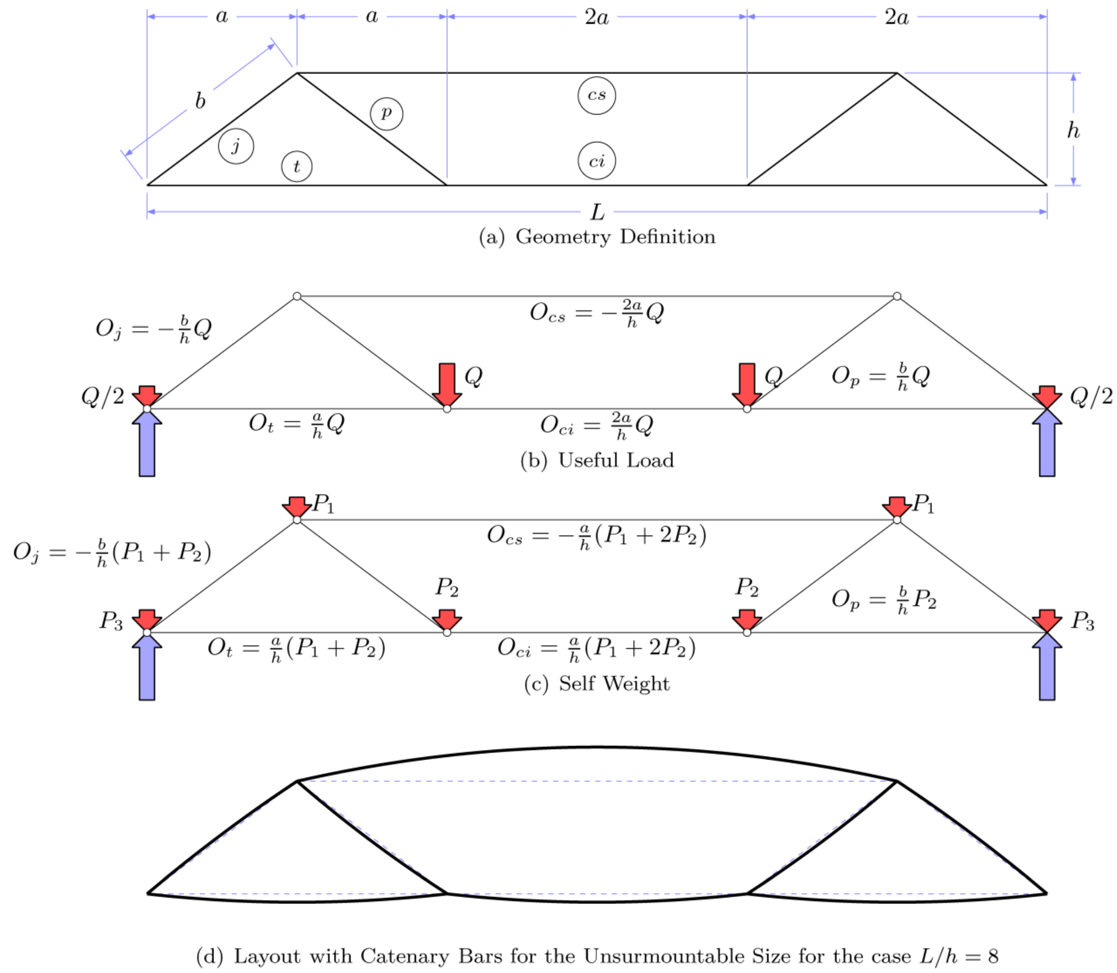


Figure 3.

To obtain the expression of the efficiency we need besides to know the value of P_3:

$$P_3 = \omega_t N_t + \omega_j N_j \tag{49}$$

Hence $r = Q/(Q + P_1 + P_2 + P_3)$.

To obtain the structural scope $\mathcal{L}$ we must solve the equation $|\mathbf{H} - \mathbf{\Omega}| = 0$, determining the value of coefficients ω for each type of bar. In the case of straight bars, for given values of h/l for solid and hollow rectangular cross-sections and additionally for given values of b/h for hollow cross-sections, we determine directly (Eq. 27) the values of such coefficients ω_i, that depend only on $L/\mathcal{A}$. From the solutions of the equation, we choose the lowest real value, which corresponds to the value $\mathcal{L}/\mathcal{A}$ for the form. In the case of catenary bars, for each step of an iterative process depending on the value of L, we calculate numerically the value of the coefficients ω_{ij} (Eq. 47) for each bar, working with lengths relatives to $\mathcal{A}$, and the resulting solution is the value of $L/\mathcal{A}$ for which $|\mathbf{H}-\mathbf{\Omega}|=0$. The sign of the coefficients ω is chosen according to the sign of the internal forces due to useful loads $\mathbf{Q}$.

3.1. A concrete case

We resolve the concrete problem of Figure 3 when the depth/span ratio is $1/8$, i.e. the proportion or slenderness λ of the structure equal to 8.

Table 1 shows the results for different cases of solid and hollow rectangular sections, beside the case of catenary bars, obtained with the proposed method. It is also include real values of structural scope in each case for a normal steel.

From the definition (5) and data from Figure 3 the Michell number in this case is $\nu=89/54$ and following GA rule the result is $\mathcal{L}/\mathcal{A} \approx 1/\nu = 0.6067$, and the scope for steel is 1,395 m. Comparing with data of Table 1, it is clear that GA rule offers a correct order of magnitude, but not a precise figure for the structural scope (the insurmountable size).

Table 1. Scopes for the structure of the example with $\boldsymbol{\lambda}=\mathbf{8}$. In the right column appear the values of the scope using normal steel with 2.3 km of material structural scope

h/l	*b/h*	$\mathcal{L}/\mathcal{A}$	$\mathcal{L}(\text{steel})(\text{m})$
Rectangular cross-sections, straight bars			
0.10		0.177	407
0.20		0.299	688
Hollow rectangular cross-sections, straight bars			
0,10	0.50	0.264	607
0,10	1.00	0.299	688
0.20	0.50	0.397	913
0.20	1.00	0.430	989
Circular cross-section, catenary bars			
		0.662	1,523

3.2. A family of cases: the structural schema

Given one truss–like structural form for an equilibrated set of external forces, one may consider affine changes in one specific direction of that layout [18] . In the study case of Figure 3, we select the perpendicular direction to the span L, hence that kind of transformations corresponds to changes in depth in the drawn structural layout sustaining the useful load, i.e. the drawing of that figure is a representative of an infinite family of structural forms in which only the h/L ratio varies. In this manner each value of the slenderness $\lambda=L/h$ and each size L select a member of the set. We name this set *structural schema*, following prof. Ricardo Aroca , who has proposed in his doctorate courses —never formally published— to consider the structural form as composite of four fundamental properties: size L, proportion or slenderness λ, schema (the drawing of Figure 3 interpreted without any size and with variable h), and thickness (i.e., the cross-sectional area of the bars in our case study). Our aim now is search for some fundamental relationships among slenderness, insurmountable size and efficiency of the structural schema derived from figure 3.

From Eq.(5), Michell's number for this structural schema is:

$$\nu(\lambda) = \frac{5\lambda}{27} + \frac{12}{9\lambda} \tag{50}$$

and equating its derivative to zero, we get the optimal slenderness for the case of null span L:

$$\lambda_{opt}(L = 0) = \frac{6}{\sqrt{5}} = 2.6833 \tag{51}$$

The GA rule predicts an insurmountable size for this slenderness of $1.006\mathcal{A}$. Is this value accurate? Not too much: with the proposed method the correct value is $1.1849\mathcal{A}$ for this slenderness. Is this slenderness the corresponding one to the maximum span for any form into this schema? The answer is no again: with a slenderness slightly lesser (2.455) we get an insurmountable size slightly greater of $1.1851\mathcal{A}$.

The following question arises: is the GA rule a good or useful rule? To discuss this issue, we draw the Figure 4. We picture two cases: $\lambda = 8$ and $\lambda = 2.455$ (the optimal slenderness from the point of view of the maximum insurmountable size of the schema). Apart of the numerical calculations with the proposed method (thick lines) and the GA rule (thin lines), we draw too Galileo's rule using the insurmountable size from the proposed method ("modified GA rule", thin, dashed lines).

From this figure, it is clear that GA rule underestimates the scope so that overestimates cost in poor efficient structures (with low useful load), but it is a superb approximation for small structures, i.e., efficient structures, the structures the practitioners would wish to use. Note, as an example, that most famous suspended bridges at their epoch have been of a structural size at most a 10% with respect to their insurmountable size relative to the structural scope of the material which they were built [17].

Can we improve the GA rule? A simple attempt consists in using Galileo's rule with the insurmountable size from the proposed method: this new approximation underestimate the cost in all cases, so it is worse than the original one, but yet a correct approximation of the order of magnitude.

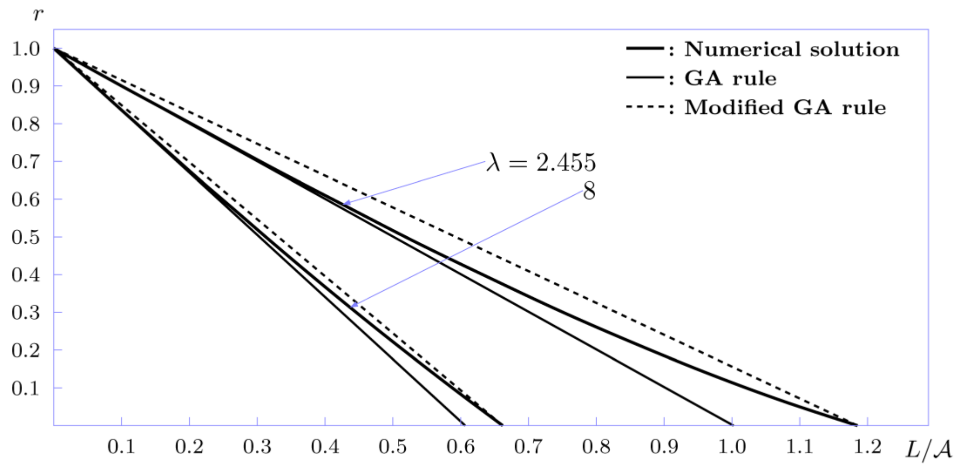


Figure 4. *Comparing obtained results of efficiency with results applying GA and modified GA rules for two slenderness cases of the example.*

Figure 5 shows a graphic relating the values of insurmountable size for the truss with catenary bars (thick line) and for the truss with beams (dashed lines) as function of λ with the function obtained from the GA rule (thin line). It results a good approximation for estimate the scope, better for higher slenderness, but even in the case of lower slenderness (near to optimal), the GA rule anticipates the value of this optimum with a great accuracy. A drawback of the GA rule is that the optimal slenderness does not varies with the span L, i.e., it does not account the difference between the optimal slenderness for $t = 0$ and $t = 1$ that we found with the proposed method: this drawback is inherent to the formulation of the rule.

It is worth of noting that GA rule does not approximate in any manner the case of beams. This is because GA rule only accounts the stress volume of the average value of normal stress in the direction χ. A similar rule appropriate for trusses formed with beams must be formulate accounting that the normal stress varies with bending due to self-weight, and the corresponding stress volume will be then greater, as from definition (4) we must integrate the absolute value of the stress times the differential of volume.

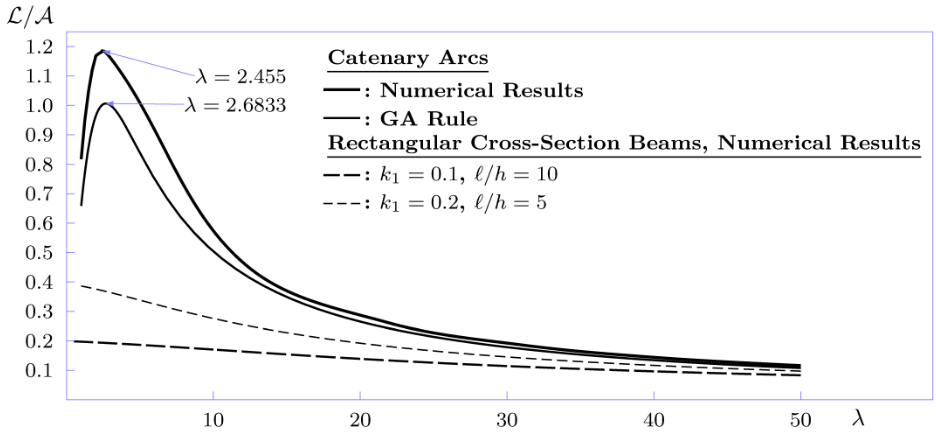


Figure 5. *Scopes for the schema of the example depending on slenderness.*

4. THE PRACTICAL USEFULNESS OF GALILEO'S RULE

Recall the concrete case of 3.1 and Table 1. Let us suppose that a solution for the Maxwell problem of Figure 3(b) is to be found with the geometry of Figure 3(a) working with normal steel and solid rectangular cross-section. We cannot use now GA rule, but we can obtain order of magnitude with Galileo rule and the structural scope in Table 1.

If the size of the problem is 50 m, we can estimate that the efficiency of the structure will be approx. $1 - 50/407 = 0.88$, i.e., 88%; further more we can estimate the self-weight as $1 - 1/0.88 = 0.14$, i.e., up to 14% of the useful load of the problem. So we can try a preliminary design with a total load of 1.14 times useful load using standards design procedures.

But if the size of the problem would be 150 m, for example, the figures would be now 62% and 59%, i.e., the self-weight of the structure is more than half the useful load and probably we would feel compelled to search for a better form, perhaps reducing the original slenderness of 8, as a way of reducing the cost of the solution.

Of course, if the size is small enough, say 10 m, the structural size will be 0.025, the efficiency of 98%, and the self-weight negligible (a standard assumption for real, not large structures). Galileo's rule justifies in this manner the normal assumption about self-weight in normal structures built with contemporary structural materials. But the rule, qualitatively, also it is important when we tackle with historic or contemporary structures built with material of small strength, like brickwork (masonry), earth-brick (adobe), etc. For example, earth-brick has structural scope about 20 m (although it varies very much depending on the quality of the mud mixture used), hence in the case of a medieval castle with walls of 12 m height, the self-weight can represent 67% of the total load, providing that the design would be strict. Actually, as the medieval design was not strict in any way, the self-weight frequently is the unique load worth of consideration, as the roof and floor loads are negligible generally.

Let us outline that the above procedure consist in a few and very simple calculations at the first step of the design procedure, providing we have at our disposal the kind of data of the Table 1 for the Maxwell problem under consideration. The teaching of this practice and the associated design theory are included into the curriculum for graduate students of the School of Architecture of Madrid since three decades ago.

5. CONCLUSIONS

In this paper we have presented a numerical method to deal with the self-weight load before the cross-sections of its bars are known. When applied to well definite Maxwell problems the method can be used to resolve concrete structural form determining the thickness of its bars, its total self-weight and its structural efficiency. Besides, the method can be used to explore the space of solutions associated to the Maxwell problem and to a definite structural schema, determining the relationships between size, slenderness, insurmountable size, and efficiency of each solution in the search space.

With this method we have checked the GA rule (i.e., Galileo-Aroca rule), an approximated but simple formulation to determine the insurmountable size and efficiency of structural forms or schemata, outlining its advantages and drawbacks, and suggesting further research to resolve the latter ones.

The proposed method could be improved in several ways: (i) incorporating a more realistic model of the stress distribution for beams (straight bars with bending); (ii) managing different values for tension and compression allowable stress; (iii) considering the local buckling of beams and compressed catenary arcs. In this way, this work can be considered as a reconsideration of the seminal problem of a theory of structural design, formulated by Galileo in 1638, and the approximate solution envisaged by Ricardo Aroca for first time in the eighties of the last century. We hope that this work can contribute to new and promising researches in the future.

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