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A conforming mixed flnite element method for the coupling of °uid °ow with porous media °ow

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Abstract

We consider a porous media entirely enclosed within a °uid region, and present a well posed conforming mixed flnite element method for the corresponding coupled problem. The interface conditions refer to mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law, which yields the introduction of the trace of the porous media pressure as a suitable Lagrange multiplier. The flnite element subspaces deflning the discrete formulation employ Bernardi-Raugel and Raviart-Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for the Lagrange multiplier. We show stability, convergence, and a priori error estimates for the associated Galerkin scheme. Finally, we provide several numerical results illustrating the good performance of the method and conflrming the theoretical rates of convergence.

1 Introduction

The interest in developing e–cient numerical methods for approximating the solution to the coupling of °uid °ow (modelled by the Stokes equation) with porous media °ow (modelled by the Darcy equation) has been increasing lately (see, e.g. [3], [6], [9], [12], and the references therein). In particular, the mathematical theory and the associated numerical analysis of a mixed variational formulation was recently provided in [9]. There, the coupling across the interface is determined by the Beavers-Joseph-Saffman conditions, which yields the introduction of the trace of the porous media pressure as a suitable Lagrange multiplier. In addition, well posedness of the corresponding continuous formulation and a detailed analysis of a nonconforming mixed flnite element method are given in [9]. We

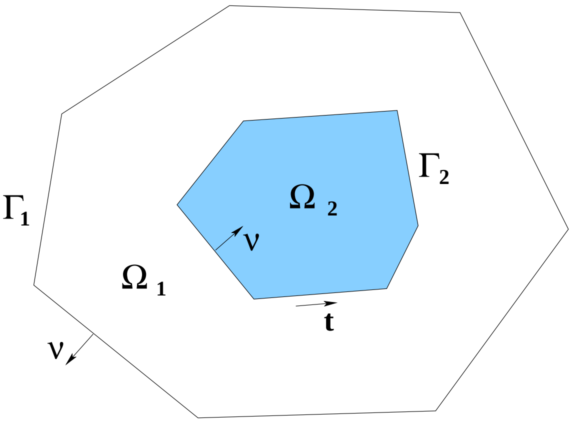

Figure 1: Geometry of the problem.

remark that the nonconformity of this discrete scheme arises from the fact that the Lagrange multiplier is approximated by piecewise constants functions, which are certainly not contained in the Sobolev space for the traces on the interface. A similar formulation to [9] is studied in [6].

In this paper we consider for simplicity a particular case of the model from [9], which is given by a porous media entirely enclosed within the fluid region, and introduce a new conforming mixed finite element method. Up to the author's knowledge, the method proposed here is the first one which is conforming for the original formulation in [9] (see also (2.1) below). Other conforming methods are proposed in [9], but for an alternative formulation. Now, in order to describe the geometry we let Ω_2 be a bounded and simply connected domain in R^2 with polygonal boundary Γ_2, and let Ω_1 be the annular region bounded by Γ_2 and another closed polygonal curve Γ_1 whose interior contains $\overline{\Omega}_2$ (see Figure 1). Then, the transmission problem consists of an incompressible viscous fluid occupying Ω_1, which flows back and forth across Γ_2 into a porous media living in Ω_2 and saturated with the same fluid.

In what follows, $\mu > 0$ is the viscosity of the fluid and K is a symmetric and uniformly positive definite tensor in Ω_2 representing the permeability of the porous media divided by the viscosity. We also assume that there exists $C > 0$ such that $\|K(x)\,z\| \le C\,\|z\|$ for almost all $x \in \Omega_2$, and for all $z \in R^2$. Then, the constitutive equations are given by the Stokes and Darcy laws, respectively, that is

$$\sigma_1(u_1; p_1) \;=\; -\,p_1\,I \,+\, 2\,\mu\,e(u_1) \quad \text{in} \quad \Omega_1\,; \quad \text{and} \quad u_2 \;=\; -\,K\,\nabla\,p_2 \quad \text{in} \quad \Omega_2\,;$$

where $(u_1; u_2)$ and $(p_1; p_2)$ denote the velocities and pressures in the corresponding domains, I is the identity matrix of $R^{2\times 2}$, $\sigma_1(u_1; p_1)$ is the stress tensor, and

$$e(u_1) \;:=\; \frac{1}{2}\left(\nabla\,u_1 + (\nabla\,u_1)^t\right)$$

is the strain tensor. Hereafter, given any normed space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. Also, the superscript ^{t} stands for the transpose matrix. Hence, given $f_1 \in [L^2(\Omega_1)]^2$ and $f_2 \in L^2(\Omega_2)$ such that $\int_{\Omega_2} f_2 = 0$, the coupled problem reads: Find (u_1, u_2) and (p_1, p_2) such that

$$\left\{\begin{array}{rcllll} -\operatorname{div}\sigma_1(u_1,p_1) & = & f_1 & \text{in} & \Omega_1 & \text{(conservation of momentum)}\,; \\ \operatorname{div} u_1 & = & 0 & \text{in} & \Omega_1 & \text{(conservation of mass)}\,; \\ u_1 & = & 0 & \text{on} & \Gamma_1 & \text{(no slip)}\,; \\ \operatorname{div} u_2 & = & f_2 & \text{in} & \Omega_2 & \text{(conservation of mass)}\,; \\ u_1\cdot\nu & = & u_2\cdot\nu & \text{on} & \Gamma_2 & \text{(conservation of mass)}\,; \\ \big(\sigma_1(u_1,p_1)\nu\big)\cdot\nu & = & -\,p_2 & \text{on} & \Gamma_2 & \text{(balance of normal forces)}\,; \\ -\frac{\kappa}{\nu}\big(\sigma_1(u_1,p_1)\,\nu\big)\cdot t & = & u_1\cdot t & \text{on} & \Gamma_2 & \text{(Beavers-Joseph-Saffman law)}\,; \end{array}\right. \tag{1.1}$$

where ν is the unit outward normal to Ω_1, t is the tangential vector on Γ_2, $\kappa > 0$ is the friction constant, and the Beavers-Joseph-Saffman law establishes that the slip velocity along Γ_2 is proportional to the shear stress along Γ_2 (assuming also, based on experimental evidences, that $u_2\cdot t$ is negligible). We refer to [2], [8], and [13] for further details on this interface condition.

Throughout the rest of the paper we utilize the standard terminology for Sobolev spaces, norms, and seminorms, employ 0 to denote a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

2 The continuous formulation

We put $\Omega := \Omega_1 \cup \Gamma_2 \cup \Omega_2$ and define the spaces

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_{\Omega} q = 0 \right\},$$

$$[H^1_{\Gamma_1}(\Omega_1)]^2 := \left\{ v_1 \in [H^1(\Omega_1)]^2 : \; v_1 = 0 \text{ on } \Gamma_1 \right\},$$

and

$$H(\operatorname{div};\Omega_2) := \left\{ v_2 \in [L^2(\Omega_2)]^2 : \; \operatorname{div} v_2 \in L^2(\Omega_2) \right\}.$$

In addition, we let

$$\mathbf{H} := [H^1_{\Gamma_1}(\Omega_1)]^2 \times H(\operatorname{div};\Omega_2) \quad \text{and} \quad \mathbf{Q} := L_0^2(\Omega) \times H^{1/2}(\Gamma_2)$$

endowed with the product norms $\|v\|_{\mathbf{H}} := \|v_1\|_{[H^1(\Omega_1)]^2} + \|v_2\|_{H(\operatorname{div};\Omega_2)}$ for all $v := (v_1, v_2) \in \mathbf{H}$, and $\|(q,\xi)\|_{\mathbf{Q}} := \|q\|_{L^2(\Omega)} + \|\xi\|_{H^{1/2}(\Gamma_2)}$ for all $(q,\xi) \in \mathbf{Q}$. Also, we denote $u := (u_1, u_2)$, $p := \left\{\begin{array}{ll} p_1 & \text{in } \Omega_1 \\ p_2 & \text{in } \Omega_2 \end{array}\right.$, and introduce the Lagrange multiplier

$$\lambda := p_2 = -\big(\sigma_1(u_1,p_1)\nu\big)\cdot\nu \quad \text{on} \quad \Gamma_2.$$

Hence, proceeding as in [9], we find that the mixed variational formulation of (1.1) reads: Find $(\mathbf{u},(p,\varphi)) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},(p,\varphi)) &= \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1 \qquad \forall\, \mathbf{v} := (\mathbf{v}_1,\mathbf{v}_2) \in \mathbf{H}, \\ b(\mathbf{u},(q,\xi)) &= -\int_{\Omega_2} f_2\, q \qquad \forall\, (q,\xi) \in \mathbf{Q}, \end{aligned} \tag{2.1}$$

where $a : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ and $b : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$ are the bilinear forms defined by

$$a(\mathbf{u},\mathbf{v}) := 2\,\mu \int_{\Omega_1} \mathbf{e}(\mathbf{u}_1) : \mathbf{e}(\mathbf{v}_1) + \frac{\nu}{\kappa} \int_{\Gamma_2} (\mathbf{u}_1 \cdot \mathbf{t})\,(\mathbf{v}_1 \cdot \mathbf{t}) + \int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2 \cdot \mathbf{v}_2\,,$$

$$b(\mathbf{v},(q,\xi)) := -\int_{\Omega_1} q\,\mathrm{div}\,\mathbf{v}_1 - \int_{\Omega_2} q\,\mathrm{div}\,\mathbf{v}_2 + \langle \mathbf{v}_1 \cdot \boldsymbol{\nu} - \mathbf{v}_2 \cdot \boldsymbol{\nu}, \xi \rangle_{\Gamma_2}\,,$$

with $\langle \cdot,\cdot \rangle_{\Gamma_2}$ being the duality pairing of $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$ with respect to the $L^2(\Gamma_2)$-inner product.

We employ the classical Babuška-Brezzi theory to prove that (2.1) is well posed.

Theorem 2.1 *There exists a unique* $(\mathbf{u},(p,\varphi)) \in \mathbf{H} \times \mathbf{Q}$ *solution to* (2.1)*. In addition, there exists* $C > 0$*, depending on* β*,* α*, and the boundedness constants for* a *and* b*, such that*

$$\|(\mathbf{u},(p,\varphi))\|_{\mathbf{H}\times\mathbf{Q}} \le C \left\{ \|\mathbf{f}_1\|_{[L^2(\Omega_1)]^2} + \|f_2\|_{L^2(\Omega_2)} \right\}.$$

3 The Galerkin formulation

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be regular triangulations of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, by triangles T of diameter h_T, and assume that the vertices of $\mathcal{T}_1$ and $\mathcal{T}_2$ coincide on the interface Γ_2. We let $h := \max\{h_1, h_2\}$, where $h_i := \max\{\, h_T : T \in \mathcal{T}_i\}$ for each $i \in \{1,2\}$. Then, for each $T \in \mathcal{T}_2$ we let $RT_0(T)$ be the local Raviart-Thomas space of lowest order, that is

$$RT_0(T) := \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\},$$

where $\mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a generic vector of R^2. In addition, for each $T \in \mathcal{T}_1$ we let $BR(T)$ be the local Bernardi-Raugel space (see [4], [7]), that is

$$BR(T) := [P_1(T)]^2 \oplus \mathrm{span}\{\lambda_2\lambda_3\boldsymbol{\eta}_1, \lambda_1\lambda_3\boldsymbol{\eta}_2, \lambda_1\lambda_2\boldsymbol{\eta}_3\},$$

where $\{\lambda_1, \lambda_2, \lambda_3\}$ are the baricentric coordinates of T, and $\{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3\}$ are the unit outward normals to the opposite sides of the corresponding vertices of T. Hereafter, given a non-negative integer k and a subset S of R^2, $P_k(S)$ stands for the space of polynomials defined on S of degree $\le k$. Hence, we define the following finite element subspaces for the velocities and the pressure:

$$H_{h_1} := \left\{ v \in [C(\bar{\Omega}_1)]^2 : \ v|_T \in BR(T) \quad \forall\, T \in \mathcal{T}_1 ; \quad v = 0 \text{ on } \Gamma_1 \right\} ;$$

$$H_{h_2} := \left\{ v \in H(\mathrm{div}; \Omega_2) : \ v|_T \in RT_0(T) \quad \forall\, T \in \mathcal{T}_2 \right\} ;$$

$$Q_h := \left\{ q \in L^2(\Omega) : \ q|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_1 \cup \mathcal{T}_2 \right\} ; \qquad Q_{h;0} := Q_h$$