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EXTREMAL GRAPHS WITHOUT TOPOLOGICAL COMPLETE SUBGRAPHS*

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Abstract. The exact values of the function $ex(n;TK_p)$ are known for $\lceil \frac{2n+5}{3} \rceil \le p < n$ (see [Cera, Diánez, and Márquez, *SIAM J. Discrete Math.*, 13 (2000), pp. 295–301]), where $ex(n;TK_p)$ is the maximum number of edges of a graph of order n not containing a subgraph homeomorphic to the complete graph of order p. In this paper, for $\lceil \frac{2n+6}{3} \rceil \le p < n-3$, we characterize the family of extremal graphs $EX(n;TK_p)$, i.e., the family of graphs with n vertices and $ex(n;TK_p)$ edges not containing a subgraph homeomorphic to the complete graph of order p.

Key words. extremal graph theory, topological complete subgraphs

AMS subject classifications. 05C35, 05C70

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1. Introduction.

The study of the function $ex(n;TK_p)$—i.e., the maximum number of edges of a graph of order n not containing a subgraph homeomorphic to K_p, where K_p is the complete graph with p vertices—is one of the most general extremal problems, as pointed out by Bollobas in [1]. Exact values for this function are known only in some cases, as can be seen in Table 1.1.

TABLE 1.1
Exact values of the function $ex(n;TK_p)$.

p	$ex(n;TK_p)$	Reference
3	$n-1$	
4	$2n-3$	[3]
5	$3n-6$	[4], [8], [9]
⋮	⋮	⋮
$\lceil \frac{2n+5}{3} \rceil \le p < \lceil \frac{3n+2}{4} \rceil$	$\binom{n}{2} - (5n-6p+3)$	[2]
$\lceil \frac{3n+2}{4} \rceil \le p < n$	$\binom{n}{2} - (2n-2p+1)$	[2]

The aim of this work is to characterize a family of extremal graphs $EX(n;TK_p)$ for appropriate values of n and p, i.e., the set of graphs of order n, with $ex(n;TK_p)$

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edges and not containing any subgraph homeomorphic to K_p. Actually, we characterize the family $EX(n;TK_p)$ for $\lceil\frac{2n+6}{3}\rceil \le p < n-3$:

$$EX(n;TK_p) = \begin{cases} (3n-4p+2)\overline{K_3} + (6p-4n-3)\overline{K_2} & \text{for } \lceil\frac{2n+6}{3}\rceil \le p < \lceil\frac{3n+2}{4}\rceil, \\ K_{4p-3n-2} + (2n-2p+1)\overline{K_2} & \text{for } \lceil\frac{3n+2}{4}\rceil \le p < n-3. \end{cases}$$

2. Definitions and notation. Given a graph H and a set $\{v_1,\ldots,v_q\}$ of vertices of H, we denote by $H_0 = H$ and by H_k for $k = 1,\ldots,q$ the induced subgraph in H by the set of vertices $V(H) - \{v_1,\ldots,v_k\}$. We denote by $\Delta(H)$ the maximum degree of the graph H and by $\delta_H(v)$ the degree of the vertex v in the graph H. The complement graph of H will be denoted by $\overline{H}$.

Let q and s be a pair of nonnegative integers; $\mathcal{C}_q^s$ denotes the set of graphs H such that there exists a set $\{v_1,\ldots,v_q\}$ of vertices of H verifying the following:

(1) $\delta_{H_{j-1}}(v_j) \ge \delta_{H_j}(v_{j+1})$ for $j = 1,\ldots,q-1$.

(2) For each positive integer h, if there exists $k \in \{1,\ldots,q\}$ and $v \in H_k$ such that $\delta_{H_k}(v) \ge h$, then $\delta_{H_j}(v_{j+1}) \ge h$ for all $j = 1,\ldots,k$.

(3) H_q has at most s edges (i.e., $|E(H_q)| \le s$).

The next results show different conditions to guarantee that a graph belongs to the family described above (see [2]).

LEMMA 2.1 (see [2]). *Let H be a graph with n vertices. Then, for any $q \le n$, there exists s such that H is in $\mathcal{C}_q^s$.*

When $s = q$, we know sufficient conditions for the edges of a graph to belong to the class $\mathcal{C}_q^q$.

LEMMA 2.2 (see [2]). *Let n and q be two positive integers, with $q < n$. If H is a graph with n vertices and $2q$ edges, then*

1. $H \in \mathcal{C}_q^q$,
2. $\delta_{H_q}(v) \le 1$ *for* $v \in V(H_q)$.

LEMMA 2.3 (see [2]). *Let q and k be two positive integers with $k \le q-2$. Let H be a graph with $4q-k+1$ vertices and $2q+k+1$ edges. Then $H \in \mathcal{C}_q^q$.*

Notation and terminology not given here can be found in [1] and [2].

3. The family of extremal graphs. In this section, we will characterize the family $EX(n;TK_p)$ for $\lceil\frac{2n+6}{3}\rceil \le p < n-3$. This problem is equivalent to characterizing $EX(n;TK_{n-q})$ for $n \ge 4q+2$ with $q \ge 4$ (case $\lceil\frac{3n+2}{4}\rceil \le p < n-3$) and $n = 4q-k+1$ with $q \ge 5$, $0 \le k \le q-5$ (the case $\lceil\frac{2n+6}{3}\rceil \le p < \lceil\frac{3n+2}{4}\rceil$).

In order to avoid excessive repetition, we define the graphs $\mathcal{H}(n;TK_{n-q})$:

$$\mathcal{H}(n;TK_{n-q}) = \begin{cases} K_{n-(4q+2)} + (2q+1)\overline{K_2} & \text{for } n \ge 4q+2, \\ (k+1)\overline{K_3} + (2(q-k)-1)\overline{K_2} & \text{for } n = 4q-k+1,\ 0 \le k \le q-5. \end{cases}$$

For $n \ge 4q+2$, a graph G belongs to the family $\{\mathcal{H}(n;TK_{n-q})\}$ if G has n vertices and $\overline{G}$ is formed by $2q+1$ nonadjacent edges (see Figure 3.1).

For $n = 4q-k+1$ with $q \ge 5$ and $0 \le k \le q-5$, a graph G belongs to the family $\{\mathcal{H}(n;TK_{n-q})\}$ if it has $4q-k+1$ vertices and $\overline{G}$ is formed by $k+1$ nonadjacent triangles and $2(q-k)-1$ nonadjacent edges, as Figure 3.2 shows.

In the next two sections, we will prove the following theorem.

THEOREM 3.1. $EX(n;TK_p) = \{\mathcal{H}(n;TK_p)\}$ *for* $\lceil\frac{2n+6}{3}\rceil \le p < n-3$.

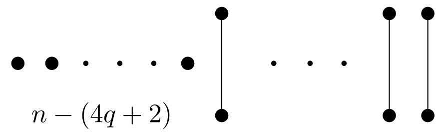

FIG. 3.1. *Structure of* $\overline{G}$ *for* $n \geq 4q+2$.

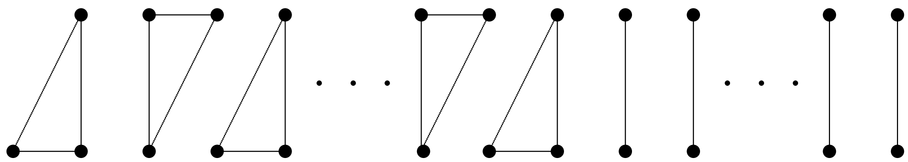

FIG. 3.2. *Structure of* $\overline{G}$ *for* $n = 4q-k+1$.

4. Case $\lceil \frac{3n+2}{4} \rceil \leq p < n-3$. The aim of this section is to prove Theorem 3.1 when n and p are related by the expression $\lceil \frac{3n+2}{4} \rceil \leq p < n-3$.

PROPOSITION 4.1. *Let n and p be two positive integers such that $\lceil \frac{3n+2}{4} \rceil \leq p < n-3$. It is verified that*

$$EX(n;TK_p) = \{\mathcal{H}(n;TK_p)\}.$$

In order to provide this proposition, we need some previous results. First, we recall the following results about the function $ex(n;TK_{n-q})$ (see [2]).

THEOREM 4.2 (see [2]). *Let n and q be two positive integers. If $n \geq 4q+2$, then*

$$ex(n;TK_{n-q}) = \binom{n}{2} - (2q+1).$$

Also, we recall that, given a graph H and $v \in H$, the set of vertices adjacent to v in H is denoted by $\Gamma(v)$ (see [1]). Given a bipartite graph B whose classes are X and Y with $|X| \leq |Y|$, we say that B has a complete matching if there exists a set of nonadjacent edges in B with cardinality $|X|$. If we need to show the existence of a complete matching in a bipartite graph, then we can use Hall's condition.

THEOREM 4.3 (see [5]). *Given a bipartite graph with classes X and Y, if $|\Gamma(A)| \geq |A|$ for all $A \subseteq X$, where $\Gamma(A) = \bigcup_{v\in A} \Gamma(v)$, then there exists a complete matching.*

The next result asserts that for any graph $G \in EX(n;TK_{n-q})$ its complement graph $\overline{G}$ is extremal for $\mathcal{C}_q^{q+1}$ in the sense that $\overline{G} \in \mathcal{C}_q^{q+1}$ and $\overline{G} \notin \mathcal{C}_q^q$.

LEMMA 4.4. *Let n and q be two nonnegative integers with $q \geq 4$ and $n \geq 4q+2$. For every graph G from the family of graphs $EX(n;TK_{n-q})$, we have*

$$\overline{G} \in \mathcal{C}_q^{q+1} - \mathcal{C}_q^q.$$

Proof. Let G be a graph such that $G \in EX(n;TK_{n-q})$. The graph G does not contain a subgraph homeomorphic to K_{n-q}, so by Theorem 4.2, we know that

$$|E(G)| = \binom{n}{2} - (2q+1).$$

Hence, $|E(H)| = 2q+1$, where $H = \overline{G}$.

By Lemma 2.1, there exists an integer s such that $H \in \mathcal{C}_q^s$. This means that there exists a subset $\{v_1, \ldots, v_q\}$ of vertices of G verifying $|E(H_q)| \leq s$, where $H_q = H - \{v_1, \ldots, v_q\}$. If $s \leq q+1$, then $H \in \mathcal{C}_q^{q+1}$. Otherwise $(s > q+1)$, let H^* be the graph obtained from H by removing one of the edges of the subgraph H_q. The graph H^* has $n \geq 4q+2$ vertices and $2q$ edges, and applying Lemma 2.2 results in $H^* \in \mathcal{C}_q^q$. Furthermore, by the construction of the graph H^*, the set of vertices chosen to prove that H^* belongs to the class of graphs $\mathcal{C}_q^q$ is the same as the one we chose previously in H; thus $|E(H_q)| \leq q+1$ and $H \in \mathcal{C}_q^{q+1}$.

Now we will prove that the number of edges of H_q may not be equal to or less than q, i.e., $H \notin \mathcal{C}_q^q$. Suppose that $H \in \mathcal{C}_q^q$. This means there exists a set of vertices $\{v_1, \ldots, v_q\}$ guaranteeing this assertion. Let $e_1 = (a_1, b_1), \ldots, e_s = (a_s, b_s)$ be the edges of H_q with $1 \leq s \leq q$.

We consider the bipartite graph B whose classes are $X = \{e_1, \ldots, e_s\}$ and $Y = \{v_1, \ldots, v_q\}$ such that e_i is adjacent to v_j in B if the path $a_i v_j b_i$ exists in G. We note that if there exists a complete matching in B, then we have that G contains a subgraph homeomorphic to K_{n-q}. Now Hall's condition implies the existence of a complete matching. Thus, we will prove that $|\Gamma(A)| \geq |A|$ for each $A \subseteq X$.

Let $A = \{e_i\}$ be a subset of X with $|A| = 1$ for $i \in \{1, \ldots, s\}$. If $|\Gamma(A)| = 0$, then e_i is nonadjacent to any vertex of the set $\{v_{q-2}, v_{q-1}, v_q\}$ in B. Hence, no vertex $v \in \{v_{q-2}, v_{q-1}, v_q\}$ is adjacent to both a_i and b_i in G. Consequently, $\delta_{H_{q-1}}(a_i) \geq 2$ or $\delta_{H_{q-1}}(b_i) \geq 2$ and, furthermore, $\delta_{H_{q-3}}(a_i) \geq 3$ or $\delta_{H_{q-3}}(b_i) \geq 3$. Thus, using property (2) of the definition of $\mathcal{C}_q^q$, we obtain that $\delta_{H_{j-1}}(v_j) \geq 3$ for $j = 1, \ldots, q-2$ and $\delta_{H_{j-1}}(v_j) \geq 2$ for $j = q-1, q$. Therefore, since $s \geq 1$ we have that

$$|E(H)| \geq 3(q-2) + 2 \cdot 2 + s \geq 2q+2$$

for $q \geq 3$. But this is not possible since $|E(H)| = 2q+1$.

We consider $A = \{e_i, e_j\} \subseteq X$ for $i, j \in \{1, \ldots, s\}$ with $i \neq j$, and we suppose $|\Gamma(A)| \leq 1$. This means that at least three vertices of the set $\{v_{q-3}, v_{q-2}, v_{q-1}, v_q\}$ are nonadjacent to e_i and to e_j in B. Taking into account property (2) of the definition of $\mathcal{C}_q^q$, we have that $\delta_{H_{j-1}}(v_j) \geq 3$ for $j = 1, \ldots, q-3$, $\delta_{H_{j-1}}(v_j) \geq 2$ for $j = q-2, q-1$ and $\delta_{H_{q-1}}(v_q) \geq 1$ (see Figure 4.1). Hence,

$$|E(H)| \geq 3(q-3) + 2 \cdot 2 + 1 + s \geq 2q+2$$

for $q \geq 4$, and this is a contradiction, as in the previous case.

Let m be an integer with $3 \leq m \leq s$. Let A be the set of vertices $\{e_{i_1}, \ldots, e_{i_m}\} \subseteq \{e_1, \ldots, e_s\}$ with $i_1 < i_2 < \cdots < i_m$. If $|\Gamma(A)| \leq m-1$, then there

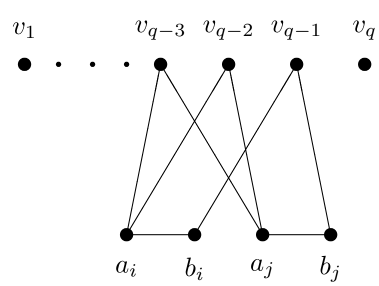

FIG. 4.1. *Possible structure of H for the most unfavorable case for $A = \{e_i, e_j\}$.*

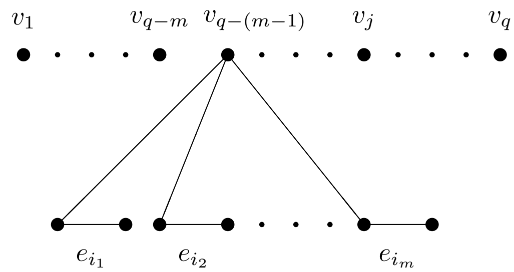

FIG. 4.2. *Possible structure of H for the most unfavorable case for $3 \leq m \leq s$.*

exists $i \in \{q-(m-1), \dots, q\}$ in such a way that v_i is not adjacent to any vertex of the set A in the graph B. By applying condition (2) of the definition of $\mathcal{C}_q^q$, we obtain that $\delta_{H_{q-m}}(v_{q-(m-1)}) \geq m$ and, therefore, $\delta_{H_{j-1}}(v_j) \geq m$ for $1 \leq j \leq q-(m-1)$ (see Figure 4.2). Furthermore, $\delta_{H_{j-1}}(v_j) \geq 1$ for $q-(m-2) \leq j \leq q$ and $|E(H_q)| = s \geq m$. Consequently,

$$\begin{aligned} |E(H)| &\geq m(q-(m-1)) + m - 1 + s \\ &\geq mq - m^2 + 3m - 1. \end{aligned}$$

Since $E(H) = 2q+1$, we have that $2q+1 \geq mq - m^2 + 3m - 1$ and, therefore, $q \leq \frac{m^2-3m+2}{m-2} \leq m-1 < m \leq s$, but this is not possible. Therefore, $|\Gamma(A)| \geq |A|$ for each $A \subseteq X$. Thus, by Hall's condition, there exists a complete matching in B and, thereby, the graph G contains a subgraph homeomorphic to K_{n-q}. This is not possible, and the result follows. $\square$

Now we can prove Proposition 4.1.

Proof of Proposition 4.1. It is equivalent to prove that

$$EX(n;TK_{n-q}) = \{\mathcal{H}(n;TK_{n-q})\}$$

for $q \geq 4$ and $n \geq 4q+2$.

Let G be a graph belonging to $\{\mathcal{H}(n;TK_{n-q})\}$ with $n \geq 4q+2$. It is easy to check that G does not contain a subgraph homeomorphic to K_{n-q}. Furthermore, by denoting $|E(G)|$ as the number of edges of G, we have that

$$|E(G)| = ex(n;TK_{n-q}) = \binom{n}{2} - (2q+1).$$

Thus, by Theorem 4.2, G is maximal on edges and

$$\{\mathcal{H}(n;TK_{n-q})\} \subseteq EX(n;TK_{n-q}).$$

In order to prove that $EX(n;TK_{n-q}) \subseteq \{\mathcal{H}(n;TK_{n-q})\}$, let G be a graph belonging to $EX(n;TK_{n-q})$, and we set $H = \overline{G}$. By Theorem 4.2 we have that $|E(H)| = 2q+1$. By Lemma 2.1, we know there exists s such that $H \in \mathcal{C}_q^s$. Let $\{v_1, \dots, v_q\}$ be a set of q vertices guaranteeing this property. We know that there exists a vertex $v \in H_q$ such that $\delta_{H_q}(v) \geq 1$, because otherwise H_q is empty and $H \in \mathcal{C}_q^q$. But this is not possible because, by Lemma 4.4, we know that $H \notin \mathcal{C}_q^q$. If $\delta(v_1) \geq 2$, then $|E(H_q)| \leq 2q+1-(2+q-1) = q$ and therefore $H \in \mathcal{C}_q^q$, a contradiction. Therefore, $\delta(v_1) \leq 1$.

Thus, as v_1 is the vertex of maximum degree in H, we have that $\delta(v) \leq 1$ for all $v \in H$, and then the graph H is formed by $2q+1$ nonadjacent edges. Therefore, the result follows. $\square$

5. Case $\lceil \frac{2n+6}{3} \rceil \le p < \lceil \frac{3n+2}{4} \rceil$. In this section, we will characterize the family of extremal graphs $EX(n; TK_{n-q})$ for $n = 4q - k + 1$ with $0 \le k \le q - 5$ in such a way that we will show that $EX(n; TK_{n-q}) = \{\mathcal{H}(n; TK_{n-q})\}$, applying techniques based on the same ideas as in the previous section.

THEOREM 5.1. *Let n and p be two positive integers with $\lceil \frac{2n+6}{3} \rceil \le p < \lceil \frac{3n+2}{4} \rceil$. Then*

$$EX(n; TK_p) = \{\mathcal{H}(n; TK_p)\}.$$

In order to prove this result, we also need to recall some results about the function $ex(n; TK_{n-q})$ (see [2]).

LEMMA 5.2 (see [2]). *Let k be a nonnegative integer and H be a graph with maximum degree 2 and at least $3k + 1$ vertices of maximum degree. Then there exist at least $k + 1$ nonadjacent vertices with degree 2.*

THEOREM 5.3 (see [2]). *Let n, k, and q be three nonnegative integers with $0 \le k \le q - 4$ and $n = 4q - k + 1$. It is verified that*

$$ex(n; TK_{n-q}) = \binom{n}{2} - (2q + k + 2).$$

Now we will show, as in Lemma 4.4, that if $G \in EX(n; TK_{n-q})$ with $n = 4q-k+1$, then $\overline{G} \in \mathcal{C}_q^{q+1}$ but $\overline{G} \notin \mathcal{C}_q^q$.

LEMMA 5.4. *Let k, n, and q be three nonnegative integers such that $q \ge 5$, $0 \le k \le q - 5$, and $n = 4q - k + 1$. If $G \in EX(n; TK_{n-q})$, then*

$$\overline{G} \in \mathcal{C}_q^{q+1} - \mathcal{C}_q^q.$$

Proof. Let G be a graph belonging to $EX(n; TK_{n-q})$. This graph does not contain a graph homeomorphic to K_{n-q}, and by Theorem 5.3 we know that

$$|E(G)| = \binom{n}{2} - (2q + k + 2).$$

Thus, $H = \overline{G}$ has $2q + k + 2$ edges.

Let H^* be the graph obtained from H by removing one edge, similar to what we have done in Lemma 4.4. Since H^* is a graph formed by $4q - k + 1$ vertices and $2q + k + 1$ edges, then applying Lemma 2.3 yields $H^* \in \mathcal{C}_q^q$, and then

$$H \in \mathcal{C}_q^{q+1}.$$

Now we will show that $H \notin \mathcal{C}_q^q$. To the contrary, suppose $H \in \mathcal{C}_q^q$ and let $\{v_1, \ldots, v_q\}$ be a set of vertices of H guaranteeing that $H \in \mathcal{C}_q^q$. Let $e_1 = (a_1, b_1), \ldots,$ $e_s = (a_s, b_s)$ be the edges of H_q with $s \le q$. We consider the bipartite graph B constructed as in Lemma 4.4, i.e., the graph whose classes are $X = \{e_1, \ldots, e_s\}$ and $Y = \{v_1, \ldots, v_q\}$ in such a way that e_i is adjacent to v_j if the path $a_i v_j b_i$ exists in the graph G. In this case, if we show the existence of a complete matching in B, then we would have that G contains a subgraph homeomorphic to K_{n-q}. Therefore, we will show that $|\Gamma(A)| \ge |A|$ for each $A \subseteq X$.

If $|A| = m = 1$, by reasoning as in the proof of Lemma 4.4, we have that

$$|E(H)| \ge 3(q - 2) + 4 + s = 3q + s - 2 \ge 3q - 1.$$

Since $k \leq q-4$, it is verified that $3q-1 \geq 2q+k+4-1 > 2q+k+2$, but this is not possible.

For $m = 2$, by considering as done previously, we have that

$$|E(H)| \geq 3(q-3)+4+1+s = 3q-4+s \geq 3q-2.$$

Taking into account that $k \leq q-5$, it is verified that $|E(H)| > 2q+k+2$, and this is a contradiction.

We consider $m = 3$. Let $A = \{e_{i_1}, e_{i_2}, e_{i_3}\}$ be a subset of vertices of X with $1 \leq i_1 < i_2 < i_3 \leq s$. If $|\Gamma(A)| \leq 2$, then there exists $i \in \{q-2, \ldots, q\}$ in such a way that v_i is not adjacent to any vertex of the set A in the graph B. Hence, by applying property (2) of the definition of $\mathcal{C}_q^q$, we have that $\delta_{H_{q-3}}(v_{q-2}) \geq 3$. Thus,

$$|E(H)| \geq 3(q-2)+2+s \geq 3q-1 > 2q+k+2$$

since $k \leq q-4$.

In general, if $4 \leq m \leq s$, then we consider A as the set of vertices $\{e_{i_1}, \ldots, e_{i_m}\} \subseteq \{e_1, \ldots, e_s\}$ with $i_1 < i_2 < \cdots < i_m$. If $|\Gamma(A)| \leq m-1$, then there exists $i \in \{q-(m-1), \ldots, q\}$ in such a way that v_i is not adjacent to any vertex of the set A in the graph B. Hence, as in the proof of Lemma 4.4, we have that $\delta_{H_{q-m}}(v_{q-(m-1)}) \geq m$ and, therefore,

$$|E(H)| \geq m(q-(m-1))+m-1+s \geq mq-m^2+3m-1.$$

But $|E(H)| = 2q+k+2 \leq 3q-3$ for $k \leq q-5$. Thus, $3q-3 \geq mq-m^2+3m-1$ and, thereby, $q \leq m - \frac{2}{m-3} < m$, but this is not possible.

Thus, using Hall's condition, there exists a complete matching in B, and consequently, G contains a subgraph homeomorphic to K_{n-q}, but this is not possible. Hence, $H \notin \mathcal{C}_q^q$ and the result follows. ∎

The next result is devoted to proving the existence of nonadjacent triangles in graphs with maximum degree 2 and the prescribed number of vertices of maximum degree.

LEMMA 5.5. *Let r be a nonnegative integer, and let H be a graph with maximum degree 2. If H has $3r+3$ vertices of degree 2 and $r+1$ of them form an independent set, then H contains $r+1$ nonadjacent triangles.*

Proof. We apply induction on r. For $r = 0$ the result is obvious, because the triangle is the unique graph formed by 3 vertices of degree 2 and all of them are adjacent among themselves.

Now suppose that $r+1 \geq 2$ and the result holds for r. Let H be a graph with $3(r+1)+3 = 3(r+2)$ vertices of degree 2, and let $w_1, \ldots, w_{r+2}$ be $r+2$ nonadjacent vertices of H.

If there exist $i,\ j \in \{1, \ldots, r+2\}$ with $i \neq j$ such that $\Gamma(w_i) \cap \Gamma(w_j) \neq \emptyset$, then $|\bigcup_{k=1}^{r+2}\{\Gamma(w_k) \cup w_k\}| < 3(r+2)$. Thus, there exists $w \in H$ with degree 2 nonadjacent to w_i for all i. Hence, $\{w, w_1, \ldots, w_{r+2}\}$ is a set of $r+3$ nonadjacent vertices of degree 2, but this is a contradiction. Therefore, $\Gamma(w_i) \cap \Gamma(w_j) = \emptyset$ for all $i \neq j$. Furthermore, if $w \in H$ is adjacent to any w_i for $i \in \{1, \ldots, r+2\}$, then w has degree 2; otherwise, since the number of vertices of degree 2 is $3(r+2)$, there exists $v \in H$ with degree 2 nonadjacent to w_i for all i, and we have seen above that this is not possible.

Now, let a and b be the vertices adjacent to w_{r+2}. If the edge (a, b) does not belong to H, we have that $\{w_1, \ldots, w_{r+1}, a, b\}$ is a set of $r+3$ nonadjacent vertices of degree 2. Thus, the vertices w_1, a, and b form a triangle.

Denote by H^* the graph obtained from H, removing the previous triangle. Therefore, H^* is a graph with $3r+3$ vertices of degree 2, and $r+1$ of them are nonadjacent; by induction hypothesis, H^* contains $r+1$ nonadjacent triangles. Thus, H contains $r+2$ nonadjacent triangles. ∎

To finish this section, we give the proof of Theorem 5.1, using the previous results.

Proof of Theorem 5.1. It is equivalent to show that

$$EX(n; TK_{n-q}) = \{\mathcal{H}(n; TK_{n-q})\}$$

for $n = 4q - k + 1$ with $q \geq 5$, $0 \leq k \leq q - 5$.

Let G be a graph belonging to the set $\{\mathcal{H}(n; TK_{n-q})\}$. By checking the structure of this graph G, it is easy to prove that G does not contain a subgraph homeomorphic to K_{n-q}. Since $|E(G)| = ex(n; TK_{n-q}) = \binom{n}{2} - (2q + k + 2)$, we have that $G \in EX(n; TK_{n-q})$.

In order to show that $EX(n; TK_{n-q}) \subseteq \{\mathcal{H}(n; TK_{n-q})\}$, let G be a graph belonging to $EX(n; TK_{n-q})$. We denote by $H = \overline{G}$. By Theorem 5.3, $|E(H)| = 2q + k + 2$. First, we will prove that $\Delta(H) \leq 2$. Suppose the contrary, that $\Delta(H) \geq 3$.

By applying Lemma 5.4, we have $H \in \mathcal{C}_q^{q+1} - \mathcal{C}_q^q$. Hence, there exists a subset of vertices $\{v_1, \ldots, v_q\}$ of H guaranteeing this property. Furthermore, $|E(H_q)| = q + 1$. We claim there exists $j \in \{1, \ldots, q\}$ such that $\Delta(H_{j-1}) \geq 3$ and $\Delta(H_j) \leq 2$, because otherwise we have $\delta_{H_{i-1}}(v_i) \geq 3$ for each $1 \leq i \leq q$, and

$$|E(H)| \geq 3q + (q + 1) > 2q + k + 2,$$

but this is not possible. Now we distinguish the cases $j \geq k + 1$ and $j \leq k$.

For $j \geq k+1$, we consider the fact that $\Delta(H_{j-1}) \geq 3$ and $\Delta(H_j) \leq 2$. Taking into account property (2) of the definition of $\mathcal{C}_q^{q+1}$ and $|E(H_q)| > 0$, we have $\delta_{H_{i-1}}(v_i) \geq 3$ for $1 \leq i \leq j$ and $\delta_{H_{i-1}}(v_i) \geq 1$ for $j + 1 \leq i \leq q$. Hence,

$$|E(H_q)| \leq 2q + k + 2 - (3j + (q - j)) \leq q - j + 1 \leq q.$$

But this is not possible since $|E(H_q)| = q + 1$.

For $j \leq k$, we have that $\delta_{H_{i-1}}(v_i) \geq 3$ for $1 \leq i \leq j$. If $\Delta(H_k) \leq 1$, then $2|E(H_k)| \leq |V(H_k)|$ and

$$4q - 2k + 1 = |V(H_k)| \geq 2|E(H_k)| \geq 2(q - k + q + 1) = 4q - 2k + 2,$$

and this is a contradiction. Thus, $\Delta(H_k) = 2$ and $\delta_{H_{i-1}}(v_i) \geq 2$ for $j + 1 \leq i \leq k$. Hence,

$$|E(H_q)| \leq 2q + k + 2 - (3j + 2(k - j + 1) + (q - k + 1)) = q - j + 1 \leq q,$$

and this not possible. Thus, $\Delta(H) \leq 2$.

Since $2|E(H)| > |V(H)|$, we have $\Delta(H) \geq 2$ and, consequently, $\Delta(H) = 2$.

Next we are going to study the structure of H. On the one hand, if H has at least $3(k+1)+1$ vertices of degree 2, then by Lemma 5.2 we have that $k+2$ of those vertices $\{w_1, \ldots, w_{k+2}\}$ are nonadjacent. Let $w_{k+3}, \ldots, w_q$ be $q - (k+2)$ vertices of H such that the set $\{w_1, \ldots, w_{k+2}, w_{k+3}, \ldots, w_q\}$ verifies properties (1) and (2) of the definition of $\mathcal{C}_q^s$. For this set of vertices, we have that

$$|E(H_q)| \leq 2q + k + 2 - (2(k + 2) + q - (k + 2)) = q,$$

and therefore, $H \in \mathcal{C}_q^q$, a contradiction. Thus, H has at most $3k+3$ vertices of degree 2. On the other hand, if we denote by n_i the number of vertices of degree i in H, we have that

$$\left.\begin{aligned} 2n_2 + n_1 &= 2(2q+k+2) \\ n_2 + n_1 + n_0 &= 4q - k + 1 \end{aligned}\right\}.$$

Thus, $n_2 = 3k + 3 + n_0 \geq 3k + 3$ and the number of vertices of degree 2 in H is $n_2 = 3k + 3$.

Furthermore, as we have shown previously, H may not have $k + 2$ nonadjacent vertices of degree 2. Since H has $3k+3 \geq 3k+1$ vertices of degree 2, by Lemma 5.2 we have that H has at least $k + 1$ nonadjacent vertices. Hence, H has maximum degree 2 and $3k + 3$ vertices of degree 2, and $k + 1$ of them are nonadjacent. Therefore, by applying Lemma 5.5, H contains $k + 1$ nonadjacent triangles. Additionally, $n_0 = 0$, $n_1 = 4q - 4k - 2$, and the result follows. □

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