ELSEVIER Discrete Mathematics 161 (1996) 45-52

DISCRETE MATHEMATICS

The center of an infinite graph[1]

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Received 16 February 1995

Abstract

In this note we extend the notion of the center of a graph to infinite graphs. Thus, a vertex is in the center of the infinite graph G if it is in the center of an increasing family of finite subgraphs covering G. We give different characterizations of when a vertex is in the center of an infinite graph and we prove that any infinite graph with at least two ends has a center.

1. Introduction

The notion of the center of a tree, introduced by Jordan 1869 [7], and the later introduction of the concept of distance in graphs by Cayley [2] have allowed to develop a theory with a wide field of application, mainly in what is called Location Theory.

On the other hand, König considered in 1916 [8] the notion of infinite graph. Since then, the theory of infinite graphs have been developed following the general theory of (finite) graphs. Thus, we find works dealing with transversality [5, 11], matching [9, 10], planarity [4], etc. in finite graphs (see the excellent survey of Thomassen [12]).

It is possible (and very often, convenient) to imagine an infinite graph as an increasing family of finite graph; in fact, most of the properties that are preserved in that family can be deduced studying the equivalent properties of the infinite graph that the union of the elements of the family defines. Thus, from now on an infinite graph will be a locally finite countable graph. In spite of this consideration, we have not found references in the literature about the concepts derived from the distance in

[1] This work is a part of the Ph.D. thesis 'El centro de una familia creciente de grafos finitos' [3] presented in the University of Seville in December 1994.

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SSDI 0012-365X(95)00274-X

infinite graphs. That is the aim of this paper, more concretely, we are going to define the center of an infinite graph, studying its main properties.

Of course, since the classical definition of the center in finite graphs is based in the eccentricity of the vertices and this notion has no sense in the infinite case, the first problem in our path is to extend the concept of the center to infinite graphs. But, keeping in mind our representation of infinite graphs as an increasing family of finite graphs (and based on the classical argument to find the center of a tree given by Jordan), we can say that a vertex v of an infinite graph G *is in the center of the graph* if there is an increasing sequence of finite subgraphs, $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \subseteq H_n \subseteq \cdots$ such that $\bigcup_n H_n = G$ and $v \in C(H_n)$ for all $n \in N$.

We will use the notations and terminology of [1, 6].

To characterize the center of an infinite graph we will use the notion of Freudenthal ends.

Let G be an infinite graph and let W_1 and W_2 be two subgraphs of G, homeomorphic to R_+. W_1 and W_2 define the same end of Freudenthal if for any finite subgraph K of G, there is a path between W_1 and W_2 in $G - K$. The number of Freudenthal ends of the graph G is denoted by $e(G)$.

2. Location in infinite graphs

Let G be an infinite graph, to prove that a vertex v is in the center of G is to find an increasing family of finite subgraphs where the eccentricity of v is the smallest of all. The problem is to find the family, if that family exists. The following proposition solves the search of the family:

Proposition 2.1. *Let G be an infinite graph and let v be a vertex of G. The following conditions are equivalent:*

1. $v \in C(G)$.
2. $v \in C(G_n^v)$ *for all* $n \in N$, *where*

$$G_n^v = \langle \{u \in V(G)/d(u,v) \leqslant n\} \rangle,$$

with $\langle A \rangle$ denoting the subgraph of G induced by A.

Proof. It is obvious that condition 2 implies condition 1. We are going to prove the other implication. Since $v \in C(G)$, there exists a family of finite subgraphs $\{H_n\}_{n \in N}$ with $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \subseteq H_n \subseteq \cdots$ such that $\bigcup_n H_n = G$ and $v \in C(H_n)$ for all $n \in N$. Given $n \in N$, there is $k \in N$ such that $G_n^v \subseteq H_k$. We are going to prove that $v \in C(G_n^v)$. If $v \notin C(G_n^v)$ then there exists $u \in V(G_n^v)$ such that $e_{G_n^v}(u) < e_{G_n^v}(v) = n$. Therefore, we have that $e_{G_n^v}(u) < e_{G_n^v}(v) = n \leqslant e_{H_k}(v) \leqslant e_{H_k}(u)$. Hence, there is $w \in V(H_k)$ such that $d_{H_k}(u, w) \geqslant n$. Let P be the path in H_k of length $d_{H_k}(v, w)$. Let v^* be a vertex of $P \cap G_n^v$ so that the degree of v^* is smaller in G_n^v than in H_k. The vertex v^* splits P in two paths P_1 *y* P_2, where $v \in V(P_1)$. Let Q be a path giving $d_{H_k}(u, v^*)$, then

$d_{H_k}(u, v^*) \leqslant d_{G_n^v}(u, v^*) \leqslant e_{G_n^v}(u) < n$. However, Q followed of P_2 is a path that joins u with w which has length l smaller than $d_{H_k}(v, w)$. Thus, we have

$$e_{H_k}(u) = d_{H_k}(u, w) \leqslant l < d_{H_k}(v) \leqslant e_{H_k}(v).$$

Therefore, $v \notin C(H_k)$. □

With the help of Proposition 2.1 and knowing the number of Freudenthal ends of the graph we can determine when an infinite tree has a center. Moreover, we can determine the vertices which are in the center.

In the same way as in finite graphs, it is not difficult to characterize the center of a graph when this graph is a tree. So, an infinite tree T has a center if and only if $e(T) \geqslant 2$. And, the vertices of the center are easily to find. A vertex is in the center of T if and only if there is not an edge separating it from the ends. And, two adjacent vertices are in the center of T if and only if the complement of the edge they define has no finite components.

These results can be summed up in a theorem. To formulate this theorem we define for an infinite graph G and a vertex v of G, a *ray starting at* v as a subgraph of G homeomorphic to R_+, where the first vertex is v. We are going to denote $\alpha = \{w_n\}_{n \in N \cup \{0\}}$ as a ray starting at a vertex v so that the distance of w_n to v in the ray is n.

Theorem 2.2. *Let v be a vertex of an infinite tree T. Then, $v \in C(T)$ if and only if there are two disjoint rays starting at v in T.*

Fig. 1 shows an infinite tree where the vertices u and v are in the center. w is not the center because there is an edge separating w from the ends.

Once having studied the problem of location in trees, we focus on the general problem of finding the center of an infinite graph.

The first idea, thinking of Theorem 2.2, is: 'a vertex v is in the center of an infinite graph if there are two rays starting at v such that the distance between them is growing'. Nevertheless, in Fig. 2 the vertex v is not in the center, but there are two rays that the distance between them is growing, although not quickly enough.

Proposition 2.3. *Let v be a vertex of an infinite graph G. If there are two rays $\alpha_1 = \{w_n\}_{n \in N}$ and $\alpha_2 = \{w_{-n}\}_{n \in N}$ starting at v so that $d_G(w_n, w_{-n}) \geqslant 2n - 1$, then v is in the center of G.*

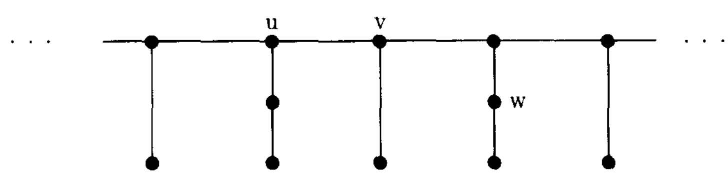

Fig. 1. In this tree the vertices u and v are in the center while w is not.

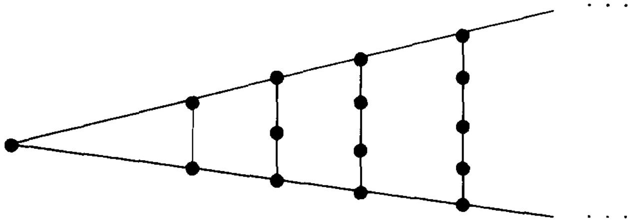

Fig. 2. This graph does not have a center.

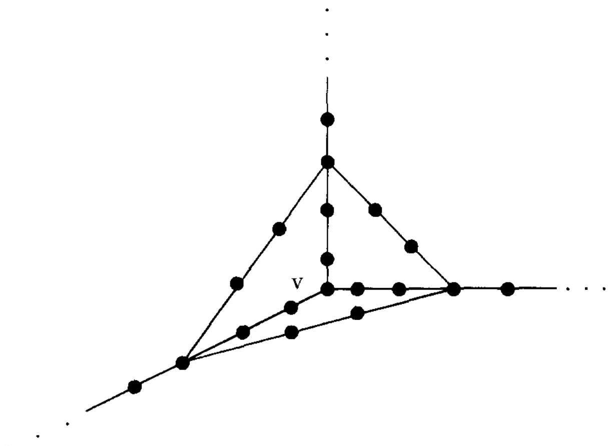

Fig. 3. v is in the center but two rays do not exist according to conditions of Proposition 2.3.

Proof. Proposition 2.1 is sufficient to prove that $v \in C(G_n^v)$ for all $n \in N$. By the conditions of the graph G_n^v, $e_{G_n^v}(v) = n$. Suppose that there is $u \in V(G_n^v)$ such that $e_{G_n^v}(u) = p < e_{G_n^v}(v)$. We take the vertices w_n and w_{-n} that are in the rays α_1 and α_2, respectively. Then,

$$\begin{aligned} 2n - 1 &\leqslant d_G(w_n, w_{-n}) \leqslant d_{G_n^v}(w_n, w_{-n}) \\ &\leqslant d_{G_n^v}(w_n, u) + d_{G_n^v}(u, w_{-n}) \leqslant 2p < 2n, \end{aligned}$$

which leads to a contradiction. Therefore, $v \in C(G_n^v)$. $\square$

Fig. 3 shows that the condition given in Proposition 2.3 is not necessary.

3. Location of the center in terms of the ends of the graph

We are going to prove that if G is an infinite graph, such that $e(G) \geqslant 2$, then G has a center, and that it is possible to characterize the vertices that are in the center.

Theorem 3.1. *If G is a graph with $e(G) \geqslant 2$, then G has a center.*

Proof. In the proof of this Theorem, we will use König's infinity lemma [9] in a way proposed by Erdős (see [4]). Let H be a finite graph separating the ends, such that $G - H$ only has infinite components. We call $H_n = \{v \in V(G)/d_G(u, H) \leqslant n\}$, where $d_G(u, H) = \min_{v \in V(H)} d_G(u, v)$. For every graph of the sequence $(H_n)_{n \in N \cup \{0\}}$, where $H_0 = H$, we consider a shorter path joining two components of $G - H_n$. We denote ϕ_n to be a path of minimum length joining two components of $G - H_n$. As H_0 is finite among all $\{\phi_n \cap H_0\}$ there is a configuaration repeated infinity many times. We deal with the subsequence of $\{\phi_n\}$ containing that configuration and we call it $\{\phi_n^0\}$. Considering $\{\phi_n^0 \cap H_1\}$, we have a sequence of paths contained within H_1 that is finite, therefore there is a configuration appearing infinity many times. We deal with the sequence $\{\phi_n^0\}$ containing the configuration mentioned above and we call it $\{\phi_n^1\}$. By repeating this process we build a 2-way path ϕ such that for all n, $\phi \cap H_n$ gives a path of minimum length joining two components of $G - H_n$. Given $v \in V(\phi)$, two rays exist under the conditions of Proposition 2.3, and thus v is in the center. □

We call *central 2-path* to a subgraph W of G homeomorphic to R that such all its vertices are in the center of G. As a Corollary of the last proof we have:

Corollary 3.2. *Any graph with at least two ends has a central 2-path.*

If $e(G) = 1$, no necessary and sufficient condition for the existence of a center in G has yet been reached. Moreover, once a vertex of the center is found we do not know any condition on the existence of other. For one end, the existence of a vertex in the center does not necessarily imply the existence of a central 2-path. In Fig. 4 we have an example of a graph G, $e(G) = 1$, with an infinite number of vertices in the center, where no single central 2-path exists.

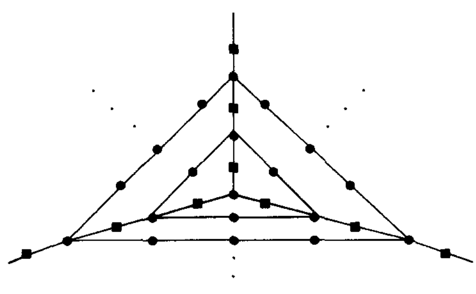

Fig. 4. Only the vertices marked by circles are in the center.

4. An order on the vertices of the graph

Definition 4.1. Let G be a graph. A relation in the vertex set of G is defined by: '$u < v$, if $u = v$ or if there is $n \in N$ such that $G^u_n \subseteq G^v_{n-1}$' (see Fig. 5).

If there is $n \in N$ such that $G^u_n \subseteq G^v_{n-1}$ then for all $m \geqslant n$, $G^u_m \subseteq G^v_{m-1}$.

As a consequence of Proposition 2.1, it is straightforward to check that the relation given in the definition above is a partial order in the vertex set of G.

Definition 4.2. Let G be a graph. An increasing chain of vertices is defined as a sequence of vertices $\{v_n\}_{n \in N}$ of G such that $v_n < v_{n+1}$ for all $n \in N$.

We can localize and characterize the center of a graph using the relation of the order defined above.

Proposition 4.3. *Let G be an infinite graph, then:*

1. *$v \in C(G)$ if and only if there is no $u \in V(G)$ with $v < u$.*
2. *G does not have a center if and only if there exists an increasing chain of vertices in G.*

Proof. (1) Let u be a vertex of the center of G. We suppose that there is a vertex v of G such that $u < v$. Thus, $k \in N$ such that $G^u_k \subseteq G^v_{k-1}$, in particular $d(u, v) \leqslant k - 1$. Therefore, $v \in V(G^v_k)$. Let $w \in V(G^v_k)$, since $G^u_k \subseteq G^v_{k-1}$ then $d(w, v) \leqslant k - 1$. However, the eccentricity of v in the graph G^u_k is smaller than the eccentricity of u.

Conversely, let u be a vertex of G such that no v with $u < v$ exists. We suppose that u is not in the center of G, thus $k \in N$ such that u is not in the center of graph G^u_k, but since G^u_k is a finite graph, v exists in the center of G^u_k. Thus, $e_{G^u_k}(v) < e_{G^u_k}(u) = k$ giving $e_{G^u_k}(v) \leqslant k - 1$. Let $w \in V(G^u_k)$, since v is in the center of G^u_k, $d(w, v) \leqslant e_{G^u_k}$, $d(w, v) \leqslant e_{G^u_k}(v) \leqslant k - 1$. Then $w \in V(G^v_{k-1})$. Thus, $u < v$ and this is a contradiction with the hypothesis.

(2) We suppose that G does not have a center. Let u be a vertex, as it is not in the center, by the first part of this Theorem v_1 exists such that que $u < v_1$. Given v_1 for the same previous reason, v_2 exists such that $u < v_1 < v_2$. Reiterating the process we form an increasing chain.

Conversely, we suppose that there is an increasing chain $\{v_n\}_{n \in N}$. If $v_k < v_{k+1}$ then n_k exists such that $G^{v_k}_{n_k} \subseteq G^{v_{k+1}}_{n_k-1}$. Without losing the generality we suppose that

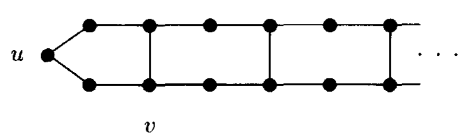

Fig. 5. In this graph $u < v$.

$\bigcup_{k\in N} G_{n_k}^{v_k} = G$. Let u be a vertex of G, then k_0 exists such that $u \in V(G_{n_{k_0}}^{v_{k_0}})$. So, $G_m^u \subseteq G_{n_{k_0}+m}^{v_{k_0}}$ for all $m \in N$. And thus,

$$G_m^u \subseteq G_{n_{k_0}+m}^{v_{k_0}} \subseteq G_{n_{k_0}+m-1}^{v_{k_0+1}} \subseteq \cdots \subseteq G_{m-1}^{v_{k_0+n_{k_0}+1}},$$

then $u < v_{k_0+n_{k_0}+1}$, so u is not in the center of G. □

Given this characterization of the center of a graph in terms of the order on the vertices, we are going to prove that an infinite graph never has finitely many vertices in the center. This result has been proved (Corollary 3.2) for the graphs with at least two ends. We are going to study the graph with one end. In the first place, we prove that there are no finite graphs with only one vertex in the center.

Proposition 4.4. *No infinite graph exists with only one vertex in the center.*

Proof. We suppose that G is an infinite graph with only one vertex in the center, denoted v. Thus, we know that $e(G) = 1$. Given v as the only vertex in the center, we know that:

1. It does not exist $u \in V(G)$ such that $v < u$.
2. Given $u \neq v$, since $u \notin C(G)$, $w \in V(G)$ exists such that $u < w$.

It can easily be proved that $u < v$ for all $u \in V(G)$. We consider a ray $\alpha = \{v_n\}_{n\in N\cup\{0\}}$ starting at v such that $d_G(v, v_k) = k$ for all $k \in N$. If we consider the vertex v_1, $n \in N$ exists such that $G_n^{v_1} \subseteq G_{n-1}^{v}$. Given v_n we observe that:

1. $d(v_1, v_n) = n - 1$ thus $v_n \in (G_n^{v_1})$.
2. $d(v_n, v) = n$ thus $v_n \notin V(G_{n-1}^{v})$.

So, this is a contradiction. □

Once having proved the no existence of infinite graphs with only one vertex in the center, we are going to prove that there are not infinite graphs with a finite number of vertices in the center either.

Corollary 4.5. *No infinite graph exists with a finite number of the vertices in the center.*

Proof. We suppose that G is an infinite graph where $C = \{v_1, \ldots, v_k\}$ is the center set of G. We consider the graph $\tilde{G} = (\tilde{V}, \tilde{A})$. Where $\tilde{V} = (V - C) \cup \{v^*\}$ and $\tilde{A} = \{xy \in A / x, y \notin C\} \cup \{xv^* / x \notin C$ and there is $y \in C$ such that $xy \in A\}$, where v^* represents the equivalence class formed by the vertices of the set C in $\tilde{G}$. Let $u \in V(\tilde{G})$, $u \neq v^*$, since $u \in V(G)$ and $v_i \in C$ exists such that $u < v_i$ in G. Thus, $u < v^*$ in $\tilde{G}$. So, $\tilde{G}$ is an infinite graph with only one vertex in the center. This contradicts Proposition 4.4. □

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