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ALMOST PERIODIC AND ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS OF LIÉNARD EQUATIONS

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Abstract. The aim of this paper is to study the almost periodic and asymptotically almost periodic solutions on $(0, +\infty)$ of the Liénard equation

$$x'' + f(x)x' + g(x) = F(t),$$

where $F : \mathbb{T} \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$) is an almost periodic or asymptotically almost periodic function and $g : (a, b) \to \mathbb{R}$ is a strictly decreasing function. We study also this problem for the vectorial Liénard equation.

We analyze this problem in the framework of general non-autonomous dynamical systems (cocycles). We apply the general results obtained in our early papers [3, 7] to prove the existence of almost periodic (almost automorphic, recurrent, pseudo recurrent) and asymptotically almost periodic (asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent) solutions of Liénard equations (both scalar and vectorial).

1. **Introduction.** In this paper we study the existence of almost periodic and asymptotically almost periodic solutions of the Liénard equation

$$x'' + f(x)x' + g(x) = F(t), \tag{1}$$

where $F : \mathbb{T} \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}_+ := [0, +\infty)$ or $\mathbb{R} := (-\infty, +\infty)$) is a continuous or locally integrable function and $f, g : (a, b) \to \mathbb{R}$ ($-\infty \le a < b \le +\infty$) are locally Lipschitz continuous functions.

We assume that the following conditions are fulfilled:

(i) g is strictly decreasing;
(ii) $f(x) \ge 0$ for all $x \in (a, b)$;
(iii) F is almost periodic (respectively, almost automorphic, recurrent, pseudo recurrent) or asymptotically almost periodic (respectively, asymptotically recurrent, asymptotically pseudo recurrent).

The typical equation of type (1) is

$$x'' + cx' + \frac{1}{x^\alpha} = F(t),$$

where $c \ge 0$, $\alpha > 0$ and $F : \mathbb{T} \to \mathbb{R}$ is an almost periodic or asymptotically almost periodic function.

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In the periodic case (i.e., when F is periodic), the dynamics of equation (1) was intensively studied by P. Martínez-Amores and P. J. Torres [13] and J. Campos and P. J. Torres [2]. For the almost periodic case (i.e. for almost periodic F) these results were generalized by P. Cieutat in [8]. The almost automorphic and asymptotically almost automorphic solutions of equation (1) were studied by P. Cieutat *et al.* [9], while the existence of pseudo almost periodic solutions of equation (1) was analyzed by El Hadi Ait Dads *et al.* [9].

Our main result in the present paper states that, when the function F is τ–periodic (respectively, quasi periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent), if equation (1) admits a solution which is bounded on $\mathbb{R}_+$, then it has a unique τ–periodic (respectively, quasi periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent) solution, and every solution of (1), bounded on $\mathbb{R}_+$, is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent). We obtain also an analog of this result when the function F is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent). These results are new and contain, as particular cases, some of the results cited above.

We present our results in the framework of general non-autonomous dynamical systems (cocycles) and we apply our abstract theory developed in [3, 7] to Liénard differential equations (both scalar and vectorial).

The paper is organized as follows.

In Section 2, we collect some notions (global attractor, minimal set, point/compact dissipativity, non-autonomous dynamical systems with convergence, quasi periodicity, Levitan/Bohr almost periodicity, almost automorphy, recurrence, pseudo recurrence, Poisson stability, etc) and facts from the theory of dynamical systems which will be necessary in this paper. We give here also some results concerning a special class of non-autonomous dynamical system (NDS): the so-called NDS with weak convergence. We give a generalization of the notion of convergent NDS. On the one hand, this type of NDS is very close to NDS with convergence (because they conserve some properties of convergent systems) and larger than that of convergent systems. On the other hand, we analyze the class of compact dissipative NDS with nontrivial Levinson center.

Section 3 is devoted to the existence of almost periodic (almost automorphic, recurrent, pseudo recurrent) and asymptotically almost periodic (asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent) solutions of Liénard equation (1).

In Sections 4 we present some results about S^p– asymptotically almost periodic (asymptotically almost periodic in the sense of Stepanoff) solutions of Liénard equation (1).

Finally, Sections 5 is devoted to study the problem of almost periodicity (respectively, almost automorphy, recurrence, pseudo recurrence) and asymptotically almost periodicity (respectively, asymptotically almost automorphy, asymptotically recurrence, asymptotically pseudo recurrence) of solutions for the vectorial Liénard equation.

2. **Nonautonomous Dynamical Systems with Convergence.** Let us start by recalling some concepts and notations about the theory of non-autonomous dynamical systems which will be necessary for our analysis.

2.1. **Compact Global Attractors of Dynamical Systems.** Let (X,ρ) be a metric space, $\mathbb{R}$ be the group of real numbers, $\mathbb{R}_+$ be the semi-group of nonnegative real numbers, $\mathbb{T}$ be one of the two sets $\mathbb{R}$ or $\mathbb{R}_+$.

A *dynamical system* is a triplet $(X,\mathbb{T},\pi)$, where $\pi:\mathbb{T}\times X\to X$ is a continuous mapping satisfying the following conditions: $\pi(0,x)=x$ $(\forall x\in X)$ and $\pi(s,\pi(t,x))=\pi(s+t,x)$ $(\forall t,\tau\in\mathbb{T}$ and $x\in X)$. When $\mathbb{T}=\mathbb{R}_+$ (respectively, $\mathbb{R}$), the dynamical system $(X,\mathbb{T},\pi)$ is called a *semi-flow* (respectively, *flow*).

The function $\pi(\cdot,x):\mathbb{T}\to X$ is called a *motion* passing through the point x at the moment $t=0$ and the set $\Sigma_x:=\pi(\mathbb{T},x)$ is called the *trajectory* of this motion.

A nonempty set $M\subseteq X$ is called *positively invariant* (*negatively invariant*, *invariant*) with respect to the dynamical system $(X,\mathbb{T},\pi)$ or, simply, positively invariant (negatively invariant, invariant), if $\pi(t,M)\subseteq M$ $(M\subseteq\pi(t,M),\pi(t,M)=M)$ for every $t\in\mathbb{T}$.

A closed positively invariant set, which does not contain any own closed positively invariant subset, is called *minimal*.

It is easy to see that every positively invariant minimal set is invariant.

The dynamical system $(X,\mathbb{T},\pi)$ is called:

- *point dissipative* if there exists a nonempty compact subset $K\subseteq X$ such that for every $x\in X$
$$\lim_{t\to+\infty}\rho(\pi(t,x),K)=0; \tag{2}$$
- *compact dissipative* if there exists a nonempty compact subset $K\subseteq X$ such that
$$\lim_{t\to+\infty}\rho(\pi(t,x),K)=0$$
uniformly with respect to x on compact subsets of X.

Let $(X,\mathbb{T},\pi)$ be compact dissipative and K be a compact set attracting every compact subset of X. Let us set
$$J:=\omega(K):=\bigcap_{t\ge0}\overline{\bigcup_{\tau\ge t}\pi(\tau,K)}. \tag{3}$$

It can be shown [5, Ch.I] that the set J defined by equality (3) does not depend on the choice of the attracting set K, but is characterized only by the properties of the dynamical system $(X,\mathbb{T},\pi)$ itself. The set J is called the *Levinson center* of the compact dissipative dynamical system $(X,\mathbb{T},\pi)$.

2.2. **Non-Autonomous Dynamical Systems with Convergence.** Recall that given two dynamical systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$, a triplet $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$, where h is a homomorphism from $(X,\mathbb{T}_1,\pi)$ onto $(Y,\mathbb{T}_2,\sigma)$, is called a *non-autonomous dynamical system*. The non-autonomous dynamical system $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be *convergent* (see [5]) if the following conditions are fulfilled:

(i) the dynamical systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ are compact dissipative;

(ii) the set $J_X\bigcap X_y$ contains no more than one point for all $y\in J_Y$, where $X_y:=h^{-1}(y):=\{x\in X\mid h(x)=y\}$ and J_X (respectively, J_Y) is the Levinson center of the dynamical system $(X,\mathbb{T}_1,\pi)$ (respectively, $(Y,\mathbb{T}_2,\sigma)$).

Thus, a non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is convergent, if the systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ are compact dissipative with Levinson centers J_X and J_Y respectively, and J_X possesses "trivial" sections, i.e., $J_X \bigcap X_y$ consists of a single point for all $y\in J_Y$. In this case the Levinson center J_X of the dynamical system $\langle (X,\mathbb{T}_1,\pi)$ is a copy (an homeomorphic image) of the Levinson center J_Y of the dynamical system $(Y,\mathbb{T}_2,\sigma)$. Thus, the dynamics on J_X is the same as on J_Y.

Remark 2.1. We note that convergent systems are in some sense the simplest dissipative dynamical systems. If Y is compact, invariant, $\mathbb{T}_2=\mathbb{R}$, $\langle (X,\mathbb{T}_1,\pi)$, $(Y,\mathbb{T}_2,\sigma)$, $h\rangle$ is a convergent non-autonomous dynamical system and J_X is the Levinson center of $(X,\mathbb{T}_1,\pi)$, then $(J_X,\mathbb{T}_2,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ are homeomorphic. Although the Levinson center of a convergent system can be completely described, it may be sufficiently complicated.

Recall [7] that the point $x\in X$ is called *asymptotically* τ*–periodic* (respectively, *asymptotically quasi periodic*, *asymptotically Bohr almost periodic*, *asymptotically recurrent*, *asymptotically pseudo recurrent*), if there exists a τ-periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent) point $p\in X$ such that $\lim\limits_{t\to+\infty}\rho(\pi(t,x),\pi(t,p))=0$.

2.3. **Non-Autonomous Dynamical Systems with Weak Convergence.** In this section we will study a class of non-autonomous dynamical systems which is very close to convergent systems, but possessing a non-trivial global attractor. This means that this class of non-autonomous systems will conserve almost all properties of convergent systems, but will have a "nontrivial" global attractor J_X, i.e., there exists at least one point $y\in J_Y$ such that the set $J_X\bigcap X_y$ contains more than one point.

A non-autonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be *weak convergent*, if the following conditions hold:

(i) the dynamical systems $(X,\mathbb{T}_1,\pi)$ and $(Y,\mathbb{T}_2,\sigma)$ are compact dissipative with Levinson centers J_X and J_Y respectively;

(ii) it follows that
$$\lim_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0,$$
for all $x_1,x_2\in J_X$ with $h(x_1)=h(x_2)$.

Given $x\in X$, let us denote by $\mathfrak{M}_x:=\{\{t_n\}\subseteq\mathbb{T}:$ such that the sequence $\{\pi(t_n,x)\}$ converges$\}$ and $\mathfrak{L}_x:=\{\{t_n\}\in\mathfrak{M}_x:\ t_n\to+\infty\}$. Similarly, for any $y\in Y$, denote by $\mathfrak{M}_y:=\{\{t_n\}\subseteq\mathbb{T}:$ such that the sequence $\{\sigma(t_n,y)\}$ converges$\}$ and $\mathfrak{L}_y:=\{\{t_n\}\in\mathfrak{M}_y:\ t_n\to+\infty\}$.

Remark 2.2. *1. Recall that the point $x\in X$ is called* [14]-[16] *comparable (respectively, uniformly comparable) by the character of recurrence with the point $y\in Y$, if $\mathfrak{N}_y\subseteq\mathfrak{N}_x$ (respectively, $\mathfrak{M}_y\subseteq\mathfrak{M}_x$), where $\mathfrak{N}_x:=\{\{t_n\}\in\mathfrak{M}_x$, such that $\{\pi(t_n,x\}\to x\}\}$.*

2. The notions of comparability and uniform comparability of motions by the character of recurrence play a very important role [14]-[16] *in the study of stability in the sense of Poisson (in particular, periodicity, quasi-periodicity, almost periodicity, almost automorphy, recurrence, etc) of solutions of differential equations with Poisson stable coefficients.*

Recall [7] that the point $x \in X$ is called *comparable with $y \in Y$ by the character of recurrence in infinity* if $\mathfrak{L}_y \subseteq \mathfrak{L}_x$.

Remark 2.3. *Note that the notion of comparability by the character of recurrence in infinity plays an important role* [7] *in the problem of existence of asymptotically almost periodic solutions of differential equations with asymptotically almost periodic coefficients.*

The next theorem contains sufficient conditions ensuring asymptotical stationarity (asymptotical periodicity, asymptotical almost periodicity, etc) for points which are comparable by the character of recurrence in infinity.

Theorem 2.4. [7] *Suppose that the following conditions hold:*

(i) $(X, \mathbb{T}_1, \pi)$ *and* $(Y, \mathbb{T}_2, \sigma)$ *are two dynamical systems;*
(ii) *the point $y \in Y$ is asymptotically stationary (respectively, asymptotically τ–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent);*
(iii) *the point x is comparable with $y \in Y$ by the character of recurrence in infinity.*

Then, the point x is also asymptotically stationary (respectively, asymptotically τ–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system. Denote by $\Omega_X := \overline{\bigcup\{\omega_x |\ x \in X\}}$, where ω_x is the ω-limit set of the point x. The following results, which have been proved in [3, 4], ensure the existence of compact minimal sets for point dissipative non-autonomous dynamical systems, as well as the existence of stationary (asymptotically periodic, asymptotically quasi-periodic, asymptotically almost periodic, etc...) points in the space X.

Corollary 2.5. [3, 4] *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system such that the following conditions hold:*

(i) *the dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are point dissipative;*
(ii) Ω_Y *is a compact minimal set;*
(iii)
$$\lim_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$$
holds for all $x_1, x_2 \in X$ with $h(x_1) = h(x_2)$;
(iv) *for every $y \in \Omega_Y$, the set $L_{\tilde{X}} \bigcap \tilde{X}_y$ contains at most one point, where $L_{\tilde{X}} := \{x \in \tilde{X} :$ there exists at least one entire motion $\gamma(\cdot) = \pi(\cdot, x)$ through the point x such that $\gamma(\mathbb{R}) \subseteq \tilde{X}$ and $\gamma(\mathbb{R})$ is relatively compact$\}$, where $\tilde{X} := h^{-1}(\Omega_Y)$.*

Then, there exists a unique compact minimal set $M \subseteq X$ such that

(i) *the section $M \bigcap X_y$ of the set M consists of a single point m_y for all $y \in Y$;*
(ii) $\Omega_X = M$;
(iii) *every point $x \in X$ is comparable with $h(x)$ by the character of recurrence in infinity.*

Corollary 2.6. [3, 4] *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system such that the following conditions hold:*

(i) *the dynamical system $(X, \mathbb{T}, \pi)$ is point dissipative;*
(ii) *there exists a point $y_0 \in Y$ such that $Y := H^+(y_0) := \overline{\{\sigma(t, y_0) :\ t \in \mathbb{T}_+\}}$;*

(iii) *the point* y_0 *is asymptotically stationary (respectively, asymptotically* τ*–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent);*

(iv)
$$\lim_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0$$
holds for all $x_1,x_2\in X$ *with* $h(x_1)=h(x_2)$*;*

(v) *for every* $y\in\Omega_Y$ *the set* $L_{\tilde{X}}\bigcap \tilde{X}_y$ *contains at most one point, where the sets* $L_{\tilde{X}}$ *and* $\tilde{X}_y$ *are the ones defined in the previous corollary.*

Then, there exists a unique compact minimal set $M\subseteq X$ *such that*

(i) *the section* $M\bigcap X_y$ *of the set* M *consists of a single point* m_y *for all* $y\in Y$*;*
(ii) $\Omega_X=M$*;*
(iii) *every point* $x\in X$ *is asymptotically stationary (respectively, asymptotically* τ*–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent).*

2.4. **Pseudo Recurrent Dynamical Systems with Convergence.** A non-autonomous dynamical system $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is said to be *uniformly stable in the positive direction on compact subsets* of X if, for arbitrary $\varepsilon>0$ and compact subset $K\subset X$, there is $\delta=\delta(\varepsilon,K)>0$ such that inequality $\rho(x_1,x_2)<\delta$ $(x_1,x_2\in K, h(x_1)=h(x_2))$ implies that $\rho(\pi(t,x_1),\pi(t,x_2))<\varepsilon$ for $t\in\mathbb{T}_1^+$, where $\mathbb{T}_1^+:=\{t\in\mathbb{T}_1:\ t\ge 0\}$.

Denote by $X\dot{\times}X=\{(x_1,x_2)\in X\times X\mid h(x_1)=h(x_2)\ \}$. If there exists a function $V:X\dot{\times}X\to\mathbb{R}_+$ with the following properties:

(i) V is continuous;
(ii) V is positive defined, i.e., $V(x_1,x_2)=0$ if and only if $x_1=x_2$;
(iii) $V(\pi(t,x_1),\pi(t,x_2))\le V(x_1,x_2)$ for all $(x_1,x_2)\in X\dot{\times}X$ and $t\in\mathbb{T}_1^+$,

then the non-autonomous dynamical system $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is called (see [5, 6], [12], and [17]) V*–monotone*.

Let (X,h,Y) be a fiber space, i.e., X and Y be two metric spaces and $h:X\to Y$ be a homomorphism from X onto Y. The subset $M\subseteq X$ is said to be *conditionally relatively compact*, if the pre-image $h^{-1}(Y')\bigcap M$ of every relatively compact subset $Y'\subseteq Y$ is a relatively compact subset of X, in particular, $M_y:=h^{-1}(y)\bigcap M$ is relatively compact for every y. The set M is called *conditionally compact* if it is closed and conditionally relatively compact.

Example 2.7. *Let* K *be a compact space,* $X:=K\times Y$*,* $h=pr_2:X\to Y$*, then the triplet* (X,h,Y) *is a fiber space. The space* X *is conditionally compact, but not compact.*

Denote by $\mathcal{K}:=\{a\in C(\mathbb{R}_+,\mathbb{R}_+)|\ a(0)=0,\ a \text{ is strictly increasing}\}$.

Recall that the dynamical system $(X,\mathbb{T}_1,\pi)$ is called *asymptotically compact* if for every positively invariant bounded subset $M\subseteq X$ there exists a nonempty compact subset $K\subseteq X$ such that
$$\lim_{t\to+\infty}\beta(\pi(t,M),K)=0,$$
where $\beta(A,B):=\sup\limits_{a\in A}\rho(a,B)$ and $\rho(a,B):=\inf\limits_{b\in B}\rho(a,b)$.

Now, we state two results proved in [3, 4] which provide some sufficient conditions ensuring the convergence character of a non-autonomous dynamical systems, as well as the existence of periodic (respectively, quasi-periodic, almost periodic, etc) points in the fiber of a periodic (respectively, quasi-periodic, almost periodic, etc) point.

Theorem 2.8. *Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{R},\sigma),h\rangle$ be a non-autonomous dynamical system satisfying the following conditions:*

1. *the dynamical system $(Y,\mathbb{R},\sigma)$ is pseudo recurrent;*
2. *the dynamical system $(X,\mathbb{T},\pi)$ is asymptotically compact;*
3. *there exists a point $x_0\in X_{y_0}$ with relatively compact positive semi-trajectory $\Sigma^+_{x_0}:=\{\pi(t,x_0):\ t\ge 0\}$;*
4. *the non-autonomous dynamical system $\langle (X,\ \mathbb{T},\ \pi),(Y,\ \mathbb{R},\ \sigma),\ h\rangle$ is V–monotone;*
5. *for all $(x_1,x_2)\in L_X\dot{\times}L_X\setminus\Delta_X$ (where $\Delta_X:=\{(x,x):\ x\in X\}$) there exists a positive number $t_0=t_0(x_1,x_2)\in\mathbb{T}$ such that $V(\pi(t_0,x_1),\pi(t_0,x_2))<V(x_1,x_2)$;*
6. *there are functions $a,b\in\mathcal{K}$ such that $Im(a)=Im(b)$ and $a(\rho(x_1,x_2))\le V(x_1,x_2)\le b(\rho(x_1,x_2))$ for all $(x_1,x_2)\in X\dot{\times}X$.*

Then, the following statements take place:

(i) *the NDS $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is convergent;*
(ii) *$J_X=\omega_{x_0}$;*
(iii) *$h(J_X)=Y$.*

Corollary 2.9. *Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{R},\sigma),h\rangle$ be a non-autonomous dynamical system such that:*

(i) *the dynamical system $(Y,\mathbb{R},\sigma)$ is transitive, i.e., there exists a point $y_0\in Y$ such that $H(y_0)=Y$;*
(ii) *the point y_0 is τ–periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent);*
(iii) *the dynamical system $(X,\mathbb{T},\pi)$ is asymptotically compact;*
(iv) *there exists a point $x_0\in X_{y_0}$ with relatively compact positive semi-trajectory $\Sigma^+_{x_0}:=\{\pi(t,x_0):\ t\ge 0\}$;*
(v) *the non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),(Y,\mathbb{R},\sigma),h\rangle$ is V–monotone;*
(vi) *for all $(x_1,x_2)\in L_X\dot{\times}L_X\setminus\Delta_X$ (where $\Delta_X:=\{(x,x):\ x\in X\}$) there exists a positive number $t_0=t_0(x_1,x_2)\in\mathbb{T}$ such that $V(\pi(t_0,x_1),\pi(t_0,x_2))<V(x_1,x_2)$;*
(vii) *there are functions $a,b\in\mathcal{K}$ such that $Im(a)=Im(b)$ and $a(\rho(x_1,x_2)\le V(x_1,x_2)\le b(\rho(x_1,x_2))$ for all $(x_1,x_2)\in X\dot{\times}X$.*

Then,

(i) *there exists a unique τ–periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent) point $x_0\in X_{y_0}:=\{x\in X:\ h(x)=y_0\}$;*
(ii) *every point $x\in X$ is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically pseudo recurrent).*

3. **Almost periodic and asymptotically almost periodic solutions of Liénard equations.** Consider the following Liénard equation

$$x''+f(x)x'+g(x)=F(t), \tag{4}$$

where $F:\mathbb{T}\to\mathbb{R}$ is a continuous function and $f,g:I\to\mathbb{R}$ ($I:=(a,b)$ with $-\infty\le a<b\le+\infty$) are locally Lipschitz continuous functions. We assume that the functions f,g and F satisfy the following conditions:

(i) g is strictly decreasing;
(ii) $f(x) \geq 0$ for all $x \in I$;
(iii) $\sup_{t\in\mathbb{T}} |F(t)| < +\infty$.

As we already pointed out, the typical example for equation (4) is given by

$$x'' + cx' + \frac{1}{x^\alpha} = F(t),$$

where c is a nonnegative constant, $\alpha > 0$ and $F : \mathbb{R} \to \mathbb{R}$ is a periodic (respectively, quasi periodic, almost periodic, almost automorphic, recurrent) function.

Recall that a function $\varphi : \mathbb{T} \to I$ is said to be *bounded* if $Q := \overline{\varphi(\mathbb{T})}$ is a compact subset of I.

Remark 3.1. A continuous function $\varphi : \mathbb{T} \to I$ is bounded if and only if $[m_\varphi, M_\varphi] \subset I$, where $m_\varphi := \inf_{t\in\mathbb{T}} \varphi(t)$ and $M_\varphi := \sup_{t\in\mathbb{T}} \varphi(t)$.

Denote by $C_b(\mathbb{T}, \mathbb{R})$ the set of all continuous functions $F : \mathbb{T} \to \mathbb{R}$ with norm $||F|| := \sup_{t\in\mathbb{T}} |F(t)| < +\infty$.

The following results are well known.

Lemma 3.2. [8] *Let $I := (t_0, +\infty)$ with $t_0 = -\infty$ or $t_0 \in \mathbb{R}$ and $F \in C_b(\mathbb{T}, \mathbb{R})$. If $\varphi(t)$ is a solution of equation* (4) *which is bounded on $\mathbb{R}_+$ (respectively, bounded on $\mathbb{R}$), then the derivatives $\varphi'(t)$ and $\varphi''(t)$ are also bounded on $\mathbb{R}_+$ (respectively, bounded on $\mathbb{R}$).*

Denote by $\varphi(t, u, v, F)$ the unique solution of equation (4) satisfying the initial conditions $\varphi(0, u, v, F) = u$ and $\varphi'(0, u, v, F) = v$. We have the following theorem.

Theorem 3.3. [1, 8] *The following statements result to be true:*

(i) *if $F \in C_b(\mathbb{R}_+, \mathbb{R})$, then for any pair of solutions $\varphi(t, u_i, v_i, F)$ $(i = 1, 2)$ of equation (4), which are bounded on $\mathbb{R}_+$, we have*

$$\lim_{t\to+\infty} (|\varphi(t, u_1, v_1, F) - \varphi(t, u_2, v_2, F)| + |\varphi'(t, u_1, v_1, F) - \varphi'(t, u_2, v_2, F)|) = 0;$$

(ii) *if $F \in C_b(\mathbb{R}, \mathbb{R})$, then equation* (4) *admits at most one solution which is bounded on $\mathbb{R}$.*

Remark 3.4. Note that Theorem 3.3 remains true if we replace the condition $F \in C_b(\mathbb{T}, \mathbb{R})$ ($\mathbb{T} = \mathbb{R}_+$ for item (i) and $\mathbb{T} = \mathbb{R}$ for item (ii)) by $F \in S^p(\mathbb{T}, \mathbb{R})$, where $S^p(\mathbb{T}, \mathbb{R})$ is the space of all functions $\varphi \in L^p_{loc}(\mathbb{T}, \mathbb{R})$ satisfying the condition $|\varphi|_{S^p} := \sup_{t\in\mathbb{T}} (\int_t^{t+1} |\varphi(s)|^p ds)^{1/p} < +\infty$ and $p \geq 1$. This statement may be proved with a slight modification of the proof of Theorem 3.3.

Denote by $C(\mathbb{T}, \mathbb{R})$ the set of all continuous functions $F : \mathbb{T} \to \mathbb{R}$ endowed with the compact-open topology, by F_τ the τ–shift of F ($\tau \in \mathbb{T}$), that is, $F_\tau(t) := F(t+\tau)$ for all $t \in \mathbb{T}$, and $(C(\mathbb{T}, \mathbb{R}), \mathbb{T}, \sigma)$ the shift dynamical system (Bebutov's dynamical system), i.e., $\sigma(\tau, F) := F_\tau$ for all $\tau \in \mathbb{T}$ and $F \in C(\mathbb{T}, \mathbb{R})$.

It is said that the function $F \in C(\mathbb{T}, \mathbb{R})$ *possesses the property* (S) *(for example, periodicity, almost periodicity, recurrence, asymptotically almost periodicity and so on)*, if the motion $\sigma(\tau, F)$, generated by the function F in the shift dynamical system $(C(\mathbb{T}, \mathbb{R}), \mathbb{T}, \sigma)$, possesses this property.

The solution $\varphi(t)$ of equation (4) is called [7, 14, 15, 16] *compatible (respectively, uniform compatible) by the character of recurrence*, if the motion $\sigma(t,(\varphi,\varphi'))$, generated by by function $(\varphi,\varphi') \in C(\mathbb{R},\mathbb{R}) \times C(\mathbb{R},\mathbb{R})$ is comparable (respectively, uniform comparable) by the character of recurrence with the motion $\sigma(\tau, F)$, i.e., $\mathfrak{N}_F \subseteq \mathfrak{N}_{(\varphi,\varphi')}$ (respectively, $\mathfrak{M}_F \subseteq \mathfrak{M}_{(\varphi,\varphi')}$), $\mathfrak{M}_F := \{\{t_n\} : \text{the sequence } \{\sigma(t_n, F)\}$ is convergent$\}$, $\mathfrak{L}_F := \{\{t_n\} \in \mathfrak{M}_F : \text{such that } t_n \to +\infty \text{ as } n \to \infty\}$ and φ' is the derivative of the function φ.

Example 3.5. Denote by $y = x'$, then equation (4) can be reduced to the following equivalent first order system

$$\begin{cases} x' = y \\ y' = -g(x) - f(x)y + F(t). \end{cases} \tag{5}$$

Along with system (5), consider its H–class, i.e., the family of systems

$$\begin{cases} x' = y \\ y' = -g(x) - f(x)y + G(t), \end{cases} \tag{6}$$

where $G \in H(F)$.

Recall that we denote by $\varphi(t,u,v,F)$ the unique solution of equation (4) satisfying the initial conditions $\varphi(0,u,v,F) = u$ and $\varphi'(0,u,v,F) = v$ and defined on $\mathbb{R}_+$ (or on $\mathbb{R}$). Then, $(\varphi(t,u,v,F), \varphi'(t,u,v,F))$ is the unique solution of system (5) with the initial data $(\varphi(0,u,v,F), \varphi'(0,u,v,F)) = (u,v) \in \mathbb{R}^2$. Let $Y = H(F) := \overline{\{\sigma(t,F) :\ t \in \mathbb{R}\}}$ and let $(Y,\mathbb{R},\sigma)$ be the shift dynamical system on $H(F)$, induced by Bebutov's dynamical system $(C(\mathbb{R},\mathbb{R}),\mathbb{R},\sigma)$. We set $W := \mathbb{R}^2 \times H(F)$, $\tilde{X} := \{((u,v),G) \in W :$ there exists a unique solution $\varphi(t,u,v)$ of equation (4) through the point $(u,v) \in \mathbb{R}^2$ at the initial moment $t = 0$ and defined on $\mathbb{R}_+\}$, $\pi(t,((u,v),G)) := (\varphi(t,u,v,G),\ \varphi'(t,u,v,G))$ for all $t \in \mathbb{R}_+$ and $((u,v),G) \in \tilde{X}$, where $(\varphi(t,u,v,G), \varphi'(t,u,v,G))$ is the unique solution of system (6) with initial data $(\varphi(0,u,v,G), \varphi'(0,u,v,G)) = (u,v)$. Let now $\varphi(t,u,v,F)$ be a solution of equation (4) bounded on $\mathbb{R}_+$, and we denote by $X = H^+((u,v,F)) := \overline{\{(\varphi(\tau,u,v,F), \varphi'(\tau,u,v,F), F_\tau) :\ \tau \in \mathbb{R}_+\}}$, where $F_\tau := F(\cdot + \tau)$. From Lemma 3.2 it follows that the set X is shift invariant, i.e., $\pi(t,X) \subseteq X$ for all $t \in \mathbb{R}_+$ and $X \subseteq \tilde{X}$. Let $h = pr_2 : X \mapsto Y$ be the second projection of X onto Y, then the triplet $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is a non-autonomous dynamical system, generated by equation (4) (or system of equations (5)) and the solution $\varphi(t,u,v,F)$.

A solution φ of equation (4) is called *compatible by recurrence in infinity* [7], if $\mathfrak{L}_F \subseteq \mathfrak{L}_{(\varphi,\varphi')}$, where φ' is the derivative of the function φ and $\mathfrak{L}_\varphi := \{\{t_n\} \in \mathfrak{M}_\varphi : t_n \to +\infty\}$.

Theorem 3.6. *Suppose that $F \in C(\mathbb{R}_+,\mathbb{R})$ and F is asymptotically recurrent. Then, every solution $\varphi(t,u,v)$ of equation* (4)*, which is bounded on $\mathbb{R}_+$, is compatible by recurrence in infinity.*

Proof. Consider the non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by equation (4) and its solution $\varphi(t,u,v,F)$ (see Example 3.5). Since F is asymptotically recurrent, the dynamical system $(Y,\mathbb{R},\sigma)$ is compact dissipative and its Levinson center $J_Y = \omega_F$ is a compact minimal set, where ω_F is the ω–limit set of the point $F \in C(\mathbb{R},\mathbb{R})$ in the shift dynamical system $(Y,\mathbb{R},\sigma)$. By Lemma 3.2, the set $X = H^+((u,v,F)) \subseteq \mathbb{R}^2 \times C(\mathbb{R},\mathbb{R})$ is compact. Let now

$G \in H^+(F) := \{F_\tau : \tau \in \mathbb{R}_+\}$ and $\varphi(t, u_i, v_i, G)$ $(i = 1, 2;\ (u_i, v_i, G) \in X)$ be two solutions of system (6). According to Theorem 3.3 we have

$$\lim_{t \to +\infty} (|\varphi(t, u_1, v_1, G) - \varphi(t, u_2, v_2, G)| + |\varphi'(t, u_1, v_1, G) - \varphi'(t, u_2, v_2, G)|) = 0.$$

On the other hand if $G \in \Omega_F$ (ω-limit set of the function F), then $G \in C_b(\mathbb{R}, \mathbb{R})$ and by Theorem 3.3, system (6) admits at most one solution which is bounded on $\mathbb{R}$. Now to finish the proof it is sufficient to apply Corollary 2.5. □

Corollary 3.7. *Suppose that $F \in C(\mathbb{R}, \mathbb{R})$ and F is asymptotically stationary (respectively, asymptotically τ–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent). Then, every solution $\varphi(t, u, v, F)$ of equation (4), which is bounded on $\mathbb{R}_+$, is asymptotically stationary (respectively, asymptotically τ–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent).*

Proof. This statement follows from Theorem 3.6 and Corollary 2.6. □

4. S^p–asymptotically almost periodic solutions.

4.1. **Dynamical systems of shifts in the space $L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu)$.** Let $(\mathbb{T}; \mathfrak{B}; \mu)$ be a measurable space in which μ is a Radon measure, and $\mathfrak{B}$ a Banach space with norm $|\cdot|$.

Let $1 \le p \le +\infty$. By $L^p(\mathbb{T}; \mathfrak{B}, \mu)$ we denote the space of all measurable functions (classes of functions) $f : \mathbb{T} \to \mathfrak{B}$ such that $|f| \in L^p(\mathbb{T}; \mathbb{R}; \mu)$, where $|f|(s) = |f(s)|$. The space $L^p(\mathbb{T}; \mathfrak{B}; \mu)$ is endowed with the norm

$$||f||_{L^p} = (\int_{\mathbb{T}} |f(s)|^p d\mu(s))^{1/p} \quad \text{and} \quad ||f||_\infty = \text{ess sup}_{s \in \mathbb{T}} |f(s)|. \tag{7}$$

$L^p(\mathbb{T}; \mathfrak{B}; \mu)$ with norm (7) is a Banach space.

Denote by $L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu)$ the set of all function $f : \mathbb{T} \to \mathfrak{B}$ such that $f_l \in L^p([-l, l] \cap \mathbb{T}; \mathfrak{B}; \mu)$ for every $l > 0$, where f_l is the restriction of the function f onto $[-l, l] \cap \mathbb{T}$.

In the space $L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu)$ we define the following family of semi-norms $||\cdot||_{l,p}$:

$$||f||_{l,p} = ||f_l||_{L^p([-l,l] \cap \mathbb{T}; \mathfrak{B}; \mu)} \quad (l > 0). \tag{8}$$

The semi-norms considered in (8) define a metrizable topology on $L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu)$. The metric given by this topology can be defined, for instance, by

$$d_p(\varphi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{||\varphi - \psi||_{n,p}}{1 + ||\varphi - \psi||_{n,p}}, \qquad \text{for } \varphi, \psi \in L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu)$$

Let us define a mapping $\sigma : L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu) \times \mathbb{T} \to L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu)$ as follows: $\sigma(f, \tau) = f^{(\tau)}$ for all $f \in L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu)$ and $\tau \in \mathbb{T}$, where $f^{(\tau)}(s) := f(s + \tau)$ $(s \in \mathbb{T})$. Then, the triplet $(L^p_{loc}(\mathbb{T}; \mathfrak{B}; \mu), \mathbb{T}, \sigma)$ is a dynamical system (see [7, ChI]).

4.2. **Stepanoff asymptotically almost periodic solutions.** Let us start by defining the concept of Stepanoff asymptotically almost period solutions.

Definition 4.1. *A function $\varphi \in L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu)$ is called* S^p almost periodic (almost periodic in the sense of Stepanoff)*, if the motion $\sigma(\cdot, \varphi)$ is almost periodic in the dynamical system $(L^p_{loc}(\mathbb{R}; \mathfrak{B}; \mu), \mathbb{R}, \sigma)$. Analogously, it can be defined the asymptotic S^p almost periodicity of functions.*

Recall now two interesting results from [7, ChI].

Theorem 4.2. *Let $\varphi \in L^p_{loc}(\mathbb{R};\mathfrak{B};\mu)$. The following statements are equivalent:*

1) *φ is S^p almost periodic;*
2) *for every $\varepsilon > 0$ there exists $l > 0$ such that on every segment of length l in $\mathbb{R}$ there is a number τ for which*
$$\int_t^{t+1} |\varphi(s+\tau) - \varphi(s)|^p ds < \varepsilon^p$$
for all $t \in \mathbb{R}$;
3) *from an arbitrary sequence $\{t_n\} \subset \mathbb{R}$ there can be extracted a subsequence $\{t_{k_n}\}$ such that the sequence $\{\varphi^{(t_{k_n})}\}$ uniformly converges in the space $L^p_{loc}(\mathbb{R};\mathfrak{B};\mu)$, i.e., there exists a function $\tilde{\varphi} \in L^p_{loc}(\mathbb{R};\mathfrak{B};\mu)$ such that*
$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} |\varphi(s+t_{k_n}) - \tilde{\varphi}(s)|^p ds = 0.$$

Theorem 4.3. *Let $\varphi \in L^p_{loc}(\mathbb{R}_+;\mathfrak{B};\mu)$. The following statements are equivalent:*

1) *the function φ is asymptotically S^p almost periodic, i.e., the motion $\sigma(\cdot,\varphi)$ is asymptotically almost periodic in the dynamical system $(L^p_{loc}(\mathbb{R}_+;\mathfrak{B};\mu)$, $\mathbb{R}$, $\sigma)$;*
2) *there exist an S^p almost periodic function p and a function $\omega \in L^p_{loc}(\mathbb{R}_+;\mathfrak{B};\mu)$ such that $p \in L^p_{loc}(\mathbb{R};\mathfrak{B};\mu)$, $\varphi = p + \omega$ and $\lim\limits_{t\to+\infty} \int_t^{t+1} |\omega(s)|^p ds = 0$;*
3) *for every $\varepsilon > 0$ there exist numbers $\beta \geq 0$ and $l > 0$ such that on every segment of length l there is a number τ for which*
$$\int_t^{t+1} |\varphi(\tau+s) - \varphi(s)|^p ds < \varepsilon^p$$
for all $t \geq \beta$ and $t + \tau \geq \beta$;
4) *from every sequence $\{t_n\}$, $t_n \to +\infty$, we can extract a subsequence $\{t_{k_n}\}$ such that the sequence $\{\varphi^{(t_{k_n})}\}$ converges uniformly with respect to $t \in \mathbb{R}_+$ in the space $L^p_{loc}(\mathbb{R}_+;\mathfrak{B};\mu)$, i.e., there exists a function $\tilde{\varphi} \in L^p_{loc}(\mathbb{R}_+;\mathfrak{B};\mu)$ such that*
$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}_+} \int_t^{t+1} |\varphi(s+t_{k_n}) - \tilde{\varphi}(s)|^p ds = 0.$$

Lemma 4.4. *Let $I := (t_0, +\infty)$ with $t_0 = -\infty$ or $t_0 \in \mathbb{R}$, $F \in L^p_{loc}(\mathbb{R}_+;\mathbb{R};\mu)$ and $\sup\limits_{t\in\mathbb{T}}(\int_t^{t+1} |F(t)|^p)^{1/p} < +\infty$ ($p \geq 1$). If $\varphi(t)$ is a solution of equation* (4) *which is bounded on $\mathbb{R}_+$ (respectively, bounded on $\mathbb{R}$), then the derivatives $\varphi'(t)$ and $\varphi''(t)$ are also bounded on $\mathbb{R}_+$ (respectively, bounded on $\mathbb{R}$).*

Proof. We omit the proof because it is a slight modification of the proof of Lemma 3.2. □

Lemma 4.5. [7, ChI] *If $F \in L^p_{loc}(\mathbb{R}_+;\mathbb{R})$ and F is asymptotically recurrent, then $\sup\limits_{t\in\mathbb{R}_+} (\int_t^{t+1} |F(t)|^p)^{1/p} < +\infty$ ($p \geq 1$).*

Now we can prove the following results.

Theorem 4.6. *Suppose that $F \in L^p_{loc}(\mathbb{R}_+;\mathbb{R})$ and F is asymptotically recurrent. Then, every solution $\varphi(t,u,v)$ of equation (4), which is bounded on $\mathbb{R}_+$, is compatible by recurrence in infinity.*

Proof. This statement can be proved using the same arguments as in the proof of Theorem 3.3 (see also Remark 3.4), Theorem 3.6 and using lemmas 4.4 and 4.5. □

Corollary 4.7. *Suppose that $F \in L^p_{loc}(\mathbb{R}_+;\mathfrak{B})$ and F is asymptotically S^p–stationary (respectively, asymptotically S^p τ–periodic, asymptotically S^p quasi-periodic, asymptotically S^p almost periodic, asymptotically S^p almost automorphic, asymptotically S^p recurrent). Then, every solution $\varphi(t,u,v)$ of equation (4), which is bounded on $\mathbb{R}_+$, is asymptotically stationary (respectively, asymptotically τ–periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent).*

Proof. The proof follows from Theorem 3.6 and Corollary 2.6. □

Remark 4.8. We would like to stress that in Corollary 4.7 the function F is asymptotically almost periodic in the sense of Stepanoff, but the solution $\varphi(t,u,v,F)$ is asymptotically almost periodic in the sence of Fréchet [10, 11].

Remark 4.9. When the function F is S^p asymptotically almost periodic, a particular case of Corollary 3.7 was established by Ait Dads et al. [1]. Namely, they proved this statement in the case when $F(t) = P(t) + \Omega(t)$ for all $t \in \mathbb{R}_+$, where $P \in C(\mathbb{R},\mathbb{R})$ is a Bohr almost periodic function and $\Omega \in C(\mathbb{R}_+,\mathbb{R})$ with the following properties: $\sup\limits_{t\in\mathbb{R}_+} |\Omega(t)| < +\infty$ and $\lim\limits_{t\to+\infty} \int_t^{t+1} |\Omega(s)|ds = 0$. It is evident, that the function F with the properties listed above is S^p (with $p = 1$) asymptotically almost periodic, but the inverse statement is not true.

5. **Convergence of Forced Vectorial Liénard Equations.**

Let $(Y,\mathbb{R},\sigma)$ be a two-sided dynamical system. Consider the following vectorial Liénard equation

$$u''(t) + \frac{d}{dt}[\nabla F(u(t))] + Cu(t) = f(\sigma(t,y)), \tag{9}$$

where $f \in C(Y,\mathbb{R}^m)$, $C : \mathbb{R}^m \mapsto \mathbb{R}^m$ is a symmetric and nonsingular linear operator, and ∇F denotes the gradient of the convex function F on $\mathbb{R}^m$. If the operator C is positive definite, then we introduce the product space $\mathbb{R}^m \times \mathbb{R}^m \simeq \mathbb{R}^{2m}$ endowed with the inner product associated to the quadratic form Q given by

$$Q(u,v) := |u|^2 + \langle C^{-1}v, v\rangle.$$

Remark 5.1. Note that the inequality

$$\alpha(|u|^2 + |v|^2) \le Q(u,v) \le \beta(|u|^2 + |v|^2)$$

holds for all $(u,v) \in \mathbb{R}^{2m}$, where $\alpha := \max(1, ||C||^{-1})$ and $\beta := \max(1, ||C^{-1}||)$, where $||C||$ is the norm of operator C.

Equation (9) can be re-written in the form

$$U'(t) + G(\sigma(t,y), U(t)) = 0, \tag{10}$$

where $G \in C(Y \times \mathbb{R}^{2m}, \mathbb{R}^{2m})$ and the partial function $G(y,\cdot,\cdot)$ is strictly monotone for each $y \in Y$ with respect to the inner product associated to Q, i.e., for each $y \in Y$

$$\langle G_1(y,u_1,v_1) - G_1(y,u_2,v_2), u_1 - u_2\rangle \\ +\langle C^{-1}(G_2(y,u_1,v_1) - G_2(y,u_2,v_2)), v_1 - v_2\rangle > 0,$$

for all $(u_i,v_i) \in \mathbb{R}^{2m}$ $(i = 1,2)$ such that $(u_1,v_1) \neq (u_2,v_2)$. Indeed, by letting $v(t) := u'(t) + \nabla F(u(t))$; $U(t) := (u(t),v(t))$, equation (9) reduces to

$$U'(t) + \nabla\Phi(U(t)) + JU(t) = \mathcal{F}(\sigma(t,y))$$

where $\Phi(u,v) := F(u)$, $J := \begin{pmatrix} 0 & -I \\ C & 0 \end{pmatrix}$ and $\mathcal{F}(y) := (0,f(y))$.

It is known the following result.

Lemma 5.2. [8] *Let I be the interval $[0,+\infty)$ or the whole real line $\mathbb{R}$. Let $f(t) := f(\sigma(t,y))$ $(\forall t \in I)$ be bounded on I. If $u \in C^2(I,\mathbb{R}^n)$ is a solution of equation (9) which is bounded on I, then u' and u'' are also bounded on I.*

We can now prove our main result on this model.

Theorem 5.3. *Suppose that the following conditions are fulfilled:*

(i) *the dynamical system $(Y,\mathbb{R},\sigma)$ is pseudo recurrent;*
(ii) *there exists $y_0 \in Y$ is such that $H(y_0) = Y$;*
(iii) *the equation (9) admits a solution u_0 which is bounded on $\mathbb{R}_+$.*

Then,

(i) *the equation (9) is convergent, i.e., the non-autonomous dynamical system (cocycle) generated by (9) is convergent;*
(ii) *if the point $y_0 \in Y$ is a τ-periodic (quasi periodic, almost periodic in the sense of Bohr, almost automorphic, recurrent, pseudo recurrent) point, then equation (9) has a unique τ-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) solution u such that $\mathfrak{M}_{y_0} \subseteq \mathfrak{M}_u$;*
(iii) *every solution of equation (9), which is bounded on $\mathbb{R}_+$, is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent).*

Proof. Consider equation (10). By Lemma 5.2, it admits a solution $U_0(t) := (u_0(t), u_0'(t))$ $(t \in \mathbb{R})$ which is bounded on $\mathbb{R}_+$. Denote by φ the cocycle associated to equation (9), where $\varphi(t,x,y)$ is the solution of equation (9) with initial condition $\varphi(0,x,y) = x$ and $x = (u,v)$. Let $X = Y \times \mathbb{R}^n$, $(X,\mathbb{R}_+,\pi)$ be the skew-product dynamical system and $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be the non-autonomous dynamical system generated by equation (10). Denote by $V : X \to \mathbb{R}^+$ the function defined by $V(y,(u_1,v_1),(u_2,v_2)) := \frac{1}{2}(\langle u_1 - u_2, u_1 - u_2\rangle + \langle C^{-1}(v_1 - v_2), v_1 - v_2\rangle)$ for all $(y,(u_i,v_i)) \in X := Y \times \mathbb{R}^n$ $(i = 1,2)$, then

$$V(\sigma(t,y),\varphi(t,x_1,y),\varphi(t,x_2,y)) = \\ \frac{1}{2}(\langle\varphi_1(t,x_1,y) - \varphi_1(t,x_2,y)),\varphi_1(t,x_1,y) - \varphi_1(t,x_2,y)\rangle + \\ \langle C^{-1}(\varphi_2(t,x_1,y)) - \varphi_2(t,x_2,y),\varphi_2(t,x_2,y) - \varphi_2(t,x_2,y))\rangle$$

(where $\varphi := (\varphi_1, \varphi_2)$). Since

$$\frac{dV(\sigma(t,y),\varphi(t,x_1,y),\varphi(t,x_2,y))}{dt} =$$

$$\langle G_1(\sigma(t,y),\varphi_1(t,x_1,y)) - G_1(\sigma(t,y),\varphi_1(t,x_2,y)), \varphi_1(t,x_1,y) - \varphi_1(t,x_2,y)\rangle +$$

$$\langle C^{-1}(G_2(\sigma(t,y),\varphi_1(t,x_1,y)) - G_2(\sigma(t,y),\varphi_1(t,x_2,y))), \varphi_2(t,x_1,y) - \varphi_2(t,x_2,y)\rangle,$$

then, by (11), one has $V(\sigma(t,y),\varphi(t,x_1,y),\varphi(t,x_2,y)) < V(y,x_1,x_2)$ for all $y \in Y$, x_1, $x_2 \in X$ ($x_1 \neq x_2$) and $t > 0$.

Let u_0 be a solution of equation (9) which is bounded on $\mathbb{R}_+$, then by Lemma 5.2 $U_0 := (u_0, u_0')$ is the solution of equation (10) which is bounded on $\mathbb{R}_+$. To finish the proof, it is sufficient to refer to Theorem 2.8 and Corollary 2.9. □

Example 5.4. *We consider the equation*

$$x'' + p(x)x' + ax = f(\sigma(t,y)), \tag{11}$$

where $p \in C(\mathbb{R},\mathbb{R})$, $f \in C(Y,\mathbb{R})$ and a is a positive number. Denote by $u = x$ and $v = u' + \mathcal{F}(u)$, where $\mathcal{F}(u) = \int_0^u p(s)ds$, then we obtain the system

$$\begin{cases} u' = v - \mathcal{F}(u) \\ v' = -au + f(\sigma(t,y)). \end{cases}$$

Theorem 5.5. *Assume the following conditions on the dynamical system and functions appearing in Example 5.4:*

(i) *the dynamical system $(Y,\mathbb{R},\sigma)$ is pseudo recurrent;*
(ii) *there exists a Poisson stable point $y_0 \in Y$ such that $H(y_0) = Y$;*
(iii) *$p(x) \geq 0$ for all $x \in \mathbb{R}$ and $\int_\alpha^\beta p(s)ds > 0$ for all $\alpha < \beta$ ($\alpha, \beta \in \mathbb{R}$);*
(iv) *the equation (11) admits a solution u_0 which is bounded on $\mathbb{R}_+$.*

Then,

(i) *if the point y_0 is τ-periodic (respectively, quasi periodic, almost periodic in the sense of Bohr, almost automorphic, recurrent, pseudo recurrent), then (9) has a unique τ-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) solution u such that $\mathfrak{M}_{y_0} \subseteq \mathfrak{M}_u$.*
(ii) *every solution of equation (9), which is bounded on $\mathbb{R}_+$, is asymptotically τ–periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo recurrent).*

Proof. Denote by $F(x) := \int_0^x \int_0^\eta p(s)dsd\eta$, then $p(x)x' = \frac{dF'(x)}{dt}$ and $F''(x) = p(x) \geq 0$. Note that $\langle \int_0^{x_1} p(s)ds - \int_0^{x_2} p(s)ds, x_1 - x_2\rangle = |(x_1 - x_2)\int_{x_1}^{x_2} p(s)ds| > 0$ for all $x_1 \neq x_2$ ($x_1, x_2 \in \mathbb{R}$). Now our statement follows from Theorem 5.3. □

Remark 5.6. Note that Theorem 5.5 remains true also if $p(x) \geq 0$ (for all $x \in \mathbb{R}$) without the condition $\int_\alpha^\beta p(s)ds > 0$ for all $\alpha < \beta$ ($\alpha, \beta \in \mathbb{R}$). This statement follows from Theorem 3.3 and Corollary 3.7.

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