Stabilisation of differential inclusions and PDEs without uniqueness by noise*

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Abstract

We prove that the asymptotic behaviour of partial differential inclusions and partial differential equations without uniqueness of solutions can be stabilised by adding some suitable Itô noise as an external perturbation. We show how the theory previously developed for the single-valued cases can be successfully applied to handle these set-valued cases. The theory of random dynamical systems is used as an appropriate tool to solve the problem.

1 Introduction

The stabilising and destabilising effects produced by noisy terms in the evolution of single valued deterministic systems is now very well known as the literature on this topic reveals (see [1], [4], [5], [13], [18], [20], [22], and the references therein). The importance of these effects in the understanding of the long time behaviour of real systems is now out of any doubt. Indeed, if we assume that the real world is non-deterministic (what seems to be a very sensible fact) and we approximate a real model by a deterministic one, we can find that on some occasions the appearance of noise in the deterministic models could produce dramatic changes in the behaviour (see, e.g. [1], [4], [20]).

However, in many real situations, the real models are better described if we consider some multi-valued or set-valued features. For instance, many systems described by differential equations does not have uniqueness of solutions, or even the problem is better modelled by a differential inclusion. So far, we still have not seen in the literature any paper dealing with the

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effects produced by noise in the asymptotic behaviour of multivalued dynamical systems (in the direction developed in the papers mentioned above).

Our aim in this paper is to start an investigation on this topic. In fact, we aim to show how some of the techniques already developed and successfully applied in the single-valued case can also be adapted to handle with some multivalued situations.

The objective of our investigation in this paper is twofold.

On the one hand, we want to show how a very simple multiplicative noise (in the Itô sense) can produce a stabilising effect on the solutions of a deterministic partial differential inclusion. This is an important fact when we want to stabilise an unstable system by acting on it with an external forcing term. Although there exists a controversy on the use of different kinds of noise (Itô versus Stratonovich), we will not go deeper in this discussion and simply show, in Section 2, a first easy way to produce stabilisation of a deterministic partial differential inclusion. It may be possible to obtain the same result with a much more complicate noisy term, and even one may try to get stabilisation by using linear Stratonovich noise (as in [13]), or much general additive noise (as in [4]), but these will be the topics for some future papers.

On the other hand, we will show in Section 3 how the theory of random dynamical systems can be a suitable and helpful tool in the analysis of the effects produced by noise in some deterministic multivalued dynamical systems. In fact, we will consider a partial differential equation (of reaction-diffusion type) without uniqueness of solutions, and which generates a multi-valued or set-valued dynamical system but without having a global attractor. Then, if we add a high-intensity multiplicative linear Itô noise, we will prove that the stochastic model generates a random dynamical system which possesses a random attractor. This reflects a regularising/stabilising effect of the noise. Even more, in some situations and for a higher intensity of the noise this random attractor becomes a single random point (random equilibria).

2 Stabilising evolution inclusions

2.1 Setting of the problem

Let V be a separable and reflexive Banach space (with norm $||\cdot||$ and inner product $((\cdot,\cdot))$), and consider a Hilbert space H (with norm $|\cdot|$ and inner product $(\cdot,\cdot)$). If we identify H with its dual space, then we can identify H with a subspace of V', so that we have

$$V \hookrightarrow H \hookrightarrow V',$$

where the previous inclusions are continuous and dense. We will denote by $||\cdot||_*$ the norm in V' and by $\langle\cdot,\cdot\rangle$ the duality product between V and V'.

Now, let us consider the following stochastic evolution inclusion in the Ito sense

$$\begin{cases} \dfrac{du\,(t)}{dt} \in Au\,(t) + F\,(u\,(t)) + \sum_{i=1}^{d} B_i u(t)\,\dfrac{dw_i\,(t)}{dt},\ 0 \leq t < +\infty, \\ u\,(0) = u_0 \in H, \end{cases} \tag{1}$$

where $w_1, w_2, ..., w_d$ are mutually independent standard Wiener processes over the same filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $B_i : H \to H$ is a linear operator for $i = 1, ..., d$, $A : V \to V'$

is a linear A operator which is the infinitesimal generator of a strongly continuous semigroup (i.e. of class C_0) denoted by $S(t)$. As we are interested in analysing the behaviour of the variational solutions of (1) (see below for the definition), we need to assume some additional hypotheses ensuring their existence. To be more precise, we need the following assumptions:

Coercivity: There exist $\alpha > 0, \lambda \in \mathbb{R}$ such that

$$-2\langle Au, u\rangle + \lambda|u|^2 \geq \alpha||u||^p, \quad \text{for all} \ \ u \in V, \tag{2}$$

where $p > 1$ is fixed.

Boundedness: There exists $\beta > 0$ such that

$$||Au||_* \leq \beta||u||^{p-1}, \quad \text{for all} \ \ u \in V. \tag{3}$$

Notice that, in the case $p = 2$, condition (2) implies that operator A is the generator of a strongly continuous semigroup (see Dautray and Lions [15, page 388]).

On the other hand, we assume that $F : H \to 2^H$ satisfies:

(F1) F has closed, bounded, convex, non-empty values.

(F2) There exists $C > 0$ such that

$$\text{dist}_H\left(F(u), F(v)\right) \leq C|u - v|, \forall u, v \in H,$$

where $\text{dist}_H(\cdot,\cdot)$ denotes the Hausdorff distance between bounded sets.

(F3) $F(0) = 0$.

Under the preceding assumptions (in fact without assuming (2), (3) and (F3)), Theorem 2.1 in Da Prato and Frankowska [16] ensures the existence of at least one solution $u(\cdot)$ of (1) for any random variable $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}, H)$ with some $p > 2$. By such a solution we mean an adapted process $u(\cdot)$ taking values in H and such that:

1. $u(\cdot, \omega)$ is continuous for $\mathbb{P}$-a.a. $\omega \in \Omega$.
2. For any $T > 0$, $u(\cdot)$ is a mild solution, on the interval $[0, T]$, of the problem

$$\begin{cases} du(t) &= Au(t)\,dt + f(t)\,dt + \sum_{i=1}^d B_i u(t) dw_i(t), \\ u(0) &= u_0, \end{cases} \tag{4}$$

in other words, we have for all $t \in [0, T]$,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds + \sum_{i=1}^d \int_0^t S(t-s)B_i u(s) dw_i(s),$$

being $f(\cdot)$ an adapted process such that

$$f(s,\omega) \in F(u(s,\omega)), \text{ for a.a. } (s,\omega) \in (0,T) \times \Omega,$$

$$\mathbb{E}\left(\int_0^T |f(s)|^2 ds\right) < \infty.$$

Observe that, for the selection $f(\cdot)$, the unique mild solution to (4) is given by $u(\cdot)$.

Remark 2.1 *Although we have imposed that the initial value u_0 belongs to the space $L^p(\Omega, \mathcal{F}_0, \mathbb{P}, H)$ for some $p > 2$, there are some interesting cases in which it is possible to take $p = 2$ (see Da Prato and Frankowska [16] for more details on this point).*

However, in order to apply Itô's formula (or to make an appropriate change of variable) we need to handle a stronger concept of solution, say, either the so-called strong solution or the variational concept of solution. In this paper we consider the latter one.

In addition to the previous assumptions we do assume now (2) and (3). Then (see Pardoux [21]), for any $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}, H)$ $(p > 2)$, there exists a unique *variational solution* of problem (4). In other words, there exists a stochastic process $v(\cdot)$ which belongs to $L^p(\Omega \times (0,T); V) \cap L^2(\Omega; C(0,T; H))$ and that satisfies the equation in (4) in the sense of V', i.e., it follows that, for all $t \in [0,T]$,

$$v(t) = u_0 + \int_0^t (Av(s) + f(s))\ ds + \sum_{i=1}^{d} \int_0^t B_i v(s)\ dw_i(s), \quad \text{for } \mathbb{P} - \text{a.a. } \omega \in \Omega, \tag{5}$$

where the equality is understood in the sense of V'. Now, taking into account that the variational solution (when it exists) is also a mild solution (see, e.g. Caraballo [3]) we have that, for an initial datum $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}, H)$, and for the selection $f(\cdot)$ in (4), we can ensure the existence of a unique variational solution which is also a solution of (1) in the sense of Da Prato and Frankowska. So, from now on, when we deal with a solution of (1) we will be always referring to this one.

2.2 Stabilisation by a linear one-dimensional noise

An important task in the asymptotic behaviour of dynamical systems is the analysis of the stability properties. As it has been mentioned in the Introduction, on some occasions it might be very important to act on a system so that its stability properties can be improved. For instance, the original (deterministic or stochastic) system may be unstable and after our action it becomes stable (or being already stable, we might be able of improving their stability, say, we increase the approaching speed of solutions, etc....). Although this problem has been extensively studied in the literature in the single-valued case, as far as we know, it still has not been considered in a set-valued framework. It is our aim here to establish some preliminary results which can serve as the basis for further investigations in this field.

As it has been noted in the single-valued case, in order to produce a stabilization effect on deterministic (and even stochastic) systems one does not need to perturb the model with a very general noise (provided it is considered in the Itô sense). In fact, a very simple multiplicative one is enough. However, if we consider the noise in the sense of Stratonovich, the analysis requires of a much more complicate structure in the noise (see, e.g. [13] and [4] for a detailed discussion on this topic). Nevertheless, we do not aim to go deeper in this direction in the present paper, since this will be the topic of some future work.

Now we can establish our main stabilisation result

Theorem 2.2 *Assume that $B_2 = \cdots = B_d = 0$ and B_1 is given by $B_1 v = \sigma v$ for $v \in H$, and $\sigma \in \mathbb{R}$. Under the preceding hypotheses, for a large enough σ^2 such that $\gamma = \sigma^2 - \lambda - 2C > 0$ (λ and C are the constants appearing in (2) and (F2)), there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 0$ and a random variable $T(\omega) \geq 0$, such that for any initial datum $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}, H)$ ($p > 2$), any of its corresponding solutions $u(\cdot)$ of problem (1) satisfies*

$$|u(t,\omega)|^2 \leq e^{-\gamma t/2} |u_0(\omega)|^2 \quad \text{for all } t \geq T(\omega), \text{ a.s.} \tag{6}$$

Proof. Let us fix $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$. Then, there exists at least a variational solution $u(\cdot)$ of (1). This means that there exists a selection $f(t) \in F(u(t))$ such that $u(\cdot)$ is solution of (5), which can be rewritten as

$$\begin{cases} du(t) = (Au(t) + f(t))\, dt + \sigma u(t) dw_1(t) \\ u(0) = u_0. \end{cases} \tag{7}$$

Then, if we perform the change of variable $v(t) = \mathrm{e}^{-\sigma w_1(t)} u(t)$, thanks to our assumptions, it follows that the process v solves the random problem

$$\begin{cases} \dfrac{dv(t)}{dt} = Av(t) + \widetilde{f}(t) - \frac{\sigma^2}{2} v(t) \\ v(0) = u_0, \end{cases}$$

where $\widetilde{f}(t) = \mathrm{e}^{-\sigma w_1(t)} f(\mathrm{e}^{\sigma w_1(t)} v(t)) \in \mathrm{e}^{-\sigma w_1(t)} F(\mathrm{e}^{\sigma w_1(t)} v(t))$. Then, taking into account that (F2)-(F3) imply

$$|\widetilde{f}(t)| \leq C|v(t)|,$$

it follows from (2) that

$$\begin{aligned} \frac{d|v(t)|^2}{dt} &= 2\left\langle v(t), Av(t) + \widetilde{f}(t) - \frac{\sigma^2}{2} v(t) \right\rangle \\ &\leq (\lambda - \sigma^2)|v(t)|^2 + 2(v(t), \widetilde{f}(t)) \\ &\leq (\lambda - \sigma^2 + 2C)|v(t)|^2 \\ &= -\gamma |v(t)|^2, \end{aligned}$$

and, consequently,

$$|v(t,\omega)|^2 \leq |u_0(\omega)|^2 \mathrm{e}^{-\gamma t}, \quad \text{for all} \quad t \geq 0, \quad \text{and all } \omega \in \Omega.$$

Thus,

$$|u(t,\omega)|^2 \leq |u_0(\omega)|^2 e^{-\gamma t + \sigma w_1(t)}, \quad \text{for all} \quad t \geq 0, \quad \text{and all } \omega \in \Omega.$$

Noticing now that $\lim_{t\to\infty} \dfrac{w_1(t)}{t} = 0$, $\mathbb{P} - a.s.$, there exists $\Omega_0 \subset \Omega, \mathbb{P}(\Omega_0) = 0$, such that for $\omega \notin \Omega_0$ there exists $T(\omega)$ satisfying, for all $t \geq T(\omega)$,

$$\frac{\sigma w_1(t)}{t} \leq \frac{\gamma}{2},$$

and, therefore,

$$|u(t,\omega)|^2 \leq |u_0(\omega)|^2 e^{-\gamma t/2}, \quad \text{for all} \quad t \geq T(\omega), \quad \text{and all } \omega \notin \Omega_0.$$

■

Remark 2.3 *As we have already mentioned, this result illustrates how we can stabilise a possible unstable system by adding a linear multiplicative noise. It is also possible to develop a similar theory to the existing one in the single-valued case in order to establish sufficient conditions ensuring the stability properties of the stochastic inclusion (1) with even nonlinear operators in the diffusion term (see, e.g. [5]). Moreover, we could also analyse the existence of random attractors for the set-valued dynamical system generated by the stochastic evolution inclusion (with very special kinds of noise: multiplicative or additive) and compare with the deterministic situation. However, instead of doing this with a differential inclusion we will carry out the problem in the next section but working with a partial differential equation without uniqueness of solutions, which also yields to a set-valued dynamical system.*

3 Stabilising a PDE without uniqueness properties

As far as we know, the analysis carried out in the literature concerning the effects of noise in the asymptotic behaviour of deterministic systems has been done in several directions: stabilization of constant solutions (equilibria) by different types of noise (see, e.g., [1], [7], [5], [13]), improvement of the stability of already stable solutions (see, e.g., [5]), existence of exponentially stable stationary solutions for stochastic perturbations of deterministic models ([6]), existence of random attractors for special stochastic perturbations (additive or multiplicative linear noise) of deterministic models possessing global (deterministic) attractors ([8], [4]), and also comparing the structure of the deterministic attractor with the corresponding random one, which can yield to prove that both attractors have "more or less" the same complexity or structure ([9]), or the random one can be somehow simpler ([8], [4]). But the common fact in all the cases we know is that the deterministic problem already possesses a global attractor. However, there is a research line which remains unexplored, and which is very important in our opinion. We are referring to the problem of analyzing if the appearance (or addition) of certain kind of noise in a deterministic model which does not have a global attractor, could ensure the existence of a non-trivial random attractor for the stochastically perturbed one. Later, it could be also interesting to investigate when this random attractor becomes a single (fixed) equilibrium.

Although we could have developed our theory in a single-valued framework, we have preferred to proceed directly in a more general set-valued context in the case of a reaction-diffusion equation (eventually) without uniqueness of solutions.

In what follows we will first recall some definitions and results from the theory of random attractors for set-valued random dynamical systems. Next, and before setting our reaction-diffusion model, we will exhibit a "simple" example of an ordinary differential equation which does not have a global attractor, and will show how the addition of a high intensity linear noise in the sense of Itô ensures the existence of a non-trivial random attractor. If the noise is considered in the sense of Stratonovich, the random perturbed model will not have a random attractor (same behaviour than the deterministic equation) no matter how large is the intensity of the noise. Finally, we will establish the general results for our reaction-diffusion equation without uniqueness.

Of course, these are the first results on this direction and much more analysis has to be done in the future. We hope that this paper could be considered, at least, as a stimulating reason to continue investigating on this topic.

3.1 Preliminaries on set-valued random dynamical systems and random attractors

We summarize the main concepts and results from the theory of random attractors of set-valued (or multi-valued) random dynamical systems developed in the papers [10], [11], [12].

Let (X, d_X) be a complete and separable metric space with the Borel σ-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega$ a measure preserving group of transformations in Ω such that the map $(t, \omega) \mapsto \theta_t\omega$ is measurable and satisfying

$$\theta_0 = Id; \qquad \theta_{t+s} = \theta_t \circ \theta_s = \theta_s \circ \theta_t, \quad \text{for} \;\; t, s \in \mathbb{R}.$$

The set $\mathbb{R}$ is endowed with its Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Definition 3.1 *A set valued map $G : \mathbb{R}^+ \times \Omega \times X \to C(X)$ ($C(X)$ denotes the set of non-empty closed subsets of X) is called a set-valued or multi-valued random dynamical system (MRDS) if is measurable (see Aubin and Frankowska [2], definition 8.1.1) and satisfies*

i) $G(0, \omega) = Id$ *on* X;

ii) $G(t + s, \omega)x = G(t, \theta_s\omega)G(s, \omega)x$ *(cocycle property) for all* $t, s \in \mathbb{R}^+, x \in X, \omega \in \Omega$

Remark 3.2 *Observe that we will use the notation $G(t, \omega)x$ instead of $G(t, \omega, x)$.*

Remark 3.3 *Throughout this paper all assertions about ω are assumed to hold on a θ-invariant set of full measure (unless otherwise stated).*

Using the notation and assumptions from [10] and [11], we have the following definition.

Definition 3.4 *A closed random set $\omega \mapsto \mathcal{A}(\omega)$ is said to be a global random attractor of the MRDS G if:*

i) $G(t, \omega)\mathcal{A}(\omega) \supseteq \mathcal{A}(\theta_t\omega)$, *for all* $t \geq 0$, $\mathbb{P} - a.s$ *(that is, it is negatively invariant);*

ii) *for all $D \subset X$ bounded,*

$$\lim_{t\to+\infty} dist(G(t, \theta_{-t}\omega)D, \mathcal{A}(\omega)) = 0;$$

iii) $\mathcal{A}(\omega)$ *is compact* $\mathbb{P} - a.s.$

Let us now establish two assumptions which will be crucial in the following theorem.

(H1) There exists an absorbing random compact set $B(\omega)$, that is, for $\mathbb{P}-$almost all $\omega \in \Omega$ and every bounded set $D \subset X$, there exists $t(\omega, D)$ such that for all $t \geq t(\omega, D)$

$$G(t, \theta_{-t}\omega)D \subset B(\omega). \tag{8}$$

(H2) $G(t,\omega): X \to C(X)$ is upper semicontinuous, for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$.

Theorem 3.5 *(see [10], [11]) Let assumptions* $(H1)-(H2)$ *hold , the map* $(t,\omega) \mapsto \overline{G(t,\omega)D}$ *be measurable for all deterministic bounded sets* $D \subset X$*, and the map* $x \in X \mapsto G(t,\omega)x$ *have compact values. Then,*

$$\mathcal{A}(\omega) := \overline{\bigcup_{\substack{D\subset X\\ bounded}} \Lambda_D(\omega)} \tag{9}$$

is a global random attractor for G (measurable with respect to $\mathcal{F}$*). It is unique and the minimal closed attracting set.*

Moreover, if the map $x \longmapsto G(t,\omega)x$ *is lower semicontinuous for each fixed* (t,ω)*, then the global random attractor* $\mathcal{A}(\omega)$ *is strictly invariant, i.e.,* $G(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega)$*, for all* $t \geq 0$*.*

Remark 3.6 *Although it is possible to refer the attractor to attract a universe of random sets instead of attracting only deterministic bounded sets (see, e.g. [6]), this last universe is enough for our purposes.*

In what follows, we will consider as our base probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical one generated by a standard two-sided real Wiener process W_t $(t \in \mathbb{R})$. In other words, we consider the Wiener probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) \mid \omega(0) = 0\},$$

equipped with the Borel $\sigma-$algebra $\mathcal{F}$, the Wiener measure $\mathbb{P}$, and the usual uniform convergence on bounded sets of $\mathbb{R}$. Recall that $W_t(\omega) := \omega(t)$ and that the flow θ_t is defined as

$$(\theta_t\omega)(s) = \omega(t+s) - \omega(t), \quad \text{for} \quad t, s \in \mathbb{R}.$$

3.2 A motivating single-valued example: Stabilisation to a random attractor

To illustrate our later analysis, we would like to consider now an example given by an ordinary differential equation which does not possess a global attractor and show the different effects that a very simple noise can produce on the systems depending on the interpretation given to the noise (Itô or Stratonovich).

Consider the following initial value (autonomous) problem

$$\begin{cases} \dot{x}(t) = x(t) + 1 \\ x(0) = x_0, \end{cases} \tag{10}$$

with $x_0 \in \mathbb{R}$. As the solution of (10) is

$$x(t; 0, x_0) = -1 + (x_0 + 1)e^t, \tag{11}$$

it is clear that the dynamical system generated by the equation in (10) does not possess a global attractor.

First, let us assume that a linear Itô noise appears in the equation, i.e, let us consider the following problem

$$\begin{cases} \dot{x}(t) = x(t) + 1 + \sigma x(t)\dfrac{dW_t}{dt} \\ x(s) = x_0, \end{cases} \tag{12}$$

where $\sigma \in \mathbb{R}$ represents the intensity of the noise and $s \in \mathbb{R}$ is the initial time. It can be easily seen (see, e.g., [14], [8]) that this equation generates a single-valued random dynamical system. More precisely, we can first transform (12) into an equivalent problem but for a stochastic equation in the sense of Stratonovich, namely

$$\begin{cases} \dot{x}(t) = (1 - \frac{\sigma^2}{2})x(t) + 1 + \sigma x(t) \circ \dfrac{dW_t}{dt} \\ x(s) = x_0, \end{cases} \tag{13}$$

where the $\circ$ denotes the Stratonovich sense for the stochastic term. Now, in order to obtain the expression for the random dynamical system generated by this equation, we perform a suitable change of variables which transforms our stochastic equation into a random one. Indeed, for a fixed realisation of our Wiener process, i.e., for a fixed $\omega \in \Omega$, and setting $y(t) = e^{-\sigma W_t(\omega)}x(t)$, we obtain

$$\begin{cases} \dot{y}(t) = (1 - \frac{\sigma^2}{2})y(t) + e^{-\sigma W_t(\omega)} \\ y(s) = y_s = e^{-\sigma W_s(\omega)}x_0, \end{cases} \tag{14}$$

whose solution is explicitly given by

$$y(t; s, \omega, y_s) = e^{(1-\frac{\sigma^2}{2})(t-s)}e^{-\sigma W_s(\omega)}x_0 + e^{(1-\frac{\sigma^2}{2})t}\int_s^t e^{-(1-\frac{\sigma^2}{2})r}e^{-\sigma W_r(\omega)}dr.$$

Therefore, the random dynamical system $G(t, \omega)$ generated by our problem is defined as

$$G(t, \omega)x_0 = e^{\sigma W_t(\omega)}y(t; 0, \omega, x_0).$$

If we now choose σ with absolute value large enough so that $1 - \frac{\sigma^2}{2} < 0$, then it is possible to take limits when $s \to -\infty$ (observe that the resulting improper integral below is well defined) yielding

$$\lim_{s\to-\infty} y(t; s, \omega, y_s) = e^{(1-\frac{\sigma^2}{2})t}\int_{-\infty}^t e^{-(1-\frac{\sigma^2}{2})r}e^{-\sigma W_r(\omega)}dr,$$

and, consequently

$$\lim_{s\to-\infty} e^{\sigma W_t(\omega)}y(t; s, \omega, y_s) = e^{(1-\frac{\sigma^2}{2})t}e^{\sigma W_t(\omega)}\int_{-\infty}^t e^{-(1-\frac{\sigma^2}{2})r}e^{-\sigma W_r(\omega)}dr. \tag{15}$$

Denoting now

$$\mathcal{A}(\omega) = \int_{-\infty}^0 e^{-(1-\frac{\sigma^2}{2})r}e^{-\sigma W_r(\omega)}dr,$$

it is straightforward to check that $x(t, \omega) := \mathcal{A}(\theta_t\omega)$ is a stationary solution of the stochastic equation in (13), and the following equality holds

$$\mathcal{A}(\theta_t\omega) = e^{(1-\frac{\sigma^2}{2})t}e^{\sigma W_t(\omega)}\int_{-\infty}^t e^{-(1-\frac{\sigma^2}{2})r}e^{-\sigma W_r(\omega)}dr.$$

Therefore, using the arguments in [14], it can be easily deduced from (15) that $A(\omega)$ is a non-trivial random attractor for our problem (12). So, this kind of Itô noise has produced a stabilisation effect to a random attractor.

However, if we consider the noise in the Stratonovich sense from the very beginning, i.e. if we consider

$$\begin{cases} \dot{x}(t) = x(t) + 1 + \sigma x(t) \circ \dfrac{dW_t}{dt} \\ x(s) = x_0, \end{cases} \tag{16}$$

then, we obtain (repeating the previous analysis) that

$$x(t; s, \omega, y_s) = e^{t-s} e^{\sigma(W_t(\omega) - W_s(\omega))} x_0 + e^t e^{\sigma W_t(\omega)} \int_s^t e^{-r} e^{-\sigma W_r(\omega)} dr$$

and we now cannot take limits as $s \to -\infty$ because the resulting improper integral has no sense now. This and the fact that the first term is not bounded as $s \to -\infty$ imply that the stochastic equation in the sense of Stratonovich does not possess a random attractor for any $\sigma \in \mathbb{R}$. Consequently, the contribution of the Itô noise to the dissipativity of the problem is determining to obtain the stabilisation to the random attractor.

3.3 Setting of the problem

Consider the following stochastic PDE in the Itô sense

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \Delta u(t,x) + f(u(t,x)) = h(x) + \sigma u(t,x) \dfrac{dW_t(t)}{dt}, \ t_0 \leq t < +\infty, \ x \in \mathcal{O}, \\ u(t,x) = 0 \quad \text{, for } t > t_0, \quad x \in \partial\mathcal{O}, \\ u(t_0, x) = u_0(x), \quad \text{for } x \in \mathcal{O}, \end{cases} \tag{17}$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open set with boundary $\partial\mathcal{O}$ regular enough, $u_0, h \in L^2(\mathcal{O})$, and W_t is the two-sided real Wiener process ($t \in \mathbb{R}$) above.on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

As usual we will denote by $|\cdot|$ and $(\cdot,\cdot)$ the norm and inner product in the space $L^2(\mathcal{O})$, by $||\cdot||$ and $((\cdot,\cdot))$ the corresponding ones in the Sobolev space $H_0^1(\mathcal{O})$, and by $\langle\cdot,\cdot\rangle$ the duality product between $H_0^1(\mathcal{O})$ and its topological dual $H^{-1}(\mathcal{O})$ with norm $||\cdot||_*$.

Recall that

$$H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow H^{-1}(\mathcal{O}),$$

where the inclusions are dense and compact and

$$\lambda_1 |u|^2 \leq ||u||^2, \quad \text{for} \ \ u \in H_0^1(\mathcal{O}),$$

where λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\mathcal{O})$.

The hypotheses for the nonlinear term f are the following

f.1) $f \in C(\mathbb{R}; \mathbb{R})$.

f.2) $|f(u)| \leq \gamma |u| + c, \quad c \in \mathbb{R}, \ \gamma > 0.$

Proposition 3.7 *Under assumption f.2) there exist $\alpha, M \in \mathbb{R}$ such that*

$$f(u)u \geq \alpha u^2 - M, \quad \text{for all } u \in \mathbb{R}. \tag{18}$$

Proof. The proof follows by a straightforward application of Young's inequality. The constants α and M will depend on the different ways in which we can apply that inequality. One of them provides $\alpha = -\gamma - \frac{1}{2}$, $M = \frac{c^2}{2}$. ■

Observe that, under the preceding assumptions on the function f, the deterministic problem (i.e. (17) with $\sigma = 0$) may not possess a global attractor. We will show that, for a σ^2 large enough, the MRDS generated by the stochastic problem will have a random attractor. This can be interpreted as a regularizing effect produced by the noise on the deterministic problem.

To construct the MRDS generated by (17), we will perform again the change of variables we used in the precedent motivating example, after having rewritten the problem in its equivalent Stratonovich formulation. Indeed, observe that we can rewrite (17) as

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \Delta u(t,x) + f(u(t,x)) = h(x) - \frac{1}{2}\sigma^2 u(t,x) + \sigma u(t,x) \circ \dfrac{dW_t}{dt}, \; t_0 \leq t < +\infty, \; x \in \mathcal{O}, \\ u(t,x) = 0, \text{for } t > t_0, \quad x \in \partial\mathcal{O}, \\ u(t_0,x) = u_0(x), \quad \text{for } x \in \mathcal{O}. \end{cases} \tag{19}$$

Given again a fixed realisation of our Wiener processes $W_t(\omega)$, we use the following change of variable $v = ue^{-\sigma W_t}$ and obtain a random PDE

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} - \Delta v(t,x) + e^{-\sigma W_t} f\left(v(t,x)e^{\sigma W_t}\right) = h(x)e^{-\sigma W_t} - \frac{1}{2}\sigma^2 v(t,x), \; t_0 \leq t < +\infty, \; x \in \mathcal{O}, \\ v(t,x) = 0, \text{for } t > t_0, \quad x \in \partial\mathcal{O}, \\ v(t_0,x) = u_0(x)e^{-\sigma W_{t_0}}, \quad \text{for } x \in \mathcal{O}. \end{cases} \tag{20}$$

Denoting now

$$g_\omega(t,v) = e^{-\sigma W_t} f(ve^{\sigma W_t}),$$

we can prove the following estimates.

Proposition 3.8 *Under the previous assumptions on f, there exist constants $M_\omega, c_\omega \in \mathbb{R}$ such that*

$$g_\omega(t,v)v \geq \alpha v^2 - M_\omega, \tag{21}$$

$$|g_\omega(t,v)| \leq \gamma|v| + c_\omega. \tag{22}$$

Proof. The proof follows easily from f.2) and (18). In fact, the constants M_ω and c_ω are given by

$$M_\omega = e^{-2\sigma W_t} M, \quad c_\omega = e^{-\sigma W_t} c.$$

■

Denote also $h_\omega(t,x) = h(x)\, e^{-\sigma W_t} \in L^2\left(t_0,T;L^2(\mathcal{O})\right)$ for any $T > t_0$. Then (20) can be rewritten as

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + g_\omega(v) = h_\omega - \frac{1}{2}\sigma^2 v, \ t_0 \le t < +\infty, \\ v(t) = 0 \text{ on } \partial\mathcal{O}, \\ v(t_0) = v_0 = u_0 e^{-\sigma W_{t_0}}, \end{cases} \tag{23}$$

where from now on we will omit the arguments in the function v when no confusion is possible.

Property (22) implies that $g_\omega(t,v) \in L^2(0,T;L^2(\mathcal{O}))$ if $v \in L^2(0,T;L^2(\mathcal{O}))$.

Definition 3.9 *The function $v = v(t,x) \in L^2(t_0,T;H_0^1(\mathcal{O})))$ is called a solution of (23) on (t_0,T), if for arbitrary $\eta \in H_0^1(\mathcal{O})$,*

$$\frac{\partial}{\partial t}(v,\eta) + ((v,\eta)) + (g_\omega(t,v),\eta) + \frac{\sigma^2}{2}(v,\eta) - (h_\omega,\eta) = 0, \tag{24}$$

in the sense of scalar distributions on (t_0,T).

It follows from this definition that $\dfrac{dv}{dt} = \triangle v - \frac{\sigma^2}{2}v - g_\omega(t,v) + h_\omega(t) \in L^2\left(t_0,T;H^{-1}(\Omega)\right)$. Hence, equality (24) is equivalent to the following one:

$$\int_{t_0}^T \left\langle \frac{dv}{dt},\xi \right\rangle dt + \int_{t_0}^T ((v,\xi))\, dt + \frac{\sigma^2}{2}\int_{t_0}^T (v,\xi)\, dt + \int_{t_0}^T (g_\omega(t,v),\xi)\, dt = \int_{t_0}^T (h_\omega,\xi)\, dt, \tag{25}$$

for all $\xi \in L^2(t_0,T;H_0^1(\Omega))$.

Since $\dfrac{dv}{dt}$ belongs to $L^2(t_0,T;H^{-1}(\mathcal{O}))$, an arbitrary solution of (23) belongs to $C([t_0,T],L^2(\mathcal{O}))$ [23, p.261]. Also, the map $t \mapsto \|u(t)\|^2$ is absolutely continuous on $[0,T]$ and a.e. on $[t_0,T]$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|v(t)|^2 &= \left\langle \frac{dv}{dt}, v \right\rangle \\ &= -\|v(t)\|^2 - \frac{\sigma^2}{2}|v(t)|^2 - (g_\omega(t,v(t)),v(t)) + (h_\omega(t),v(t)). \end{aligned} \tag{26}$$

The property $v \in C([t_0,T],L^2(\mathcal{O}))$ allows us to consider the Cauchy problem

$$v(t,x)|_{t=t_0} = v_0(x) \in L^2(\mathcal{O}).$$

Theorem 3.10 *For any $\omega \in \Omega$ and $T > t_0$ there exists at least one solution for problem* (20) *on $[t_0,T]$.*

Proof. The existence of a solution will be proved by the Galerkin approximation method. Let $\{w_j\}_{j=1}^\infty \subset H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ be an orthonormal basis in $L^2(\mathcal{O})$, which consists of the eigenfunctions of $-\triangle$ in $H_0^1(\mathcal{O})$. This basis is complete in $H_0^1(\mathcal{O})$. We denote by $[w_1,...,w_N]$ the space spanned by $\{w_j\}_{j=1}^N$, and by $P_N : L^2(\mathcal{O}) \mapsto [w_1,...,w_N]$ the orthoprojector defined by

$$P_N v = \sum_{j=1}^N (v,w_j)w_j, \ \forall\ u \in L^2(\mathcal{O}).$$

For every $N \geq 1$ we consider $v^N = v^N(t,x) = \sum_{j=1}^{N} c_j^N(t) w_j(x)$, where the unknown functions $\{c_j^N(\cdot)\}_{j=1}^N$ satisfy the following system of ordinary differential equations:

$$\begin{cases} \frac{d}{dt} c_j^N + \lambda_j c_j^N + \frac{\sigma^2}{2} c_j^N + (g_\omega(t, \sum_{j=k}^{N} c_k^N(t) w_k(x)), w_j) = (h_\omega, w_j), \quad j = 1, ...N, \\ v^N(0) = v_0^N \quad \left(\to v_0, \quad \text{in } L^2(\mathcal{O})\right), \end{cases} \tag{27}$$

where λ_i are the eigenvalues of $-\Delta$ in $H_0^1(\mathcal{O})$.

From well-known results on the existence of solutions for ordinary differential equations, for every $N \geq 1$ there exists $T_N > t_0$ such that there exists at least a solution of (27), say v^N, on $[t_0, T_N]$. Now we deduce a priori estimates, which will guarantee that $T_N = T$. If follows from (21) that

$$\begin{aligned} \frac{d}{dt} \left|v^N(t)\right|^2 + 2\|v^N(t)\|^2 &\leq 2|(h_\omega(t), v^N(t))| + 2M_\omega \mu(\mathcal{O}) - 2\alpha \left|v^N(t)\right|^2 \\ &\leq K_1 \left|v^N(t)\right|^2 + K_2(|h_\omega(t)|^2 + 1), \end{aligned}$$

where the constants $K_1, K_2 > 0$ do not depend on N. It follows from the Gronwall lemma that

$$\left|v^N(t)\right| \leq K_3, \text{ for all } t \in [t_0, T],$$

$$\int_{t_0}^{T} \|v^N(t)\|^2 dt \leq K_3,$$

where $K_3 > 0$ does not depend on N. We obtain that $T_N = T$ and also that the sequence $\{v^N\}$ is bounded in $L^2(t_0, T; H_0^1(\mathcal{O})) \cap L^\infty(t_0, T; L^2(\mathcal{O}))$. It follows from these inequalities and (22) that $g_\omega(t, v^N)$ is bounded in $L^2\left(t_0, T; L^2(\mathcal{O})\right)$. Further, in view of (27) and the equality $P_N(\triangle v^N) = \triangle v^N$ (which is true because our basis consists of the eigenfunctions of $-\Delta$ in $H_0^1(\mathcal{O})$) we have

$$v_t^N = \triangle v^N - \frac{\sigma^2}{2} v^N - P_N g_\omega(t, v^N) + P_N h_\omega(t),$$

so that the sequence $\{v_t^N\}$ is bounded in $L^2(t_0, T; H^{-1}(\mathcal{O}))$. Therefore by the Compactness Lemma (see [19]) there is a function $v = v(t,x) \in L^2(0, T; H_0^1(\mathcal{O})) \cap L^\infty(0, T; L^2(\mathcal{O}))$ such that, up to a subsequence,

$$\begin{gathered} v^N \to v \text{ weakly in } L^2(t_0, T; H_0^1(\mathcal{O})) \text{ and weakly star in } L^\infty(t_0, T; L^2(\mathcal{O})), \\ v^N \to v \text{ strongly in } L^2(t_0, T; L^2(\mathcal{O})), \\ v^N(t) \to v(t) \text{ in } L^2(\mathcal{O}) \text{ for a.a. } t \in (t_0, T), \\ v^N(t,x) \to v(t,x) \text{ for a.a. } (t,x) \in (t_0, T) \times \mathcal{O}. \end{gathered} \tag{28}$$

Hence $g_\omega(t, v^N(t,x)) \to g_\omega(t, v(t,x))$ for a.a. $(t,x) \in (t_0, T) \times \mathcal{O}$. Since the sequence $\{g_\omega(t, v^N)\}$ is bounded in $L^2(t_0, T; L^2(\mathcal{O}))$, up to a subsequence $g_\omega(t, v^N) \to \chi$ weakly in $L^2(t_0, T; L^2(\mathcal{O}))$. Then we have that $g_\omega(t, v^N) \to \chi = g_\omega(t, v)$ weakly in $L^2(t_0, T; L^2(\mathcal{O}))$ (see [19]). This fact and (28) allow us to pass to the limit in the equality

$$-\int_0^T (v^N, w_j)\eta_t dt + \int_0^T \left(((v^N, w_j)) + \frac{\sigma^2}{2} \left(v^N, w_j\right) + (g_\omega(t, v^N), w_j) - (h_\omega, w_j) \right) \eta dt = 0 \tag{29}$$

for each fixed $j \geq 1$. In view of the completeness of $\{w_j\}$ in $H_0^1(\mathcal{O})$ we deduce that the limit function u satisfies (24). Thus, $v \in L^2(0,T;H_0^1(\mathcal{O}))$ is a solution of (23). Further, in a standard way by using the Ascoli-Arzelà theorem and the compact embedding $L^2(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ we obtain that $v^N \to v$ in $C\left([t_0,T], H^{-1}(\mathcal{O})\right)$. In particular, $v^N(0) \to v(0)$ in $H^{-1}(\mathcal{O})$, but we have also that $v^N(0) \to v_0$ in $L^2(\mathcal{O})$, so that $v(0) = v_0$. ■

Notice that all the solutions whose existence is ensured by the previous theorem can be extended to be defined for all $t \geq t_0$, by simply concatenating solutions. This is important in order to construct the dynamical system. Indeed, for any $u_0 \in L^2(\mathcal{O})$ and $\omega \in \Omega$, let us denote by $\mathcal{D}(\omega,u_0)$ the set of all solutions (globally defined in time) v of (20) corresponding to $t_0 = 0$, and set

$$G(t,\omega)u_0 = \bigcup_{v\in\mathcal{D}(\omega,u_0)} e^{\sigma W_t(\omega)} v(t). \tag{30}$$

Then, G is a set-valued random dynamical system. The proof follows the same lines that the ones in [12], so the reader is referred to this reference for the details.

We also note that if we additionally assume that $f \in C^1(\mathbb{R};\mathbb{R})$ and the derivative f' is bounded below, i.e.

$$f'(u) \geq -C, \tag{31}$$

then it is easy to see that $\frac{\partial}{\partial v} g_\omega(t,v) \geq -C$, also. Then in a standard way one can check that we have uniqueness of the Cauchy problem, and the map G defines a single-valued random dynamical system, i.e. it is a cocycle. We note that the result that we prove in the following section is also new for this case.

It is also important to point out that under condition (31) it is possible to obtain the existence and uniqueness of the Cauchy problem for the original stochastic equation (17) (see e.g. [21]). However, without this additional assumption we failed to obtain a theorem on existence of solutions due to some difficulties in the proof of the measurability of the solution.

3.4 Existence of a random attractor

We can now prove the existence of a random attractor for the set-valued random dynamical system defined in (30). We will check that the assumptions in Theorem 3.5 hold.

Proposition 3.11 *Under the conditions f.1)-f.2), if σ is such that $\beta := \lambda_1 + \alpha + \frac{\sigma^2}{2} > 0$, then there exists $r_1 : \Omega \to \mathbb{R}^+$ measurable such that, for all $R > 0$ and all $\omega \in \Omega$, there exists $t(\omega,R) > 1$ satisfying*

$$\|G(t-1,\theta_{-t}\omega)B_R\| \leq r_1(\omega), \quad \textit{for all} \;\; t \geq t(\omega,R),$$

where B_R denotes the ball in H centered at 0 and with radius R.

Proof. Let v be a solution of (20). From (26) and (21) we obtain

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2 + \|v(t)\|^2 + \alpha|v(t)|^2 - Me^{-2\sigma W_t} \leq |h|\,|v(t)|e^{-\sigma W_t} - \frac{1}{2}\sigma^2|v(t)|^2,$$

and, consequently

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2 + \left(\lambda_1 + \alpha + \frac{1}{2}\sigma^2\right)|v(t)|^2 - Me^{-2\sigma W_t} \leq |h|\,|v(t)|e^{-2\sigma W_t}. \tag{32}$$

By Young's inequality we deduce the existence of positive constants $c_1, c_2 > 0$, such that

$$\frac{d}{dt}|v(t)|^2 + \beta|v(t)|^2 \leq (c_1|h|^2 + c_2)e^{-2\sigma W_t}.$$

By the Gronwall lemma in $[t_0, -1]$, with $t_0 \leq -1$ we get

$$|v(-1)|^2 \leq e^{-\beta(-1-t_0)}|u_0|^2 e^{-2\sigma W_{t_0}} + e^{\beta}\int_{t_0}^{-1} e^{\beta s}(c_1|h^2| + c_2)e^{-2\sigma W_s})\,ds.$$

Thus, given $R > 0$ and considering the bounded set B_R, there exists $t(\omega, R) \leq -1$ such that, for all $t_0 \leq t(\omega, R)$ and all $u_0 \in B_R$ we have that

$$|v(-1)|^2 \leq r^2(\omega),$$

with

$$r^2(\omega) = e^{\beta}(1 + \int_{-\infty}^{-1} e^{\beta s}(c_1|h^2| + c_2)e^{-2\sigma W_s})\,ds).$$

It is enough to choose $t(\omega, R)$ such that

$$e^{\beta(1+t_0)}e^{-2\sigma W_{t_0}}|R|^2 \leq 1, \text{ for all } t_0 \leq t(\omega, R),$$

since

$$e^{\beta t}e^{-2\sigma W_t} \to 0 \quad \text{as} \quad t \to -\infty, \quad \mathbb{P} - a.s.$$

The measurability of $r(\omega)$ follows, from example, as in Crauel and Flandoli [14].

Let $y \in G(-t_0 - 1, \theta_{t_0}\omega)B_R$. Then, $y = e^{\sigma W_{-1} - \sigma W_{t_0}}z(-t_0 - 1)$, being z a solution of (20) with $z(0) = u(0) \in B_R$ and with $\theta_{t_0}\omega$ instead of ω. Then,

$$\frac{\partial z}{\partial t} - \Delta z + e^{-\sigma(W_{t+t_0} - W_{t_0})} f\left(ze^{\sigma(W_{t+t_0} - W_{t_0})}\right) = he^{-\sigma(W_{t+t_0} - W_{t_0})} - \frac{1}{2}\sigma^2 z. \tag{33}$$

If we now make the change $v(t) = z(t)e^{-\sigma W_{t_0}}$ we get

$$\frac{\partial v}{\partial t} - \Delta v + e^{-\sigma W_{t+t_0}} f\left(ve^{\sigma W_{t+t_0}}\right) = he^{-\sigma W_{t+t_0}} - \frac{1}{2}\sigma^2 v, \tag{34}$$

with $v(0) = u_0 e^{-\sigma W_{t_0}}$. If we write $\tau = t + t_0$ we have

$$\frac{\partial v}{\partial \tau} - \Delta v + e^{-\sigma W_{\tau}} f\left(ve^{\sigma W_{\tau}}\right) = he^{-\sigma W_{\tau}} - \frac{1}{2}\sigma^2 v, \tag{35}$$

and $v(t_0) = u_0 e^{-\sigma W_{t_0}}$. Thus, by the previous arguments

$$|v(-1)|^2 \leq r^2(\omega), \text{ for all } t_0 \leq t(\omega, B_R),$$

so that

$$|y|^2 \leq e^{2\sigma W_{-1}} r^2(\omega) = r_1^2(\omega).$$

■

Proposition 3.12 *Given $u_n \to u_0$ weakly in $L^2(\mathcal{O})$, $t_n \to t_0$ and $\omega_n \to \omega_0$, with $y_n \in G(t_n,\omega_n)u_n$, there exists a subsequence y_{n_k} converging to $y_0 \in G(t_0,\omega_0)u_0$. As a consequence, the maps $(t,\omega,x) \mapsto G(t,\omega)x$, $(t,\omega) \mapsto \overline{G(t,\omega)D}$ (where D is any bounded set of $L^2(\mathcal{O})$) are measurable, G possesses an absorbing random compact set, has compact values, and $G(t,\omega)$ is upper semicontinuous for all $(t,\omega) \in \mathbb{R}\times\Omega$.*

Proof. Take $T > 0$ such that $t_n, t_0 \in [0,T]$. For $y_n \in G(t_n,\omega_n)u_n$ there exist solutions of (20) $v_n(\cdot)$ with $v_n(0) = u_n$ such that $y_n = e^{\sigma W_{t_n}} v_n(t_n)$.

If we multiply (20) by v_n (see (26)) using (21) we get, for $W_t^n = W_t(\omega_n)$,

$$\frac{1}{2}\frac{d}{dt}|v_n(t)|^2 + \|v_n(t)\|^2 \le M_1 e^{-2\sigma W_t^n}(|h|^2+1) + (-\alpha - \frac{\sigma^2}{2} + \frac{1}{2})|v_n|^2.$$

As $\omega_n \to \omega_0$ and $t \in [0,T]$ we have a uniform bound

$$|W_t^n| \le K, \quad \forall t \in [0,T],$$

so that

$$\frac{d}{dt}|v_n(t)|^2 + 2\|v_n(t)\|^2 \le M_2 + M_3|v_n|^2$$

and so

$$|v_n(t)|^2 + 2\int_s^t \|v_n(s)\|^2 \le M_2(t-s) + M_3\int_s^t |v_n(s)|^2\, ds. \tag{36}$$

Now, by Gronwall's lemma, we obtain that v_n are bounded in $L^\infty(0,T;L^2(\mathcal{O}))\cap L^2(0,T;H_0^1(\mathcal{O}))$.

Now, taking subsequences if necessary, and using the compactness Lemma [19], we have that

$$\begin{aligned}
&v_n \to v \text{ weakly in } L^2(0,T;H_0^1(\mathcal{O})), \text{ and strongly in } L^2(0,T;L^2(\mathcal{O})),\\
&v_n \to v \text{ weakly star in } L^\infty(0,T;L^2(\mathcal{O})),\\
&v_n(t) \to v(t) \text{ strongly in } L^2(\mathcal{O}), \text{ for almost all } t \in [0,T],\\
&\frac{dv_n}{dt} \to \frac{dv}{dt} \text{ weakly in } L^2(0,T;H^{-1}(\mathcal{O})),\\
&v_n(t,x) \to v(t,x) \text{ for almost all } (t,x) \in [0,T]\times\mathcal{O}.
\end{aligned}$$

Thus,

$$g_n(t,x) = e^{-\sigma W_t^n} f(e^{\sigma W_t^n} v_n(t,x)) \to g(t,x) = e^{-\sigma W_t} f(e^{\sigma W_t} v(t,x)) \text{ for a.a. } (t,x) \in [0,T]\times\mathcal{O}.$$

Moreover, we also have that $|g_n|_{L^2(0,T;H)} \le C$, so that arguning as in the proof of Theorem 3.10 we have $g_n \to g$ weakly in $L^2(0,T;L^2(\mathcal{O}))$. Thus, $v(\cdot)$ is a weak solution of (20). Let us see that $v(0) = v_0$. Since $L^2(\mathcal{O}) \hookrightarrow H^{-1}(\mathcal{O})$ compactly, from the Ascoli-Arzelà theorem we get that $v_n \to v$ in $C([0,T];H^{-1}(\mathcal{O}))$. Then, for every $t_n \to t_0$ we have that $v(t_n) \to v(t_0)$ strongly in $H^{-1}(\mathcal{O})$. Thanks to the uniform bound $|v_n(t_n)| \le C$, a standard argument gives $v(t_n) \to v(t_0)$ weakly in $L^2(\mathcal{O})$. In particular,

$$v_n(0) \to v(0) = v_0 = u_0 \text{ weakly in } L^2(\mathcal{O}). \tag{37}$$

If we finally prove that $v(t_n) \to v(t_0)$ strongly in $L^2(\mathcal{O})$ we would finish the proof, since then

$$y_n \to e^{\sigma W_{t_0}} v(t_0) \in G(t_0,\omega_0)u_0.$$

Now, from (37) we get that

$$|v(t_0)| \leq \liminf_{n \to +\infty} |v_n(t_n)|.$$

Let us finally prove that

$$\limsup_{n \to +\infty} |v_n(t_n)| \leq |v(t_0)|. \tag{38}$$

Let us define the non-increasing (see (36)) and continuous functions

$$\begin{aligned} J_n(t) &= |v_n(t)|^2 - M_2 t - M_3 \int_0^t |v_n(s)| \, ds, \\ J(t) &= |v(t)|^2 - M_2 t - M_3 \int_0^t |v(s)| \, ds. \end{aligned}$$

As $v_n(t) \to v(t)$ in $L^2(\mathcal{O})$, for almost all $t \in [0,T]$, and $v_n \to v$ strongly in $L^2(0,T;L^2(\mathcal{O}))$, we have that $J_n(t) \to J(t)$ for almost all $t \in [0,T]$. Moreover, given $0 < t_m < t_0$ such that $v_n(t_m) \to v(t_0)$ in $L^2(\mathcal{O})$, using that J_n are non-increasing and J is continuous, we get that given $\varepsilon > 0$ there exist t_m and $N(\varepsilon, t_m)$ such that

$$\begin{aligned} J_n(t_n) - J(t_0) &\leq J_n(t_n) - J_n(t_m) + J_n(t_m) - J(t_m) + J(t_m) - J(t_0) \\ &\leq J_n(t_n) - J_n(t_m) + |J_n(t_m) - J(t_m)| + \varepsilon \\ &\leq 0 + \varepsilon + \varepsilon, \end{aligned}$$

if $n \geq N$ (note that $t_n > t_m$ for n large).

Thus,

$$\limsup_{n \to +\infty} J_n(t_n) = \limsup_{n \to +\infty} |v_n(t_n)|^2 - M_2 t_0 - M_3 \int_0^{t_0} |v(s)|^2 \, ds \leq J(t_0),$$

from which we have (38), and so the result holds.

As a consequence, $G(t,\omega)$ has compact values. A standard argument (see, for instance, Corollary 7 in [24]) implies that the map $x \to G(t,\omega)x$ is upper semicontinuous. Finally, the measurability of the maps $(t,\omega,x) \mapsto G(t,\omega)x$, $(t,\omega) \mapsto \overline{G(t,\omega)D}$ and the existence of an absorbing random compact set follow from Lemma 2 in Kapustyan [17] and Proposition 3.11. ■

We finally have, as a consequence of Proposition 3.12 and Theorem 3.5, the following theorem:

Theorem 3.13 *There exists a compact random attractor associated to* (17).

As we remarked before, we have obtained that the random equation (23) possesses a random attractor, despite the fact that equation (17) in the deterministic case (i.e. with $\sigma = 0$) may not possess a global attractor, which can be interpreted as a regularizing effect produced by the noise on the deterministic problem. This result is also new in the single-valued setting (which appears assuming condition (31)), which is a particular case of our theorem. It is interesting to point out that in the single-valued framework the existence of the global random attractor could be established by using for example the arguments of [14]. However, the proof of the existence of a compact absorbing set is not suitable for the set-valued case, as we do not have enough regularity of solutions in order to justify the estimates in H^1 spaces.

3.5 Stabilisation to a single equilibrium

We have just proved that the addition of noise can imply the existence of a random attractor for a stochastic perturbation of a deterministic model which could not have it previously. Although we can interpret this result as a kind of stabilisation or regularization for the deterministic model, it does not say anything about the stability or attractivity of the equilibria for the deterministic system (if they exist). It may happens, as we will show below, that a deterministic equation possesses an equilibrium (or more than one) which is not (or not known to be) stable, then the appearance of a high intensity noisy term can ensure the existence of a random attractor which is given by a single deterministic point (which is a steady state solution) and which attracts any other solution. In other words, some improvement has been produced in the behaviour of the system. Let us illustrate this with the following example.

Let us assume that the constant $c = 0$ in assumption $f.2)$ and that $h = 0$ (for simplicity). Then, it follows that $f(0) = 0, \alpha = -\gamma$ and $M = 0$, and therefore $u \equiv 0$ is an equilibrium of the deterministic and the stochastic problems, i.e. (17) with $\sigma = 0$ and $\sigma \neq 0$.

Then, arguing as in the proof of Proposition 3.11, we obtain from (32)

$$\frac{d}{dt}|v(t)|^2 + 2\left(\lambda_1 - \gamma + \frac{1}{2}\sigma^2\right)|v(t)|^2 \leq 0.$$

Denoting $\beta_1 = 2\left(\lambda_1 - \gamma + \frac{1}{2}\sigma^2\right)$, we obtain for any $u_0 \in H$ and any $t_0 \leq t$

$$|v(t)|^2 \leq \mathrm{e}^{-\beta_1(t-t_0)-2\sigma W_{t_0}}|u_0|^2$$

and

$$|u(t)|^2 \leq \mathrm{e}^{-\beta_1(t-t_0)+2\sigma(W_t - W_{t_0})}|u_0|^2.$$

Then, observe the following facts:

- If $\lambda_1 - \gamma > 0$, the null solution of the deterministic problem attracts any other solution starting in any initial point u_0. Futhermore, there exists a global attractor which is given by $\mathcal{A} = \{0\}$. Then for any $\sigma \in \mathbb{R}$, the stochastically perturbed system also possesses a random attractor which is given by $\mathcal{A}(\omega) = \{0\}$.
- If $\lambda_1 - \gamma < 0$, then the deterministic problem may not have a global attractor, and even the steady state null solution can be unstable (in fact, it is known to be unstable when, for instance, we are in the linear case, i.e., $f(u) = -\gamma u$). In this case, with a stochastic perturbation of sufficently large intensity σ, namely, for σ such that $\beta_1 > 0$, we can ensure that there exists a random attractor which is again given by $\mathcal{A}(\omega) = \{0\}$. This means that any solution starting at any point should approaches the steady state solution as $t_0 \to -\infty$ (pullback sense) and, what is more important, in the forward sense, i.e. when $t \to \infty$.

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