Regularization and relaxation tools for interface coupling

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Resumen

We analyze a relaxation method for approximating the coupling of two Euler systems at a fixed interface and more generally for approximating fluid systems.

1. Introduction

We have been considering, in a series of papers, the coupling of systems, both from a theoretical and from a numerical point of views. The actual problem arises from the coupling of codes which model liquid-vapor flows and the systems under consideration are systems of conservation laws of hyperbolic nature.

The coupling problem might be interpreted as solving conservation laws with discontinuous coefficients, in which case the flux is assumed to be continuous at the interface. Many works have been recently devoted to the study of conservation laws with discontinuous coefficients, see [1] [4] and the references therein. These conservative approaches involve rather naturally some interface entropy condition along the line of discontinuity.

However, we have followed an approach where the coupling is a priori non conservative, which we have named *state coupling* since (in general) it ensures the continuity or *transmission* of the state variables, as opposed to *flux coupling*. In some cases the two may coincide as we will see. In state coupling, we may chose to *transmit* the conservative variables say $\mathbf{u}$, or any set of state variables $\mathbf{v}$ such that $\mathbf{u} \to \mathbf{v}$ is an admissible change of variables. We will then speak of $\mathbf{v}$−state coupling (we refer to [2] for detailed definitions).

The precise coupling conditions, introduced in [10], impose that two boundary value problems be well-posed. Since boundary value problems for hyperbolic systems are a difficult subject (see [11]), these coupling conditions cannot always be explicited, and

moreover may lead to ill-posed problems. However, we can justify our approach from a theoretical point of view in some cases. In fact, the state-coupling approach can handle different models of coupling including flux coupling (via relaxation).

First, the coupling conditions can indeed be explicited in the case of Lagrangian systems whose particular structure has enabled us to exhibit a special set of transmitted variables for rather general fluid systems in [2]. In particular, for the Euler system in Lagrangian coordinates, transmitting the primitive variables ($\mathbf{v} = (\varrho, u, p)$, ϱ density, u velocity, p pressure) yields the continuity of u, p at the interface and thus results in a conservative coupling [2]. The case of gas dynamics in Eulerian coordinates is more complex since the eigenvalues may change sign (another way of expressing the difficulty is that the flux does not make an admissible change of variables).

However, if we choose to transmit the conservative variables, or any set of variables which does not yield continuity of the flux, the coupling is non conservative and moreover there is no natural criteria such as an entropy condition at the interface to select a unique *admissible* solution. We may think of some regularization procedure to treat the problem.

Dafermos regularization was introduced in [5] in the scalar case. This regularization allows the approximation of coupled Riemann problems by smooth profiles and enlights the possible discontinuous behavior of the limit solutions at the interface, but does not give uniqueness as expected in the nonconservative case.

In another direction, we have a now classical numerical procedure for approximating the problem which in some cases is justified by convergence results. Indeed, the *two flux finite volume method* introduced in [10] provides a natural tool for the numerical coupling associated to our state coupling approach. It is based on finite volume methods characterized by their numerical flux and a key point lies in the definition of the two fluxes which express the transmission at the interface of coupling.

We have also used approximation by relaxation systems in a global numerical procedure. In [3] we introduce a larger but simpler system on which a state coupling procedure is applied. It ensures a conservative numerical coupling of two Euler systems (in the isentropic case) which avoids resonance (though we do not have uniqueness). This very powerful tool can be used for rather general fluid systems as we will see below. Note also that in the scalar case, we can link the results to those of [4] for conservation laws with discontinuous coefficients. This link should exist for systems which enter into the framework of [12].

We first adress the coupling of two Euler systems (with energy) and then investigate the extension of our relaxation approach to general fluid systems [9]. We present the Lagrangian coordinate case and forget the coupling for this part. However, the results extend to systems in Eulerian coordinates thanks to Wagner's equivalence result. The approach can also be done directly in Eulerian coordinates but the computations are obviously simpler with the Lagrangian form. For details we refer to [8].

2. Relaxation solver for flux-coupling of Euler systems

We want to ensure a conservative coupling of two Euler systems

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \quad x \in \mathbb{R}, t > 0 \\ \partial_t(\varrho e) + \partial_x((\varrho e + p)u) = 0, \end{cases} \tag{1}$$

differing by their pressure law at a fixed interface $x = 0$

$$p = p(x) = p_\alpha(\varrho, \varepsilon) = \tilde{p}_\alpha(\tau, s), \alpha = L \text{ in } x < 0, \alpha = R \text{ in } x > 0, \quad (2)$$

or equivalently $p(x) \equiv (1 - H(x))p_L + H(x)p_R$, and for some given initial condition, say $\mathbf{u}_0(x)$. This problem is a simple model for coupling two codes which simulate the same kind of multiphase flow but take into account different closure laws. In (2), p_α is a given function expressed either in terms of density and internal energy (ϱ, ε) or equivalently (with a tilde) in terms of specific volume and entropy (τ, s) (with $\tau = 1/\varrho$ and the physical specific entropy is $-s$), satisfying usual assumptions derived from laws of thermodynamics, in particular we can write $\varepsilon = \tilde{\varepsilon}(\tau, s)$ with $\tilde{\varepsilon}_\tau = -p < 0$, $\tilde{\varepsilon}_s = -T < 0$, and $\tilde{\varepsilon}(\tau, s)$ strictly convex (see [11]). The internal energy satisfies $\varepsilon = e - u^2/2$, the set of states is $\{\mathbf{u} = (\varrho, \varrho u, \varrho e); \varrho > 0, u \in \mathbb{R}, e - u^2/2 > 0\}$. Discontinuous solutions of (1) are required to satisfy an inequality $\partial_t \varrho s + \partial_x(\varrho s u) \leq 0$, which becomes an equality for smooth solutions

$$\partial_t \varrho s + \partial_x(\varrho s u) = 0. \quad (3)$$

2.1. The relaxation system

We approximate the solutions of (1) by way of the relaxation system

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) = 0 \\ \partial_t \varrho s + \partial_x(\varrho s u) = 0, \\ \partial_t(\rho \mathcal{T}) + \partial_x(\varrho \mathcal{T} u) = \lambda \rho(\tau - \mathcal{T}), \end{cases} \quad (4)$$

with a singular perturbation, where $\lambda > 0$ stands for the relaxation parameter. The set of states for (4) is $\{\mathbf{U} = (\varrho, \varrho u, \varrho s, \varrho \mathcal{T})^T; \varrho > 0, u \in \mathbb{R}, s > 0, \mathcal{T} > 0\}$. The closure relation for Π is

$$\Pi = \Pi_\alpha(\mathbf{U}) = \tilde{\Pi}_\alpha(\tau, s, \mathcal{T}) \equiv \tilde{p}_\alpha(\mathcal{T}, s) + a^2(\mathcal{T} - \tau), \alpha = L \text{ in } x < 0, \alpha = R \text{ in } x > 0, \quad (5)$$

i.e., the exact pressure law $\tilde{p}$ of (2) modified by a correction term. We will note $\mathbf{F}_\alpha(\mathbf{U}) = (\varrho u, \varrho u^2 + \Pi, \varrho s u, \varrho \mathcal{T} u)^T$. In (5), a is a positive constant which plays the role of a Lagrangian sound speed and is required to upper bound the exact sound speed

$$a^2 > -\partial_\tau \tilde{p}_\alpha(\mathcal{T}, s), \quad (6)$$

for all the states $\mathbf{U}$ under consideration, this is Whitham's or subcharacteristic condition. Formally as the relaxation parameter $\lambda \to \infty$, $\mathcal{T} - \tau \to 0$, so that $\Pi \to p$ and we indeed recover system (1) at equilibrium where the energy equation has been replaced by the entropy (3). In order to recover a consistent approximation of (1), we follow the approach developped in [7].

The system (4) without source term ($\lambda = 0$) when the law p_α (p_L or p_R) is used on the whole line (i.e. without coupling) has four linearly degenerate fields $u - a\tau, u, u, u + a\tau$, and explicit solutions for Riemann problems which we note $\mathbf{W}_\alpha(x/t; \mathbf{U}_\ell, \mathbf{U}_r)$. A coupled Riemann problem (CRP) is then a Cauchy problem for (4 with $\lambda = 0$) (5), corresponding to a piecewise constant initial data

$$\mathbf{U}(x, 0) = \begin{cases} \mathbf{U}_\ell, & x < 0, \\ \mathbf{U}_r, & x > 0, \end{cases} \quad (7)$$

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and a coupling condition associated to a transmitted set of variables say $\mathbf{V}$. The solution of a CRP depends on the choice of $\mathbf{V}$.

We have proved existence and uniqueness of the solution of the CRP with transmission of $\mathbf{V} = (\tau, u, \Pi, \Sigma)$, corresponding to data *at equilibrium*, i.e., such that $\mathcal{T}_\alpha = \tau_\alpha$, $\alpha = L, R$. The proof assumes (6) is satisfied for all the states occuring in the solution of the Riemann problem. The law $\mathcal{T}_\alpha(\tau, \Pi, s)$ is obtained by inverting the relation $\tilde{p}_\alpha(\mathcal{T}, s) + a^2\mathcal{T} = \Pi + a^2\tau$, for a satisfying (6) and this yields that $\mathbf{V}$ is an admissible change of variables obtained by composing $(\tau, u, s, \mathcal{T})^T \to (\tau, u, s, \Pi)^T$ and $(\tau, u, s, \Pi)^T \to (\tau, u, \Sigma, \Pi)^T$, and we may write $\mathbf{U} = \varphi_\alpha(\mathbf{V})$.

Theorem 1. *Assuming (6), for any given equilibrium states $\mathbf{U}_\ell, \mathbf{U}_r$, the coupled Riemann problem for (4), with transmission of $\mathbf{V}$ admits a unique solution $\mathbf{W}_c^{\mathbf{V}}(x/t; \mathbf{U}_\ell, \mathbf{U}_r)$ which coincides with the solution of the classical Riemann problem in each half-space*

$$\mathbf{W}_c^{\mathbf{V}}(x/t; \mathbf{U}_\ell, \mathbf{U}_r) = \begin{cases} \mathbf{W}_L(x/t; \mathbf{U}_\ell, \varphi_\ell(\mathbf{V}_r)), & x < 0, \\ \mathbf{W}_R(x/t; \varphi_R(\mathbf{V}_\ell), \mathbf{U}_r), & x > 0. \end{cases} \tag{8}$$

Moreover, ϱu, $\varrho u^2 + \Pi$ and $(\varrho\Sigma + \Pi)u$, are continuous at the interface.

The solution we obtain for the above CRP expressed in terms of $\mathbf{V}$ coincides with the solution of the Riemann problem of the conservative system

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) & = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) & = 0, \\ \partial_t(\varrho\Pi) + \partial_x((\varrho\Pi + a^2)u) & = 0, \\ \partial_t(\varrho\Sigma) + \partial_x((\varrho\Sigma + \Pi)u) & = 0, \end{cases} \tag{9}$$

where we have replaced the third equation in (4, $\lambda = 0$) by an equation on Π which is easy to verify. In the last one, the energy Σ is defined by

$$\Sigma(\tau, u, s, \mathcal{T}) = \tilde{\varepsilon}(\mathcal{T}, s) + u^2/2 + (\tilde{\Pi}^2(\tau, s, \mathcal{T}) - \tilde{p}^2(\mathcal{T}, s))/2a^2. \tag{10}$$

Since all the characteristic fields are linearly degenerate, the two systems are equivalent. For expressing the solution in terms of $\mathbf{U}$, we use two changes of variables $\mathcal{T}_\alpha$ and s (see [8] for details).

System (4) is endowed with an entropy (the energy Σ). The relaxation procedure, outside coupling, satisfies some stability principles. First we have a minimization principle.

Proposition 1. *For a given $\mathbf{u} = (\varrho, \varrho u, \varrho s)^T$, let us note $\mathbf{U}^{eq} = (\varrho, \varrho u, \varrho s, 1) = (\mathbf{u}, 1)$. We have, noting $\mathbf{U} = (\mathbf{u}, \varrho\mathcal{T})$,*

$$\varrho e = \varrho\Sigma(\mathbf{U}^{eq}) = min_{\mathcal{T} \in K}\varrho\Sigma(\mathbf{U}). \tag{11}$$

Here, K is a compact interval so that $\tau \in \overset{\circ}{K}$, and we take $a^2 > \max_{\mathcal{T} \in K}(-\partial_\tau\tilde{p}(\mathcal{T}, s))$. Then a Chapman-Enskog type dissipativity principle is valid. Introducing a first order correction term for $\mathcal{T}$ in inverse powers of λ: $\mathcal{T}^{(\lambda)} = \tau^{(\lambda)} + \lambda^{-1}\tau_1^{(\nu)} + \mathcal{O}(\lambda^{-2})$ (if we emphasize the dependence on λ of the solution of (4) and then drop the superscript λ), yields,

$$\partial_t\varrho u + \partial_x(\varrho u^2 + \tilde{p}(\tau, s)) = \lambda^{-1}\partial_x((\partial_\tau\tilde{p}(\tau, s) + a^2)\tau\partial_x u) + \mathcal{O}(\lambda^{-2}).$$

This last equation is a dissipative approximation of the momentum equation of (1) if the stability or Whitham criterion (6) holds.

2.2. The global relaxation solver

The aim of this section is to ensure the conservative coupling of two Euler systems (1) differing by their pressure law (2). In [3], we have introduced a global relaxation approximation of the coupling problem in the isentropic case and we follow the same approach. The scheme associates a relaxation procedure and finite volume numerical schemes.

We first give the main lines of the relaxation part of the scheme which involves a fractional step method to advance the solution in time from t_n to $t_{n+1} = t_n + \Delta t$ with three steps: reconstruction, evolution, projection. We describe the main lines of the procedure for $n = 0$, since they are similar at all other time. Let $\mathbf{u}_0(x) = (\varrho_0, \varrho_0 u_0, \varrho_0 e_0)^T(x)$ be an initial datum for system (1), (2):

1. Define the extended initial datum $\mathbf{U}_0(x) = (\varrho_0, \varrho_0 u_0, \varrho_0 s_0, \varrho_0 \mathcal{T}_0)^T(x)$ for the relaxation system (4), where $s_0 = s(\mathbf{u}_0)$ and defining $\mathcal{T}_0 \equiv 1/\varrho_0$: $\mathbf{U}_0$ is at equilibrium.

2. Solve (approximately) the Cauchy problem (4), (5) with the initial data $\mathbf{U}_0$ for $t \in (0, \Delta t]$ and with appropriate $\mathbf{V}-$coupling condition, we obtain $\mathbf{U}_1^-(x) = \mathbf{U}(x, \Delta t)$.

3. Project $\mathbf{U}_1^- = (\varrho_1, \varrho_1 u_1, \varrho_1 s_1, \varrho_1 \mathcal{T}_1)^T$ on the equilibrium set of system (4) (instantaneous relaxation) to get $\mathbf{U}_1 = (\varrho_1, \varrho_1 u_1, \varrho_1 s_1, 1)$.

4. Define $\mathbf{u}_1(x) = (\varrho_1, \varrho_1 u_1, \varrho_1 e_1)^T(x)$.

In steps 2 and 3 we have solved the coupling of the relaxation systems (4) by an operator splitting method. In the evolution step 2, we will use a Godunov solver, this results in our global relaxation solver (GRS). In fact step 4 is straightforward at the computational level but requires a careful analysis to justify the scheme for $\mathbf{u} = (\varrho, \varrho u, \varrho e)^T$ from $\mathbf{U} = (\varrho, \varrho u, \varrho s, \varrho \mathcal{T})^T$. Moreover $\mathbf{u}_0(x)$ is first discretized

$$\mathbf{u}_{j+1/2}^0 = \int_{j\Delta x}^{(j+1)\Delta x} \mathbf{u}_0(x) dx.$$

Let us summarize the resulting global relaxation method for approximating the coupling of two Euler systems (1) differing by their pressure law (2). In each half space, we use a Godunov solver, and at the interface too. Starting from an initial condition $\mathbf{u}_0(x)$ discretized by $\mathbf{u}_{j+1/2}^0 = (\varrho, \varrho u, \varrho e)_{j+1/2}^0, j \in \mathbb{Z}$, setting $\mu = \Delta t / \Delta x$,
- Define $\mathbf{U}_{j+1/2}^n = ((\varrho, \varrho u, \varrho s)_{j+1/2}^n, 1)^T$ the extended equilibrium state.
- Solve the Riemann problems $\mathbf{W}_\alpha(0; \mathbf{U}_{j-1/2}^n, \mathbf{U}_{j+1/2}^n)$, $\alpha = L$ for $j < 0, \alpha = R$ for $j > 0$. Define the usual numerical fluxes of Godunov's method

$$\mathbf{G}_{\alpha,j}^{God,n} = \mathbf{F}_\alpha(\mathbf{W}_\alpha(0; \mathbf{U}_{j-1/2}^n, \mathbf{U}_{j+1/2}^n)), \; j \neq 0. \tag{12}$$

Solve the CRP (8) with data $\mathbf{U}_{-1/2}^n, \mathbf{U}_{+1/2}^n$ and prescribed coupling condition, we get an explicit formula for $\mathbf{W}_c^{\mathbf{V}}(x/t; \mathbf{U}_{-1/2}^n, \mathbf{U}_{+1/2}^n)$, and define the corresponding numerical flux

$$\mathbf{G}_{\alpha,0}^{God,n} = \mathbf{F}_\alpha(\mathbf{W}_c^{\mathbf{V}}(0; \mathbf{U}_{-1/2}^n, \mathbf{U}_{+1/2}^n)). \tag{13}$$

- Update $\mathbf{U}_{j+1/2}^{n+1-}$ by Godunov's scheme, we get what we call the Godunov scheme with $\mathbf{V}-$state coupling

$$\begin{cases} \mathbf{U}_{j-1/2}^{n+1} = \mathbf{U}_{j-1/2}^n - \mu\left(\mathbf{G}_{L,j}^{God,n} - \mathbf{G}_{L,j-1}^{God,n}\right), & j \leq 0, n \geq 0, \\ \mathbf{U}_{j+1/2}^{n+1} = \mathbf{U}_{j+1/2}^n - \mu\left(\mathbf{G}_{R,j+1}^{God,n} - \mathbf{G}_{R,j}^{God,n}\right), & j \geq 0, n \geq 0. \end{cases} \tag{14}$$

- Keep the two first components for the two first components of $\mathbf{u}_{j+1/2}^{n+1}$. This results in a conservative discrete equation for $(\varrho, \varrho u)_{j+1/2}^{n+1}$. Here, we skip the details of step 4 which follows [7] for defining the energy in the global solver so as to obtain a standard conservative scheme with an energy component of the flux. Moroever, using (11), an entropy inequality can be established.

Proposition 2. *The global relaxation solver satisfies a discrete entropy inequality*

$$\varrho s_{j+1/2}^{n+1} \leq \varrho s_{j+1/2}^n - \mu\left(\mathcal{G}_{\varrho s,j+1}^n - \mathcal{G}_{\varrho s,j}^n\right), \tag{15}$$

where the discrete entropy flux $\mathcal{G}_{\varrho s,j}^n = \mathcal{G}_{\alpha,\varrho s,j}^n$ is defined by the third component of Godunov's flux (12) ($j \neq 0$) (resp. of (13), for $j = 0$).

Finally, we have a formal Lax-Wendroff-type convergence result. Denoting by $\mathbf{u}_\Delta$ the piecewise constant function associated in a classical way to the scheme, we prove

Theorem 2. *Assume that $\mathbf{u}_\Delta$ is bounded in $\mathbb{L}^\infty(\mathbb{R}+\times\mathbb{R})$ and that the scheme converges in the sense that $\mathbf{u}_\Delta \to \mathbf{u}$ in $\mathbb{L}^1_{loc}(\mathbb{R}+;\mathbb{L}^1_{loc}(\mathbb{R}))$ and a.e. with $\mathbf{u}_\Delta(0,t) \to \mathbf{u}(0,t)$ in $\mathbb{L}^1_{loc}(\mathbb{R})$ and a.e. Then the limit $\mathbf{u}$ is solution of system (1) (2) with initial condition $\mathbf{u}_0$.*

3. Relaxation for Lagrangian systems

3.1. The form of general Lagrangian systems

We consider systems of n conservation laws in Lagrangian coordinates which we write $\partial_t\mathbf{u} + \partial_x\mathbf{f}(\mathbf{u}) = \mathbf{0}$, ($x$ stands for a mass variable) that meet some common properties (we refer to [9] for a detailed description):

(i) they are endowed with a strictly convex entropy $s(\mathbf{u})$, with null associated entropy flux, so that for smooth solutions

$$\partial_t s = 0.$$

(ii) $\mathbf{u}$ is made of $n-1-d$ *state* variables $\mathbf{w}$ and d *velocity* variables $\mathbf{v}$. The last component of $\mathbf{u}$ is the total energy which we will denote e

$$u_n \equiv e = \frac{1}{2}|\mathbf{v}|^2 + \epsilon \tag{16}$$

where the internal energy $\epsilon(\mathbf{w}, s)$ is a *state* variable, then s is also a *state* variable. We will assume that $s(\mathbf{u})$, satisfies $\frac{\partial s}{\partial e}(\mathbf{u}) \equiv s_e(\mathbf{u}) < 0$. The model is then called a *fluid model*. We also assume (iii) Galilean invariance, and (iv) reversibility in time for smooths solutions. Then, such a system can be written in a canonical form (see [2]), involving the entropy variables that symmetrize the system $\mathbf{u}^* \equiv s'(\mathbf{u})^T = (s_{u_1}, \cdots, s_{u_{n-1}}, s_e)^T$. Here we will only use the following form (equivalent for smooth solutions) of the equations

$$\begin{cases} \partial_t\mathbf{w} - N\partial_x\mathbf{v} = \mathbf{0}, \\ \partial_t\mathbf{v} - N^T\partial_x\phi(\mathbf{w}, s) = \mathbf{0}, \\ \partial_t s = 0, \end{cases} \tag{17}$$

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where N is a constant $d \times (n-1-d)$ matrix, $\mathbf{v} \in \mathbb{R}^d$, is the velocity vector as defined above, $\mathbf{w} \in \mathbb{R}^{n-d-1}$ represents the state variable vector, and $\phi(\mathbf{w}, s) = e_{\mathbf{w}}(\mathbf{u})$, note that $e_{\mathbf{v}}(\mathbf{u}) = \mathbf{v}$. We use the notation $e_{\mathbf{w}} \equiv \nabla_{\mathbf{w}} e$ for the vector of partial derivatives $(e_{w_1}, e_{w_2}, ..., e_{w_{n-d-1}})$, and $e_{w_i} = \frac{\partial e}{\partial w_i}$.

Finally the spectrum of $\mathbf{f}'(\mathbf{u})$ is symmetric and there are at least $n - 2d$ null eigenvalues, and $n - 2d \geq 1$ (there is at least one which is associated to the conservation of entropy). We may reduce even further the system in order to distinguish all the null eigenvalues (see [8]). Here we assume for simplicity that N is a square invertible matrix and $n = 2d + 1$.

3.2. The relaxation system

Consider, again for simplicity, the isentropic case, $s = s_0$, set $\epsilon(\mathbf{w}) = \epsilon(\mathbf{w}, s_0)$, $\epsilon' = e_{\mathbf{w}}$, then (17) reads

$$\begin{cases} \partial_t \mathbf{w} - N \partial_x \mathbf{v} = \mathbf{0}, \\ \partial_t \mathbf{v} - N^T \partial_x \epsilon'(\mathbf{w}) = \mathbf{0}. \end{cases} \tag{18}$$

We introduce a larger relaxation system

$$\begin{cases} \partial_t \mathbf{w} - N \partial_x \mathbf{v} = \mathbf{0}, \\ \partial_t \mathbf{v} - N^T \partial_x \mathcal{X} = \mathbf{0}, \\ \partial_t \mathcal{W} = \lambda(\mathbf{w} - \mathcal{W}), \end{cases} \tag{19}$$

with some initial condition $\mathbf{U}_0(x) = (\mathbf{w}_0, \mathbf{v}_0, \mathcal{W}_0)(x)$, where

$$\mathcal{X} = \mathcal{X}(\mathbf{w}, \mathcal{W}) = \epsilon'(\mathcal{W}) + \theta'(\mathbf{w}) - \theta'(\mathcal{W}), \tag{20}$$

$\mathcal{W} \in \mathbb{R}^d$, $\theta : \mathbb{R}^d \to \mathbb{R}$ and we note either $\theta_{\mathbf{w}}$ or θ' its derivative and $(.,.)$ the scalar product in $\mathbb{R}^d$. For the system with entropy, all the computations can be done, only replacing the derivatives by partial derivatives, for instance $\theta = \theta(\mathbf{w}, s)$ will depend on s (see again [8] for details). In (20), θ will be chosen in order to assume entropy dissipation. By assumption $\epsilon_{\mathbf{w},\mathbf{w}} = \epsilon''$ is symmetric positive definite and we assume $\theta_{\mathbf{w},\mathbf{w}} = \theta''$ is also symmetric positive definite. For example we may take $\theta(\mathbf{w}) = (\mathbf{w}, \Lambda \mathbf{w})$ with Λ a positive definite matrix, even in the simplest case Λ may be diagonal with positive entries. Formally, as the relaxation parameter $\lambda \to \infty$, $\mathbf{w} - \mathcal{W} \to 0$ and at equilibrium, $\mathcal{X}(\mathbf{w}, \mathbf{w}) = \epsilon'(\mathbf{w})$, we recover system (18).

In order to justify the relaxation procedure, we introduce an energy $\Sigma(\mathbf{w}, \mathbf{v}, \mathcal{W})$,

$$\Sigma(\mathbf{w}, \mathbf{v}, \mathcal{W}) = \frac{1}{2}|\mathbf{v}|^2 + \epsilon(\mathcal{W}) + \theta(\mathbf{w}) - \theta(\mathcal{W}) + ((\epsilon' - \theta')(\mathcal{W}), \mathbf{w} - \mathcal{W}), \tag{21}$$

which coincides with e at equilibrium, i.e. $\Sigma(\mathbf{w}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}|\mathbf{v}|^2 + \epsilon$, and should be dissipated. More precisely, we have for fixed $\mathbf{w}, \mathbf{v}$

$$e(\mathbf{w}, \mathbf{v}) = \min_{\mathcal{W} \in K} \Sigma(\mathbf{w}, \mathbf{v}, \mathcal{W})$$

i.e., the minimum of Σ is attained at equilibrium $\mathcal{W} = \mathbf{w}$. This holds under the following assumption

(H) $\qquad\qquad\qquad\qquad \theta''(\mathcal{W}) - \epsilon''(\mathcal{W})$ is positive definite

for all $\mathcal{W} \in K$ (where K denotes a compact set of the phase space such that $\mathbf{w} \in \mathring{K}$).

3.3. Approximation results

As in [6], following [13][14] we have convergence results for smooth solutions of (19) to smooth solutions of (18). As in these papers, we address the periodic case and denote by $H^s(\mathbb{T})$ the Sobolev space of functions with period 1. If we apply the results of [13], we can prove a local in time result, without assuming that the initial data is close to equilibrium.

Theorem 3. *Let $s \geq 2$ and consider an initial data $\mathbf{U}_0 = (\mathbf{w}_0, \mathbf{v}_0, \mathcal{W}_0)$ in $H^{s+2}(\mathbb{T})$ that takes values in a compact subset. There exist θ, with θ'' constant, and $T > 0$ such that*

- $\forall \lambda > 0, \exists \mathbf{U}^{(\lambda)} = (\mathbf{w}^{(\lambda)}, \mathbf{v}^{(\lambda)}, \mathcal{W}^{(\lambda)}) \in \mathcal{C}([0,T], H^s(\mathbb{T}))$ unique solution of (19),

- system (18) admits a unique solution $(\mathbf{w}^e, \mathbf{v}^e) \in \mathcal{C}([0,T], H^{s+2}(\mathbb{T}))$ with initial data $(\mathbf{w}_0, \mathbf{v}_0)$,

- $(\mathbf{v}^{(\lambda)}, \mathbf{u}^{(\lambda)})$ converges towards $(\mathbf{w}^e, \mathbf{v}^e)$ in $\mathcal{C}([0,T], H^s(\mathbb{T}))$ as $\lambda \to \infty$.

Note that θ'' constant, which corresponds to the constant a in Whitham's condition (6), is a little restrictive but important for the applications.

We can also follow [14], and prove the existence of a (unique global smooth) solution to (19), under an initial data close to an equilibrium state $\mathbf{U}_e$. The proof of these results (see [8]) relies on checking that our system satisfies the structural properties stated in [13][14].

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