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ASYMPTOTIC BEHAVIOUR OF A NON-AUTONOMOUS LORENZ-84 SYSTEM

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Dedicado a la Memoria de nuestro querido Maestro y Amigo Pepe Real,
con toda nuestra Gratitud.

Abstract. The so called Lorenz-84 model has been used in climatological studies, for example by coupling it with a low-dimensional model for ocean dynamics. The behaviour of this model has been studied extensively since its introduction by Lorenz in 1984. In this paper we study the asymptotic behaviour of a non-autonomous Lorenz-84 version with several types of non-autonomous features. We prove the existence of pullback and uniform attractors for the process associated to this model. In particular we consider that the non-autonomous forcing terms are more general than almost periodic. Finally, we estimate the Hausdorff dimension of the pullback attractor. We illustrate some examples of pullback attractors by numerical simulations.

1. **Introduction and setting of the problem.** Weather and climate prediction are difficult tasks, because of the complexity of the atmospheric evolution. Most of the real world models concerning the atmosphere involve a large number of variables and parameters. Therefore, it is practically impossible to perform detailed studies of their dynamical properties. There is experimental evidence [31] that low-dimensional attractors appear in some hydrodynamical flows just after the onset of turbulence. As a consequence, low-dimensional models have attracted the attention of meteorologists, mathematicians and physicists over the last decades.

In recent years, a great deal of interest has been focused on studying the complexity of nonlinear dynamical systems. The Lorenz classical model for thermal convection in the atmosphere [23] was the first chaotic system discovered and has been one of the most extensively investigated. As a modification of this model of turbulence generation, in [24] Lorenz derived a simple but powerful model based on the 'general circulation" of the atmosphere. This model proposed by Lorenz in 1984 is a Galerkin truncation of the Navier-Stokes equations and gives the simplest approximation to the general atmospheric circulation at midlatitude.

Lorenz-84 model has been used in climatological studies and its behaviour has been studied extensively. In this paper we analyze the non-autonomous Lorenz-84 model with non-autonomous

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forcing terms. The model is defined by the following three ordinary differential equations

$$\left.\begin{aligned} x' &= -ax - y^2 - z^2 + aF(t), \\ y' &= -y + xy - bxz + G(t), \\ z' &= -z + bxy + xz, \end{aligned}\right\} \tag{1}$$

with initial condition

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0, \tag{2}$$

where $t_0 \in \mathbb{R}$, $a > 0$, $b \in \mathbb{R}$ and $F, G : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions such that satisfy

$$\int_{-\infty}^{t} e^{ls} F^2(s) ds < +\infty, \int_{-\infty}^{t} e^{ls} G^2(s) ds < +\infty \quad \forall t \in \mathbb{R}, \tag{3}$$

where $l := min\{a, 1\}$.

F and G represent thermal forcings: F is meant to be the symmetric cross-latitude external-heating contrast and G the asymmetric heating contrast between oceans and continents, and a and b are positive parameters ($a < 1$ and $b > 1$). Conventionally $a = \frac{1}{4}$ and $b = 4$. The variable x represents the intensity of the symmetric globe-encircling westerly wind current and also the poleward temperature gradient which is assumed to be in permanent equilibrium with it. The variables y and z represent the cosine and sine phases of a chain of superposed large-scale eddies, which transport heat poleward at a rate proportional to the square of their amplitude, and transport no angular momentum at all.

In [24], some early ideas concerning the general circulation of the atmosphere are reviewed. Moreover, Lorenz proved that for different intensities of the axially symmetric and asymmetric thermal forcing, Lorenz-84 model may possess one or two stable steady-state solutions, one or two stable periodic solutions, or irregular (aperiodic) solutions. Numerical and analytical explorations of model introduced by Lorenz in 1984 can be found for instance in [26], [27] and [30], and the bifurcation diagram of this model has been analysed in [29]. In [25], Lorenz pointed out that F and G should be allowed to vary periodically during a year. In particular, F should be larger in winter than in summer. However, in his numerical study he kept G fixed, identifying $(F, G) = (6, 1)$, and $(F, G) = (8, 1)$, with summer, respectively winter, conditions. He introduced a periodical variation of the parameter F between summer and winter conditions. In [3] the authors considered that the differential equations of Lorenz-84 model are subject to periodic forcing term, where the period is one year and different types of strange attractors are found in four regions (chaotic ranges). Moreover the related routes to chaos are discussed.

There are basically two ways to define attraction of a compact and invariant non–autonomous set for a process on a metric space. The first, and perhaps more obvious, corresponds to the attraction in the sense of Lyapunov stability, which is called *forward attraction*, and involves a moving target, while the second, called *pullback attraction*, involves a fixed target set with progressively earlier starting time. In general, these two types of attraction are independent concepts, while for the autonomous case, they are equivalent. Physically, the pullback attractor provides a way to assess an asymptotic regime at time t (the time at which we observe the system) for a system starting to evolve from the remote past. The pullback dynamics contains interesting dynamical properties, which allow us to understand the forward attraction. We would like to emphasize, for instance, Theorem 3.20 in [14], which provides some information about a form of forwards convergence of the cocycle mapping, which is different from that in the definition of a forward attractor. In this theorem, within the framework of cocycle dynamical systems, it is proved that the closure of the union of the fibers of the pullback attractor is forward attracting under the assumption that there exists a compact pullback absorbing set B for the skew product flow such that the cocycle mapping on B is contained in B (see [7], [14] for more details). The first aim of this paper is to show the existence of a pullback and a uniform attractor for the process associated to (1)-(2). The fact that F and G are non-autonomous are the main novelties of our problem.

The dynamics induced by the class of periodic, almost periodic or almost automorphic continuous functions is not robust to small changes in the forcing term in the sense that a bounded entire solution corresponding to a perturbed forcing term may not belong to this class. Then, Kloeden and Rodrigues presented in [15] an alternative extension of periodic and almost periodic functions. Namely, the introduce the class of functions consisting of uniformly continuous functions, defined on the real line and taking values in a Banach space, with the property that a bounded entire solution of a non autonomous ODE belongs to this class when the forcing term does.

The fact that the forcing term belongs to the class more general than almost periodic in a non-autonomous systems means that the external force of the phenomena modeled possesses different types of intensity along time. For example, a tornado is a meteorological problem which can have different steps: initially, the intensity of the external force can be periodic for a certain period of time, then the type of intensity may change and can become constant for another period of time, later on, it may become periodic again but with a different period than the first one, and continue in this way recursively. Then, this phenomena can be modeled by a non–autonomous system where the forcing term belongs to this class more general than almost periodic. For this reason, we also consider that (1) includes forcing terms which belong to this class of functions introduced by Kloeden and Rodrigues in [15] .

On the other hand, the theory of topological dimension [13], developed in the first half of the 20th century, is of little use in giving the scale of dimensional characteristics of attractors. The point is that the topological dimension can take integer values only. Hence the scale of dimensional characteristics compiled in this manner turns out to be quite poor. For investigating attractors, the Hausdorff dimension of a set is much better. This dimensional characteristic can take any nonnegative value. In [1], a simple estimate for the dimension of attractors of Lorenz system has been obtain. In [19] the authors interpret the Hausdorff measure as an analogue of a Lyapunov function, and they estimate the Hausdorff dimension of strange attractors, particularly the (generalized) Lorenz systems. Recently in [22] Lyapunov-type functions are introduced into upper estimates for the Hausdorff dimension of negatively invariant sets of cocycles. For this purpose, the methods proposed in [2, 19, 20, 21] are further developed. The second aim of this paper is to estimate the Hausdorff dimension of the pullback attractor of (1)-(2) using the recent method proposed by Leonov *et al.* in [22].

The structure of the paper is as follows. In Section 2 we briefly recall some abstract results about the theory of pullback and uniform attractors. Some sufficient conditions ensuring the existence of such type of attractors for (1)-(2) are collected in Section 3. We consider that the differential equations of non-autonomous Lorenz-84 model are subjected to periodic forcing terms in Section 4. In Section 5 we consider that (1) includes forcing terms which belong to a class of functions more general than almost periodic. We use recent results proposed by Kloeden and Rodrigues [15] to prove that the solution of (1)-(2) belongs to this class when the forcing terms do. Finally, in Section 6 we estimate the Hausdorff dimension of the pullback attractor associated to (1)-(2). We obtain estimates that are similar to those for the autonomous case (cf. [19]) and illustrate these results with some numerical simulations.

2. Abstract results on Pullback and Uniform Attractors.

In this section we recall some abstract results on the theory of pullback attractors (see [5, 6, 7]) and we establish some results on the theory of uniform attractors (see [7, 10]).

2.1. Processes and attractors.

Let (X, d_X) be a metric space, and let us denote $\mathbb{R}^2_d = \{(t,t_0) \in \mathbb{R}^2 : t_0 \le t\}$.

A process on X is a mapping U such that $\mathbb{R}^2_d \times X \ni (t,t_0,x) \mapsto U(t,t_0)x \in X$ with $U(t_0,t_0)x = x$ for any $(t_0,x) \in \mathbb{R} \times X$, and $U(t,r)(U(r,t_0)x) = U(t,t_0)x$ for any $t_0 \le r \le t$ and all $x \in X$.

Definition 1. *Let U be a process on X. U is said to be continuous if for any pair $t_0 \le t$, the mapping $U(t,t_0) : X \to X$ is continuous.*

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. Let $\mathcal{D}$ be a nonempty set of parameterized families of nonempty bounded sets $\widehat{D} = \{D(t) = D \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $D \subset X$ is a bounded set. In what follows, we will consider a fixed universe of attraction $\mathcal{D}$ and throughout our analysis the concepts of absorption and attraction will be referred to this fixed universe.

Definition 2. *It is said that $\widehat{D}_0 \subset \mathcal{P}(X)$ is pullback absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\widehat{t}_0(t,\widehat{D}) \le t$ such that*

$$U(t,t_0)D(t_0) \subset D_0(t) \quad \text{for all } t_0 \le \widehat{t}_0(t,\widehat{D}).$$

We denote by $\mathrm{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_X(\mathcal{O}_1,\mathcal{O}_2) = \sup_{x\in\mathcal{O}_1} \inf_{y\in\mathcal{O}_2} d_X(x,y) \quad \text{for } \mathcal{O}_1,\, \mathcal{O}_2 \subset X.$$

Definition 3. *It is said that* $\widehat{D}_0 \subset \mathcal{P}(X)$ *is pullback attracting if*

$$\lim_{t_0 \to -\infty} \mathrm{dist}_X(U(t,t_0)D(t_0), D_0(t)) = 0 \quad \textit{for all } \widehat{D} \in \mathcal{D}, \quad t \in \mathbb{R}.$$

There exists now a wide literature on pullback attractors (see, e.g., [16, 17, 28]), but we would like to emphasize that these notions take the final time as fixed and moves the initial time backwards towards $-\infty$. Note that this does not mean that we are moving backwards in time, but we consider the state of the system at time t that had begun in earlier and earlier initial instants t_0, i.e., $t_0 \to -\infty$.

Definition 4. *Let* $\widehat{D}_0(t) \subset \mathcal{P}(X)$*. This family is said to be invariant with respect to the process* U *if*

$$U(t,t_0)D_0(t_0) = D_0(t) \quad \textit{for all } t_0 \leq t.$$

Denote the omega-limit set of $\widehat{D}$ by

$$\Lambda(\widehat{D},t) := \bigcap_{s \leq t} \overline{\bigcup_{t_0 \leq s} U(t,t_0)D(t_0)}^X \quad \text{for all } t \in \mathbb{R}, \tag{4}$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

Definition 5. *The family of compact sets* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *is said to be a pullback attractor associated to the continuous process* U *if is invariant, attracts every* $\{D(t)\} \in \mathcal{D}$ *and minimal in the sense that if* $\{C(t)\}_{t\in\mathbb{R}}$ *is another family of closed attracting sets, then* $\mathcal{A}(t) \subset C(t)$ *for all* $t \in \mathbb{R}$.

The general result on the existence of pullback attractor is a generalization of the abstract theory for autonomous dynamical systems [32]:

Theorem 6. *[Crauel et al.* [11]*, Schmalfuss* [28]*] Assume that there exists a family of compact pullback absorbing sets* $\{B(t)\}_{t\in\mathbb{R}}$*. Then, the family* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *defined by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda\left(\widehat{D},t\right)}^X, \tag{5}$$

is the pullback attractor, where $\Lambda\left(\widehat{D},t\right)$ *is the omega-limit set at time* t *of* $\widehat{D} \in \mathcal{D}$*, where* $\mathcal{D}$ *is the universe of fixed nonempty bounded subsets of* X*.*

Another approach to the asymptotic dynamics of non-autonomous equations, the uniform attractor, has been developed by Chepyzhov and Vishik [10]. The theory of uniform attractors can be developed for a single non-autonomous process (see [9, 10]).

Definition 7. *A set* $K \subseteq X$ *is said to be uniformly (with respect to* $t_0 \in \mathbb{R}$*) attracting for the process* $\{U(t,t_0)\}$ *on* X *if for all* $t_0 \in \mathbb{R}$ *and for any bounded set* $B \subset X$*,*

$$\lim_{T\to+\infty} \left(\sup_{t_0\in\mathbb{R}} \mathrm{dist}_X(U(T+t_0,t_0)B, K) \right) = 0. \tag{6}$$

Respectively, the process $\{U(t,t_0)\}$ *is said to be uniformly asymptotically compact (with respect to* $t_0 \in \mathbb{R}$*) if there exists a compact uniformly (with respect to* $t_0 \in \mathbb{R}$*) attracting set of* $\{U(t,t_0)\}$*.*

Definition 8. *A closed set* $\mathcal{A}_1 \subseteq X$ *is said to be a uniform (with respect to* $t_0 \in \mathbb{R}$*) attractor for a process* $\{U(t,t_0)\}$ *if it is the minimal closed uniformly (with respect to* $t_0 \in \mathbb{R}$*) attracting set for this process. Minimality is meant in the sense that any closed attracting set is contained in* $\mathcal{A}_1$*.*

Theorem 9. *[Chepyzhov and Vishik* [8, 10]*, Haraux* [12]*] If a process* $\{U(t,t_0)\}$ *is uniformly asymptotically (with respect to* $t_0 \in \mathbb{R}$*) compact, then it has the uniform (with respect to* $t_0 \in \mathbb{R}$*) attractor* $\mathcal{A}_1$*. The set* $\mathcal{A}_1$ *is compact in* X*.*

To describe the structures of uniform attractors and to perform a comparison with the pullback attractor we introduce the notions of complete trajectory of a process, kernel of a process and kernel section (the terminology is due to Chepyzhov and Vishik [9, 10]).

Definition 10. *A map* $u : \mathbb{R} \to X$ *is called a complete trajectory of a process* $U(t,t_0)$ *if*

$$U(t,t_0)u(t_0) = u(t) \quad \textit{for all } t \geq t_0, \quad t, t_0 \in \mathbb{R}.$$

Definition 11. *The kernel $\mathbb{K}$ of a process $U(t,t_0)$ consists of all of its bounded complete trajectories of the process $U(t,t_0)$.*

Definition 12. *The set*
$$\mathcal{K}(s)=\{u(s):u(\cdot)\in\mathbb{K}\}$$
is said to be the kernel section at a time moment $t=s$, $s\in\mathbb{R}$.

These kernel sections are, essentially, the fibres of the pullback attractor: if $U(\cdot,\cdot)$ is a process that has a pullback attractor $\mathcal{A}$, then any backwards bounded trajectory is contained in $\mathcal{A}(t)$, and we can deduce that if $\mathcal{A}(\cdot)$ is bounded then $\mathcal{A}(t)=\mathcal{K}(t)$ (see for instance [7]). Observe that Theorem 9 implies the existence of a (fixed) compact attracting set K for $U(\cdot,\cdot)$, so that, from (6) and Theorem 3.11 in [7] it also implies the existence of a pullback attractor, which is then uniformly included in $\mathcal{K}$. Just as $\mathcal{A}(t)$ must contain $\mathcal{K}(t)$ for each t, the uniform attractor must contain the union of all the kernel sections (see [7]).

Lemma 13. *If $U(\cdot,\cdot)$ has a uniform attractor $\mathcal{A}_1$, then*
$$\bigcup_{t\in\mathbb{R}}\mathcal{K}(t)\subseteq\mathcal{A}_1.$$

2.2. **The structures of attractors for periodic processes.** In this subsection we investigate attractors for periodic processes. Let $\{U(t,t_0)\}$ be a periodic process, and let T be its period, i.e.,
$$U(t+T,t_0+T)=U(t,t_0)\quad\forall t\geq t_0,t_0\in\mathbb{R}.$$
We can now state a theorem about attractors of periodic processes.

Theorem 14. *[Chepyzhov and Vishik [9]] Let $\{U(t,t_0)\}$ be a periodic uniformly (with respect to $t_0\in\mathbb{R}$) asymptotic compact and $(X\times\mathbb{T}^1)-$continuous process, where $\mathbb{T}^1=\mathbb{R}$ (mod T). Then, the process $\{U(t,t_0)\}$ has a uniform (with respect to $t_0\in\mathbb{R}$) attractor, $\mathcal{A}_1$, and is given by*
$$\mathcal{A}_1=\bigcup_{\sigma\in[0,T)}\mathcal{K}(\sigma),$$
where $\mathcal{K}(\sigma)$ is the kernel section of the process $\{U(t,t_0)\}$ at time $t=\sigma$.

3. **Pullback and uniform attractors.** Since the functions on the right hand side of (1) are locally Lipschitz with respect to x, y, and z, then for any $t_0\in\mathbb{R}$ and any $(x_0,y_0,z_0)\in\mathbb{R}^3$ there exists a unique local solution of the model (1)-(2), denoted by $u(t;t_0,u_0):=(x(t;t_0,(x_0,y_0,z_0)),$ $y(t;t_0,(x_0,y_0,z_0)),z(t;t_0,(x_0,y_0,z_0)))$, and this solution is a global solution one (8) is proved.

3.1. **Pullback Attractor.** In this section, we will show the existence of a pullback attractor in $\mathbb{R}^3$ of our problem (1)-(2). First, thanks to the uniqueness of solution of (1)-(2), we can define a process $\{U(t,t_0),t_0\leq t\}$ in $\mathbb{R}^3$, by
$$U(t,t_0)u_0=u(t;t_0,u_0)\quad\forall u_0\in\mathbb{R}^3. \tag{7}$$
The process defined by (7) is continuous in $\mathbb{R}^3$.

Proposition 15. *Assume that $a>0$ and $b\in\mathbb{R}$. Then for any initial condition $u_0\in\mathbb{R}^3$, the solution u of (1)-(2) satisfies*
$$|u(t;t_0,u_0)|^2\leq e^{-l(t-t_0)}\,|u_0|^2+ae^{-lt}\int_{-\infty}^{t}e^{ls}F^2(s)ds+e^{-lt}\int_{-\infty}^{t}e^{ls}G^2(s)ds, \tag{8}$$
for all $t\geq t_0$, where $l:=min\{a,1\}$.

Proof. We deduce that
$$\frac{d}{dt}\,|u(t)|^2=-2(ax^2+y^2+z^2)+2axF(t)+2yG(t).$$
We have
$$2axF(t)\leq ax^2+aF^2(t),$$
and
$$2yG(t)\leq y^2+G^2(t).$$
Then, we can deduce
$$\frac{d}{dt}\,|u(t)|^2+l\,|u(t)|^2\leq aF^2(t)+G^2(t), \tag{9}$$
where $l:=min\{a,1\}$.

Multiplying (9) by e^{lt}, we obtain that

$$\frac{d}{dt}\left(e^{lt}\left|u(t)\right|^2\right) \le ae^{lt}F^2(t) + e^{lt}G^2(t).$$

Integrating between t_0 and t

$$\begin{aligned} e^{lt}\left|u(t)\right|^2 &\le e^{lt_0}\left|u_0\right|^2 + a\int_{t_0}^t e^{ls}F^2(s)ds + \int_{t_0}^t e^{ls}G^2(s)ds \\ &\le e^{lt_0}\left|u_0\right|^2 + a\int_{-\infty}^t e^{ls}F^2(s)ds + \int_{-\infty}^t e^{ls}G^2(s)ds, \end{aligned} \tag{10}$$

whence (8) follows. □

We consider the universe of fixed nonempty bounded subsets of $\mathbb{R}^3$. Now, we prove that there exists a pullback absorbing family for the process $U(t, t_0)$ defined by (7).

Proposition 16. *Assume that $a > 0$ and $b \in \mathbb{R}$. Let F, G satisfy (3). Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{\mathbb{R}^3}(0, \rho_l(t))$, where $\rho_l(t)$ is the nonnegative number given by*

$$\rho_l^2(t) = 1 + ae^{-lt}\int_{-\infty}^t e^{ls}F^2(s)ds + e^{-lt}\int_{-\infty}^t e^{ls}G^2(s)ds,\ \forall t \in \mathbb{R}, \tag{11}$$

is pullback absorbing family for the process U defined by (7).

Proof. Let $D \subset \mathbb{R}^3$ be bounded. Then, there exists $d > 0$ such that $|u_0| \le d$ for all $u_0 \in D$. Thanks to Proposition 15, we deduce that for every $t_0 \le t$ and any $u_0 \in D$,

$$\begin{aligned} \left|U(t,t_0)u_0\right|^2 &\le e^{-lt}e^{lt_0}\left|u_0\right|^2 + ae^{-lt}\int_{-\infty}^t e^{ls}F^2(s)ds + e^{-lt}\int_{-\infty}^t e^{ls}G^2(s)ds \\ &\le e^{-lt}e^{lt_0}d^2 + ae^{-lt}\int_{-\infty}^t e^{ls}F^2(s)ds + e^{-lt}\int_{-\infty}^t e^{ls}G^2(s)ds. \end{aligned}$$

If we consider $T(t, D) := l^{-1}\log(e^{lt}d^{-2})$, we have

$$\left|U(t,t_0)u_0\right|^2 \le 1 + ae^{-lt}\int_{-\infty}^t e^{ls}F^2(s)ds + e^{-lt}\int_{-\infty}^t e^{ls}G^2(s)ds,$$

for all $t_0 \le T(t, D)$ and for all $u_0 \in D$.

Consequently the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{\mathbb{R}^3}(0, \rho_l(t))$ is pullback absorbing for the process U defined by (7). □

Now, as a direct consequence of the preceding results and Theorem 6, we have the existence of the pullback attractor for the process U defined by (7).

Theorem 17. *Under the assumptions in Proposition 16, the process U defined by (7) possesses a pullback attractor $\mathcal{A}$, which is given by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda\left(\widehat{D}, t\right)}. \tag{12}$$

3.2. **Uniform Attractor.** Now, we suppose that F and G are translations bounded in $L^2_{loc}(\mathbb{R};\mathbb{R})$, i.e.,

$$\sup_{t\in\mathbb{R}}\int_t^{t+1} F^2(s)ds < \infty,\ \sup_{t\in\mathbb{R}}\int_t^{t+1} G^2(s)ds < \infty\quad. \tag{13}$$

In this subsection, using Theorem 9, we will prove that, under the assumption (13), the process $\{U(t, t_0)\}$ has a uniform (with respect to $t_0 \in \mathbb{R}$) attractor.

Remark 18. *Observe that assumption (13) implies (3).*

Proposition 19. *Assume that $a > 0$ and $b \in \mathbb{R}$. Let F, G satisfy (13). Then, the process U defined by (7) is uniformly (with respect to $t_0 \in \mathbb{R}$) asymptotically compact.*

Proof. Let $D \subset \mathbb{R}^3$ be bounded, and as in the proof of Proposition 16, let $d > 0$ such that $|u_0| \leq d$ for all $u_0 \in D$. Using (10), we have for any $u_0 \in D$

$$\begin{aligned}|u(t;t_0,u_0)|^2 &\leq e^{-l(t-t_0)}\,|u_0|^2 + a\int_{t_0}^t e^{-l(t-s)}F^2(s)ds + \int_{t_0}^t e^{-l(t-s)}G^2(s)ds\\ &\leq e^{-l(t-t_0)}d^2 + a\int_{t_0}^t e^{-l(t-s)}F^2(s)ds + \int_{t_0}^t e^{-l(t-s)}G^2(s)ds,\end{aligned} \tag{14}$$

for all $t \geq t_0$. We estimate the integrals on the right-hand side of (14), taking into account (13),

$$\begin{aligned}\int_{t_0}^t e^{-l(t-s)}F^2(s)ds &\leq \int_{-\infty}^t e^{-l(t-s)}F^2(s)ds \leq \sum_{n\geq 0}\int_{t-(n+1)}^{t-n} e^{-l(t-s)}F^2(s)ds\\ &\leq \sum_{n\geq 0} e^{-nl}\int_{t-(n+1)}^{t-n} F^2(s)ds = C_1(1-e^{-l})^{-1},\end{aligned}$$

where $C_1 := \sup_{t\in\mathbb{R}}\int_t^{t+1} F^2(s)ds < \infty$. Similarly, we have

$$\int_{t_0}^t e^{-l(t-s)}G^2(s)ds \leq C_2(1-e^{-l})^{-1},$$

where $C_2 := \sup_{t\in\mathbb{R}}\int_t^{t+1} G^2(s)ds < \infty$.

Then, we can deduce that there exists a positive constant C_l such that

$$|u(t;t_0,u_0)|^2 \leq e^{-l(t-t_0)}d^2 + C_l.$$

Replacing t by $t + t_0$, we have

$$|u(t+t_0;t_0,u_0)|^2 \leq e^{-lt}d^2 + C_l,$$

and if we consider $t \geq T(D) := \frac{\log d^2}{l}$, we obtain

$$|u(t+t_0;t_0,u_0)|^2 \leq 1 + C_l,$$

for all t_0 and for all $u_0 \in D$.

Then, the set $B_0 := \overline{B}_{\mathbb{R}^3}(0, 1 + C_l)$ is compact and uniformly (with respect to $t_0 \in \mathbb{R}$) attracting for the process U defined by (7). Therefore, the process U is uniformly (with respect to $t_0 \in \mathbb{R}$) asymptotically compact. □

We can now state a theorem about the existence of a uniform attractor of (1)-(2). Taking into account Theorem 9 and Lemma 13, we can deduce the following result.

Theorem 20. *Under the assumptions in Proposition 19, the process U defined by (7) has a uniform attractor $\mathcal{A}_1$, which is compact in $\mathbb{R}^3$. Moreover, we have the following relation:*

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \subseteq \mathcal{A}_1, \tag{15}$$

where $\mathcal{A}(t)$ is given by (12).

Remark 21. *Note that if b depends continuously on time, the previous results also hold true.*

4. Attractors for Periodic Equations.

We consider (1)-(2) with $a > 0$, $b \in \mathbb{R}$ and T-periodic continuous functions $F, G : \mathbb{R} \mapsto \mathbb{R}$.

4.1. Pullback Attractor.

In this subsection we show that when we have periodic nonlinear terms we obtain that the pullback attractor is a periodic pullback attractor. We observe that F and G satisfy (3). In fact F and G satisfy (13).

Then, under the assumptions in Proposition 16, the process defined by (7) has a pullback attractor $\mathcal{A}$ which is given by (12).

Corollary 22. *Assume that F and G are T-periodic continuous functions. Then the process U defined by (7) is periodic with period T, that is*

$$U(t,t_0)u_0 = U(t+T, t_0+T)u_0,$$

for all $t_0, t \in \mathbb{R}$, and the pullback attractor $\mathcal{A}(\cdot)$ is also periodic with period T.

Proof. We can deduce that $(X(\cdot;t_0,u_0),Y(\cdot;t_0,u_0),Z(\cdot;t_0,u_0)) := (x(\cdot+T;t_0+T,u_0),y(\cdot+T;t_0+T,u_0),z(\cdot+T;t_0+T,u_0))$ is the unique solution of (1) with initial value u_0 at $t=t_0$ because

$$\begin{aligned}\frac{dX}{dt}(t) = \frac{dx}{dt}(t+T) = \frac{dx}{d\tau}(\tau) &= -ax(\tau) - y^2(\tau) - z^2(\tau) + aF(\tau) \text{ where } \tau = t+T \\ &= -aX(t) - Y^2(t) - Z^2(t) + aF(t),\end{aligned}$$

by T-periodicity of F,

$$\begin{aligned}\frac{dY}{dt}(t) = \frac{dy}{dt}(t+T) = \frac{dy}{d\tau}(\tau) &= -y(\tau) + x(\tau)y(\tau) - bx(\tau)z(\tau) + G(\tau) \text{ where } \tau = t+T \\ &= -Y(t) + X(t)Y(t) - bX(t)Z(t) + G(t),\end{aligned}$$

by T-periodicity of G, and

$$\begin{aligned}\frac{dZ}{dt}(t) = \frac{dz}{dt}(t+T) = \frac{dz}{d\tau}(\tau) &= -z(\tau) + bx(\tau)y(\tau) + x(\tau)z(\tau) \text{ where } \tau = t+T \\ &= -Z(t) + bX(t)Y(t) + X(t)Z(t).\end{aligned}$$

Hence, we have

$$U(t,t_0)u_0 = U(t+T,t_0+T)u_0,$$

for all $t \geq t_0$.

Replacing t_0 by $t_0 - t$, where $t \geq 0$, and t by t_0, we thus have

$$U(t_0,t_0-t)u_0 = U(t_0+T,t_0+T-t)u_0,$$

so, by (4),

$$\begin{aligned}\Lambda(\widehat{D},t_0) &:= \bigcap_{s\leq t_0} \overline{\bigcup_{t_0-t\leq s} U(t_0,t_0-t)D(t_0-t)} \\ &= \bigcap_{s\leq t_0} \overline{\bigcup_{t_0-t\leq s} U(t_0+T,t_0+T-t)D(t_0-t)} = \Lambda(\widehat{D},t_0+T),\end{aligned}$$

and then

$$\mathcal{A}(t_0) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda\left(\widehat{D},t_0\right)} = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda\left(\widehat{D},t_0+T\right)} = \mathcal{A}(t_0+T).$$

Hence, $\mathcal{A}(\cdot)$ is also T-periodic. □

In Figure 1 we exhibit a simulation showing the pullback attractor for (1)-(2) with F and G periodic functions. In this simulation, we used the following parameters and initial conditions values: $a = \frac{1}{4}$, $b = 4$, $F(t) = cos(t)$, $G(t) = cos(t)$, $x(-2000) = 0$, $y(-2000) = -0.2$ and $z(-2000) = 0.5$.

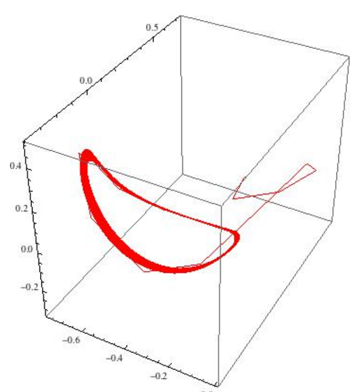

FIGURE 1. Numerical solution $(x(t), y(t), z(t))$

Remark 23. *Note that if* b *is a* T*-periodic continuous function* $b : \mathbb{R} \mapsto \mathbb{R}$*, the previous results also hold. In Figure 2 we present a simulation showing the pullback attractor for (1)-(2) with* b*,* F *and* G *periodic functions. We used the following parameters and initial conditions values:* $a = \frac{1}{4}$*,* $b(t) = 4 + 0.5cos(t)$*,* $F(t) = cos(t)$*,* $G(t) = cos(t)$*,* $x(-2000) = 0$*,* $y(-2000) = -0.2$ *and* $z(-2000) = 0.5$*.*

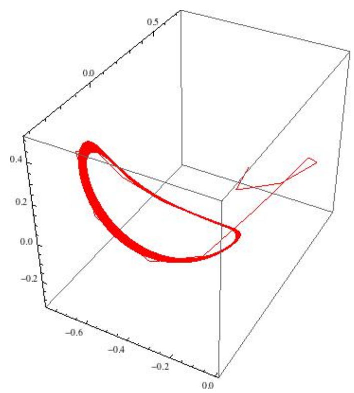

FIGURE 2. Numerical solution $(x(t), y(t), z(t))$

Now, we consider that F and G have different periods. For $a = \frac{1}{4}$ and $b = 4$, we have the following simulations:

- In Figure 3 we present a simulation showing the pullback attractor for (1)-(2) where we used the following non-autonomous forcing terms and initial conditions values: $F(t) = cos(t)$, $G(t) = cos(2t)$, $x(-2000) = 0$, $y(-2000) = -0.2$ and $z(-2000) = 0.5$.
- In Figure 4 we consider $F(t) = cos(t)$, $G(t) = cos(t/2)$ and the following initial conditions values: $x(-3000) = 0$, $y(-3000) = -0.2$ and $z(-3000) = 0.5$.
- In Figure 5 we consider $F(t) = cos(t)$, $G(t) = cos(\sqrt{2}t)$ and the following initial conditions values: $x(-500) = 0$, $y(-500) = -0.2$ and $z(-500) = 0.5$.

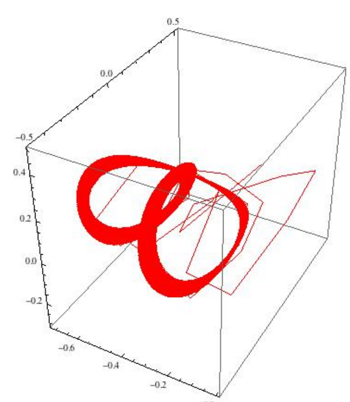

FIGURE 3. Numerical solution $(x(t), y(t), z(t))$

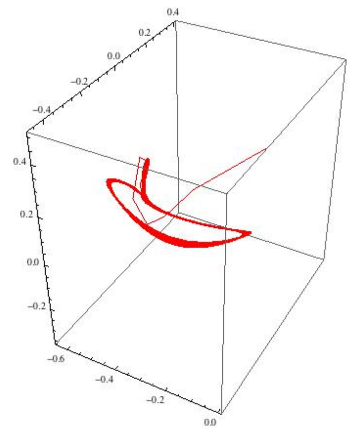

FIGURE 4. Numerical solution $(x(t), y(t), z(t))$

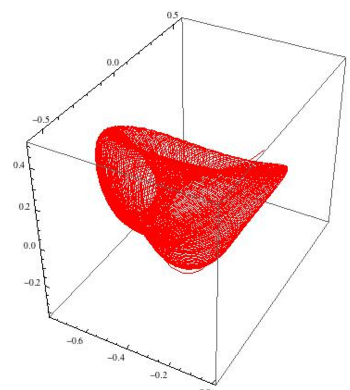

FIGURE 5. Numerical solution $(x(t), y(t), z(t))$

4.2. **Uniform Attractor.** In this subsection we show that when we have periodic terms we obtain a relation between the uniform attractor and the pullback attractor.

Proposition 24. *The process U defined by (7) is $(\mathbb{R}^3 \times \mathbb{T}^1, \mathbb{R}^3)$- continuous.*

Proof. We have to prove that for all fixed $t_0 \in \mathbb{R}$, $t \geq t_0$, the mapping $(u, t) \mapsto U(t, t_0)u$ is continuous from $\mathbb{R}^3 \times \mathbb{T}^1$ into $\mathbb{R}^3$. By the continuous dependence of solutions of (1)-(2) on initial values, we have that as the coefficients of (1) are locally Lipschitz, then the process U defined by (7) is $(\mathbb{R}^3 \times \mathbb{T}^1, \mathbb{R}^3)$- continuous. □

We can now state a theorem about the existence of a uniform attractor of (1)-(2). Taking into account Theorem 14, we can deduce the following result.

Theorem 25. *Assume that $a > 0$, $b \in \mathbb{R}$ and F, G are T-periodic continuous functions. Then, the set*

$$\mathcal{A}_1 = \bigcup_{\sigma \in [0,T)} \mathcal{A}(\sigma),$$

is a uniform (with respect to $t_0 \in \mathbb{R}$) attractor for the process U defined by (7), where $\{\mathcal{A}(\sigma)\}_{\sigma \in \mathbb{R}}$ is the pullback attractor of the process U defined by (7).

5. **Pullback attractors for a class of ODEs more general than almost periodic.** In this section we use a new class of functions and we generalize some results about periodic solutions of (1)-(2). We use recent results due to Kloeden and Rodrigues [15], where the authors introduced a class of functions which has the property that a bounded temporally global solution of a nonautonomous ordinary differential equation belongs to this class when the forcing terms do. Let $BUC(\mathbb{R}, \mathbb{R}^3)$ denotes the space of bounded and uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}^3$, with the supremum norm. We consider as in [15] the following class of functions,

$$\mathcal{F} := \{f \in BUC(\mathbb{R}, \mathbb{R}^3) : f \text{ has precompact range } \mathcal{R}(f)\}.$$

The class $\mathcal{F}$ includes periodic functions. We now consider the class $\mathcal{F}_{ODE}$ defined by

$$\mathcal{F}_{ODE} := \{f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3;\ \text{is uniformly continuous in } t \in \mathbb{R},\ \text{uniformly in } (x, y, z)$$
$$\text{in compact subsets } C \subset \mathbb{R}^3, \text{with precompact range } \mathcal{R}_C(f)\},$$

where

$$\mathcal{R}_C(f) := \bigcup_{(x,y,x)\in C} \{f(t, x, y, z), t \in \mathbb{R}\}.$$

Functions in the class $\mathcal{F}$ belong trivially to the class $\mathcal{F}_{ODE}$. For our problem, we consider

$$f_1(t, x, y, z) := -ax - y^2 - z^2 + aF(t), \tag{16}$$
$$f_2(t, x, y, z) := -y + xy - bxz + G(t), \tag{17}$$
$$f_3(t, x, y, z) := -z + bxy + xz, \tag{18}$$

and we suppose that

$$F, G \in BUC(\mathbb{R}, \mathbb{R}). \tag{19}$$

Proposition 26. *Under assumption (19), f_1, f_2 and f_3 defined by (16)-(18) belong to the class $\mathcal{F}_{ODE}$.*

Proof. We prove that $f_1 \in \mathcal{F}_{ODE}$. For f_2 the proof is similar and f_3 trivially belongs to $\mathcal{F}_{ODE}$ since it does not depend on t.

We have that

$$|f_1(t_1, x, y, z) - f_1(t_2, x, y, z)| \le a\,|F(t_1) - F(t_2)|\,.$$

Thus, using (19), we deduce that given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|t_1 - t_2| \le \delta(\varepsilon)$ then

$$|f_1(t_1, x, y, z) - f_1(t_2, x, y, z)| \le \varepsilon,$$

therefore f_1 is uniformly continuous in $t \in \mathbb{R}$.

Finally, thanks to (19), in particular we have that F and G are bounded functions in $t \in \mathbb{R}$, and we can deduce that $\mathcal{R}_C(f_1)$ is precompact, where $C \subset \mathbb{R}^3$ is a compact subset. Therefore, $f_1 \in \mathcal{F}_{ODE}$. □

Then, we can write (1) as

$$\frac{du}{dt} = f(t, u),\ \ t \in \mathbb{R}, \tag{20}$$

with initial condition

$$u(t_0) = u_0, \tag{21}$$

where $u(t; t_0, u_0) := (x(t; t_0, (x_0, y_0, z_0)), y(t; t_0, (x_0, y_0, z_0)), z(t; t_0, (x_0, y_0, z_0)))$, $t_0 \in \mathbb{R}$ and $f(t, u) := (f_1(t, x, y, z), f_2(t, x, y, z), f_3(t, x, y, z))$ belongs to the class $\mathcal{F}_{ODE}$.

Thanks to Lemma 8 in [15], on account of the following Proposition, the components sets of the pullback attractor and its entire solutions are in fact uniformly continuous.

Proposition 27. *Under assumption (19), problem (20)-(21) generates a process which possesses a pullback attractor $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ such that $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is precompact.*

Proof. Taking into account (19) we deduce that F, G satisfy (3) and (13). Then, thanks to Theorem 17, there exists the pullback attractor for the process defined by (7). On the other hand, using Theorem 20, we have (15). Therefore, $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is bounded and therefore $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ is precompact. □

Lemma 28. *Under the assumptions in Proposition 27 we have that (ϕ_1, ϕ_2, ϕ_3) belongs to the class $\mathcal{F}$ for every entire solution (ϕ_1, ϕ_2, ϕ_3) of the problem (20)-(21) taking values in the pullback attractor.*

In Figure 6 we present a simulation showing the pullback attractor for (20)-(21) where F and G satisfy (19). We used the following parameters and initial conditions values: $a = \frac{1}{4}$, $b = 4$, $F(t) = 1 + e^{-|t|}$, $G(t) = cos(t)$, $x(-2000) = 0$, $y(-2000) = -0.2$ and $z(-2000) = 0.5$.

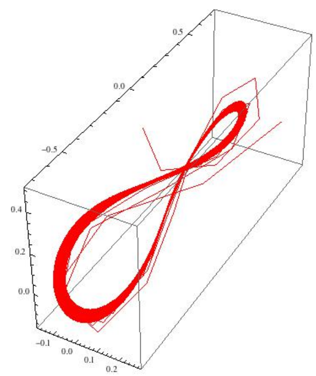

FIGURE 6. Numerical solution $(x(t), y(t), z(t))$

Remark 29. *Due to the uniformity properties imposed of the forcing terms, one may expect that the pullback attractor might also be forward attracting. This is what happens in the example exhibited in the paper* [15]*, where each fiber of the pullback attractor is formed by a single point. However, the techniques used in our case to prove the existence of the pullback attractor do not allow us to state that it is also true here, and a more sophisticated analysis may be necessary.*

6. **Upper Estimates for the Hausdorff Dimension of the Pullback Attractor.** In this section we obtain an upper bound for the Hausdorff dimension of the pullback attractor of the process defined by (7). For this purpose, we use a method proposed by Leonov *et al.* in [22] in the framework of cocycle dynamical systems.

Assume that $F, G \in BUC(\mathbb{R}, \mathbb{R})$ and satisfy the following additional conditions:

(H1) Boundedness in time, i.e., there exist nonnegative constants F_0 and G_0 such that

$$|F(t)| \leq F_0, \quad |G(t)| \leq G_0, \quad \text{for all } t \in \mathbb{R}. \tag{22}$$

(H2) The hull of the function f denoting the right-hand side of (1), is a compact metric space, i.e., $\mathcal{H}(f) = \overline{\{f(t + \cdot, \cdot) : t \in \mathbb{R}\}}$ is a compact metric space.

Notice that if F and G are almost periodic functions, then F and G satisfy (22) and the hull $\mathcal{H}(f)$ is a compact metric space where the closure is taken in the uniform convergence topology (see [10, 14] for more details).

In Section 3 we have proved that the solution mapping of (1)-(2) defines a process given by (7) which has a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}} \subset \mathbb{R}^3$ defined by (12).

Also we can obtain a cocycle by considering

$$\left.\begin{aligned} v' &= \mathbb{F}(\sigma_t p, v), \\ v(0) &= v_0 \in \mathbb{R}^3, \end{aligned}\right\} \tag{23}$$

where $p \in \mathcal{H}(f)$, $\mathbb{F}(p, v) := p(0, v)$ and σ is defined as the shift mapping $\sigma_t : \mathcal{H}(f) \mapsto \mathcal{H}(f)$ given by

$$\sigma_t \widetilde{f} := \widetilde{f}(\cdot + t, \cdot),$$

for $t \in \mathbb{R}$ and $\widetilde{f} \in \mathcal{H}(f)$.

Then, the cocycle generated by (23) is given by

$$\varphi(t, p)v_0 = v(t; p, v_0),$$

where $v(t;p,v_0)$ denotes the solution of (23) with initial value v_0 at $t=0$. If we take $p=f\in\mathcal{H}(f)$, then

$$\varphi(t,f)v_0 = v(t;f,v_0),$$

and (23) becomes

$$\left.\begin{aligned} v' &= \sigma_t f(0,v),\\ v(0) &= v_0\in\mathbb{R}^3, \end{aligned}\right\} \tag{24}$$

i.e.,

$$\left.\begin{aligned} v' &= f(t,v),\\ v(0) &= v_0\in\mathbb{R}^3, \end{aligned}\right\} \tag{25}$$

and we have

$$\varphi(t,f)v_0 = U(t,0)v_0.$$

Then, our problem (1)-(2) generates a cocycle $(\{\varphi(t,p)\cdot\}_{p\in\mathcal{H}(f),t\in\mathbb{R}},\mathbb{R}^3)$ over the base flow $(\{\sigma_t\}_{t\in\mathbb{R}},\mathcal{H}(f))$, where

$$\varphi(t,\sigma_s f)v_0 = U(t+s,s)v_0. \tag{26}$$

Now, to estimate the Hausdorff dimension of the pullback attractor associated to the process defined by (7), we will use Theorem 2 in [22], which is stated in the framework of cocycle dynamical systems. Then, for the cocycle generated by our system, we need to verify:

i) There exists a family of compact sets $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(f)}$ which is negatively invariant for the cocycle defined by (26), i.e.

$$\widetilde{\mathcal{A}}(\sigma_t p)\subset\varphi(t,p)\widetilde{\mathcal{A}}(p), \text{ for all } p\in\mathcal{H}(f), t\geq 0.$$

ii) There exists a compact set $\widetilde{K}\subset\mathbb{R}^3$ such that

$$\overline{\bigcup_{p\in\mathcal{H}(f)}\widetilde{\mathcal{A}}(p)}\subset\widetilde{K}.$$

iii) There exists a continuous function $V:\mathcal{H}(f)\times\mathbb{R}^3\to\mathbb{R}$ with derivatives $\frac{d}{dt}V(\sigma_t p,\varphi(t,p)u_0)$ along a given trajectory such that

$$\lambda_1(\sigma_t p,\varphi(t,p)u_0)+\lambda_2(\sigma_t p,\varphi(t,p)u_0)+s\lambda_3(\sigma_t p,\varphi(t,p)u_0)+\frac{d}{dt}V(\sigma_t p,\varphi(t,p)u_0)<0, \tag{27}$$

for all $t\in\mathbb{R}$, $u_0\in\widetilde{K}$, $p\in\mathcal{H}(f)$ and $s\in(0,1]$, where λ_i with $i=1,2,3$ are the eigenvalues of the symmetrized Jacobian matrix of the right-hand side of (1) arranged in nonincreasing order $\lambda_1\geq\lambda_2\geq\lambda_3$.

Using the pullback attractor, $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, associated to the process defined by (7), we define the family $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(f)}$ by

$$\widetilde{\mathcal{A}}(p)=\begin{cases}\mathcal{A}(s) & \text{if } p=\sigma_s f,\\ \{x\in\mathbb{R}^3 : x=\lim_{t_n\to+\infty}x_{t_n},\ x_{t_n}\in\mathcal{A}(t_n)\} & \text{if } p\neq\sigma_s f,\end{cases} \tag{28}$$

where $s\in\mathbb{R}$ and $p\in\mathcal{H}(f)$.

The set $\widetilde{\mathcal{A}}(p)$ is compact for any $p\in\mathcal{H}(f)$. Moreover, the family $\{\widetilde{\mathcal{A}}(p)\}_{p\in\mathcal{H}(f)}$ is negatively invariant. Indeed, if $p=\sigma_s f$, taking into account (26) and the fact that $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is invariant for the process U defined by (7), we obtain that $\varphi(t,p)\widetilde{\mathcal{A}}(p)=\widetilde{\mathcal{A}}(\sigma_t p)$ for all $t\geq 0$. If $p\neq\sigma_s f$, then $p=\lim_{t_n\to+\infty}\sigma_{t_n}f$ and it is easy to see that $\varphi(t,p)\widetilde{\mathcal{A}}(p)\supseteq\widetilde{\mathcal{A}}(\sigma_t p)$.

On the other hand, we can consider the following compact set

$$\widetilde{K}:=\overline{\bigcup_{t\in\mathbb{R}}\mathcal{A}(t)}\subset\mathbb{R}^3, \tag{29}$$

and we have that

$$\overline{\bigcup_{p\in\mathcal{H}(f)}\widetilde{\mathcal{A}}(p)}\subset\widetilde{K},$$

and, consequently, condition i)-ii) hold.

We can now establish our result on the estimate of the Haussdorff dimension of the pullback attractor for our model. We denote by $dim_H K$ the Hausdorff dimension of K.

Theorem 30. *Assume that $a > 0$ and $b \in \mathbb{R}$, and that $F, G \in BUC(\mathbb{R}, \mathbb{R})$ satisfy (H1)-(H2). Then the pullback attractor of the process U defined by (7) satisfies*

$$dim_H \mathcal{A}(t) \leq 3 - \frac{2(a+2-F_0)}{a+1+F_0+m}, \tag{30}$$

for all $t \in \mathbb{R}$, where m is a positive number given by

$$m := \sqrt{(a-1)^2 + \frac{[2(1-a)+F_0(1+ab^2+a)]^2}{4a(b^2+1)} + \frac{1}{4}G_0^2 a(\frac{1}{a}+b^2+1)^2}.$$

Proof. We need to verify iii).

It is easy to see that the eigenvalues of the symmetrized Jacobian matrix of the right-hand side of (1) are

$$x-1,\ \frac{1}{2}\left\{x-1-a \pm \sqrt{(1-x-a)^2+(b^2+1)(y^2+z^2)}\right\}.$$

Hence, condition (27) can be written in the form

$$\begin{aligned} &x(3+s) - 2 - (1+a)(1+s) \\ &+ (1-s)\sqrt{(1-x-a)^2+(b^2+1)(y^2+z^2)} + 2\frac{d}{dt}V_p(t,x,y,z) < 0, \end{aligned} \tag{31}$$

for all $t \in \mathbb{R}$, $(x,y,z) \in \widetilde{K}$ and $p \in \mathcal{H}(f)$. Here,

$$V_p(t,x,y,z) \equiv V(\sigma_t p, \varphi(t,p)(x,y,z))$$

is a function defined for $(x,y,z) \in \widetilde{K}$, $p \in \mathcal{H}(f)$, and $t \in \mathbb{R}$ by the relation

$$V(\sigma_t p, x, y, z) := \frac{k}{4}(1-s)(x^2+y^2+z^2) + \frac{1}{2a}(3+s)x,$$

where k is a varied parameter. Then

$$\begin{aligned} \frac{d}{dt}V_p =& \frac{k}{2}(1-s)(-ax^2+axF(t)-y^2+yG(t)-z^2) \\ &+ \frac{1}{2a}(3+s)(-ax-y^2-z^2+aF(t)), \end{aligned}$$

and inequality (31) is equivalent to the following

$$-2-(1+a)(1+s)+(1-s)\vartheta(t,x,y,z)+(3+s)F(t) < 0, \tag{32}$$

where

$$\begin{aligned} \vartheta(t,x,y,z;k) :=& \sqrt{(1-x-a)^2+(b^2+1)(y^2+z^2)} \\ &+ k(-ax^2+axF(t)-y^2+yG(t)-z^2). \end{aligned}$$

Let us denote

$$m := \inf_k \max_{t,x,y,z} \vartheta(t,x,y,z;k).$$

Using (22) we have iii) from (32), and due to Theorem 2 in [22] we obtain

$$dim_H \widetilde{\mathcal{A}}(p) \leq 2 + \frac{m+3F_0-a-3}{a+1+F_0+m} = 3 - \frac{2(a+2-F_0)}{a+1+F_0+m}, \tag{33}$$

for all $p \in \mathcal{H}(f)$.

We have

$$\begin{aligned} \vartheta(t,x,y,z;k) =& -\left(\gamma\sqrt{(1-x-a)^2+(b^2+1)(y^2+z^2)} - \frac{1}{2\gamma}\right)^2 \\ &+ \gamma^2\left[(1-x-a)^2+(b^2+1)(y^2+z^2)\right] + \frac{1}{4\gamma^2} \\ &+ k(-ax^2-y^2-z^2+axF(t)+yG(t)), \end{aligned}$$

where $\gamma \neq 0$ is a varied parameter. Further,

$$\begin{aligned}\vartheta(t,x,y,z;k) \le& \gamma^2(a-1)^2 + \frac{1}{4\gamma^2} - (ka-\gamma^2)x^2 - (2\gamma^2(1-a) - kaF(t))x \\ &- (k-(b^2+1)\gamma^2)y^2 - (-kG(t))y - (k-(b^2+1)\gamma^2)z^2 \\ =& -(ka-\gamma^2)\left[x + \frac{2\gamma^2(1-a)-kaF(t)}{2(ka-\gamma^2)}\right]^2 + \frac{\left[2\gamma^2(1-a)-kaF(t)\right]^2}{4(ka-\gamma^2)} \\ &- (k-(b^2+1)\gamma^2)\left[y + \frac{-kG(t)}{2(k-(b^2+1)\gamma^2)}\right]^2 + \frac{[-kG(t)]^2}{4(k-(b^2+1)\gamma^2)} \\ &- (k-(b^2+1)\gamma^2)z^2 + \gamma^2(a-1)^2 + \frac{1}{4\gamma^2}.\end{aligned}$$

If we take the varied parameters k and γ such that $ka - \gamma^2 > 0$ and $k - (b^2+1)\gamma^2 > 0$, then

$$\vartheta(t,x,y,z;k) \le \frac{\left[2\gamma^2(1-a)-kaF(t)\right]^2}{4(ka-\gamma^2)} + \frac{[-kG(t)]^2}{4(k-(b^2+1)\gamma^2)} + \gamma^2(a-1)^2 + \frac{1}{4\gamma^2}.$$

Let us take

$$k = \gamma^2(\frac{1}{a} + b^2 + 1),\ \gamma^2 = \frac{1}{2\sqrt{(a-1)^2 + \frac{\left[2(1-a)+F_0(1+ab^2+a)\right]^2}{4a(b^2+1)} + \frac{aG_0^2(1/a+b^2+1)^2}{4}}}.$$

Then, taking into account (22) we have

$$\vartheta(t,x,y,z;k) \le \sqrt{(a-1)^2 + \frac{[2(1-a)+F_0(1+ab^2+a)]^2}{4a(b^2+1)} + \frac{1}{4}G_0^2 a(\frac{1}{a}+b^2+1)^2},$$

and (28) and (33) imply (30). □

Remark 31. *For $a = \frac{1}{4}$, $b = 4$, $F(t) = cos(t)$ and $G(t) = cos(t)$, from the estimate (30), we obtain $dim_H \mathcal{A}(t) \le 2.68$ for all $t \in \mathbb{R}$.*

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