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AN ALTERNATIVE MODEL FOR WAVE PROPAGATION IN ANISOTROPIC IMPEDANCE-MATCHED METAMATERIALS

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Abstract—The propagation of light in an anisotropic impedance-matched metamaterial is studied in the frame of geometrical optics. We prove that directions of fields $\bar{D}$, $\bar{B}$ and $\bar{v}$ (ray velocity) are a triad of conjugate directions with respect to the inverse relative dielectric permittivity tensor and constitutes a local basis, whose reciprocal one is formed by directions of $\bar{E}$, $\bar{H}$ fields and wave-vector $\bar{k}$. Consequently, both dual bases are intrinsically related to the physical properties of medium. We have identified these bases with direct and reciprocal bases of a curvilinear coordinates system, showing that physics defines geometry. This identification provides a powerful tool to solve two kinds of problems (direct and inverse ones) that currently arise: In direct problems, medium properties are given and it suffices to know $\tilde{\varepsilon} = \tilde{\mu}$ tensor at every point, to obtain the wave structure. In inverse problems, medium properties must be found for the rays to propagate along prescribed trajectories. The procedure is applied to an illustrating example.

1. INTRODUCTION

When dealing with laws of propagation of monochromatic plane waves, the so called "optical transformation theory" in stating an equivalence principle between electromagnetic parameters of a physical medium expressed in Cartesian coordinates and their analogues in vacuum obtained from a curvilinear coordinate transformation, puts in correspondence the Euclidean inhomogeneous and anisotropic physical

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space, with another homogeneous, isotropic and matter free space, but endowed with a locally flat metric. Thus, the ray trajectory in the physical medium is associated with a geodesic curve in virtual space.

Every analogy in Physics is a guide that enable us to predict hidden relations and harmonies. Thus, if we consider that stream lines in irrotational fluid motion are the analogues of rays in geometrical optics, the invisibility problem in optics becomes similar to that of the flux of a fluid stream in the presence of an obstacle. Analogy and principle of equivalence allow, in flat problems, to recover the old theory of complex potential (conformal mapping) and, consequently, mathematical analysis is considerably simplified [1].

Nevertheless,"geometrical physics" in these media can be understood, in our opinion, not only as a direct consequence ("all occurs as if") of the invariance of Maxwell equations in coordinate transformations [2–4], but anything else: physical properties of a medium intrinsically define the most adequate geometry.

In general, at visible frequencies (430–790 THz), natural media exhibit a homogeneous and isotropic magnetic behaviour together with an inhomogeneous and anisotropic dielectric one. Certain metamaterials (to which theories developed and described in this paper can be applied) when subjected to external actions, obey linear constitutive equations: $\bar{D} = \varepsilon_0 \tilde{\varepsilon} \bar{E}$; $\bar{B} = \mu_0 \tilde{\mu} \bar{H}$, with equal relative permittivity and permeability tensors, ($\tilde{\varepsilon} = \tilde{\mu}$). The specificity of behaviour laws enable us to endow the space with a metric biunivocally linked to the metamaterial physics. These properties can not be applied to natural media with dielectric anisotropy and magnetic isotropy [5].

In the present alternative treatment, it has been proved that in any point of these metamaterials (in absence of charges and currents) the wave front (or the ray) is getting distorted in such a way that $\bar{D}$, $\bar{B}$ fields and ray velocity $\bar{v}$ (or $\bar{E}$, $\bar{H}$ and $\bar{k}$) constitute a local basis of conjugated directions with respect to relative permittivity or permeability tensor evaluated at that point.

2. ALTERNATIVE ANALYTICAL MODEL

In this section, from relative permittivity (or permeability) tensor and from laws of evolution in the medium, we are intended to establish vector bases consisting of conjugated directions. These bases, intrinsically related to physical properties, are put in correspondence with local vector bases that, in curvilinear coordinates, are going to define the medium geometry.

2.1. Conjugate Directions

Let $\tilde{I}$ be a symmetric tensor of second order, with components I_{ij} referred to a Cartesian reference frame $\{\bar{i}_1, \bar{i}_2, \bar{i}_3\}$ and let $\hat{\nu}$ and $\hat{\mu}$ be any two directions. The bound vector to $\hat{\nu}$ by the tensor [6] is vector $\bar{\Gamma}_\nu$ defined as $\bar{\Gamma}_\nu = \tilde{I} \cdot \hat{\nu}$.

The intrinsic component, σ_ν of vector $\bar{\Gamma}_\nu$ is equal to its projection onto vector $\hat{\nu}$ in such a way that $\sigma_\nu = \bar{\Gamma}_\nu \cdot \hat{\nu}$. The projection of vector $\bar{\Gamma}_\nu$ onto $\hat{\mu}$ is $\bar{\Gamma}_\nu \cdot \hat{\mu}$ and as a consequence of Cauchy theorem [7] we arrive to: $\bar{\Gamma}_\nu \cdot \hat{\mu} = \bar{\Gamma}_\mu \cdot \hat{\nu}$. Two directions $\hat{\nu}$ and $\hat{\mu}$ are named conjugate if [8, 9]:

$$\bar{\Gamma}_\nu \cdot \hat{\mu} = \bar{\Gamma}_\mu \cdot \hat{\nu} = 0 \tag{1}$$

Figure 1 provides a simple geometrical interpretation: A relevant property of Cauchy quadric (defined by $\sigma_\nu = const$) [7] is that its normal at every point is collinear with bound vector $\bar{\Gamma}_\nu$ for any direction $\hat{\nu}$. The orthogonality of vectors $\bar{\Gamma}_\nu$ and $\hat{\mu}$ implies the relation of conjugation.

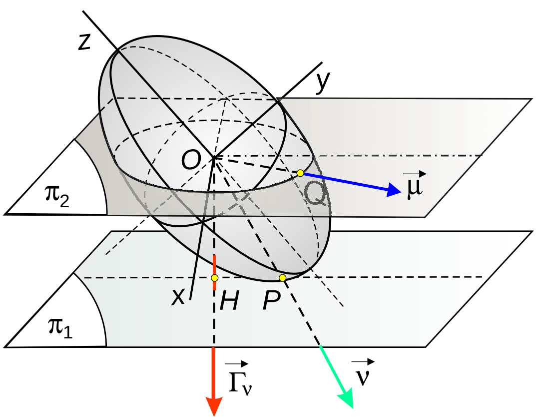

Figure 1. Cauchy's quadric associated with a second-order tensor $\tilde{I}$. The orthogonality of vectors $\bar{\Gamma}_\nu = \tilde{I} \cdot \hat{\nu}$ and $\hat{\mu}$ implies that unit vectors $\hat{\nu}$ and $\hat{\mu}$ are conjugate directions. Vector $\hat{\mu}$ lies in plane π_2 parallel to the tangent plane of the ellipsoid at "P".

It must be noted that principal directions of a tensor are also conjugate directions. Conjugation arises as a broader concept than that of principal directions (a particular case, where $\bar{\Gamma}_\nu$ is collinear with $\hat{\nu}$), but keeping the algebraic property of diagonalisation of tensors, if expressed in covariant components.

2.2. Laws of Evolution

The double Fourier transform $\bar{f}(\omega, \bar{k})$ of a vector field $\bar{F}(t, \bar{r})$, function of both time and position is:

$$\bar{f}\left(\omega, \bar{k}\right) = \int dt d^3\bar{r}\bar{F}(t, \bar{r}) e^{i\left(\omega t - \bar{k}\cdot\bar{r}\right)} \tag{2}$$

and it can be shown that [10]:

$$\frac{\partial \bar{F}}{\partial t} \rightarrow -i\omega f(\omega, \bar{k}); \quad \nabla \cdot \bar{F}(t, \bar{r}) \rightarrow i\bar{k} \cdot f(\omega, \bar{k}) \tag{3}$$

$$\nabla \times \bar{F}(t, \bar{r}) \rightarrow i\bar{k} \times f(\omega, \bar{k}) \tag{4}$$

Applying these expressions to Maxwell equations for source-free media, we have:

$$\bar{k} \times \bar{E} = \omega\bar{B}; \quad \bar{k} \times \bar{H} = -\omega\bar{D} \tag{5}$$

$$\bar{k} \cdot \bar{D} = 0; \qquad \bar{k} \cdot \bar{B} = 0 \tag{6}$$

Bearing in mind that wave vector $\bar{k}$ (wave-front gradient) and ray velocity $\bar{v}$ (collinear with Poynting vector), are given by [11]:

$$\bar{k} = \frac{(\bar{D} \times \bar{B})\omega}{W}; \quad \bar{v} = \frac{(\bar{E} \times \bar{H})}{W} \tag{7}$$

(where W is the electromagnetic energy density), from Maxwell equations [12] or from Expression (7), the following equations in $\bar{v}$-domain arise:

$$\bar{v} \times \bar{B} = -\bar{E}; \quad \bar{v} \times \bar{D} = \bar{H} \tag{8}$$

$$\bar{v} \cdot \bar{E} = 0; \qquad \bar{v} \cdot \bar{H} = 0 \tag{9}$$

In order to complete the above equations, we add some specific constitutive relations of artificial media (metamaterials), where relative permittivity tensor $(\tilde{\varepsilon}_{ij})$ and relative permeability one $(\tilde{\mu}_{ij})$ are identical when referred to the same vector basis; consequently the identity holds also for inverse tensors $\tilde{\varepsilon}_{ij}^{-1} = \tilde{\mu}_{ij}^{-1} = \tilde{\tau}_{ij}$ (in what follows, $\tilde{\tau}$). In other words,

$$\bar{D} = \varepsilon_0 \tilde{\varepsilon} \cdot \bar{E}; \quad \bar{E} = \frac{\tilde{\tau} \cdot \bar{D}}{\varepsilon_0}; \quad \bar{B} = \mu_0 \tilde{\varepsilon} \cdot \bar{H}; \quad \bar{H} = \frac{\tilde{\tau} \cdot \bar{B}}{\mu_0} \tag{10}$$

It is immediate to state that in these media, electromagnetic energy density is given by $W = \bar{E} \cdot \bar{D} = \bar{B} \cdot \bar{H}$; i.e., electric energy density coincides with magnetic one, and consequently, $\bar{k} \cdot \bar{v} = \omega$.

Under the assumptions of propagation of monochromatic (ω = constant) plane waves in the absence of current densities and charges, Equations (5), (6), (8), (9) and (10) enable us to derive some different geometrical structures and their corresponding physical laws of evolution.

2.2.1. Geometrical Structure of Fields

a) In order to describe the laws of evolution of fields, we choose a local base $(P, \bar{e}_i/\{i = 1, 2, 3\})$, consisting of vectors $\bar{e}_i$, collinear with $\bar{D}$, $\bar{B}$ fields and $\bar{v}$ (ray velocity), respectively. We will prove that this basis is conjugated with respect to the tensor $\tilde{\tau}$. The chosen vectors are:

$$\bar{e}_1 = \frac{\bar{D}}{|\bar{D}|}; \quad \bar{e}_2 = \frac{\bar{B}}{|\bar{B}|}; \quad \bar{e}_3 = \frac{\bar{v}}{|\bar{v}|}; \quad \text{with} \quad \bar{e}_i = h_i \hat{u}_i; \quad h_1 = h_2 = h_3 = 1 \quad (11)$$

and $\sqrt{g} = (\bar{e}_1 \times \bar{e}_2) \cdot \bar{e}_3$. Equations (8) and (9) prove that $\bar{E}$ $(\bar{H})$ is orthogonal to $\bar{B}(\bar{D})$ and $\bar{v}$, and constitutive laws (10) prove that $\bar{E}$ is collinear with $\bar{\Gamma}_{\mathbf{u_1}} = \tilde{\tau} \cdot \hat{u}_1$ (and $\bar{H}$ is collinear with $\bar{\Gamma}_{\mathbf{u_2}} = \tilde{\tau} \cdot \hat{u}_2$). Thus,

$$\frac{\varepsilon_0 \bar{E}}{|\bar{D}|} = \tilde{\tau} \cdot \hat{u}_1 = \bar{\Gamma}_{\mathbf{u_1}}; \quad \frac{\mu_0 \bar{H}}{|\bar{B}|} = \tilde{\tau} \cdot \hat{u}_2 = \bar{\Gamma}_{\mathbf{u_2}} \quad (12)$$

From the above mentioned orthogonality relations, Equation (12) and Cauchy theorem, we can deduce that $\bar{\Gamma}_{\mathbf{u_1}} \cdot \hat{u}_2 = \bar{\Gamma}_{\mathbf{u_1}} \cdot \hat{u}_3 = \bar{\Gamma}_{\mathbf{u_2}} \cdot \hat{u}_1 = \bar{\Gamma}_{\mathbf{u_3}} \cdot \hat{u}_1 = \bar{\Gamma}_{\mathbf{u_2}} \cdot \hat{u}_3 = \bar{\Gamma}_{\mathbf{u_3}} \cdot \hat{u}_2 = 0$. In other words, the electric displacement field $\bar{D}$, the magnetic induction field $\bar{B}$ and ray velocity $\bar{v}$ locally propagate along conjugated directions of the inverse relative permittivity tensor of the medium. From the above analysis, it yields the choice of the reciprocal local base $(P, \bar{e}^j/\{j = 1, 2, 3\})$, that consists of vectors collinear with the bound vectors to directions $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ and defined as:

$$\bar{e}^j = \frac{\bar{\Gamma}_{\mathbf{u_j}}}{h_j \tau_j} = \frac{\tilde{\tau} \cdot \bar{e}_j}{h_j^2 \tau_j} \longrightarrow \text{it yields } \bar{e}_i \cdot \bar{e}^j = \delta_i^j \quad (13)$$

with $\tau_j = \bar{\Gamma}_{\mathbf{u_j}} \cdot \hat{u}_j$ (τ_j coincide with the eigenvalues of the tensor, when vectors $\hat{u}_i$ are orthogonal).

In these frames of reference (bases $\bar{e}_i$ or $\bar{e}^i$), fields have only a component (covariant or contravariant). Thus,

$$\bar{D} = D^1 \bar{e}_1; \quad \bar{B} = B^2 \bar{e}_2; \quad \bar{v} = v^3 \bar{e}_3; \quad \bar{E} = E_1 \bar{e}^1; \quad \bar{H} = H_2 \bar{e}^2; \quad (14)$$

with $D^1 = |\bar{D}|/h_1$; $B^2 = |\bar{B}|/h_2$; $v^3 = |\bar{v}|/h_3$; $E_1 = |\bar{D}| h_1 \tau_1/\varepsilon_0$; $H_2 = |\bar{B}| h_2 \tau_2/\mu_0$, and the inverse relative permittivity tensor (in covariant components) is written as:

$$\tilde{\tau} = \tau_{ij} \bar{e}^i \otimes \bar{e}^j \quad \text{with} \quad \tau_{ij} = h_i^2 \tau_j \delta_i^j \quad (15)$$

With respect to the physical meaning of τ_1 and τ_2, we retain the expressions of electric and magnetic energy densities in a linear medium [11, 13]:

$$W_e = \frac{\bar{E} \cdot \bar{D}}{2}; \quad W_m = \frac{\bar{H} \cdot \bar{B}}{2} \quad (16)$$

Since $\tau_j = \bar{\Gamma}_{\mathbf{u_j}} \cdot \hat{u}_j$, we can write:

$$\tau_1 = \bar{\Gamma}_{\mathbf{u_1}} \cdot \hat{u}_1 = \frac{\varepsilon_0 \bar{E} \cdot \bar{D}}{|\bar{D}|^2} = \frac{W_e}{W_e^0}; \quad \tau_2 = \bar{\Gamma}_{\mathbf{u_2}} \cdot \hat{u}_2 = \frac{\mu_0 \bar{H} \cdot \bar{B}}{|\bar{B}|^2} = \frac{W_m}{W_m^0} \quad (17)$$

Coefficient τ_1 is the ratio between the electric energy density in the medium, W_e, and the electric energy density, W_e^0, carried by the wave if propagation were in vacuum and obviously, it coincides with the inverse relative permittivity associated with the $\hat{u}_1$ direction. The same applies to τ_2, but referred to magnetic energy density.

In order to know the physical meaning of every vector of the basis that describe covariant components of fields, it remains only to investigate what vector $\bar{e}^3$ represents. From (7), (13), (14) and taking into account the linear character of operator $\tilde{\tau}$, we can write,

$$\begin{aligned} \tilde{\tau} \cdot (\bar{E} \times \bar{H}) &= E_1 H_2 \tilde{\tau} \left(\bar{e}^1 \times \bar{e}^2 \right) = E_1 H_2 \epsilon^{123} \tilde{\tau} \cdot \bar{e}_3 = \frac{E_1 H_2}{\sqrt{g}} h_3^2 \tau_3 \epsilon^{312} \bar{e}_1 \times \bar{e}_2 \\ &= \frac{h_1^2 h_2^2 h_3^2 \tau_1 \tau_2 \tau_3}{\varepsilon_0 \mu_0 g} D^1 B^2 \left(\bar{e}_1 \times \bar{e}_2 \right) = \frac{h_1^2 h_2^2 h_3^2 \tau_1 \tau_2 \tau_3}{\varepsilon_0 \mu_0 g} \left(\bar{D} \times \bar{B} \right) \quad (18) \end{aligned}$$

where components $\epsilon^{123} = \epsilon^{312}$ of Levi-Civita tensor are equal to $1/\sqrt{g}$. From Equations (7), (13), (18), and taking into account that $\varepsilon_0 \mu_0 = 1/c^2$, we can write:

$$\tilde{\tau} \cdot \hat{v} = \frac{\tau_1 \tau_2 \tau_3 h_1^2 h_2^2 h_3^2 c^2}{g \omega} \bar{k} \Longrightarrow \bar{e}^3 = \frac{\tau_1 \tau_2 h_1^2 h_2^2 c^2}{g \omega |\bar{v}|} \bar{k} \quad (19)$$

We have shown that vector $\bar{e}^3$ has the direction of the wave-vector $\bar{k}$, and then $\bar{k} = k_3 \bar{e}^3$. To summarize,

$$\bar{D} = D^1 \bar{e}_1; \quad \bar{B} = B^2 \bar{e}_2; \quad \bar{v} = v^3 \bar{e}_3; \quad \bar{E} = E_1 \bar{e}^1; \quad \bar{H} = H_2 \bar{e}^2; \quad \bar{k} = k_3 \bar{e}^3 \quad (20)$$

Then, vectors of the reciprocal base $\bar{e}^i$ lie along $\bar{E}$, $\bar{H}$ and $\bar{k}$ directions. There is an intrinsic geometrical structure, associated with propagation of plane waves in these media in such a way that the Euclidean space is endowed, at every point P, with two dual bases defined from physical properties of the medium. Table 1 and Figure 2 show this geometrical structure.

Table 1. Direct base vectors and reciprocal base vectors obtained from tensor $\tilde{\tau}$.

Direct basis	**Reciprocal basis**
$\bar{e}_1$ collinear with $\bar{D}$	$\bar{e}^1$ colllinear with $\bar{E}$
$\bar{e}_2$ collinear with $\bar{B}$	$\bar{e}^2$ collinear with $\bar{H}$
$\bar{e}_3$ collinear with $\bar{v}$	$\bar{e}^3$ collinear with $\bar{k}$

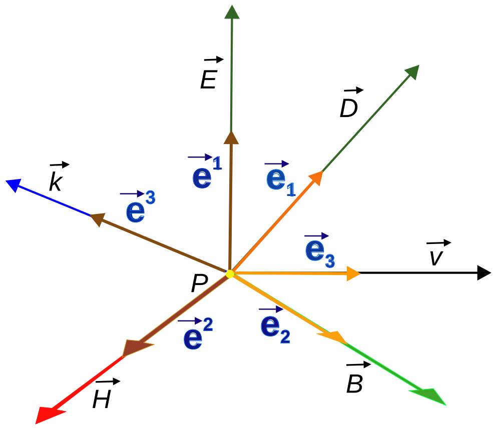

Figure 2. Local reciprocal bases $\bar{e}_i$ and $\bar{e}^i$ associated with the structure of a light plane wave propagating through an anisotropic impedance-matched medium.

b) Irrespective of anything given above, we may assume that the space is described by a system of curvilinear coordinates, q_1, q_2, q_3 related to Cartesian ones, x_1, x_2, x_3 through the equations:

$$q_j = q_j(x_1, x_2, x_3) \quad \Leftrightarrow \quad x_j = x_j(q_1, q_2, q_3) \quad \text{with} \quad j = 1, 2, 3, \tag{21}$$

that allow us to define, if the transformation is admissible [7], two reciprocal local bases at any point P as:

$$\text{Direct basis: } (P, \bar{e}_j)/\bar{e}_j = \frac{\partial \bar{r}}{\partial q_j} = \sum_{k=1}^{k=3} \frac{\partial x_k}{\partial q_j} \bar{i}_k \quad \text{with } j = \{1, 2, 3\} \tag{22}$$

$$\text{Dual basis: } (P, \bar{e}^j)/\bar{e}^j = \nabla q_j = \sum_{k=1}^{k=3} \frac{\partial q_j}{\partial x_k} \bar{i}_k \quad \text{with } j = \{1, 2, 3\} \tag{23}$$

where reciprocity relation $\bar{e}_i \cdot \bar{e}^j = \delta_i^j$ holds. Transformation rules between base vectors are given by $\bar{e}^i = g^{ij} \bar{e}_j$, $\bar{e}_i = g_{ij} \bar{e}^j$, where $g_{ij} = \bar{e}_i \cdot \bar{e}_j$ is the metric tensor.

This geometry, defined by metric tensor g_{ij}, suggests to identify local bases (11) and (13), intrinsically related to the physical properties of medium, with (22) and (23) of the just described arbitrary curvilinear coordinates system. From this correspondence, physics defines geometry.

Assuming the equality of local bases, we have a powerful tool to solve two kinds of problems that currently arise: direct and inverse ones. For direct problems, medium properties are given and it suffices to know $\tilde{\varepsilon} = \tilde{\mu}$ tensor at every point, because the method provides the wave structure. In inverse problems, we must find the medium properties for the rays to propagate along prescribed trajectories. If ray trajectory is given, i.e., geometry is known, we can determine medium properties $\tilde{\tau}$ (See Section 3: Application).

2.2.2. Ray Evolution and Fermat Principle

Unlike natural optical anisotropic media (with dielectric anisotropy and magnetic isotropy), in these media with $\tilde{\varepsilon} = \tilde{\mu}$, there is an only mode of propagation and an only ray velocity $\bar{v}$, for every wave-vector $\bar{k}$.

From Equation (19) and since in this case, $\bar{k} \cdot \bar{v} = \omega$, we obtain,

$$\bar{v} \cdot \tilde{\tau} \cdot \bar{v} = \frac{\tau_1 \tau_2 \tau_3 h_1^2 h_2^2 h_3^2 c^2}{g} \tag{24}$$

In accordance with the definition of ray index of refraction n_r [11] as $n_r = c/v$, and remembering that $\bar{v}$ is tangent to the ray trajectory: $\bar{v} = v d\bar{r}/dl = v\bar{t}$, we can write:

$$n_r^2 = \frac{c^2}{v^2} = \frac{g}{h_1^2 h_2^2 h_3^2 \tau_1 \tau_2 \tau_3} \left(\hat{u}_3 \cdot \tilde{\tau} \cdot \hat{u}_3 \right) \tag{25}$$

We have just found the expression of the ray index as:

$$n_r = \frac{\sqrt{g}}{h_1 h_2 h_3} \left(\bar{t} \cdot \frac{\tilde{\tau}}{\tau_1 \tau_2 \tau_3} \cdot \bar{t} \right)^{1/2} \tag{26}$$

with $\hat{u}_3 \equiv \bar{t} = \frac{d\bar{r}}{dl}$ with $|d\bar{r}| = dl$. Since n_r is a function both of position $\bar{r}$ and of the direction of vector $d\bar{r}/dl$, Equation (26) shows the complexity of light propagation in inhomogeneous and anisotropic media, even for the materials in discussion, for which there is an only mode of propagation, and we can solve for n_r. Unlike the isotropic case, the distance between two infinitely near points depend on their position and also on their relative orientation. Consequently, equations of evolution can be found as geodesics in a Finsler space [14], in which the line element is given by $ds = n_r(\bar{r}, d\bar{r}/dl)dl$. When values of n_r are substituted into the Fermat's variational principle in order to use Hamiltonian and Lagrangian formalism, a problem arises because conventional Legendre transform, would obtain an identically vanishing Hamiltonian from Lagrangian, which is homogeneous of degree one in $d\bar{r}/dl$. Nevertheless, Fermat principle is invariant under reparameterising, then parameter "arc length" l may be substituted by optical path, in such a way that $dl = n_r d\sigma$ [15], and a non-singular

Lagrangian arises as $L = n_r^2(\bar{r}, d\bar{r}/d\sigma)$. Consequently, equations of evolution can be found as extremals of the functional,

$$\delta \int n_r^2 d\sigma = 0 \tag{27}$$

where distance is measured by travel time (optical path).

3. APPLICATION

Let us illustrate the above theory with an application. Since direct problems have been more discussed, we deal with an inverse one: Which optical properties must the medium have for light-rays to propagate along a prescribed trajectory? For instance, we want to characterize medium properties in such a way that every plane wave that propagates along an axis, say z (wave vector $\bar{k}$ parallel to z axis), yields a ray describing an helicoidal trajectory of axis z. For that purpose, we introduce the following coordinates system (See Figure 3).

$$q_1 = \frac{z}{\theta}; \quad q_2 = \varrho; \quad q_3 = z \tag{28}$$

expressed in cylindrical coordinates: ϱ, θ, z. We see that coordinate plane q_3 = constant describes the wave-front and coordinate line q_3 is along the desired ray.

Vectors of the direct basis are:

$$\bar{e}^1 = \nabla q_1 = -\frac{z}{\varrho\theta^2}\hat{u}_\theta + \frac{1}{\theta}\hat{u}_z; \quad \bar{e}^2 = \nabla q_2 = \hat{u}_\varrho; \quad \bar{e}^3 = \nabla q_3 = \hat{u}_z \tag{29}$$

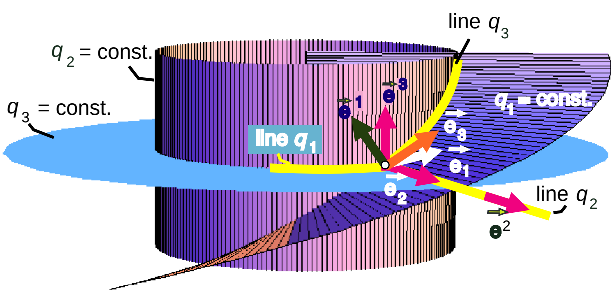

Figure 3. Coordinate surfaces and coordinate lines of curvilinear coordinates system (q_1, q_2, q_3), defined as $q_1 = z/\theta$; $q_2 = \varrho$; $q_3 = z$, where cylindrical coordinates ϱ, θ and z are used. In this case, coordinate plane q_3 = constant describes the wave-front and coordinate line q_3 is along the desired ray.

and fundamental metric tensor g^{ij} can be expressed in contravariant and covariant components, respectively, as:

$$g^{ij}=\begin{pmatrix}\frac{z^2}{\varrho^2\theta^4}+\frac{1}{\theta^2} & 0 & \frac{1}{\theta}\\ 0 & 1 & 0\\ \frac{1}{\theta} & 0 & 1\end{pmatrix}; \quad g_{ij}=(g^{ij})^{-1}=\begin{pmatrix}\frac{\varrho^2\theta^4}{z^2} & 0 & -\frac{\varrho^2\theta^3}{z^2}\\ 0 & 1 & 0\\ -\frac{\varrho^2\theta^3}{z^2} & 0 & 1+\frac{\varrho^2\theta^2}{z^2}\end{pmatrix} \tag{30}$$

From transformation rule $\bar{e}_i = g_{ij}\bar{e}^j$, vectors of the dual base $\bar{e}_i$ are obtained:

$$\bar{e}_1 = -\frac{\varrho\theta^2}{z}\hat{u}_\theta; \quad \bar{e}_2 = \hat{u}_\varrho; \quad \bar{e}_3 = \frac{\varrho\theta}{z}\hat{u}_\theta + \hat{u}_z \tag{31}$$

Since $\tilde{\tau} = h_i^2\tau_i\delta_{ij}\bar{e}^i \otimes \bar{e}^j$, tensor $\tilde{\tau}$ can be expressed in cylindrical coordinates as:

$$\tilde{\tau} = \begin{pmatrix}\tau_2 & 0 & 0\\ 0 & \tau_1 & -\alpha\tau_1\\ 0 & -\alpha\tau_1 & \tau_3+\alpha^2(\tau_1+\tau_3)\end{pmatrix}_{\hat{u}_\varrho,\hat{u}_\theta,\hat{u}_z} \tag{32}$$

where $\alpha = \varrho\theta/z$. Substitution into Equation (25) becomes:

$$n_r^2=\frac{\alpha^2\theta^2}{\alpha^2\theta^2(1+\alpha^2)\tau_1\tau_2\tau_3}\bar{t}\cdot\tilde{\tau}\cdot\bar{t}; \quad \text{where} \quad \bar{t}=\frac{d\bar{r}}{dl}=\begin{pmatrix}\frac{d\varrho}{dl}\\ \frac{1}{\varrho}\frac{d\theta}{dl}\\ \frac{dz}{dl}\end{pmatrix} \tag{33}$$

After some calculations, we get:

$$n_r^2=\frac{z^2}{(z^2+\varrho^2\theta^2)}\left\{\tau_2\left(\frac{d\varrho}{dl}\right)^2+\tau_1\left(\frac{1}{\varrho}\frac{d\theta}{dl}-\frac{\varrho\theta}{z}\frac{dz}{dl}\right)^2+\tau_3\left(1+\frac{\varrho^2\theta^2}{z^2}\right)\left(\frac{dz}{dl}\right)^2\right\} \tag{34}$$

Fermat principle would lead to the already known solution: $\hat{u}_3 \equiv \bar{t}$. Thus,

$$\hat{u}_3 = \frac{1}{\sqrt{1+\alpha^2}}\begin{Bmatrix}0\\ \alpha\\ 1\end{Bmatrix} \equiv \frac{d\bar{r}}{dl} = \begin{Bmatrix}\frac{d\varrho}{dl}\\ \varrho\frac{d\theta}{dl}\\ \frac{dz}{dl}\end{Bmatrix} \tag{35}$$

So, we get: $\varrho = a$, $\theta = \theta$, and $z = b\theta$, where ab constants, and the equation of a helix is reobtained. Then, properties of the medium are given by $\tilde{\varepsilon} = \tilde{\tau}^{-1}$, and so,

$$\tilde{\varepsilon} = \frac{1}{(z^2+\varrho^2\theta^2)}\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{\tau_1\varrho^2\theta^2+\tau_3(z^2+\varrho^2\theta^2)}{\tau_1\tau_3} & \varrho\theta z\\ 0 & \varrho\theta z & \frac{z^2}{\tau_3}\end{pmatrix}_{\hat{u}_\varrho,\hat{u}_\theta,\hat{u}_z} \tag{36}$$

4. CONCLUSIONS

We have considered the problem of propagation of light plane waves in an anisotropic impedance-matched metamaterial, under an alternative approach, based on the concept of conjugacy of directions with respect a tensor. We have shown that directions of fields $\bar{D}$, $\bar{B}$ and $\bar{v}$ (ray velocity) are a triad of conjugate directions with respect to the inverse relative dielectric permittivity tensor and constitutes a local basis, whose reciprocal one is formed by directions of $\bar{E}$, $\bar{H}$ fields and wave-vector $\bar{k}$. We noticed that there exists an intrinsic geometrical structure associated with propagation of plane waves in these media in such a way that the space is endowed, at every point P, with two reciprocal bases defined from physical properties of the medium. This structure is similar to that used in the study of crystal structures.

Since propagation of electromagnetic waves are always described in orthogonal frames, where there is no distinction between covariant and contravariant vector components, the different tensor nature of vector fields ($\bar{E}$ and $\bar{D}$) remains masked (the same for $\bar{B}$ and $\bar{H}$). We realized that these fields, together with $\bar{k}$ and $\bar{v}$ have only a component (covariant or contravariant) in these dual bases.

We have put into correspondence dual bases with direct and reciprocal bases of a curvilinear coordinates system, showing that physics defines geometry. This identification provides a powerful tool to solve current direct and inverse problems. In direct problems, medium properties are known and behaviour of rays is investigated. We have shown that it suffices to know $\tilde{\varepsilon} = \tilde{\mu}$ tensor at every point, to obtain the wave structure.

Of more interest are inverse problems (for instance, in dealing with perfect lenses, wave-guide propagation, and so on), where medium properties must be found for the rays to propagate along prescribed trajectories. The procedure has been applied to an illustrating example.

REFERENCES

1. Leonhardt, U., "Optical conformal mapping," *Science*, Vol. 312, 1777–1780, 2006.
2. Pendry, J. B., D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," *Science*, Vol. 312, 1780–1782, 2006.
3. Chen, H., C. T. Chan, and P. Sheng, "Transformation optics and metamaterials," *Nature Mater.*, Vol. 9, 387–396, 2010.
4. Leonhardt, U. and T. G. Philbin, *Geometry and Light: The Science of Invisibility*, Dover, Mineola, 2010.

5. Bellver-Cebreros, C. and M. Rodriguez-Danta, "Conjugacy and reciprocal bases in anisotropic dielectric media," *Optica Pura y Aplicada*, Vol. 44, No. 3, 571–580, 2011.
6. Bellver-Cebreros, C. and M. Rodriguez-Danta, "Analogy between the torque-free motion of a rigid body about a fixed point and light propagation in anisotropic media," *European Journal of Physics*, Vol. 30, 285–294, 2009.
7. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice Hall, Englewood Cliffs, New Jersey, 1965.
8. Salmon, G., *Treatise on Conic Sections*, AMS Chelsea Publishing, Providence, Rhode Island, 1960.
9. Zwikker, C., *The Advanced Geometry of Plane Curves and Their Applications*, Dover, Mineola, 2005.
10. Melrose, D. B. and R. C. McPhedran, *Electromagnetic Processes in Dispersive Media*, Cambridge University Press, Cambridge, 1991.
11. Born, M. and E. Wolf, *Principles of Optics*, Pergamon Press, New York, 1999.
12. Bellver-Cebreros, C. and M. Rodriguez-Danta, "Refraccion conica externa en medios biaxicos a partir de la construccion de Mohr," *Optica Pura y Aplicada*, Vol. 36, 33–37, 2003.
13. Jackson, J. D., *Classical Electrodynamics,* 3th Edition, Wiley and Sons, Oxford, 2009.
14. Joets, A. and R. Ribotta, "A geometrical model for the propagation of rays in an anisotropic inhomogeneous medium," *Optics Communications*, Vol. 107, 200–204, 1994.
15. Bellver-Cebreros, C. and M. Rodriguez-Danta, "Eikonal equation from continuum mechanics and analogy between equilibrium of a string and geometrical light rays," *American Journal of Physics*, Vol. 69, 360–367, 2001.