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Bifurcaciones de Órbitas Periódicas y Conjuntos Invariantes en Sistemas Dinámicos Lineales a Trozos.

Bifurcations of Periodic Orbits and Invariant Sets in Piecewise Linear Dynamical Systems.

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Tesis Doctoral

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Bifurcations of Periodic Orbits and Invariant Sets in Piecewise Linear Dynamical Systems.

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A Jaume.

A mis padres y a Elena.

A mis abuelos.

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Introduction

C'est par la logique qu'on démontre, c'est par l'intuition qu'on invente.

(It is by logic that we prove, but by intuition that we discover).

Henri Poincaré, *Science et Méthode*, 1908.

This year is the 100^{th} anniversary of the death of Jules Henri Poincaré (Nancy, France, 29 April 1854-Paris, France, 17 July 1912), the founding father of Dynamical Systems theory. In mathematics, he is known as *The Last Universalist*, since he excelled in all fields of the discipline during his life. As it is well-known, the motivation of part of his work was the Celestial Mechanics [82, 83] and more specifically, the three-body problem. Within these more than 100 years, the Dynamical Systems theory has become one of the most important topics of interest for the scientific community. This is mainly due to the broad field of application. Although the first applications of Poincaré's ideas were in engineering, more concretely in electronic circuits and control theory in the 20's (Appleton and Van der Pol [3, 4, 93], Cartan [27], Liénard [69], Andronov and Pontryaguin [1, 2]), nowadays the applications go from engineering to biomathematics, (such as neural networks [63, 95]) passing through financial problems and social behaviors [84].

Among dynamical systems, in the last years we have attended to the expansion of the field of Piecewise-Smooth dynamical systems. First examples of the use of piecewise-smooth functions (in particular, piecewise linear) are found in the 1937 book of Andronov, Vitt and Khaikin [1], where they used it to model electronic, mechanical and control systems (saturation functions, impacts, switching...). Since then, the capability of piecewise-smooth systems to model a multitude of phenomena has been proven. In the 2008 published book of Mario di Bernardo et al. [35] they revise the state of the art of piecewise-smooth systems and we can find a huge number of references.

In the framework of piecewise-smooth dynamical systems, there exists a class that is worth mentioning: the Piecewise Linear (PWL) Systems. As we have just said, the first examples of their use can be found in [1]. The importance of PWL systems is due, inter alia, to their ability to model faithfully real applications (neuron models [5, 33, 86], Chua's circuit [81], Colpitts's oscillator [74], Wien-Bridge oscillator [62, 76]), to reproduce bifurcations of differential systems and to show new behaviors, impossible to obtain under differentiability hypothesis (the behavior around the Teixeira singularity [90, 91], the continuous matching of two stable linear systems can be unstable [26]...). Furthermore, although the system can be integrated in each zone of linearity, which allows us to obtain explicitly some geometric and dynamical basic elements, it is not possible to obtain the general solution of the system and the classical theory of differential systems cannot be applied to PWL systems. Therefore, it is necessary creating a new theory to tackle PWL systems.

The first step to analyze PWL systems is their simplification and reduction to a canonical form [13, 22, 45, 60, 61]. In this thesis, we focus our attention, mainly, on two-zonal three-dimensional continuous and planar discontinuous PWL systems. The work is split into six chapters. In the first section of the introductory Chapter 1, we show the canonical forms of the systems object of study along this work and we do it by classifying the systems from the point of view of the control notions (observability and controllability), as it was performed in [13, 22].

In a second step, the dynamical behavior must be studied. The analysis begins usually by finding the equilibrium states, i.e., the equilibrium points of the system and their stabilities. After that, the objective is the searching of periodic behaviors, that is, periodic orbits. This is neither an easy task in a differential system in general, nor in a piecewise-smooth system. A usual technique is the construction of the so-called Poincaré map, which in the case of PWL systems is defined through the composition of some transition maps, the Poincaré half-maps [56, 59, 72], defined in each zone of linearity. The second section of Chapter 1 is devoted to defining the Poincaré map for continuous PWL systems.

As we have just commented, the analysis of periodic orbits in piecewise-smooth systems is not obvious. One of the aims of this essay is to shed light on this problem, by using different techniques to analyze the existence, bifurcations and stabilities of periodic orbits in planar and three-dimensional PWL systems.

To find periodic orbits in planar smooth systems it is well-known the Melnikov theory [8], sometimes called Malkin-Loud theory [73, 75, 78]. The most important property that a family of systems must fulfill to apply the Melnikov theory is the existence of a system of the family having a continuum of periodic orbits, homoclinic connections or heteroclininic cycles. The Melnikov method was generalized to planar continuous piecewise-smooth systems in [13]. In Chapter 2 of this thesis, we generalize the Melnikov theory to hybrid systems (mixture between a flow and a map [35]) and we apply it to discontinuous PWL systems with two zones of linearity and to continuous PWL systems with three zones.

On the other hand, we will consider three-dimensional homogeneous continuous PWL systems. In such systems, when only equilibrium point is the origin, the most significant question is the stability of this equilibrium point. This is not at all an easy question and the answer can be counterintuitive. In fact, the continuous matching of two stable linear systems can be unstable [26]. As we explain in the last section of Chapter 1, the stability of the system is related to the existence of a type of invariant objects called invariant cones. The analysis of their existence, bifurcations and stabilities is other objective of this thesis. Specifically, in Chapter 3, we analyze the invariant cones of a family of observable three-dimensional continuous PWL linear systems by studying the periodic orbits of a family of planar hybrid PWL systems, where the Melnikov theory of Chapter 2 can be applied.

The original Melnikov theory was developed for smooth planar systems, and in Chapter 2 we have generalized it to a class of non-smooth systems. In Chapter 4 we adapt it to a class of continuous three-dimensional systems. As it has been commented, one of the properties that a family of systems must fulfill to be able to adapt the ideas of the Melnikov theory is the existence of a system of the family having a continuum of periodic orbits. We have found an appropriate family of non-controllable systems in Chapter 4 and we have applied the ideas of the Melnikov theory to analyze the existence of periodic orbits.

The existence of invariant cones in observable PWL systems has been analyzed, inter alia, in [18, 21, 25]. However, as far as we know, the problem has not been tackled for the non-observable case. Chapter 5 begins with the analysis of invariant cones in three-dimensional continuous homogeneous non-observable PWL systems. Among these systems, we set the conditions for the existence of a system having a cone foliated by periodic orbits. Beyond the Melnikov theory, there exist other methods to find periodic orbits. For instance, in Chapter 5, we have adapted the method of Chapter 14 of [31] for analyzing the periodic orbits of the continuum that remain after a perturbation which makes the system observable and non-homogeneous.

The analysis of the existence of periodic orbits or invariant cones by the usual techniques in PWL systems, for instance, by computing the fixed points of the Poincaré map, needs the transversal intersection of the orbits (respectively, cones) with the separation boundary between the linearity zones. With the theory and methods that has been developed in this thesis, we have been able to study the existence of periodic orbits and invariant cones with tangential intersection to the separation boundary. Between the periodic orbits with tangential intersection with the separation boundary, we can distinguish two different cases. In a first case, the orbit remain locally in the same zone of differentiability. This case has been analyzed along the work in chapters 3 and 4. The other possible case is when the tangent orbit crosses the separation boundary. In Chapter 6, the dynamical richness that this situation fosters has been shown by analyzing a three-dimensional continuous PWL system with two zones of linearity, which is considered a PWL version of the well-known Michelson system [77], which possesses a periodic orbit with two intersection points with the separation plane, which crosses it with tangential intersection. In particular, it will be shown that the presence of this orbit is the necessary germ for the existence of the so-called noose bifurcation [58]. The noose bifurcation occurs when the curve of the family of periodic orbits that appears from a period-doubling bifurcation and the curve of the original family of periodic orbits come together and annihilate at a saddle-node bifurcation. Therefore, two of the most common bifurcations of periodic orbits (saddle-node and period-doubling) are connected by a noose-shaped curve.

To conclude the thesis, we will write a short summary and we will consider the open problems that arise from this study.

Chapter 1

Piecewise Linear Systems. Basic Concepts

In this chapter, we introduce some concepts and tools that are used in the thesis. Among other things, we will familiarize with the notation that will be employed along the work and the systems object of study.

The chapter is outlined as follows. In a first section we introduce the main part of the differential systems to be studied in this work. Continuous Piecewise Linear systems (later on, CPWL) are introduced and classified from the point of view of control theory, as it was done in [22]. Moreover, in this first section the canonical forms of planar Discontinuous Piecewise Linear systems (later on, DPWL) is considered [45]. After that, in a second section we define a useful tool to study the existence of periodic orbits and invariant manifolds in differential systems in general, but adapted to CPWL systems: the Poincaré map and the displacement function. Finally, in the last section of this chapter, we explain the motivation of studying invariant cones in three-dimensional CPWL systems and we state the basis for their study.

Let us begin by introducing some notation. For a function

$$\begin{aligned} \mathbf{x}: \quad & \mathbb{R} \longrightarrow \mathbb{R}^n, \\ & t \mapsto (x_1(t), x_2(t), \ldots, x_n(t))^T, \end{aligned}$$

let us denote by $\dot{\mathbf{x}}$ the derivative of $\mathbf{x}$ with respect to the temporal variable t.

Let $\mathbf{e}_i$ be the i-th vector of the canonical basis of $\mathbb{R}^n$ and $\mathcal{M}_{m\times n}(\mathbb{R})$ with $m,\ n \in \mathbb{N}$ the set of matrices of order $m \times n$ with elements in $\mathbb{R}$. Specifically, we denote $\mathcal{M}_n(\mathbb{R})$ the set of squared matrices of order n with elements in $\mathbb{R}$. By $\langle\cdot,\cdot\rangle$ we denote the usual dot product in $\mathbb{R}^n$ and by $\|\cdot\|$ the euclidean norm associated to this product. Furthermore, in this work, we will use the following notation, which will be truly useful in the systems object of study and that was previously used in

[13].

$$M^{\nabla}\mathbf{x} = \begin{cases} M^{-}\mathbf{x} \text{ if } x_1 \leq 0, \\ M^{+}\mathbf{x} \text{ if } x_1 > 0, \end{cases}$$

where M^{+}, $M^{-} \in \mathcal{M}_{m\times n}(\mathbb{R})$.

1.1 Piecewise Linear Systems. Observability and Controllability. Canonical forms

In a first subsection we introduce the CPWL systems that have appeared in the literature up to now, by centering our attention on those that will be studied in this thesis, whose are, mainly, three-dimensional. After that, we will introduce the canonical form of planar DPWL systems recently found in [45].

1.1.1 Continuous Piecewise Linear Systems

Definition 1.1 It is said that the autonomous equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1.1}$$

defines a two-zonal continuous piecewise linear (CPWL) system in $\mathbb{R}^n$ if there exist $\mathbf{a}_1, \mathbf{a}_2, \mathbf{v} \in \mathbb{R}^n$, with $\mathbf{v} \neq 0$, $A_1, A_2 \in \mathcal{M}_n(\mathbb{R})$ and $\delta \in \mathbb{R}$ such that

(a) $\mathbf{f}(\mathbf{x}) = \begin{cases} A_1\mathbf{x} + \mathbf{a}_1 & \text{if } \langle \mathbf{x}, \mathbf{v}\rangle + \delta \leq 0, \\ A_2\mathbf{x} + \mathbf{a}_2 & \text{if } \langle \mathbf{x}, \mathbf{v}\rangle + \delta > 0. \end{cases}$

(b) $A_1\mathbf{x} + \mathbf{a}_1 = A_2\mathbf{x} + \mathbf{a}_2$ when $\langle \mathbf{x}, \mathbf{v}\rangle + \delta = 0$.

The hyperplane $\langle \mathbf{x}, \mathbf{v}\rangle + \delta = 0$ is called the separation boundary.

Later on, the set of dynamical systems satisfying Definition 1.1 will be denoted by 2CPWL$_n$. The separation hyperplane splits the space into two regions, in which system (1.1) is linear. We will see in the next Proposition that doing an appropriate change of variables the separation boundary can be transformed into the hyperplane $\{x_1 = 0\}$.

Proposition 1.2 The CPWL system (1.1) can be written into the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{cases} A^-\mathbf{x} + \mathbf{b} & \text{if} \quad x_1 \leq 0, \\ A^+\mathbf{x} + \mathbf{b} & \text{if} \quad x_1 > 0. \end{cases} \tag{1.2}$$

where $\mathbf{b} \in \mathbb{R}^n$ and the matrices $A^+, A^- \in \mathcal{M}_n(\mathbb{R})$ satisfy the relationship

$$A^+ - A^- = (A^+ - A^-)\mathbf{e}_1\mathbf{e}_1^T.$$

Proof: Consider the Householder matrix H which verifies condition $H\mathbf{v} = (\|\mathbf{v}\|, 0, \ldots, 0)^T \in \mathbb{R}^n$. The change of variable $\mathbf{y} = H\left(\mathbf{x} + \delta\mathbf{v}/\|\mathbf{v}\|^2\right)$ allows us to write system (1.1) as

$$\dot{\mathbf{y}} = \begin{cases} HA_1H\mathbf{y} + H\left(\mathbf{a}_1 - A_1\delta\mathbf{v}/\|\mathbf{v}\|^2\right) & \text{if} \quad y_1 \leq 0, \\ HA_2H\mathbf{y} + H\left(\mathbf{a}_2 - A_2\delta\mathbf{v}/\|\mathbf{v}\|^2\right) & \text{if} \quad y_1 > 0. \end{cases}$$

Taking into account the continuity of the field we deduce, on the one hand, the coincidence of the last $n-1$ columns of the matrices HA_1H and HA_2H and, on the other hand, the equality

$$H\left(\mathbf{a}_1 - A_1\delta\mathbf{v}/\|\mathbf{v}\|^2\right) = H\left(\mathbf{a}_2 - A_2\delta\mathbf{v}/\|\mathbf{v}\|^2\right).$$

By taking $A^- = HA_1H$, $A^+ = HA_2H$ and $\mathbf{b} = H\left(\mathbf{a}_1 - A\delta\mathbf{v}/\|\mathbf{v}\|^2\right)$, we conclude the proof by renaming the variable $\mathbf{y}$ by $\mathbf{x}$. □

In the following proposition it is stated that the existence and uniqueness of solution of the initial value problem associated to systems of the class 2CPWL$_n$ is guaranteed. In fact, this result is also valid for systems with $m \in \mathbb{N}$ zones of definition, mCPWL$_n$.

Proposition 1.3 The initial value problem

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{1.3}$$

with $\mathbf{f}$ given in Definition 1.1 and $\mathbf{x}_0 \in \mathbb{R}^n$ possesses a unique solution which is defined in $\mathbb{R}$.

Proof: It is sufficient to prove that function $\mathbf{f}$ is globally Lipschitz in $\mathbb{R}^n$. (see [31, 80]).

From Proposition 1.2, it can be assumed, without loss of generality, that the field $\mathbf{f}$ takes the form (1.2).

Consider the euclidean matrix norm. For $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, the following possibilities arise.

(a) If $x_1 \geq 0$ and $y_1 \geq 0$, then

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| = \|A^+\mathbf{x} - A^+\mathbf{y}\| \leq \|A^+\| \cdot \|\mathbf{x} - \mathbf{y}\|.$$

(b) If $x_1 \leq 0$ e $y_1 \leq 0$, then

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| = \|A^-\mathbf{x} - A^-\mathbf{y}\| \leq \|A^-\| \cdot \|\mathbf{x} - \mathbf{y}\|.$$

(c) If $x_1 > 0$ and $y_1 < 0$, take

$$\mathbf{z} = \frac{y_1}{y_1 - x_1}(\mathbf{x} - \mathbf{y}) + \mathbf{y},$$

then $z_1 = 0$ and from the continuity of $\mathbf{f}$ it follows

$$\begin{aligned}
\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| &= \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{z}) + \mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{y})\| \leq \\
&\leq \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{z})\| + \|\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{y})\| = \\
&= \|A^+\mathbf{x} - A^+\mathbf{z}\| + \|A^-\mathbf{z} - A^-\mathbf{y}\| \leq \\
&\leq \|A^+\| \cdot \|\mathbf{x} - \mathbf{z}\| + \|A^-\| \cdot \|\mathbf{z} - \mathbf{y}\| \leq \\
&\leq \max\{\|A^+\|, \|A^-\|\} \cdot \|\mathbf{x} - \mathbf{y}\|.
\end{aligned}$$

Therefore,

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

where $K = \max\{\|A^+\|, \|A^-\|\}$. That is, $\mathbf{f}$ is globally Lipschitz in $\mathbb{R}^n$, so the initial value problem (1.3) possesses a unique solution defined in $\mathbb{R}$. □

In control theory, it is usual to find CPWL systems [6, 13, 22]. In particular, by taking into consideration observability and controllability notions, the equations of the system can be simplified. Subsequently, we give a definition of the observability and controllability concepts for 2CPWL$_n$ systems.

Definition 1.4 System (1.2) is said to be observable if the observability matrix

$$\mathcal{O} = \left(\mathbf{e}_1|(A^-)^T\mathbf{e}_1|((A^-)^T)^2\mathbf{e}_1|\ldots|((A^-)^T)^{n-1}\mathbf{e}_1\right)^T$$

has full rank. The system is said to be controllable if the controllability matrix

$$\mathcal{C} = \left(\mathbf{b}|A^-\mathbf{b}|(A^-)^2\mathbf{b}|\ldots|(A^-)^{n-1}\mathbf{b}\right)^T,$$

where $\mathbf{b} = (A^+ - A^-)\mathbf{e}_1$, has full rank.

From the notions of observability and controllability, it is possible to simplify the expression of CPWL systems getting canonical forms. Although the analysis of these canonical forms could be done in $\mathbb{R}^n$ as it was done in [13, 22], we prefer to focus our attention on the three-dimensional case. In this work, the main part of CPWL systems to consider are in the three-dimensional space.

When $n = 3$ system (1.2) can be written as block matrices

$$\dot{\mathbf{x}} = A^{\triangledown}\mathbf{x} + \mathbf{b} = \left(\begin{array}{c|c} a_{11}^{\triangledown} & A_{12} \\ \hline A_{21}^{\triangledown} & A_{22} \end{array}\right)\mathbf{x} + \mathbf{b}, \tag{1.4}$$

where $\mathbf{b} = (b_1, b_2, b_3)^T \in \mathbb{R}^3$, $a_{11}^+, a_{11}^- \in \mathbb{R}$, $A_{12} \in \mathcal{M}_{1\times 2}(\mathbb{R})$, $A_{21}^+, A_{21}^- \in \mathcal{M}_{2\times 1}(\mathbb{R})$ and $A_{22} \in \mathcal{M}_2(\mathbb{R})$.

In the following proposition, which was proven in [13, 22], we obtain a canonical form for system (1.4), under observability hypothesis. The obtained form will be called the Liénard form.

Proposition 1.5 The CPWL system (1.4) is observable if and only if the set of vectors $\{A_{12}, A_{12}A_{22}\} \subset \mathbb{R}^2$ is a basis of $\mathbb{R}^2$. Moreover, in such a case, there exists a linear change of variable which takes the system to the Liénard form

$$\dot{\mathbf{x}} = \begin{pmatrix} t^{\triangledown} & -1 & 0 \\ m^{\triangledown} & 0 & -1 \\ d^{\triangledown} & 0 & 0 \end{pmatrix}\mathbf{x} - a\,\mathbf{e}_3, \tag{1.5}$$

where $t^{\triangledown} = \mathrm{tr}(A^{\triangledown})$, $m^{\triangledown}$ and $d^{\triangledown} = \det(A^{\triangledown})$ are the coefficient of the characteristic polinomial of $A^{\triangledown}$ and $a \in \mathbb{R}$.

Note that, if system (1.4) is non-observable, it can be decoupled. For instance, if $A_{12} = \mathbf{0}$ system (1.4) is, from Proposition 1.5, non-observable, and its equations are

$$\begin{cases} \dot{x} &= a_{11}^{\triangledown} x + b_1, \\ \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} &= A_{21}^{\triangledown} x + A_{22} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} b_2 \\ b_3 \end{pmatrix}, \end{cases}$$

where variable x evolves independently.

If $A_{12} \neq \mathbf{0}$ and the set of vectors $\{A_{12}, A_{12}A_{22}\} \subset \mathbb{R}^2$ is linearly dependent, then after direct changes of variable [13, 22], system (1.4) is written as

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} a_{11}^{\triangledown} & -1 \\ a_{21}^{\triangledown} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix}, \\ \dot{z} &= a_{31}^{\triangledown} x + a_{32} y + a_{33} z + b_3, \end{cases} \tag{1.6}$$

and variables x and y evolve independently of z.

Under the hypothesis of observability, the loss of controllability also allows us to decouple the system. Thus, if system (1.4) is observable, by following [13, 22], it is non-controllable if and only if matrices A^- and A^+ share some eigenvalue. If the shared eigenvalue is real, for example λ, the system can be reduced to

$$\begin{cases} \dot{x} &= (t^{\triangledown} - \lambda)x - y, \\ \dot{y} &= (m^{\triangledown} - t^{\triangledown}\lambda + \lambda^2)x - z, \\ \dot{z} &= \lambda z - a, \end{cases}$$

where variable z evolves independently.

If the shared eigenvalue is complex, for example $\alpha \pm \beta$i, then the system can be written as

$$\begin{cases} \dot{x} &= (t^{\triangledown} - 2\alpha)x - y, \\ \dot{y} &= 2\alpha y - z, \\ \dot{z} &= (\alpha^2 + \beta^2)y - a, \end{cases} \tag{1.7}$$

where variables y and z evolve independently of x. Chapter 4 is focused on the study of the dynamical behavior of an observable non-controllable system of the family (1.7) and a perturbation of this situation. The main results of Chapter 4 are published in [17].

To finish this subsection, we introduce the system that will be considered in the last chapter of this thesis. It is the representantive of a particular class of observable systems. Specifically, we will consider a observable reversible divergence-free 2CPWL system with a saddle-focus equilibria in each zone of linearity.

Consider a non-homogeneous 2CPWL$_3$ system. First, under observability hypothesis, we know from Proposition 1.5 that the system can be written into the Liénard form (1.5). Without loss of generality, we can take $a = 1$,

$$\dot{\mathbf{x}} = \begin{pmatrix} t^{\triangledown} & -1 & 0 \\ m^{\triangledown} & 0 & -1 \\ d^{\triangledown} & 0 & 0 \end{pmatrix} \mathbf{x} - \mathbf{e}_3. \tag{1.8}$$

By assuming that the system is time-reversible with respect to the involution

$$\mathbf{R}(x, y, z) = (-x, y, -z), \tag{1.9}$$

that is, it is invariant under the transformation

$$(\mathbf{x}, t) \mapsto (\mathbf{R}(\mathbf{x}), -t),$$

the following conditions must be fulfilled

$$t^- = -t^+, \quad m^- = m^+, \quad \text{and} \quad d^- = -d^+.$$

On the other hand, by assuming that the system is divergence-free (i.e., volume preserving), it is a necessary condition that $t^+ = t^- = 0$. Finally, the supposition of the existence of saddle-focus equilibria in both zones of linearity assures that $d^+, m^+ > 0$. By writing $d = d^+$ and $m = m^+$, system (1.8) must be

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = mx - z, \\ \dot{z} = d|x| - 1. \end{cases} \tag{1.10}$$

The change of variables

$$X = -m^{3/2}x, \quad Y = my, \quad Z = m^{3/2}x - m^{1/2}z \quad \text{and} \quad s = m^{1/2}t,$$

transforms the system into

$$\begin{cases} X' = Y, \\ Y' = Z, \\ Z' = 1 - Y - \dfrac{d}{m^{3/2}}|X|, \end{cases} \tag{1.11}$$

where the prime stands for the derivative with respect to s. Renaming $X = x$, $Y = y$, $Z = z$, $d/m^{3/2} = c$ and $s = t$, system (1.11) is rewritten as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = 1 - y - c|x|. \end{cases} \tag{1.12}$$

In the half-space $\{x < 0\}$, the system has exactly one saddle-focus equilibrium point $\mathbf{p}_- = (-1/c, 0, 0)^T$, since the eigenvalues of the coefficient matrix of the left zone are λ, $\alpha \pm \mathrm{i}\beta$, with

$$c = \lambda(1+\lambda^2), \quad \alpha = -\frac{\lambda}{2}, \quad \beta = \frac{\sqrt{4+3\lambda^2}}{2}. \tag{1.13}$$

By the reversibility with respect to $\mathbf{R}$, there exists exactly one saddle-focus equilibrium $\mathbf{p}_+ = -\mathbf{p}_-$ in the half-space $\{x > 0\}$ whose eigenvalues are given by $-\lambda$ and $-\alpha \pm \mathrm{i}\,\beta$.

Thus, the parameter $\lambda > 0$ can be chosen as the parameter of the family and the system can be rewritten as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = 1 - y - \lambda(1+\lambda^2)|x|. \end{cases} \tag{1.14}$$

This system is a piecewise linear version of the well-known Michelson system [39, 58, 67, 77], namely,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = \rho^2 - y - \dfrac{1}{2}x^2, \end{cases} \tag{1.15}$$

where the parameter ρ is strictly positive.

Some authors have analyzed the PWL version (1.14) of the well-known Michelson system. In [19, 20, 46] the authors studied some global connections of the PWL version of the system. Particularly, they gave an analytical proof of the existence of a pair of homoclinic connections and a

T-point heteroclinic cycle. On the other hand, in [15, 46] the existence of a family of periodic orbits which intersects the separation plane at two points is proven. This family of periodic orbits ends in a special periodic orbit which crosses tangentially the separation plane. As we have commented in the Introduction of this work, in looking for periodic orbits which cross the separation plane tangentially, the usual techniques for proving the existence of periodic orbits in PWL systems cannot be applied. In [68] the authors analyze the dynamics around a flow which is tangent to the section where the Poincare map is defined (in our case, as we will see in the next section, the separation boundary). Chapter 6 of this work is devoted to analyzing in system (1.14) the behavior of periodic orbits beyond the existence of a special periodic orbit which crosses the separation plane tangentially. In particular, we find that this orbit fosters the existence of periodic orbits which intersect the separation plane at four points and we study the bifurcations that occur. Furthermore, we will see that this tangential orbit is the necessary germ for the existence of the so-called noose bifurcation [58]. The results shown in Chapter 6 are gathered in [14].

1.1.2 Planar Discontinuous Piecewise Linear Systems

After the pioneering book of Filippov [38] in the 80's and due to the demonstrated applicability of discontinuous piecewise linear (DPWL) systems, for instance, in engineering and biomathematics [35], a lot of works have recently appeared in the literature dealing with vector fields where continuity at the common boundary is not assumed, for instance, [32, 36, 47, 48, 50, 54, 71, 89].

The simplest possible configuration in DPWL systems is the planar case with two linearity regions separated by a straight line. Similarly to the continuous case, it can be assumed without loss of generality that the separation line is $x = 0$, as we enunciate subsequently.

Proposition 1.6 Every planar DPWL system can be written into the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{cases} A^-\mathbf{x} + \mathbf{b}^- & \text{if} \quad x < 0, \\ A^+\mathbf{x} + \mathbf{b}^+ & \text{if} \quad x > 0, \end{cases} \tag{1.16}$$

where $\mathbf{b}^+, \mathbf{b}^- \in \mathbb{R}^2$ and $A^+, A^- \in \mathcal{M}_2(\mathbb{R})$.

Note that, the planar DPWL systems satisfy the relationship

$$A^{+}\begin{pmatrix}0\\ y\end{pmatrix}+\mathbf{b}^{+}\neq A^{-}\begin{pmatrix}0\\ y\end{pmatrix}+\mathbf{b}^{-}.$$

In studying planar DPWL systems (1.16), as usual, the first step is searching a canonical form which copes with a sufficient broad class of systems. A first canonical form for planar DPWL systems has been introduced in [45]. In particular, the authors deduce a Liénard-like canonical form with seven parameters, which is able to cover all the cases where self-sustained oscillations are possible.

The main objective in dealing with planar DPWL systems in this thesis is the analysis of the existence of periodic orbits. In this subsection we state the Liénard canonical form of planar DPWL system introduced in Proposition 3.1 of [45],

$$\begin{aligned}&\begin{cases}\dot{x}=t^{-}x-y,\\ \dot{y}=d^{-}x-a^{-},\end{cases}\qquad \text{if } x<0,\\ &\begin{cases}\dot{x}=t^{+}x-y+b,\\ \dot{y}=d^{+}x-a^{+},\end{cases}\qquad \text{if } x>0,\end{aligned}\tag{1.17}$$

where $t^{+}, t^{-}, d^{+}, d^{-}, a^{+}, a^{-}\in\mathbb{R}$ and $b\geq 0$, being t^{+}, t^{-} and d^{+}, d^{-} the trace and the determinant of the coefficient matrices in their corresponding zone of linearity.

The discontinuity of the vector field at the separation boundary $x=0$ leads us to adopt the Filippov convection (see [38]) to define the vector field at the discontinuity line. Then, the line $x=0$ is divided into two different sets, the sliding set and the crossing set. The sliding set, where the normal components of both vector fields to the line $x=0$ have different sign, is the segment $\{(x,y) : x=0, 0\leq y\leq b\}$. On the crossing set, which is the complement of the sliding set, the normal components of both vector fields to the line $x=0$ have the same sign and so, the orbits can be concatenated in the natural way.

System (1.17) can have sliding limit cycles, i.e., limit cycles which have some point in the sliding set. However, in this work, we will focus our attention on crossing limit cycles, that is, limit cycles which do not share points with the sliding set. In Chapter 2, crossing limit cycles for the DPWL system (1.17) are analyzed by means of an adaptation of the Melnikov theory to discontinuous systems.

1.2 Poincaré map and Displacement function

To analyze the dynamical behavior of CPWL systems, it is usual to introduce a suitable Poincaré map defined in the separation boundary, by using some Poincaré half-maps. In the case of system (1.2), the separation boundary is the hyperplane $\Pi \equiv \{x_1 = 0\}$. Now, a Poincaré map for system (1.2) is defined. A detailed study of Poincaré half-maps can be found in [56, 59, 72].

For every point $\mathbf{p} = (x_1^{\mathbf{p}}, x_2^{\mathbf{p}}, \ldots, x_n^{\mathbf{p}})^T \in \mathbb{R}^n$ we denote by $\mathbf{x_p}(t) = (x_1^{\mathbf{p}}(t), x_2^{\mathbf{p}}(t), \ldots, x_n^{\mathbf{p}}(t))^T$ the solution of system (1.2) with initial condition $\mathbf{x_p}(0) = \mathbf{p}$. The corresponding orbit is denoted by $\gamma_{\mathbf{p}}$.

Let $\mathbf{p}$ be a point located at the separation boundary Π. We say that the orbit $\gamma_{\mathbf{p}}$ is transversal to the boundary in the point $\mathbf{p}$, if the vector tangent to the orbit $\gamma_{\mathbf{p}}$ in $\mathbf{p}$ is not contained in Π, i.e., if the following condition holds

$$\langle \mathbf{e}_1, A^{\triangledown}\mathbf{p} + \mathbf{b} \rangle \neq 0. \tag{1.18}$$

Due to the continuity of field (1.2), it is satisfied that $\langle \mathbf{e}_1, A^{-}\mathbf{p} + \mathbf{b} \rangle = \langle \mathbf{e}_1, A^{+}\mathbf{p} + \mathbf{b} \rangle$, for every point $\mathbf{p}$ belonging to the separation boundary. Note that, if condition (1.18) is not verified, then $\mathbf{p}$ is a point of tangency of the orbit with Π.

Let us define the left Poincaré half-map associated to system (1.2).

Definition 1.7 Assume that a point $\mathbf{p}$ belonging to the separation boundary Π verifies condition $\langle \mathbf{e}_1, A^{\triangledown}\mathbf{p} + \mathbf{b} \rangle \leq 0$. If there exists $\tau > 0$ such that $\mathbf{x_p}(\tau) \in \Pi$, then we define the left flying half-time as the positive value $\tau_{\mathbf{p}}^{-}$ such that $x_1^{\mathbf{p}}(\tau_{\mathbf{p}}^{-}) = 0$ and $x_1^{\mathbf{p}}(t) < 0$ for all $t \in (0, \tau_{\mathbf{p}}^{-})$. In such a case, we define the left Poincaré half-map in the point $\mathbf{p}$ as $P^{-}(\mathbf{p}) = \mathbf{x_p}(\tau_{\mathbf{p}}^{-})$.

Given a point $\mathbf{p}$ where the left Poincaré half-map is defined and such that both $\mathbf{p}$ and its image are not points of tangency, then the left Poincaré half-map and the left flying half-time are well-defined in a neighborhood of $\mathbf{p}$ and, moreover, they are analytic [72].

Now, we define the right Poincaré half-map associated to system (1.2).

Definition 1.8 Assume that a point $\mathbf{p}$ belonging to the separation boundary Π verifies condition $\langle \mathbf{e}_1, A^{\triangledown}\mathbf{p} + \mathbf{b} \rangle \geq 0$. If there exists $\tau > 0$ such that $\mathbf{x_p}(\tau) \in \Pi$, then we define the right flying half-time as the positive value $\tau_{\mathbf{p}}^{+}$ such that $x_1^{\mathbf{p}}(\tau_{\mathbf{p}}^{+}) = 0$ and $x_1^{\mathbf{p}}(t) > 0$ for all $t \in (0, \tau_{\mathbf{p}}^{+})$. In such a case, we define the right Poincaré half-map in the point $\mathbf{p}$ as $P^+(\mathbf{p}) = \mathbf{x}_\mathbf{p}(\tau^+_\mathbf{p})$.

Analogously, given a point $\mathbf{p}$ where the right Poincaré half-map is defined and such that both $\mathbf{p}$ and its image are not points of tangency, then the right Poincaré half-map and the right flying half-time are well-defined in a neighborhood of $\mathbf{p}$ and, moreover, they are analytic [72].

From the Poincaré half-maps, we define the Poincaré map associated to system (1.2).

Definition 1.9 The Poincaré map $P : \Pi \to \Pi$ associated to the CPWL system (1.2) is given by the composition of both Poincaré half-maps, namely,

$$P = P^+ \circ P^-.$$

Note that, the Poincaré map associated to system (1.2) is well-defined in a point $\mathbf{p} \in \Pi$ verifying $\langle \mathbf{e}_1, A^\triangledown \mathbf{p} + \mathbf{b}\rangle \leq 0$ if the left Poincaré half-map in $\mathbf{p}$ and the right Poincaré half-map in $\mathbf{q} := P^-(\mathbf{p})$ are well-defined. In this case, there exist two positive values $\tau^-_\mathbf{p}$ and $\tau^+_\mathbf{q}$ such that $x_1^\mathbf{p}(\tau^-_\mathbf{p}) = 0$, $x_1^\mathbf{q}(\tau^+_\mathbf{q}) = 0$, $x_1^\mathbf{p}(t) < 0$ for all $t \in (0, \tau^-_\mathbf{p})$ and $x_1^\mathbf{q}(t) > 0$ for all $t \in (0, \tau^+_\mathbf{q})$. The positive value $\tau_\mathbf{p} = \tau^-_\mathbf{p} + \tau^+_\mathbf{q}$ is called the flying time and the Poincaré map associated to system (1.2) in the point $\mathbf{p}$ is given by

$$P(\mathbf{p}) = \mathbf{x}_{P^-(\mathbf{p})}(\tau^+_{P^-(\mathbf{p})}) = \mathbf{x}_\mathbf{p}(\tau_\mathbf{p}).$$

Finally, given a point $\mathbf{p}$ where the Poincaré map is well-defined and such that points $\mathbf{p}$, $P^-(\mathbf{p})$ and $P(\mathbf{p})$ are not points of tangency, then the Poincaré map and the flying time are well-defined in a neighborhood of $\mathbf{p}$ and, moreover, both functions are analytic [72].

On the other hand, it is obvious that the fixed points of the Poincaré map correspond to periodic orbits of system (1.2). Thus, it makes sense to consider a function whose zeros correspond to periodic orbits of system (1.2), which will be done in the following definition.

Definition 1.10 The displacement function $d : \Pi \to \Pi$ associated to the CPWL system (1.2) is given by the difference between the Poincaré map and the identity function, namely,

$$d = P - id.$$

1.3 Invariant Cones in Homogeneous Piecewise Linear Systems

Piecewise linear systems can be used as a tool to understand some basic bifurcations that have their starting point in the change of stability of one equilibrium point [24, 40, 42, 44, 48]. In some of the previous works, it is shown that the change of the stability of one equilibrium point forces the appearance of a limit cycle. Indeed, to analyze this phenomenon it is necessary to study, in some situations, the behavior of the equilibria on the separation boundaries. The topological type of these equilibria is essential to assure (or not) the existence of limit cycles. When the system is 2CPWL$_2$, this behavior is well known [40, 44]. However, the problem is compounded for continuous three-dimensional systems. For instance, in [26] the authors prove that the continuous matching of two stable linear systems can be unstable. The instability of the origin, the unique equilibrium point of the system, can only appear when the system has an invariant cone; by contrast, the absence of invariant cones assure the stability of the origin when the matrices of the system are Hurwitzian [25, 26]. This is a generalization of Theorem 3.4 of [12]. Therefore, it is important to study the existence of invariant cones for this class of systems. We realize that these invariant manifolds can be considered, in the non-generic case where it is foliated by periodic orbits, as the center manifold for non-smooth systems [66] and so, the invariant cones have to play an important role. As a remark related, the stability of the origin can be guaranteed, as it is well known, by means of Lyapunov functions. However, it is not easy to find Lyapunov functions for PWL systems, even when the involved matrices are Hurwitzian [55, 88].

In this section the concept of invariant cone in a three-dimensional continuous PWL system is introduced. To begin with, every homogeneous 2CPWL$_3$ system can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{cases} A^{-}\mathbf{x} & \text{if} \quad x_1 \leq 0, \\ A^{+}\mathbf{x} & \text{if} \quad x_1 > 0, \end{cases} \tag{1.19}$$

where the matrices $A^{+}, A^{-} \in \mathcal{M}_3(\mathbb{R})$ satisfy the relationship $A^{+} - A^{-} = (A^{+} - A^{-})\mathbf{e}_1\mathbf{e}_1^T$.

It is possible to introduce an adequate Poincaré map defined in the separation boundary, by using some Poincaré half-maps, as it has been done in the previous section for a $n-$dimensional system. In this case, the separation boundary is the plane $\Pi \equiv \{x_1 = 0\}$.

Taking into consideration that vector field $\mathbf{f}$ of system (1.19) is homogeneous, i.e. $\mathbf{f}(\mu\mathbf{x}) = \mu\mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$ and $\mu \geq 0$, it is easy to see that the maps P^{+}, P^{-} and P are also homogeneous and hence, these maps transform straight half-lines contained in the plane $x_1 = 0$ and passing through the origin into straight half-lines contained in the plane $x_1 = 0$ that also pass through the origin (see Fig. 1.1). Thus, if the Poincaré map P possesses an invariant straight half-line line we say that system (1.19) has a two-zonal invariant cone. In Fig. 1.2 it is shown a two-zonal invariant cone and a continuum of one-zonal invariant cones. In the last case, the system has an invariant cone intersecting tangentially the separation plane.

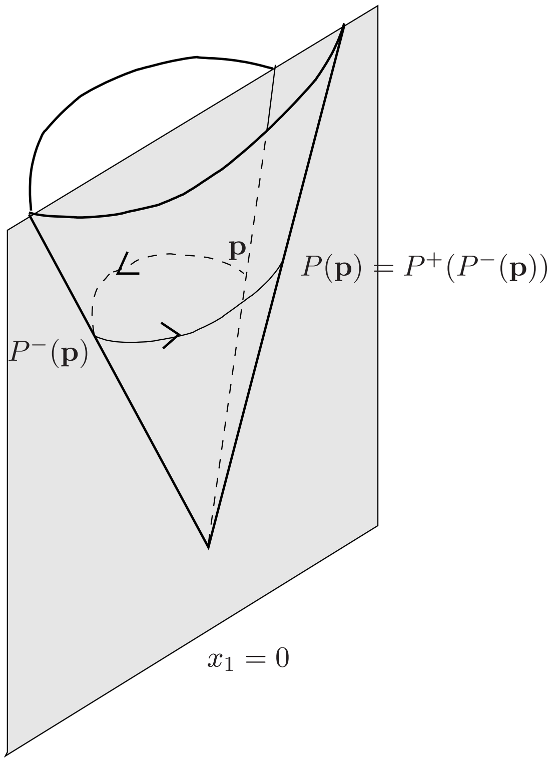


Figure 1.1: Poincaré half-maps P^+ and P^- and Poincaré map P of system (1.19).

Furthermore, it is possible to define a map $\mathcal{S}^-$ that transforms the slopes of the initial straight half-lines into the slopes of the final straight half-lines by means of P^-. Similarly, we can also define a map $\mathcal{S}^+$ by considering the slopes of initial and final straight half-lines by applying the right Poincaré half-map P^+. Hence, system (1.19) has a two-zonal invariant cone if and only if the map $\mathcal{S} = \mathcal{S}^+ \circ \mathcal{S}^-$ has a fixed point, or equivalently, the generalized eigenvalue problem $P(\mathbf{v}) = \delta \mathbf{v}$ has solution. In [65], the author studies the eigenvalue problem to give some bifurcations of periodic orbits which

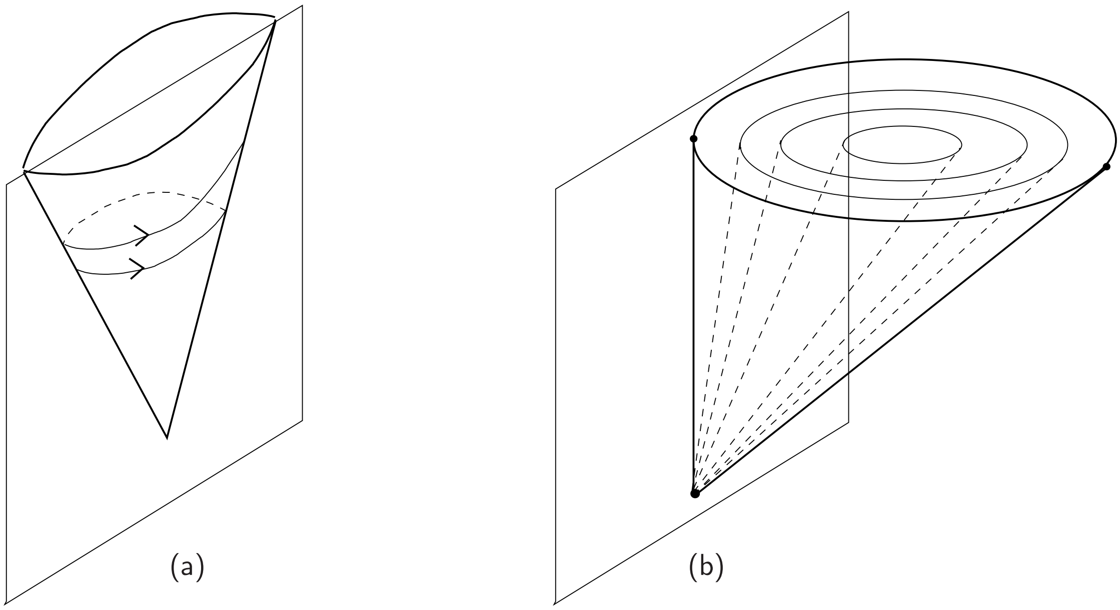


Figure 1.2: (a) A two-zonal invariant cone of system (1.19). (b) A continuum of one-zonal invariant cones of system (1.19). Note that an invariant cone intersecting tangentially the separation plane appears.

live in the invariant cones; in particular, a generalization of the Hopf bifurcation is analyzed. When the system is observable, an analysis of maps $\mathcal{S}^+$ and $\mathcal{S}^-$ can be found in [25], where the authors provide a parametric representation of these maps as functions of the flying half-times.

On the other hand, if the flow of system (1.19) is projected onto the unit sphere S^2, then the invariant cones of the system can be considered as periodic orbits of a suitable system defined on the unit sphere. To see this, by following the original work of Hadeler [52], it is just necessary to do the change of variables $\mathbf{u} = \mathbf{x}\|\mathbf{x}\|^{-1}$, $r = \|\mathbf{x}\|$, for $\mathbf{x} \neq \mathbf{0}$. Then, we obtain the system

$$\begin{cases} \dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}) - \langle \mathbf{f}(\mathbf{u}), \mathbf{u} \rangle \mathbf{u}, & \mathbf{u} \in S^2, \\ \dot{r} = \langle \mathbf{f}(\mathbf{u}), \mathbf{u} \rangle \, r, & r \geqslant 0. \end{cases}$$

Now, it is immediate to observe that the invariant cones of system (1.19) correspond one-to-one to the periodic orbits of the following continuous piecewise cubic system on S^2

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}) - \langle \mathbf{f}(\mathbf{u}), \mathbf{u} \rangle \mathbf{u}, \quad \mathbf{u} \in S^2. \tag{1.20}$$

Here, it is possible to define the periodic orbits as one-zonal and two-zonal ones. By using, for example, system (1.20), one can prove that the one-zonal invariant cones of (1.19) cannot be isolated and in this case matrix A^+ (or A^-) has complex eigenvalues with the real part of the complex eigenvalues and the real eigenvalue shared. Moreover, when system (1.19) possesses one invariant cone living in each zone of linearity, then the sphere is foliated by a continuum of periodic orbits when the traces of matrices A^+ and A^- coincide. Here, two invariant cones tangent to the separation plane appear.

Let us introduce a definition about the invariant cones.

Definition 1.11 It is said that a invariant cone of system (3.1) is hyperbolic (resp. non-hyperbolic) if its corresponding periodic orbit in S^2 is hyperbolic (resp. non-hyperbolic). In the same way, we will say that the cone is attractive (resp. repulsive, semi-attractive) if its corresponding periodic orbit is asymptotically stable (resp. unstable, semi-stable).

The works about invariant cones for systems (1.19) that we have found in the current literature [21, 25] assume observability hypothesis, thus, they study homogeneous systems written in the Liénard form (1.5). In Chapter 3 of this thesis, we extend the results obtained in [21] and we prove a conjecture about the existence of saddle-node bifurcation of invariant cones that was stated in [25]. The results of Chapter 3 have been published in [18]. With the objective of sorting out this lack of information about the existence of invariant cones in non-observable systems, we center our attention on that matter in Chapter 5 of this work.

Chapter 2

Melnikov Theory for a Class of Planar Hybrid Systems. Some Applications

In this chapter, the existence of periodic orbits in a class of planar hybrid piecewise smooth systems is studied. We understand as hybrid system, a piecewise-smooth system which is a mixture of a flow and a map. In this class, each zone of differentiability is separated by a straight line and in these zones, the dynamics is governed by a smooth system. When an orbit reaches the separation line then a reset map applies before entering the orbit in the other zone.

To find periodic orbits in smooth systems, we can use different techniques. One of them is the averaging theory [30, 70, 94]. The idea of this theory is to relate the periodic orbits of a system to the equilibrium points of an autonomous one, the averaged differential system. When a planar smooth system can be written as a perturbation of another system which has a continuum of periodic orbits, the Melnikov theory [8], sometimes called Malkin-Loud theory [73, 75, 78], can be applied. Thus, from the Poincaré map we compute a function, the so-called Melnikov function, whose zeros provide us the number and positions of the periodic orbits of the perturbed system. Moreover, the Melnikov theory is applied to study the existence of homoclinic and heteroclinic connections in perturbations of systems having global connections [51].

The method of averaging has been generalized to continuous non-smooth dynamical systems [10, 11, 13, 64] but, due to the loss of continuity of the hybrid systems in the separation boundary, as far as we know, the theory developed up to now cannot be applied to this case. The generalization of the Melnikov theory for periodic orbits in non-smooth systems has been studied in [13] with planar continuous piecewise smooth systems. The extension for planar Filippov systems can be seen in [36]. The Melnikov theory for global connections has been generalized to non-smooth systems in [7, 64]. In [49], the authors consider a Hamiltonian piecewise-defined autonomous system with a region closed by two heteroclinic connections fully covered by periodic orbits, and the splitting of the separatrices as well as the persistence of periodic orbits under a non-autonomous periodic Hamiltonian perturbation is analyzed. In the same spirit, this chapter is devoted to the extension of the Melnikov theory for periodic orbits to planar hybrid systems, which include the discontinuous ones.

The chapter is organized as follows. In a first section, the classical Melnikov Theory for planar smooth systems is introduced. Next, in the second section, the planar hybrid systems object of our study are stated. After that, Sec. 2.3 is focused on defining a Poincaré map and a displacement function for the hybrid systems presented in Section 2.2. Subsequently, in Section 2.4, the expression of the Melnikov function is deduced from the displacement function, and the main results of the chapter are stated. Finally, Section 2.5 is devoted to applying the developed theory to DPWL with two zones of linearity and CPWL systems with three zones of linearity.

The most important application of the theory developed in this chapter, will be given in Chapter 3. There, we will study the invariant cones of a observable homogeneous 2CPWL$_3$ system, taking into account their correspondence to periodic orbits of planar hybrid PWL systems with a bilinear reset map.

The main results of this chapter are enshrined in [16].

2.1 The classical Melnikov theory

Let us consider a vector field $\mathbf{f} \in C^1(\mathbb{R}^2)$ and assume that the planar system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2, \tag{2.1}$$

possesses a continuum of periodic orbits which are transversal to a section. We can assume, without loss of generality, that the section is $\Sigma = \{(0, y) : y \in I\}$, where $I \subset \mathbb{R}$ is an open interval.

Denote by $\gamma_{y_0}(t)$ the periodic orbit of system (2.1) which passes through the point $\mathbf{x}(0) = (0, y_0)$. Its period will be denoted by T_{y_0}, $y_0 \in I$.

The problem is to know which of the periodic orbits of system (2.1) remain if the system is perturbed in the following form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, \varepsilon, \boldsymbol{\mu}), \tag{2.2}$$

where $\mathbf{x} \in \mathbb{R}^2$, $\varepsilon \in \mathbb{R}$, $\boldsymbol{\mu} \in \mathbb{R}^k$ and $\mathbf{g} \in C^1$.

If the solution of system (2.2) that for $t = 0$ passes through $(0, y_0)$, with $y_0 \in I$, is denoted by

$$\mathbf{x}(t, y_0, \varepsilon, \boldsymbol{\mu}) = (x(t, y_0, \varepsilon, \boldsymbol{\mu}), y(t, y_0, \varepsilon, \boldsymbol{\mu}))^T,$$

then, the continuity and differentiability with respect to the initial conditions and parameters, let us assure that, for ε small enough, the orbit of this solution intersects again to Σ for the first time after $T(y_0, \varepsilon, \boldsymbol{\mu})$ in a point $x(T(y_0, \varepsilon, \boldsymbol{\mu}), y_0, \varepsilon, \boldsymbol{\mu})$.

We can now define the Poincaré map as

$$P(y_0, \varepsilon, \boldsymbol{\mu}) = x(T(y_0, \varepsilon, \boldsymbol{\mu}), y_0, \varepsilon, \boldsymbol{\mu})$$

and the displacement function as

$$d(y_0, \varepsilon, \boldsymbol{\mu}) = P(y_0, \varepsilon, \boldsymbol{\mu}) - y_0.$$

It is obvious that if y_0 is a zero of the displacement function d, then the solution

$$(x(t, y_0, \varepsilon, \boldsymbol{\mu}), y(t, y_0, \varepsilon, \boldsymbol{\mu}))^T$$

provides a periodic orbit of system (2.2) with a period of $T(y_0, \varepsilon, \boldsymbol{\mu})$.

Thus, the idea is to find a function $y_0 = h(\varepsilon, \boldsymbol{\mu})$ such that $d(h(\varepsilon, \boldsymbol{\mu}), \varepsilon, \boldsymbol{\mu}) = 0$. Note that $d(y_0, 0, \boldsymbol{\mu}) \equiv 0$ for all $y_0 \in I$, and so $\partial d/\partial y_0 \equiv 0$, which prevent from applying the Implicit Function Theorem for finding the function $y_0 = h(\varepsilon, \boldsymbol{\mu})$ such that the displacement function vanishes. However, this allow us to write d as

$$d(y_0, \varepsilon, \boldsymbol{\mu}) = \varepsilon D(y_0, \varepsilon, \boldsymbol{\mu}).$$

Therefore, if there exists a function h such that $D(h(\varepsilon, \boldsymbol{\mu}), \varepsilon, \boldsymbol{\mu}) = 0$, then it is satisfied that $d(h(\varepsilon, \boldsymbol{\mu}), \varepsilon, \boldsymbol{\mu}) = 0$.

Hence, the problem of finding periodic orbits in the perturbed system (2.2) has been reduced to the problem of finding implicit solutions of the equation $D(y_0, \varepsilon, \boldsymbol{\mu}) = 0$.

Then, if there exists $y_0 \in I$ such that

$$D(y_0, 0, \boldsymbol{\mu}) = \frac{\partial d}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) = 0 \quad \text{and} \quad \frac{\partial D}{\partial y_0}(y_0, 0, \boldsymbol{\mu}) = \frac{\partial^2 d}{\partial y_0 \partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) \neq 0,$$

the Implicit Function Theorem assures the existence of a solution function defined implicitly $y_0 = h(\varepsilon, \boldsymbol{\mu})$.

We call the reduced displacement function to

$$D(y_0, 0, \boldsymbol{\mu}) = \frac{\partial d}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}).$$

The Melnikov method establishes that the reduced displacement function is proportional to a function, so-called Melnikov function, which is exclusively defined from the vector fields f and g.

The Melnikov function for system (2.2) along a periodic orbit $\gamma_{y_0}(t)$ with a period of T_{y_0} is given by

$$M(y_0, \boldsymbol{\mu}) = \int_0^{T_{y_0}} e^{-\int_0^t \mathrm{div}\ \mathbf{f}(\gamma_{y_0}(s))ds} \mathbf{f}(\gamma_{y_0}(t)) \wedge \mathbf{g}(\gamma_{y_0}(t), 0, \boldsymbol{\mu}) dt, \tag{2.3}$$

where, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we define the wedge product $\mathbf{x} \wedge \mathbf{y} = x_1 y_2 - y_1 x_2$ and div $\mathbf{f}$ is the divergence of the vector field $\mathbf{f}$.

Now, by applying the Implicit Function Theorem and the Weierstrass Preparation Theorem [29], it can be verified the relationship between the number, position and multiplicity of the limit cycles of system (2.2), when ε is sufficiently small, and the number, position and multiplicity of the zeros of the Melnikov function. The mail goal of this chapter is to extrapolate the ideas of the Melnikov theory and to adapt it to a class of planar hybrid systems. In particular, the systems to be considered are introduced in the following section.

2.2 Statement of the problem

Let us consider the following piecewise smooth system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))^T, \tag{2.4}$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$ and

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{f}^-(\mathbf{x}) = (f_1^-(\mathbf{x}), f_2^-(\mathbf{x}))^T & \text{if } x < 0, \\ \mathbf{f}^+(\mathbf{x}) = (f_1^+(\mathbf{x}), f_2^+(\mathbf{x}))^T & \text{if } x > 0, \end{cases} \tag{2.5}$$

with $\mathbf{f}^+, \mathbf{f}^- \in C^r(\mathbb{R}^2), r \geq 1$ and $f_1(0, y) = f_1^+(0, y) = f_1^-(0, y)$, for all $y \in \mathbb{R}$. Note that only the continuity of the first component of vector field $\mathbf{f}$ is required.

Let us suppose that system (2.4) possesses a continuum of periodic orbits. Specifically, we will consider the following hypothesis.

Hypothesis 2.1 System (2.4) possesses a continuum of periodic orbits crossing transversally the separation straight line $\{x = 0\}$ at two points, $(0, y_0)$ and $(0, \hat{y}_0)$ with $y_0 \in I$, $\hat{y}_0 \in \hat{I}$ and $y_0 > \hat{y}_0$, where $I, \hat{I} \in \mathbb{R}$ are open intervals. Furthermore, we assume without loss of generality that the orbits are counterclockwise oriented, that is, $f_1(0, y_0) < 0$ for all $y_0 \in I$ and $f_1(0, \hat{y}_0) > 0$ for all $\hat{y}_0 \in \hat{I}$, see Fig. 2.1.

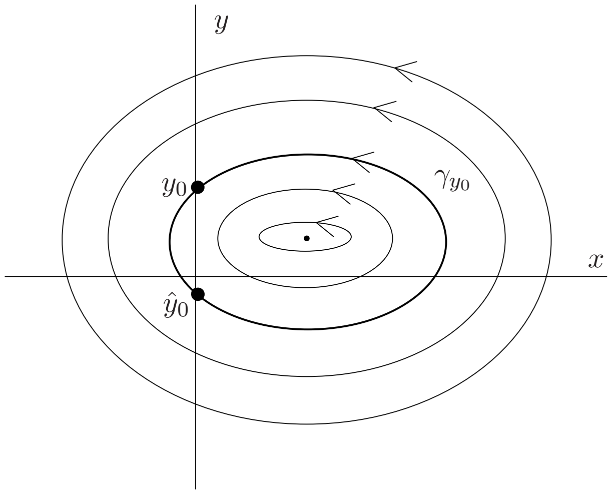


Figure 2.1: Schematic picture of the continuum of periodic orbits of system (2.4).

In the following, we will denote by $\gamma_z(t)$ the orbit of system (2.4) passing for $t = 0$ through the point $(0, z)$, with $z \in I \cup \hat{I}$.

Let us define the left half-period $T^-_{y_0}$ of the orbit $\gamma_{y_0}(t)$ as the time that the orbit $\gamma_{y_0}(t)$ spends going from y_0 to $\hat{y}_0$, the right half-period $T^+_{y_0}$ as the time that the orbit $\gamma_{y_0}(t)$ takes going from $\hat{y}_0$ to y_0 and the period T_{y_0} as the time that the orbit spends to return to y_0 from itself, i.e. $T_{y_0} = T^-_{y_0} + T^+_{y_0}$.

Because system (2.4) has a continuum of periodic orbits, it is natural to think about the number and positions of the periodic orbits that persist after a perturbation. Here, we consider the following perturbation of system (2.4)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, \varepsilon, \boldsymbol{\mu}), \tag{2.6}$$

where $\varepsilon \in \mathbb{R}$, $|\varepsilon| \ll 1$, $k \in \mathbb{N}$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{R}^k$ with the reset map

$$\begin{array}{rcc} \eta : \mathbb{R}^{k+2} & \longrightarrow & \mathbb{R} \\ (y_0, \varepsilon, \boldsymbol{\mu}) & \longmapsto & \eta(y_0, \varepsilon, \boldsymbol{\mu}) \end{array} \tag{2.7}$$

satisfying

$$\eta(y_0, 0, \boldsymbol{\mu}) = y_0 \in \mathbb{R}, \quad f_1(0, y_0) \cdot f_1(0, \eta(y_0, \varepsilon, \boldsymbol{\mu})) > 0$$

and $\eta \in C^r(\mathbb{R}^{k+2})$ with $r \geq 1$. Finally, let us assume that

$$\mathbf{g}(\mathbf{x}, \varepsilon, \boldsymbol{\mu}) = \begin{cases} \mathbf{g}^-(\mathbf{x}, \varepsilon, \boldsymbol{\mu}) & \text{if } x < 0, \\ \mathbf{g}^+(\mathbf{x}, \varepsilon, \boldsymbol{\mu}) & \text{if } x > 0, \end{cases} \tag{2.8}$$

where $\mathbf{g}^+, \mathbf{g}^- \in C^r(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^k), r \geq 1$.

Note that hybrid system (2.6) has two zones of differentiability, separated by the straight line $x = 0$. The dynamics in each zone of differentiability is governed by a smooth system and, when an orbit reaches the separation line, then before entering the system in the other zone the reset map given in (2.7) applies. For the sake of clarity, an orbit of system (2.6) visiting both smooth zones is represented in Fig. 2.2.

Although, a priori, it does not make sense to consider a solution of (2.6) with initial condition in the separation line $x = 0$, there is no problem to define a two-zonal periodic orbit for this system. In fact, according to Fig. 2.2, we have a two-zonal periodic orbit when $y_0 = \eta(y_2, \varepsilon, \boldsymbol{\mu})$.

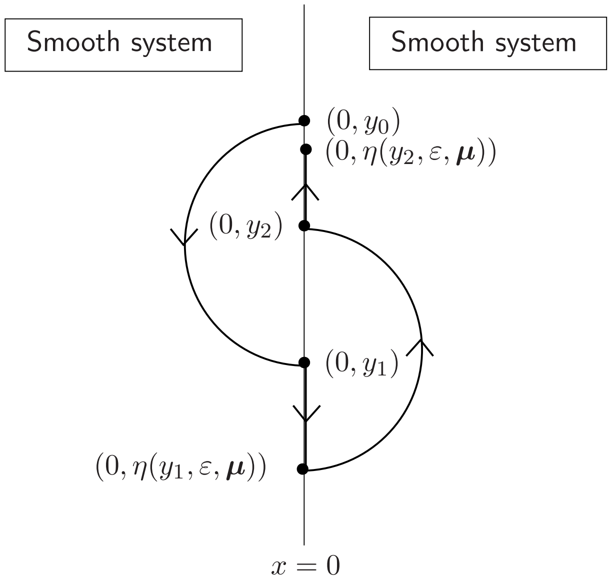


Figure 2.2: Representation of an orbit of the hybrid system (2.6) visiting both zones of differentiability, beginning at $(0, y_0)$. The orbit evolves in the zone $x < 0$ arriving at the point $(0, y_1)$, then a reset map acts getting the orbit to the point $(0, \eta(y_1, \varepsilon, \boldsymbol{\mu}))$. After that, the orbit evolves in the zone $x > 0$ up to arriving at the point $(0, y_2)$, where the reset map applies getting the orbit to the point $(0, \eta(y_2, \varepsilon, \boldsymbol{\mu}))$.

2.3 Definition of a Poincaré map and a displacement function

The first step in the definition of the Melnikov function for system (2.6) is considering some adequate Poincaré half-maps for both planar smooth systems

$$\dot{\mathbf{x}} = \mathbf{f}^-(\mathbf{x}) + \varepsilon \mathbf{g}^-(\mathbf{x}, \varepsilon, \boldsymbol{\mu}) \tag{2.9}$$

and

$$\dot{\mathbf{x}} = \mathbf{f}^+(\mathbf{x}) + \varepsilon \mathbf{g}^+(\mathbf{x}, \varepsilon, \boldsymbol{\mu}). \tag{2.10}$$

For every point $\mathbf{p} = (0, y_0)^T$ with $y_0 \in I$, let us denote by

$$\left(x\left(t, y_0, \varepsilon, \boldsymbol{\mu}\right), y\left(t, y_0, \varepsilon, \boldsymbol{\mu}\right)\right)^T$$

the solution of system (2.9) with initial condition $\mathbf{p}$. Due to Hypothesis 2.1, it is clear that for $|\varepsilon|$

sufficiently small, function $x(\cdot, y_0, \varepsilon, \boldsymbol{\mu})$ vanishes at some value in the interval $(0, +\infty)$, and then, the corresponding orbit crosses transversally the separation line. We define the left flying half-time $\tau^-(y_0, \varepsilon, \boldsymbol{\mu})$ as the positive value such that $x\left(\tau^-(y_0, \varepsilon, \boldsymbol{\mu}), y_0, \varepsilon, \boldsymbol{\mu}\right) = 0$ and $x\left(t, y_0, \varepsilon, \boldsymbol{\mu}\right) < 0$ for t belonging to the interval $(0, \tau^-(y_0, \varepsilon, \boldsymbol{\mu}))$. In such a case, the left Poincaré half-map P^- of system (2.9) at the point y_0 is defined as $P^-(y_0, \varepsilon, \boldsymbol{\mu}) = y\left(\tau^-(y_0, \varepsilon, \boldsymbol{\mu}), y_0, \varepsilon, \boldsymbol{\mu}\right)$.

For system (2.10), one can define in the same way, the right flying half-time $\tau^+(\hat{y}_0, \varepsilon, \boldsymbol{\mu})$ and the right Poincaré half-map $P^+(\hat{y}_0, \varepsilon, \boldsymbol{\mu})$.

Note that the left and the right half-periods of the orbit $\gamma_{y_0}(t)$ of the continuum of system (2.4) satisfy $T^-_{y_0} = \tau^-(y_0, 0, \boldsymbol{\mu})$ and $T^+_{y_0} = \tau^+(\hat{y}_0, 0, \boldsymbol{\mu})$, with $\hat{y}_0 = P(y_0, 0, \boldsymbol{\mu})$.

At this point, the Poincaré map and the displacement function for the hybrid system (2.6) can be defined.

Definition 2.2 Assume that system (2.4) satisfies Hypothesis 2.1. The Poincare map P of system (2.6) defined in a neighborhood of $I \times \{0\} \times \mathbb{R}^{k+1}$ is given by

$$P(y_0, \varepsilon, \boldsymbol{\mu}) = \eta(P^+(\eta(P^-(y_0, \varepsilon, \boldsymbol{\mu}), \varepsilon, \boldsymbol{\mu}), \varepsilon, \boldsymbol{\mu}), \varepsilon, \boldsymbol{\mu}). \tag{2.11}$$

The displacement function d of system (2.6) is defined by

$$d(y_0, \varepsilon, \boldsymbol{\mu}) = P(y_0, \varepsilon, \boldsymbol{\mu}) - y_0. \tag{2.12}$$

Note that P and d are functions of class C^r in their respective domains of definition. In Fig. 2.1 the Poincaré map P defined in (2.11) is represented.

Also note that if $\eta(y_0, \varepsilon, \boldsymbol{\mu}) = y_0$ for all $(y_0, \varepsilon, \boldsymbol{\mu}) \in \mathbb{R}^{k+2}$, then the Poincaré map defined in (2.11) is given by $P = P^+ \circ P^-$.

It is obvious that the periodic orbits of system (2.6) correspond one-to-one to the fixed points of the Poincaré map or, equivalently, to the zeros of the displacement function given in (4.17). Now, the stability and multiplicity of periodic orbits of system (2.6) are defined.

Definition 2.3 Let $\tilde{y}_0$ be a value such that $d(\tilde{y}_0, \varepsilon, \boldsymbol{\mu}) = 0$. We say that the corresponding two-zonal periodic orbit of system (2.6) is a hyperbolic limit cycle if the derivative of the displacement function d with respect to y_0 evaluated at $(\tilde{y}_0, \varepsilon, \boldsymbol{\mu})$ does not vanish. If this derivative is negative

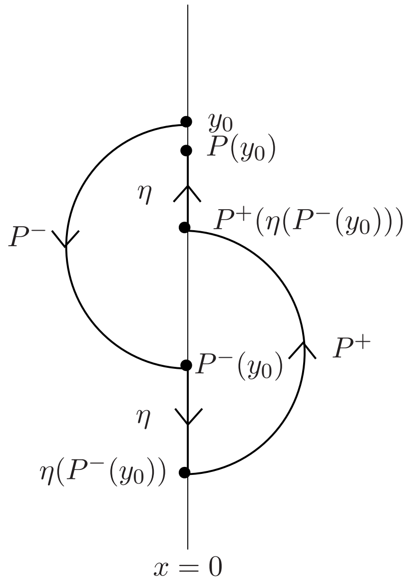


Figure 2.3: Representation of the Poincaré function defined in (2.11). The dependence with respect to ε and $\boldsymbol{\mu}$ has been removed for the sake of brevity and clarity.

(resp. positive) the limit cycle will be called stable (resp. unstable). If this derivative vanishes, then we say that the periodic orbit is non-hyperbolic.

It is said that a periodic orbit has multiplicity $m \in \mathbb{N}$ if the corresponding zero $\tilde{y}_0$ of the displacement function is a root of multiplicity m.

Once the Poincaré map and the displacement function are defined, it is possible to derive a real function of a real variable whose zeros provide us the number and position of the limit cycles that appear after the perturbation. This derivation is developed in Section 2.4.

2.4 Derivation of the Melnikov function. Statement of the main result

From the definitions of the Poincaré map and the displacement function done for the hybrid system (2.6), we deduce in this section a function whose roots give us the number and position of the periodic orbits of the continuum of system (2.4) that remain after the perturbation. This function will be called the Melnikov function by similarity between our theory and the Melnikov theory for smooth systems.

We proceed by analogy with the classical Melnikov theory written in 2.1.

In looking for periodic orbits of the perturbed system (2.6), we must determine the zeros of the displacement function d defined in (4.17). Then, for every ε sufficiently small, we must find y_0 and $\boldsymbol{\mu}$ such that

$$d(y_0, \varepsilon, \boldsymbol{\mu}) = P(y_0, \varepsilon, \boldsymbol{\mu}) - y_0 = 0.$$

Let us consider the Taylor series expansion of the displacement function at $\varepsilon = 0$

$$d(y_0, \varepsilon, \boldsymbol{\mu}) = d(y_0, 0, \boldsymbol{\mu}) + \varepsilon D(y_0, \varepsilon, \boldsymbol{\mu}).$$

From Hypothesis 2.1, it is clear that $d(y_0, 0, \boldsymbol{\mu}) = 0$, and then, solutions for $\varepsilon \neq 0$ of $d(y_0, \varepsilon, \boldsymbol{\mu}) = 0$ corresponds to solutions of equation

$$D(y_0, \varepsilon, \boldsymbol{\mu}) = 0.$$

To apply the Implicit Function Theorem we need to find y_0 and $\boldsymbol{\mu}$ such that

$$D(y_0, 0, \boldsymbol{\mu}) = 0 \quad \text{and} \quad \frac{\partial D}{\partial y_0}(y_0, 0, \boldsymbol{\mu}) \neq 0.$$

It is clear that

$$D(y_0, 0, \boldsymbol{\mu}) = \frac{\partial d}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) = \frac{\partial P}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}),$$

and so, it is enough to search for zeros of the following equation

$$\frac{\partial P}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) = 0. \tag{2.13}$$

To get it, first we look for an explicit expression of $\frac{\partial P}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu})$. Let us start by computing the characteristic exponent of the periodic orbits of the continuum of system (2.4).

Let us consider the divergence of the discontinuous and piecewise smooth function $\mathbf{f}$ as

$$\operatorname{div} \mathbf{f}(\mathbf{x}) = \begin{cases} \dfrac{\partial f_1^+}{\partial x}(\mathbf{x}) + \dfrac{\partial f_2^+}{\partial y}(\mathbf{x}) & \text{if } x > 0, \\ \dfrac{\partial f_1^-}{\partial x}(\mathbf{x}) + \dfrac{\partial f_2^-}{\partial y}(\mathbf{x}) & \text{if } x < 0. \end{cases}$$

Proposition 2.4 If system (2.4) satisfies Hypothesis 2.1, then

$$\int_0^{T_{y_0}} \operatorname{div} \mathbf{f}(\gamma_{y_0}(t))dt = 0 \quad \text{for all} \quad y_0 \in I. \tag{2.14}$$

Proof: From Hypothesis 2.1, $P(y_0, 0, \boldsymbol{\mu}) = y_0$ for all $y_0 \in I$, so,

$$\frac{\partial P}{\partial y_0}(y_0, 0, \boldsymbol{\mu}) = 1 \quad \text{for all} \quad y_0 \in I.$$

From the results given in [28] about the derivatives of the transition maps, we obtain

$$\frac{\partial P^-}{\partial y_0}(y_0, 0, \boldsymbol{\mu}) = \frac{f_1(0, y_0)}{f_1(0, P^-(y_0, 0, \boldsymbol{\mu}))} \exp\left(\int_0^{T_{y_0}^-} \operatorname{div} \mathbf{f}(\gamma_{y_0}(t))dt\right)$$

and

$$\frac{\partial P^+}{\partial \hat{y}_0}(\hat{y}_0, 0, \boldsymbol{\mu}) = \frac{f_1(0, \hat{y}_0)}{f_1(0, P^+(\hat{y}_0, 0, \boldsymbol{\mu}))} \exp\left(\int_0^{T_{y_0}^+} \operatorname{div} \mathbf{f}(\gamma_{\hat{y}_0}(t))dt\right) \tag{2.15}$$

for all $y_0 \in I$ and $\hat{y}_0 \in \hat{I}$. Since $P = P^+ \circ P^-$, $\hat{y}_0 = P^-(y_0, 0, \boldsymbol{\mu})$ and on a periodic orbit we have $y_0 = P^+(\hat{y}_0, 0, \boldsymbol{\mu})$, the proof follows from a direct application of the chain rule. □

To obtain the derivatives of the Poincaré half-maps with respect to ε for $\varepsilon = 0$, we will use the following lemma, whose proof is straightforward.

Lemma 2.5 Consider $\mathbf{F} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $\gamma(t)$ a solution of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\boldsymbol{\omega} \in C(\mathbb{R}, \mathbb{R}^2)$ and $\mathbf{x}_0(t)$ a solution of the equation

$$\dot{\mathbf{x}} = D\mathbf{F}(\gamma(t))\mathbf{x} + \boldsymbol{\omega}(t), \qquad \mathbf{x} \in \mathbb{R}^2, \tag{2.16}$$

where $D\mathbf{F}$ denotes the jacobian matrix of vector field $\mathbf{F}$.

Then, $\psi(t) = \mathbf{F}(\gamma(t)) \wedge \mathbf{x}_0(t)$ is a solution of the one-dimensional differential equation

$$\dot{\psi}(t) = \operatorname{div} \mathbf{F}(\gamma(t))\psi(t) + \mathbf{F}(\gamma(t)) \wedge \boldsymbol{\omega}(t). \tag{2.17}$$

Proposition 2.6 If system (2.4) satisfies Hypothesis 2.1, then the following statements hold.

(a) The derivative of the left Poincaré half-map P^- with respect to ε for $\varepsilon = 0$ is given by

$$\frac{\partial P^-}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) = \frac{\rho^-(T_{y_0}^-)}{f_1(0, \hat{y}_0)} \int_0^{T_{y_0}^-} \frac{\mathbf{f}^-(\gamma_{y_0}(t)) \wedge \mathbf{g}^-(\gamma_{y_0}(t), 0, \boldsymbol{\mu})}{\rho^-(t)} dt, \tag{2.18}$$

where $\hat{y}_0 = P^-(y_0, 0, \boldsymbol{\mu})$ and

$$\rho^-(t) = \exp\left(\int_0^t \operatorname{div} \mathbf{f}^-(\gamma_{y_0}(\tau)) d\tau\right). \tag{2.19}$$

(b) The derivative of the right Poincaré half-map P^+ with respect to ε for $\varepsilon = 0$ is given by

$$\frac{\partial P^+}{\partial \varepsilon}(\hat{y}_0, 0, \boldsymbol{\mu}) = \frac{\rho^+(T_{y_0}^+)}{f_1(0, y_0)} \int_0^{T_{y_0}^+} \frac{\mathbf{f}^+(\gamma_{\hat{y}_0}(t)) \wedge \mathbf{g}^+(\gamma_{\hat{y}_0}(t), 0, \boldsymbol{\mu})}{\rho^+(t)} dt, \tag{2.20}$$

where $y_0 = P^+(\hat{y}_0, 0, \boldsymbol{\mu})$ and

$$\rho^+(t) = \exp\left(\int_0^t \operatorname{div} \mathbf{f}^+(\gamma_{\hat{y}_0}(\tau)) d\tau\right). \tag{2.21}$$

Proof: We focus our attention on proving item (a). A similar reasoning allows us to prove statement (b).

Let us begin by computing the derivatives with respect to ε of the relationships

$$x(\tau^-(y_0, \varepsilon, \boldsymbol{\mu}), y_0, \varepsilon, \boldsymbol{\mu}) = 0 \quad \text{and} \quad y(\tau^-(y_0, \varepsilon, \boldsymbol{\mu}), y_0, \varepsilon, \boldsymbol{\mu}) = P^-(y_0, \varepsilon, \boldsymbol{\mu}).$$

Then, we see that the vector function

$$\mathbf{w}(t) = \left(\frac{\partial x}{\partial \varepsilon}(t, y_0, 0, \boldsymbol{\mu}), \frac{\partial y}{\partial \varepsilon}(t, y_0, 0, \boldsymbol{\mu})\right)^T$$

satisfies

$$\frac{\partial \tau^-}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) \mathbf{f}^-(0, \hat{y}_0) + \mathbf{w}(T_{y_0}^-) = \left(0, \frac{\partial P^-}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu})\right)^T \tag{2.22}$$

where $\hat{y}_0 = P^-(y_0, 0, \boldsymbol{\mu})$.

On the other hand, function $\mathbf{w}(t)$ is the solution of the initial value problem

$$\begin{cases} \dot{\mathbf{w}}(t) = D\mathbf{f}^-(\gamma_{y_0}(t))\mathbf{w}(t) + \mathbf{g}^-(\gamma_{y_0}(t), 0, \boldsymbol{\mu}), \\ \mathbf{w}(0) = (0,0)^T. \end{cases} \tag{2.23}$$

Now, from Lemma 2.5, it follows that $\psi(t) = \mathbf{f}^-(\gamma_{y_0}(t)) \wedge \mathbf{w}(t)$ satisfies

$$\begin{cases} \dot{\psi}(t) = \operatorname{div}\mathbf{f}^-(\gamma_{y_0}(t))\psi(t) + \mathbf{f}^-(\gamma_{y_0}(t)) \wedge \mathbf{g}^-(\gamma_{y_0}(t), 0, \boldsymbol{\mu}), \\ \psi(0) = 0, \end{cases} \tag{2.24}$$

The solution ψ of the linear equation (2.24) can be expressed as

$$\psi(t) = \rho^-(t)\int_0^t \frac{\mathbf{f}^-(\gamma_{y_0}(\tau)) \wedge \mathbf{g}^-(\gamma_{y_0}(\tau), 0, \boldsymbol{\mu})}{\rho^-(\tau)} d\tau,$$

where $\rho^-(t)$ is defined in (2.19).

From relationship (2.22), directly follows that

$$\psi(T^-_{y_0}) = f_1(0, y_0) \cdot \frac{\partial P^-}{\partial \varepsilon}(0, y_0, \boldsymbol{\mu}),$$

and the proof is finished. □

Now, the Melnikov function of the perturbed system (2.6) along a periodic orbit $\gamma_{y_0}(t)$ of the unperturbed system (2.4) can be defined, as follows.

Definition 2.7 If system (2.4) satisfies Hypothesis 2.1, the Melnikov function of the perturbed system (2.6) along a periodic orbit $\gamma_{y_0}(t)$ of the unperturbed system (2.4) is defined as

$$\begin{aligned} M(y_0; \boldsymbol{\mu}) = & \frac{\partial \eta}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) + \\ & \frac{1}{f_1(0, y_0)} \frac{\partial \eta}{\partial y_0}(y_0, 0, \boldsymbol{\mu}) \left[\frac{\partial \eta}{\partial y_0}(\hat{y}_0, 0, \boldsymbol{\mu}) \int_0^{T^-_{y_0}} \frac{\mathbf{f}^-(\gamma_{y_0}(t)) \wedge \mathbf{g}^-(\gamma_{y_0}(t), 0, \boldsymbol{\mu})}{\rho^-(t)} dt + \right. \\ & \left. \rho^+(T^+_{y_0}) \left(\frac{\partial \eta}{\partial \varepsilon}(\hat{y}_0, 0, \boldsymbol{\mu}) f_1(0, \hat{y}_0) + \int_0^{T^+_{y_0}} \frac{\mathbf{f}^+(\gamma_{\hat{y}_0}(t)) \wedge \mathbf{g}^+(\gamma_{\hat{y}_0}(t), 0, \boldsymbol{\mu})}{\rho^+(t)} dt \right) \right], \end{aligned} \tag{2.25}$$

where $\hat{y}_0 = P^-(y_0, 0, \boldsymbol{\mu})$,

$$\rho^-(t) = \exp\left(\int_0^t \text{div}\,\mathbf{f}^-(\gamma_{y_0}(\tau))d\tau\right), \quad \rho^+(t) = \exp\left(\int_0^t \text{div}\,\mathbf{f}^+(\gamma_{\hat{y}_0}(\tau))d\tau\right).$$

Through the Melnikov function it is possible to state results about the existence and bifurcation of limit cycles.

Theorem 2.8 Assume that unperturbed system (2.4) satisfies Hypothesis 2.1 and the functions $\mathbf{f}^+ + \varepsilon\mathbf{g}^+$, $\mathbf{f}^- + \varepsilon\mathbf{g}^-$ and η are $\mathcal{C}^r$ with $r \geq 1$. Consider $\tilde{y}_0 \in I$ and $\boldsymbol{\mu}_0 \in \mathbb{R}^k$. The following statements hold.

(a) If $M(\tilde{y}_0; \boldsymbol{\mu}_0) \neq 0$ then, there exists $\varepsilon_0 > 0$ and small enough, such that the perturbed system (2.6) does not possess periodic orbits in a neighborhood of the periodic orbit $\gamma_{\tilde{y}_0}$ for $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $|\varepsilon| < \varepsilon_0$.

(b) If $r \geq 2$, $M(\tilde{y}_0; \boldsymbol{\mu}_0) = 0$ and $q_0 := \frac{\partial M}{\partial y_0}(\tilde{y}_0; \boldsymbol{\mu}_0) \neq 0$, then, the perturbed system (2.6) has a hyperbolic limit cycle in a neighborhood of $\gamma_{\tilde{y}_0}$, for $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $|\varepsilon|$ different from zero and sufficiently small. Moreover, the limit cycle is asymptotically stable if $\varepsilon \cdot q_0 < 0$ and unstable if $\varepsilon \cdot q_0 > 0$.

(c) If $r \geq 3$, $M(\tilde{y}_0; \boldsymbol{\mu}_0) = q_0 = 0$ and there exits $j \in \{1, 2, ..., k\}$ such that

$$\frac{\partial^2 M}{\partial y_0^2}(\tilde{y}_0, \boldsymbol{\mu}_0) \neq 0 \quad \text{and} \quad \frac{\partial M}{\partial \boldsymbol{\mu}_j}(\tilde{y}_0; \boldsymbol{\mu}_0) \neq 0,$$

then, there exist a function $\boldsymbol{\mu}(\varepsilon) = \boldsymbol{\mu}_0 + O(\varepsilon)$ such that the perturbed system (2.6) has a unique limit cycle of multiplicity two in a neighborhood of $\gamma_{\tilde{y}_0}$, for $\boldsymbol{\mu} = \boldsymbol{\mu}(\varepsilon)$ and $|\varepsilon|$ different from zero and sufficiently small.

Proof: By applying the chain rule in expression (2.11), one obtains

$$\begin{aligned}\frac{\partial P}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) &= \frac{\partial \eta}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) + \frac{\partial \eta}{\partial y_0}(y_0, 0, \boldsymbol{\mu}) \cdot \\ &\cdot \left[\frac{\partial P^+}{\partial \hat{y}_0}(\hat{y}_0, 0, \boldsymbol{\mu}) \left(\frac{\partial \eta}{\partial y_0}(\hat{y}_0, 0, \boldsymbol{\mu}) \frac{\partial P^-}{\partial \varepsilon}(y_0, 0, \boldsymbol{\mu}) + \frac{\partial \eta}{\partial \varepsilon}(\hat{y}_0, 0, \boldsymbol{\mu})\right) + \frac{\partial P^+}{\partial \varepsilon}(\hat{y}_0, 0, \boldsymbol{\mu})\right].\end{aligned} \tag{2.26}$$

Substituting expression (2.15) in relationship (2.26) and taking into account Proposition 2.6, we arrive to

$$\begin{aligned}\frac{\partial P}{\partial \varepsilon}(y_0,0,\boldsymbol{\mu}) = \frac{\partial \eta}{\partial \varepsilon}(y_0,0,\boldsymbol{\mu}) + \frac{\rho^+(T^+_{y_0})}{f_1(0,y_0)}\frac{\partial \eta}{\partial y_0}(y_0,0,\boldsymbol{\mu})\cdot \\ \cdot\left[\frac{\partial \eta}{\partial y_0}(\hat{y}_0,0,\boldsymbol{\mu})\rho^-(T^-_{y_0})\int_0^{T^-_{y_0}}\frac{\mathbf{f}^-(\gamma_{y_0}(t))\wedge\mathbf{g}^-(\gamma_{y_0}(t),0,\boldsymbol{\mu})}{\rho^-(t)}dt+\right. \\ \left.\frac{\partial \eta}{\partial \varepsilon}(\hat{y}_0,0,\boldsymbol{\mu})f_1(0,\hat{y}_0)+\int_0^{T^+_{y_0}}\frac{\mathbf{f}^+(\gamma_{y_0}(t))\wedge\mathbf{g}^+(\gamma_{y_0}(t),0,\boldsymbol{\mu})}{\rho^+(t)}dt\right].\end{aligned} \tag{2.27}$$

On the other hand, it is clear that

$$\rho^-(T^-_{y_0})\cdot\rho^+(T^+_{y_0}) = 1. \tag{2.28}$$

Finally, from expressions (2.27) and (2.28), one obtains

$$\frac{\partial P}{\partial \varepsilon}(y_0,0,\boldsymbol{\mu}) = M(y_0;\boldsymbol{\mu}), \tag{2.29}$$

where $M(y_0;\boldsymbol{\mu})$ is the Melnikov function for hybrid systems given in (2.25).

The proof continues by using the Implicit Function Theorem and following similar ideas to the proofs of theorems 1.2 and 1.3 of [8]. □

We remark that the third statement of Theorem 2.8 can be extended to limit cycles of higher multiplicity. Also, the above result can be generalized to the case $f_1^+(0,y) \neq f_1^-(0,y)$ by obtaining a slightly different expression for the Melnikov function, see [16]. The expression of the Melnikov function for Filippov systems can be seen in [36]. In addition, the theory can be extended to the case with an hybrid unperturbed system and to systems with multiple zones of definition of the vector field. Nevertheless, we will not detailed these cases to keep the chapter within a reasonable length.

Obviously, when the reset map becomes the identity, (that is, $\eta(y_0,\varepsilon,\boldsymbol{\mu}) = y_0$), the Melnikov function (2.25) can be simplified, as we do in the following result.

Proposition 2.9 If system (2.4) satisfies Hypothesis 2.1 and the reset map η satisfies $\eta(y,\varepsilon,\boldsymbol{\mu}) = y$, in a neighborhood of $(I \cup \hat{I}) \times \{0\} \times \mathbb{R}^k$, then the Melnikov function defined in (2.25) becomes

$$M(y_0;\boldsymbol{\mu}) = \frac{1}{f_1(0,y_0)}\int_0^{T_{y_0}}\frac{\mathbf{f}(\gamma_{y_0}(t))\wedge\mathbf{g}(\gamma_{y_0}(t),0,\boldsymbol{\mu})}{\rho(t)}dt, \tag{2.30}$$

where $\rho(t) = \exp\left(\int_0^t \operatorname{div} \mathbf{f}(\gamma_{y_0}(\tau))d\tau\right)$.

Proof: Taking into account that $\eta(y, \varepsilon, \boldsymbol{\mu}) = y$, it follows that expression (2.25) becomes

$$\begin{aligned} M(y_0; \boldsymbol{\mu}) = \quad & \frac{1}{f_1(0, y_0)} \left[\int_0^{T_{y_0}^-} \frac{\mathbf{f}^-(\gamma_{y_0}(t)) \wedge \mathbf{g}^-(\gamma_{y_0}(t), 0, \boldsymbol{\mu})}{\rho^-(t)} dt + \right. \\ & \left. \rho^+(T_{y_0}^+) \left(\int_0^{T_{y_0}^+} \frac{\mathbf{f}^+(\gamma_{\hat{y}_0}(t)) \wedge \mathbf{g}^+(\gamma_{\hat{y}_0}(t), 0, \boldsymbol{\mu})}{\rho^+(t)} dt \right) \right], \end{aligned} \tag{2.31}$$

where $\hat{y}_0 = P^-(y_0, 0, \boldsymbol{\mu})$ and $\rho^-(t)$, $\rho^+(t)$ are given in (2.19) and (2.21), respectively.

By doing the change of variables $s = t + T_{y_0^-}$ in the second integral of (2.31) and taking into account that $\gamma_{\hat{y}_0}(s - T_{y_0^-}) = \gamma_{y_0}(s)$ and $\rho^+(T_{y_0}^+) \cdot \rho^-(T_{y_0}^-) = 1$, we get

$$\begin{aligned} \rho(T_{y_0}^+) \int_0^{T_{y_0}^+} \frac{\mathbf{f}^+(\gamma_{\hat{y}_0}(t)) \wedge \mathbf{g}^+(\gamma_{\hat{y}_0}(t), 0, \boldsymbol{\mu})}{\rho^+(t)} dt = & \\ & \int_{T_{y_0}^-}^{T_{y_0}} \frac{(\mathbf{f}^+(\gamma_{y_0}(s)) \wedge \mathbf{g}^+(\gamma_{y_0}(s), 0, \boldsymbol{\mu}))}{\rho^-(T_{y_0}^-) \cdot \rho^+(s - T_{y_0}^-)} ds. \end{aligned} \tag{2.32}$$

Since for $T_{y_0}^- \leq s \leq T_{y_0}$, we have

$$\rho^-(T_{y_0}^-) \cdot \rho^+\left(s - T_{y_0}^-\right) = \quad \exp\left(\int_0^s \operatorname{div} \mathbf{f}(\gamma_{y_0}(\tau))d\tau\right),$$

the conclusion follows. □

Note that expression (2.30) coincides, except for the leading factor $1/(f_1(0, y_0))$, with the classical Melnikov function [8], see (4.6). If additionally $\operatorname{div} \mathbf{f}(\mathbf{x}) = 0$ then, expression (2.30) can be further simplified.

Proposition 2.10 If system (2.4) satisfies Hypothesis 2.1, the reset map η satisfies $\eta(y, \varepsilon, \boldsymbol{\mu}) = y$, in a neighborhood of $(I \cup \hat{I}) \times \{0\} \times \mathbb{R}^k$, and $\operatorname{div} \mathbf{f}(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$, $x \neq 0$, then the

Melnikov function defined in (2.25) becomes

$$M(y_0;\boldsymbol{\mu}) = \frac{-1}{f_1(0,y_0)}\cdot \\ \cdot\left(\iint_{\mathrm{int}(\gamma_{y_0})} \operatorname{div}\mathbf{g}(x,y,0,\boldsymbol{\mu})\,dx\,dy + \int_{\hat{y}_0}^{y_0}\left[g_1^+(0,y,0,\boldsymbol{\mu}) - g_1^-(0,y,0,\boldsymbol{\mu})\right]dy\right). \tag{2.33}$$

Proof: Let us consider the following sets

$$\gamma_+ = \gamma_{y_0}\cap\{(x,y)\in\mathbb{R}^2 : x>0\},\quad \gamma_- = \gamma_{y_0}\cap\{(x,y)\in\mathbb{R}^2 : x<0\},$$

$$L_+ = \{(x,y)\in\mathbb{R}^2 : y=\kappa y_0+(1-\kappa)\hat{y}_0, 0\le\kappa\le 1)\}$$

and

$$L_- = \{(x,y)\in\mathbb{R}^2 : y=(1-\kappa)y_0+\kappa\hat{y}_0, 0\le\kappa\le 1)\},$$

(see Fig. 2.4). Since $\gamma_+\cup L_+$ is a closed Jordan curve, it surrounds a region $\Theta_+ =$int$\{\gamma_+\cup L_+\}$. Analogously, $\gamma_-\cup L_-$ surrounds a region $\Theta_- =$int$\{\gamma_-\cup L_-\}$. (see Fig. 2.4). If we denote the

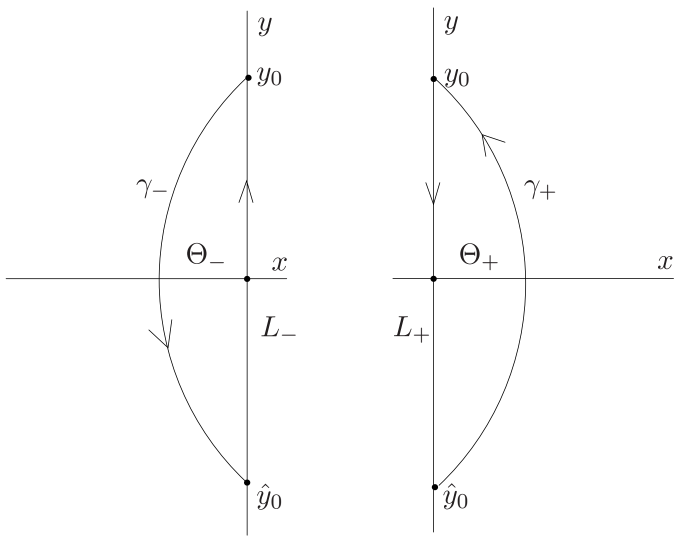


Figure 2.4: Path along the boundary of the set $\Theta_+\cup\Theta_-$ for applying the Green's Theorem to a crossing periodic orbit γ_{y_0} of system (2.4).

orthogonal of a vector field $\mathbf{h} = (h_1, h_2)^T$ as $\mathbf{h}^\perp = (h_2, -h_1)$, from expression (2.30) we can write

$$\begin{aligned} f_1(0,y_0)M(y_0;\boldsymbol{\mu}) = & \int_0^{T_{y_0}} \mathbf{f}(\gamma_{y_0}(t)) \wedge \mathbf{g}(\gamma_{y_0}(t),0,\boldsymbol{\mu})dt = \\ & \int_{\gamma_-} (\mathbf{g}^-)^\perp d\mathbf{r} + \int_{L_-} (\mathbf{g}^-)^\perp d\mathbf{r} - \int_{L_-} (\mathbf{g}^-)^\perp d\mathbf{r} + \\ & \int_{\gamma_+} (\mathbf{g}^+)^\perp d\mathbf{r} + \int_{L_+} (\mathbf{g}^+)^\perp d\mathbf{r} - \int_{L_+} (\mathbf{g}^+)^\perp d\mathbf{r}. \end{aligned} \tag{2.34}$$

Now, by using the Green's Theorem, it follows that

$$\begin{aligned} f_1(0,y_0)M(y_0;\boldsymbol{\mu}) = & -\iint_{\Theta_-} \text{div }\mathbf{g}^-(x,y,0,\boldsymbol{\mu})dxdy + \int_{\hat{y}_0}^{y_0} g_1^-(0,y,0,\boldsymbol{\mu})dy \\ & -\iint_{\Theta_+} \text{div }\mathbf{g}^+(x,y,0,\boldsymbol{\mu})dxdy - \int_{\hat{y}_0}^{y_0} g_1^+(0,y,0,\boldsymbol{\mu})dy, \end{aligned}$$

and expression (2.33) is obtained. □

Remark 2.11 Note that, expression (2.30) continues being valid for systems with multiple zones, obviously, when f_1 is continuous in $\mathbb{R}^2$ and system (2.6) has not a reset map. Furthermore, if g_1 is continuous, expression (2.33) is translated into

$$M(y_0;\boldsymbol{\mu}) = \frac{-1}{f_1(0,y_0)} \iint_{\text{int}(\gamma_{y_0})} \text{div}\,\mathbf{g}(x,y,0,\boldsymbol{\mu})\; dx\, dy, \tag{2.35}$$

expression that remains valid for systems with multiple zones.

To conclude the chapter, we will apply the developed theory to planar DPWL systems with two zones and to planar CPWL systems with three zones.

2.5 Application to planar continuous and discontinuous piecewise linear systems

In this section, we apply the developed Melnikov theory to find periodic orbits of two different types of systems. The section is divided in two subsections.

In the first one, we analyze the existence and saddle-node bifurcation of periodic orbits for two different classes of two-zonal planar DPWL systems. Subsequently, a saddle-node bifurcation of periodic orbits in a class of planar CPWL systems with three zones ($3CPWL_2$) is considered.

2.5.1 Application to two-zonal planar discontinuous piecewise linear systems

We work in this subsection with DPWL systems of the class (1.17). In order to apply the Melnikov method, we look for systems having a continuum of crossing periodic orbits. There exist different possibilities, among them, we will focus our attention on two different situations.

First, let us consider a continuous and homogeneous system whose coefficient matrices possess a pair of complex eigenvalues and they have traces with opposite sign and equal determinants, that is, we consider the system

$$\begin{cases} \dot{x} = -2\alpha x - y, \\ \dot{y} = \left(\alpha^2+1\right)x, \end{cases} \quad \text{if } x<0,$$
$$\begin{cases} \dot{x} = 2\alpha x - y, \\ \dot{y} = \left(\alpha^2+1\right)x, \end{cases} \quad \text{if } x\geqslant 0, \tag{2.36}$$

where $\alpha \in \mathbb{R}$. Note that, without loss of generality, we have assumed that the imaginary part of the eigenvalues is equal to the unity. Otherwise, we can make an appropriate rescaling in time in each zone of differentiability. System (2.36) is homogeneous, the origin is the unique equilibrium and it is surrounded by an unbounded continuum of periodic orbits with a period of $T = 2\pi$ (see [40, 44]). Specifically, the periodic solution $\gamma_{y_0}(t) = (x_{y_0}(t), y_{y_0}(t))$ with initial condition $\gamma_{y_0}(0) = (0, y_0)$, being $y_0 > 0$, is given by

$$\begin{cases} x_{y_0}(t) = -y_0 e^{-\alpha t}\sin t, \\ y_{y_0}(t) = y_0 e^{-\alpha t}\left(\alpha \sin t + \cos t\right), \end{cases} \quad \text{if} \quad 0 \leqslant t \leqslant \pi,$$

$$\begin{cases} x_{y_0}(t) = -y_0 e^{\alpha(t-2\pi)}\sin t, \\ y_{y_0}(t) = -y_0 e^{\alpha(t-2\pi)}\left(\alpha \sin t + \cos t\right), \end{cases} \quad \text{if} \quad \pi \leqslant t \leqslant 2\pi.$$

The phase plane of system (2.36) can be seen in Fig. 2.5.

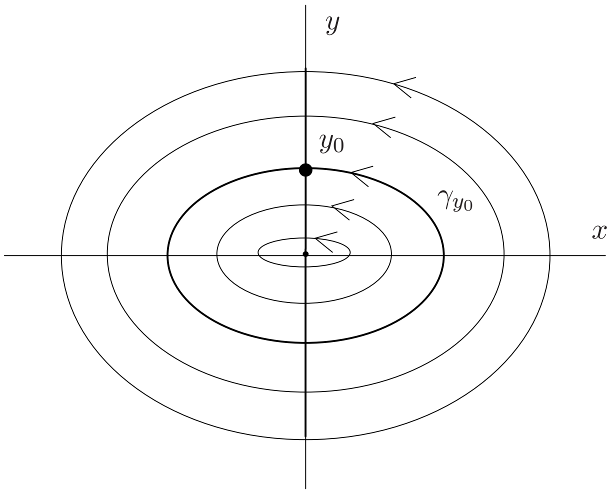


Figure 2.5: Phase plane of the unperturbed system (2.36).

Now, we consider the following perturbation of system (2.36),

$$\begin{cases} \dot{x} = 2(\varepsilon\omega - \alpha)x - y, \\ \dot{y} = ((\varepsilon\omega - \alpha)^2 + 1)x + \varepsilon A_-, \end{cases} \quad \text{if } x < 0,$$
$$\begin{cases} \dot{x} = 2\alpha x - y + \varepsilon B, \\ \dot{y} = (\alpha^2 + 1)x + \varepsilon A_+, \end{cases} \quad \text{if } x > 0, \tag{2.37}$$

where $A_-, A_+, B, \omega, \varepsilon \in \mathbb{R}$ and $|\varepsilon| \ll 1$.

By studying the Melnikov function (2.25) in this case, since Theorem 2.8, we are able to give the following result.

Proposition 2.12 If $\omega \neq 0$ and $\mathrm{sgn}\,(\omega)= \mathrm{sgn}\left(2\alpha\,(\mathrm{A}_+ + \mathrm{A}_-) - \mathrm{B}(\alpha^2+1)\right)$, then for $|\varepsilon| \neq 0$ and sufficiently small, system (2.37) possesses a hyperbolic limit cycle which is asymptotically stable if $\varepsilon \cdot \omega < 0$ and unstable if $\varepsilon \cdot \omega > 0$. Furthermore, the limit cycle is located in a neighborhood of the orbit $\gamma_{\tilde{y}_0}$, where

$$\tilde{y}_0 = \left(2\alpha\frac{A_+ + A_-}{\alpha^2+1} - B\right)\frac{1+e^{\alpha\pi}}{\omega\pi}. \tag{2.38}$$

Proof: The divergence of the vector field $\mathbf{f}$ of system (2.36) is given by

$$\text{div}\, \mathbf{f}(x, y) = \begin{cases} -2\alpha & \text{if} \quad x < 0, \\ 2\alpha & \text{if} \quad x > 0. \end{cases}$$

Some direct but tedious computations allow us to compute the expression of the Melnikov function (2.25) along a periodic orbit γ_{y_0}

$$M(y_0; \boldsymbol{\mu}) = (1 + e^{\alpha\pi}) \left(B - 2\alpha \frac{A_+ + A_-}{\alpha^2 + 1} \right) + \omega\pi y_0, \tag{2.39}$$

where $\boldsymbol{\mu} = (\alpha, A_-, A_+, B, \omega)$.

If $\omega \neq 0$, then the Melnikov function (2.39) has strictly positive zeros only when

$$\text{sgn}\,(\omega) = \text{sgn}\left(2\alpha\,(A_+ + A_-) - B(\alpha^2 + 1)\right).$$

In this case, the Melnikov function vanishes at the point

$$\tilde{y}_0 = \left(2\alpha \frac{A_+ + A_-}{\alpha^2 + 1} - B \right) \frac{1 + e^{\alpha\pi}}{\omega\pi}$$

and the proof concludes by a direct application of Theorem 2.8. □

Secondly, let us focus our attention on a different situation where the system possesses a continuum of periodic orbits. Precisely, we consider

$$\begin{array}{ll} \begin{cases} \dot{x} = -y, \\ \dot{y} = x - a, \end{cases} & \text{if } x < 0, \\ \begin{cases} \dot{x} = -y, \\ \dot{y} = x - |a|, \end{cases} & \text{if } x > 0, \end{array} \tag{2.40}$$

where $a \in \mathbb{R}$. Note that the second component of system (2.40) is discontinuous for $|a| \neq 0$. The shape of the continuum of periodic orbits crossing the straight line $x = 0$ depends on the sign of the parameter a.

For the case $a < 0$, the system has two equilibria, the point $\bar{x}^- = (a, 0)$ in the zone $x < 0$ and the point $\bar{x}^+ = (|a|, 0)$ in the zone $x > 0$. Each equilibrium point is surrounded by a bounded continuum of periodic orbits contained in its respective zone and both equilibrium points are surrounded by an unbounded continuum of periodic orbits γ_{y_0}, see Fig. 2.6.

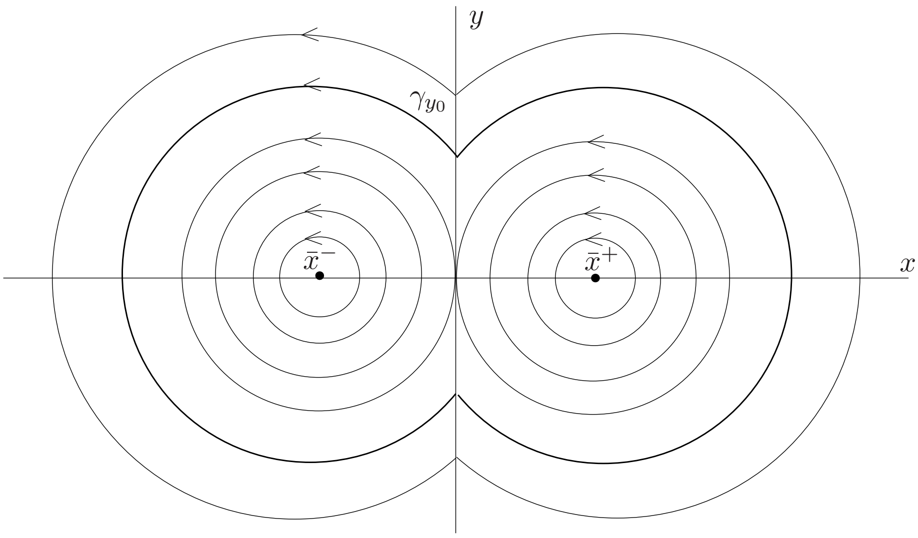


Figure 2.6: Phase plane of system (2.40) for $a < 0$.

For the case $a > 0$, the system has only one equilibrium point which is surrounded by an unbounded continuum of periodic orbits. The phase plane of system (2.40) for $a > 0$ is represented in Fig. 2.7.

We consider the following perturbation of system (2.40)

$$\begin{cases} \dot{x} = 2\sigma\varepsilon x - y, \\ \dot{y} = x - a, \end{cases} \quad \text{if } x < 0,$$
$$\begin{cases} \dot{x} = 2\sigma\varepsilon x - y - \varepsilon B, \\ \dot{y} = x - |a|, \end{cases} \quad \text{if } x > 0. \tag{2.41}$$

The analysis of system (2.41) via the Melnikov method let us state the following results about the existence of limit cycles. First, we consider the case $a < 0$.

Proposition 2.13 Consider system (2.41) with $a < 0$ and $\sigma \neq 0$. Let us define $\tilde{B} = -2a\sigma(1 + 2\pi)$. The following statements hold.

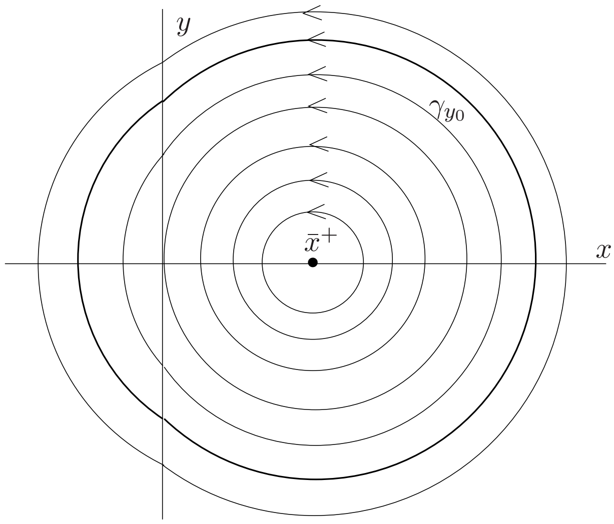


Figure 2.7: Phase plane of system (2.40) for $a^- > 0$.

(a) If $\sigma \cdot (B - \tilde{B}) > 0$, then system (2.41) has two hyperbolic limit cycles for $|\varepsilon| \neq 0$ and small enough, one asymptotically stable and the other one unstable, and they live in a neighborhood of the periodic orbits $\gamma_{\tilde{y}_0^-}$ and $\gamma_{\tilde{y}_0^+}$, respectively, where

$$\tilde{y}_0^{\pm} = \frac{B + 2a\sigma \pm \sqrt{(B + 2a\sigma)^2 - 16a^2\pi^2\sigma^2}}{4\pi\sigma}.$$

(b) If $B = \tilde{B}$, then there exist functions $\sigma(\varepsilon)$ and $B(\varepsilon)$ defined for ε sufficiently small, such that the perturbed system (2.41) with $\sigma = \sigma(\varepsilon)$ and $B = B(\varepsilon)$ has a unique periodic orbit in a neighborhood of the periodic orbit γ_{-a}, and it possesses multiplicity two.

(c) If $\sigma \cdot (B - \tilde{B}) < 0$, then for every $y_0 > 0$ there exists $\varepsilon_0 > 0$ sufficiently small such that system (2.41) has not periodic orbits for $0 < |\varepsilon| < \varepsilon_0$ near to the periodic orbit γ_{y_0}.

Proof: Since vector field of system (2.41) is divergence-free, the Melnikov function M given in (2.25) can be obtained by means of Proposition 2.10, and after some computations we get that it is given by

$$M(y_0, \sigma, a, B) = \frac{2}{y_0}\left(2a^2\pi\sigma - (2a\sigma + B)y_0 + 2\pi\sigma y_0^2\right). \tag{2.42}$$

By studying the positive roots and applying Theorem 2.8, the conclusion is straightforward. □

Next, we consider the case $a > 0$.

Proposition 2.14 Consider system (2.41) with $a > 0$ and $\sigma \neq 0$ and let us define $\hat{B} = 2a\sigma\pi$. The following statements hold

(a) If $\sigma \cdot (B - \hat{B}) > 0$, then system (2.41) has two hyperbolic limit cycles for $|\varepsilon| \neq 0$ and small enough, one asymptotically stable and the other one unstable. Furthermore, they are, respectively, in a neighborhood of the periodic orbits $\gamma_{\tilde{y}_0^-}$ and $\gamma_{\tilde{y}_0^+}$, where

$$\tilde{y}_0^{\pm} = \frac{B \pm \sqrt{B^2 - 4a^2\sigma^2\pi^2}}{2\pi\sigma}.$$

(b) If $B = \hat{B}$, then there exist functions $\sigma(\varepsilon)$ and $B(\varepsilon)$ defined for ε sufficiently small, such that the perturbed system (2.41) with $\sigma = \sigma(\varepsilon)$ and $B = B(\varepsilon)$ has a unique periodic orbit in a neighborhood of the periodic orbit γ_a and it possesses multiplicity two.

(c) If $\sigma \cdot (B - \hat{B}) < 0$, then for every $y_0 > 0$ there exists $\varepsilon_0 > 0$ sufficiently small such that system (2.41) has not periodic orbits for $0 < |\varepsilon| < \varepsilon_0$ near to the periodic orbit γ_{y_0}.

Proof: Since vector field of system (2.41) is divergence-free, it results from Proposition 2.10 that the Melnikov function M is given by

$$M(y_0, \sigma, a, B) = \frac{2}{y_0}\left(\sigma\pi y_0^2 - y_0 B + a^2\sigma\pi\right). \tag{2.43}$$

By studying positive roots and applying Theorem 2.8, the conclusion is direct. □

2.5.2 Application to three-zonal planar continuous piecewise linear systems

To complete this chapter, the Melnikov theory will be applied to 3CPWL$_2$ systems (see Remark 2.11). Specifically, the existence of a saddle-node bifurcation of three-zonal limit cycles is proven.

Let us consider the Lienard canonical form for continuous symmetric PWL systems with three zones of linearity

$$\begin{cases} \dot{x} = t_e x - y - (t_i - t_e)\mathrm{sat}(x), \\ \dot{y} = d_e x + (d_i - d_e)\mathrm{sat}(x), \end{cases} \tag{2.44}$$

where $t_i, t_e, d_i, d_e \in \mathbb{R}$ and $\mathrm{sat}(x)$ is the saturation function, that is,

$$\mathrm{sat}\,(x) = \begin{cases} \mathsf{sgn}(x) & \text{if} \quad |x| > 1, \\ x & \text{if} \quad |x| \leqslant 1. \end{cases}$$

The developed Melnikov theory allows us to get some of the results shown in [41, 92]. In particular, the saddle-node bifurcation of limit cycles conjectured in [41] and proven in [92] can be proven now through the Melnikov theory.

Let us remark that $t_i \cdot t_e < 0$ is a necessary condition for the existence of limit cycles of system (2.44), see Theorem 1.1 of [41]. We will focus our attention on the analysis of system (2.44) for t_i and t_e different from zero and small enough. Hence, we consider

$$t_i = \varepsilon \mathcal{T}_i, \quad t_e = \varepsilon \mathcal{T}_e, \quad \text{with} \quad |\varepsilon| \ll 1.$$

With this choice and doing $\varepsilon = 0$, system (2.44) becames

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = d_e x + (d_i - d_e)\mathrm{sat}(x), \end{cases} \tag{2.45}$$

and we have for ε different from zero,

$$\begin{cases} \dot{x} = -y + \varepsilon(\mathcal{T}_i - \mathcal{T}_e)\mathrm{sat}(x), \\ \dot{y} = d_e x + (d_i - d_e)\mathrm{sat}(x). \end{cases} \tag{2.46}$$

We show only the analysis for $d_i < 0$ and $d_e > 0$. In this case, system (2.45) possesses three equilibrium points, one of them is the origin, another one is found in the half-plane $x < -1$ and the last one in the half-plane $x > 1$.

In this case, according to symmetry properties of the unperturbed system (2.45), it can be proven that the system possesses three different continua of periodic orbits and a pair of homoclinic loops (see Fig. 2.8).

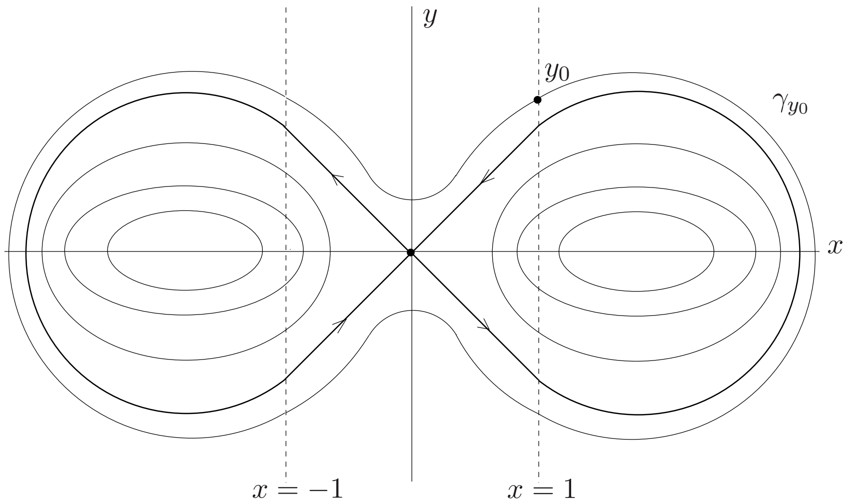


Figure 2.8: Phase plane of the unperturbed system (2.45) for $d_i < 0$ and $d_e > 0$.

The unbounded continuum of periodic orbits that surrounds the three equilibrium points of system (2.45) is represented as γ_{y_0}, for $y_0 > \sqrt{-d_i}$, where $(1, y_0)$ is the intersection point of the curve γ_{y_0} with the half straight-line $\{x = 1, y > 0\}$, see Fig. 2.8.

Due to vector field (2.45) is divergence-free, taking into account the symmetry of the system and that

$$\operatorname{div} \mathbf{g}(x, y, 0, \mathcal{T}_i, \mathcal{T}_e) = \begin{cases} \mathcal{T}_e & \text{if} \quad |x| > 1, \\ \mathcal{T}_i & \text{if} \quad |x| < 1, \end{cases}$$

it results from expression (2.35) that the Melnikov function $\tilde{M}$ is given by

$$\tilde{M}(y_0; \mathcal{T}_i, \mathcal{T}_e) = \frac{1}{y_0} \left(4\mathcal{T}_i R_i(y_0) + 4\mathcal{T}_e R_e(y_0) \right), \tag{2.47}$$

for $y_0 > \sqrt{-d_i}$, with

$$R_i(y_0) = \text{Area} \left(\text{int} \left(\gamma_{y_0} \right) \cap \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, y > 0\} \right)$$

and

$$R_e(y_0) = \text{Area} \left(\text{int} \left(\gamma_{y_0} \right) \cap \{(x, y) \in \mathbb{R}^2 : x > 1, y > 0\} \right).$$

For analyzing the roots of the Melnikov function given in (2.47), it is convenient to consider the function

$$M(y_0; \mathcal{T}_i, \mathcal{T}_e) = y_0 \tilde{M}(y_0, \mathcal{T}_i, \mathcal{T}_e) = 4\mathcal{T}_i R_i(y_0) + 4\mathcal{T}_e R_e(y_0). \tag{2.48}$$

By analyzing the function M we reproduce faithfully the results obtained in [41] and [92] about the existence and bifurcations of three-zonal limit cycles. In particular, we prove the existence of a saddle-node bifurcation of three-zonal periodic orbits for system (2.46). Let us remark that the persistence of the homoclinic loops of system (2.45) can be analyzed through the works [7, 13, 64]. From these works, it can be proven that these homoclinic connections remain if the value K given in the next result vanishes.

Proposition 2.15 Consider the perturbed system (2.46) with $d_i < 0, d_e > 0$ and $\mathcal{T}_i \cdot \mathcal{T}_e < 0$ and let us define

$$K = \mathcal{T}_i\sqrt{-d_i} + \frac{\mathcal{T}_e}{\sqrt{d_e}}\left[\frac{d_i}{d_e}(d_i - d_e)\left(\pi - \sin^{-1}\sqrt{\frac{d_e}{d_e - d_i}}\right) - d_i\sqrt{-\frac{d_i}{d_e}}\right].$$

Then, the following statements hold.

(a) Function $M(\cdot, \mathcal{T}_i, \mathcal{T}_e)$ has a unique critic point $y_0^* \in (\sqrt{-d_i}, +\infty)$.

(b) If $K \neq 0$ and $\mathrm{sgn}(\mathcal{T}_i) \neq \mathrm{sgn}(K)$, then the following statements hold.

a) If $\mathrm{sgn}(\mathcal{T}_i) = \mathrm{sgn}(M(y_0^*, \mathcal{T}_i, \mathcal{T}_e))$, then system (2.46) possesses two three-zonal hyperbolic limit cycles for $|\varepsilon| \neq 0$ and small enough, one asymptotically stable and the other one unstable. Furthermore, they are, respectively, in a neighborhood of $\gamma_{\hat{y}_0^1}$ and $\gamma_{\hat{y}_0^2}$, where $\hat{y}_0^1, \hat{y}_0^2$ are the only solutions of the equation $M(\cdot, \mathcal{T}_i, \mathcal{T}_e) = 0$ in the interval $(\sqrt{-d_i}, +\infty)$.

b) If $M(y_0^*, \mathcal{T}_i, \mathcal{T}_e) = 0$, then there exist two functions $\mathcal{T}_i(\varepsilon)$ and $\mathcal{T}_e(\varepsilon)$, defined for ε sufficiently small, such that system (2.46) with $\mathcal{T}_i = \mathcal{T}_i(\varepsilon)$ and $\mathcal{T}_e = \mathcal{T}_e(\varepsilon)$ has a unique three-zonal periodic orbit in a neighborhood of $\gamma_{y_0^*}$, and it possesses multiplicity two.

c) If $M(y_0^*, \mathcal{T}_i, \mathcal{T}_e) \neq 0$ and $\mathrm{sgn}(\mathcal{T}_i) \neq \mathrm{sgn}(M(y_0^*, \mathcal{T}_i, \mathcal{T}_e))$, then for every $y_0 > 0$ there exists $\varepsilon_0 > 0$ sufficiently small such that system (2.46) has no three-zonal periodic orbits close to the periodic orbit $\gamma_{y_0^*}$, for $0 < |\varepsilon| < \varepsilon_0$.

Proof: Let us suppose $\mathcal{T}_i > 0$, $\mathcal{T}_e < 0$. It is easy to see that function M defined in (2.48) satisfies

$$\lim_{\substack{y_0 \to \sqrt{-d_i} \\ y_0 > \sqrt{-d_i}}} M(y_0; \mathcal{T}_i, \mathcal{T}_e) = 2K,$$

where

$$K = \mathcal{T}_i\sqrt{-d_i} + \frac{\mathcal{T}_e}{\sqrt{d_e}}\left[\frac{d_i}{d_e}(d_i - d_e)\left(\pi - \sin^{-1}\sqrt{\frac{d_e}{d_e - d_i}}\right) - d_i\sqrt{-\frac{d_i}{d_e}}\right].$$

The three-zonal periodic orbits γ_{y_0}, for $y_0 > \sqrt{-d_i}$, are formed from pieces of ellipses in the exterior zones and pieces from hyperbolas in the interior zone, namely,

$$(d_e x - d_i)^2 + d_e y^2 = (d_e - d_i)^2 + d_e y_0^2 \quad \text{and} \quad d_i x^2 + y^2 = d_i + y_0^2.$$

The area of the portions of these ellipses and hyperbolas which take part in the expression of function M defined in (2.48), are given by

$$R_i(y_0) = \int_{-1}^{0} \sqrt{d_i - d_i(x+1)^2 + y_0^2}\, dx \tag{2.49}$$

and

$$R_e(y_0) = \frac{d_i^2 + d_e y_0^2}{2d_e\sqrt{d_e}} - \frac{1}{d_e}\int_0^{y_0}\left(\sqrt{d_e(y_0^2 - d_i - y^2)} + d_i\right) dy. \tag{2.50}$$

By substituting expressions (2.49) and (2.50) in (2.48) and by taking the derivative with respect to y_0, one obtains

$$\frac{\partial M}{\partial y_0}(y_0; \mathcal{T}_i, \mathcal{T}_e) = 4y_0 F(y_0),$$

where

$$F(y_0) = \frac{\mathcal{T}_i}{\sqrt{-d_i}}\sinh^{-1}\sqrt{\frac{-d_i}{y_0^2 + d_i}} + \frac{\mathcal{T}_e}{\sqrt{d_e}}\pi - \frac{\mathcal{T}_e}{\sqrt{d_e}}\sin^{-1}\sqrt{\frac{d_e y_0^2}{d_i^2 + d_e y_0^2}}.$$

Since

$$\lim_{\substack{y_0 \to \sqrt{-d_i} \\ y_0 > \sqrt{-d_i}}} F(y_0) = +\infty, \quad \text{and} \quad \lim_{y_0 \to +\infty} F(y_0) = \frac{\pi\mathcal{T}_e}{2\sqrt{d_e}} < 0,$$

we deduce that there exists a value $y_0^* > \sqrt{-d_i}$ such that $F(y_0^*) = 0$.

Due to the function

$$\frac{dF}{dy_0}(y_0) = \frac{-\mathcal{T}_i}{y_0^2 + d_i} + \frac{\mathcal{T}_e d_i}{d_i^2 + d_e y_0^2}$$

vanishes at most once in the interval $(\sqrt{-d_i}, +\infty)$, the root $y_0^* > \sqrt{-d_i}$ of function F is unique. That proves the first item.

Let us now turn to the second statement.

It is easy to see that $\lim_{y_0 \to +\infty} M(y_0; \mathcal{T}_i, \mathcal{T}_e) = -\infty$. If $K < 0$, then the following items hold.

- When $M(y_0^*; \mathcal{T}_i, \mathcal{T}_e) > 0$, then function $M(\cdot; \mathcal{T}_i, \mathcal{T}_e)$ possesses exactly two zeros $\hat{y}_0^1 < \hat{y}_0^2$ in the interval $(\sqrt{-d_i}, +\infty)$, which satisfy $\frac{\partial M}{\partial y_0}(\hat{y}_0^2; \mathcal{T}_i, \mathcal{T}_e) < 0 < \frac{\partial M}{\partial y_0}(\hat{y}_0^1; \mathcal{T}_i, \mathcal{T}_e)$, see Fig. 2.9.

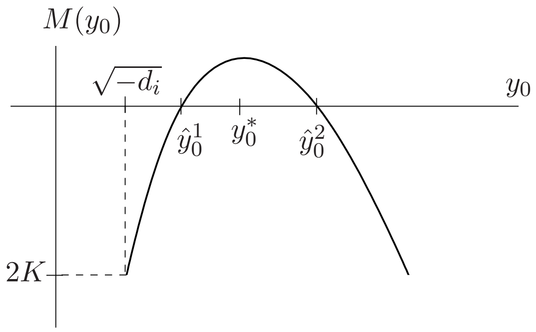


Figure 2.9: Function M defined in (2.48) for $\mathcal{T}_i > 0, \mathcal{T}_e < 0, K < 0$ and $M(y_0^*, \mathcal{T}_i, \mathcal{T}_e) > 0$.

- If $M(y_0^*; \mathcal{T}_i, \mathcal{T}_e) = 0$, then $\frac{\partial M}{\partial y_0}(y_0^*; \mathcal{T}_i, \mathcal{T}_e) = 0$ and $\frac{\partial^2 M}{\partial y_0^2}(y_0^*; \mathcal{T}_i, \mathcal{T}_e) \neq 0$, see Fig. 2.10.
- When $M(y_0^*; \mathcal{T}_i, \mathcal{T}_e) < 0$, then function $M(\cdot; \mathcal{T}_i, \mathcal{T}_e)$ does not possess zeros in the interval $(\sqrt{d_i}, +\infty)$, see Fig. 2.11.

The case $K > 0$ can be analyzed in the same manner.

From this, the conclusions of the second statement are a direct consequence of Theorem 2.8.

That concludes the proof for $\mathcal{T}_i > 0$ and $\mathcal{T}_e < 0$. In an analogous way, it would do for $\mathcal{T}_i < 0$ and $\mathcal{T}_e > 0$. □

In this chapter, we have generalized the Melnikov theory for a class of planar hybrid systems and we have applied our results to analyze the existence of periodic orbits in planar continuous

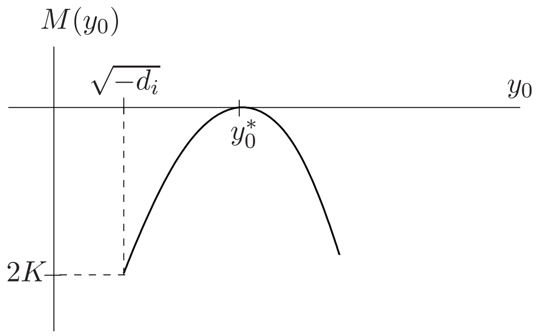


Figure 2.10: Function M defined in (2.48) for $\mathcal{T}_i > 0, \mathcal{T}_e < 0, K < 0$ and $M(y_0^*, \mathcal{T}_i, \mathcal{T}_e) = 0$.

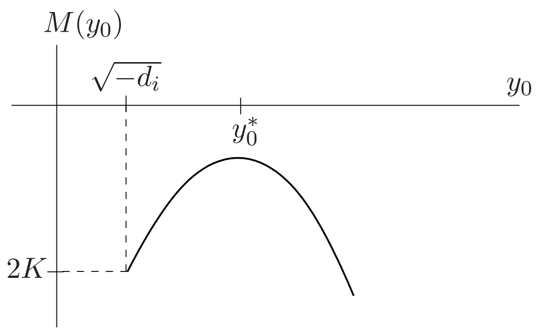


Figure 2.11: Function M defined in (2.48) for $\mathcal{T}_i > 0, \mathcal{T}_e < 0, K < 0$ and $M(y_0^*, \mathcal{T}_i, \mathcal{T}_e) < 0$.

and discontinuous PWL systems. One application of the developed theory will be done in the next chapter, where we analyze periodic orbits of a family of planar hybrid PWL systems.

Chapter 3

Invariant Cones in Three-Dimensional Observable Continuous Piecewise Linear Systems via Melnikov Theory

In the last section of Chapter 1, the motivation for the analysis of the existence of invariant cones in CPWL systems was expounded.

The principal aim of this chapter is the application of the theory that has been developed in Chapter 2 to find invariant cones in three-dimensional observable CPWL homogeneous systems. To get it, we will relate one-to-one the invariant cones in this class of systems to the periodic orbits of several planar hybrid systems.

The obtained results, among other things, extend those results given in [25] and prove the conjecture about the existence of a saddle-node bifurcation of invariant cones stated in that paper.

The chapter is organized as follows. In an introductory section, we state the problem of studying invariant cones in three-dimensional observable CPWL systems. Next, the equivalence between invariant cones in three-dimensional observable CPWL systems and periodic orbits in some planar hybrid systems is stated. The main results are written in Sec. 3.3. Subsequently, we state results of existence of invariant cones and saddle-node bifurcations of invariant cones. Finally, the last section is devoted to analyzing the two-zonal invariant cones which arise from the invariant cones of the unperturbed situation which are tangent to the separation plane $x = 0$.

The main results of this chapter are published in [18].

3.1 Introduction and preliminary results

As it was stated in Chapter 1, every homogeneous observable 2CPWL$_3$ system can be written into the Liénard form

$$\dot{\mathbf{x}} = M^{\triangledown}\mathbf{x} = \begin{pmatrix} t^{\triangledown} & -1 & 0 \\ m^{\triangledown} & 0 & -1 \\ d^{\triangledown} & 0 & 0 \end{pmatrix} \mathbf{x}, \tag{3.1}$$

where $\mathbf{x} = (x, y, z)^T$, being $t^{\pm}, m^{\pm}$ and $d^{\pm}$ the coefficients of the characteristic polynomials of matrix $M^{\pm}$.

On the other hand, in Chapter 1 it has been said that the one-zonal invariant cones of (3.1) cannot be isolated and in this case matrix M^+ (or M^-) has complex eigenvalues with the real part of the complex eigenvalues and the real eigenvalue shared. Moreover, when system (3.1) possesses one invariant cone living in each zone of linearity, then the space is foliated by invariant cones when the traces of matrices M^+ and M^- coincide. Here, two invariant cones tangent to the separation plane appear. These statements are stated in the next results and are deduced from Proposition 6 and statement (b) of Theorem 2 in [25].

Proposition 3.1 Assume that the eigenvalues of the matrices of system (3.1), M^+ and M^-, are λ^-, $\alpha^- \pm \mathrm{i}\beta^-$ and λ^+, $\alpha^+ \pm \mathrm{i}\beta^+$, respectively, with λ^-, α^-, β^-, λ^+, α^+, $\beta^+ \in \mathbb{R}$, $\beta^- > 0$ and $\beta^+ > 0$. Then, the following statements hold.

(a) If system (3.1) has a one-zonal invariant cone $\mathcal{C}$ living in the half-space $\{x \leqslant 0\}$, then $\alpha^- = \lambda^-$ and the system has a continuum of one-zonal invariant cones living in the zone $\{x \leqslant 0\}$.

(b) If system (3.1) has a one-zonal invariant cone $\mathcal{C}$ living in the half-space $\{x \geqslant 0\}$, then $\alpha^+ = \lambda^+$ and the system has a continuum of one-zonal invariant cones living in the zone $\{x \geqslant 0\}$.

Proposition 3.2 Under the hypotheses of Proposition 3.1, the three-dimensional space $\mathbb{R}^3$ is foliated by invariant cones of system (3.1) if and only if $\alpha^- = \lambda^- = \alpha^+ = \lambda^+$.

Let us introduce one remark about the invariant cones.

Remark 3.3 Under the hypothesis of complex eigenvalues of the matrices of system (3.1), in [25] is proven that when two different two-zonal invariant cones appear, they are hyperbolic and they have exchanged attractiveness. Moreover, if $(\alpha^- - \lambda^-)(\alpha^+ - \lambda^+) < 0$ and there exists one unique invariant cone, it is non-hyperbolic and semi-attractive.

The dynamical behavior considered in Proposition 3.2 provide us the germ necessary to analyze the presence of invariant cones and the existence of saddle-node bifurcations of invariant cones. The key idea is to know what invariant cones persist when this situation is perturbed.

3.2 Equivalence between invariant cones in 2CPWL$_3$ systems and periodic orbits in planar PWL hybrid systems

The correspondence one-to-one between the invariant cones of system (3.1) and periodic orbits of certain planar hybrid PWL systems is considered in this section. In the following, the characteristic polynomial of matrices M^- and M^+ will be denoted by

$$p_{M^-}(\lambda) = \det(M^- - \lambda I) = -\lambda^3 + t^-\lambda^2 - m^-\lambda + d^-$$

and

$$p_{M^+}(\lambda) = \det(M^+ - \lambda I) = -\lambda^3 + t^+\lambda^2 - m^+\lambda + d^+,$$

respectively.

Let us expose some geometrical properties of the invariant cones that may appear in the class of systems that we are studying. To begin with, if $\lambda^- \in \mathbb{R}$ is an eigenvalue of matrix M^-, then it is immediate to observe that the plane $\Pi^- \equiv (\lambda^-)^2 x - \lambda^- y + z = 0$ is an invariant manifold for the linear system $\dot{\mathbf{x}} = M^-\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^3$. Analogously, if λ^+ is a real eigenvalue of matrix M^+, then the plane $\Pi^+ \equiv (\lambda^+)^2 x - \lambda^+ y + z = 0$ is an invariant manifold for the linear system $\dot{\mathbf{x}} = M^+\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^3$. When $\lambda^- \neq \lambda^+$, then Π^- and Π^+ are not invariant manifolds of system (3.1) and when $\lambda^- = \lambda^+$, then the planes Π^- and Π^+ coincide and constitute a planar two-zonal invariant cone for system (3.1). It is clear that the invariance of these planes must restrict the sets where the invariant cones of system (3.1), if any, are located. We will say that a cone is above (resp. below) a plane if for every point (x_1, y_1, z_1) not at the origin and belonging to the cone, there exists another point (x_1, y_1, z_2) belonging to the plane such that $z_1 > z_2$ (resp. $z_1 < z_2$). Now, from Lemma 21 and Proposition 22 of [25], it has been proven that the non-planar two-zonal invariant cones of the system, if they exist, are located either above or below both planes Π^- and Π^+. Therefore, we must look for the invariant cones for system (3.1) above the planes Π^- and Π^+ or below these planes.

When one search invariant cones above Π^+, these cones correspond one-to-one to the periodic orbits of a planar continuous piecewise quadratic system, as it is established in the next result.

Proposition 3.4 Let λ^+ be a real eigenvalue of matrix M^+, then the invariant cones of system (3.1) located above the plane Π^+ are in one-to-one correspondence to the periodic orbits of the continuous planar piecewise quadratic system

$$\begin{cases} \dot{u}_1 = (t^- - \lambda^+)\, u_1 - u_2 - p_{M^-}(\lambda^+) u_1^2, \\ \dot{u}_2 = \left[m^- + (\lambda^+)^2\right] u_1 - 2\lambda^+ u_2 - p_{M^-}(\lambda^+) u_1 u_2 - 1, \end{cases} \quad \text{if } u_1 \leqslant 0,$$
$$\begin{cases} \dot{u}_1 = (t^+ - \lambda^+)\, u_1 - u_2, \\ \dot{u}_2 = \left[m^+ + (\lambda^+)^2\right] u_1 - 2\lambda^+ u_2 - 1, \end{cases} \quad \text{if } u_1 > 0. \tag{3.2}$$

Proof: It is sufficient to do the change of variables

$$u_1 = \frac{x}{(\lambda^+)^2 x - \lambda^+ y + z}, \qquad u_2 = \frac{y}{(\lambda^+)^2 x - \lambda^+ y + z}, \qquad Z^+ = (\lambda^+)^2 x - \lambda^+ y + z, \tag{3.3}$$

when $(\lambda^+)^2 x - \lambda^+ y + z > 0$. □

We can establish an analogous result for the invariant cones living above the plane Π^-.

Proposition 3.5 Let λ^- be a real eigenvalue of matrix M^-, then the invariant cones of system (3.1) located above the plane Π^- are in one-to-one correspondence to the periodic orbits of the continuous planar piecewise quadratic system

$$\begin{cases} \dot{v}_1 = (t^- - \lambda^-)\, v_1 - v_2, \\ \dot{v}_2 = \left[m^- + (\lambda^-)^2\right] v_1 - 2\lambda^- v_2 - 1, \end{cases} \quad \text{if } v_1 \leqslant 0,$$
$$\begin{cases} \dot{v}_1 = (t^- - \lambda^-)\, v_1 - v_2 - p_{M^+}(\lambda^-) v_1^2, \\ \dot{v}_2 = \left[m^+ + (\lambda^-)^2\right] v_1 - 2\lambda^- v_2 - p_{M^+}(\lambda^-) v_1 v_2 - 1, \end{cases} \quad \text{if } v_1 > 0. \tag{3.4}$$

Proof: It is sufficient to do the change of variables

$$v_1 = \frac{x}{(\lambda^-)^2 x - \lambda^- y + z}, \qquad v_2 = \frac{y}{(\lambda^-)^2 x - \lambda^- y + z}, \qquad Z^- = (\lambda^-)^2 x - \lambda^- y + z, \quad (3.5)$$

when $(\lambda^-)^2 x - \lambda^- y + z > 0$. □

Note that for $\lambda^+ = \lambda^-$, systems (3.2) and (3.4) are planar CPWL systems. Moreover, if $\lambda^+ \neq \lambda^-$, then system (3.2) is linear for $u_1 \geqslant 0$ and quadratic for $u_1 \leqslant 0$; and system (3.4) is linear for $v_1 \leqslant 0$ and quadratic for $v_1 \geqslant 0$.

Taking into account that piecewise quadratic systems (3.2) and (3.4) have been obtained from system (3.1) by means of the changes of variables given in (3.3) and (3.5), respectively, we can conclude that both continuous piecewise quadratic systems must be equivalent in a suitable region of $\mathbb{R}^2$. Indeed, it is possible to obtain the change of variable which transforms system (3.2) into system (3.4) as well as the suitable region. From expressions (3.3) and (3.5), one obtains

$$Z^+ - Z^- = \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right) x - y\right]$$

and so,

$$\frac{Z^-}{Z^+} = 1 - \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right)\frac{x}{Z^+} - \frac{y}{Z^+}\right].$$

Hence, since $u_1 = x/Z^+$, $u_2 = y/Z^+$, $Z^+ > 0$, $Z^- > 0$ and $Z^-/Z^+ = u_1/v_1 = u_2/v_2$, we deduce that

$$1 - \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right) u_1 - u_2\right] > 0,$$

$$v_1 = \frac{u_1}{1 - (\lambda^+ - \lambda^-)\left[(\lambda^+ + \lambda^-) u_1 - u_2\right]}$$

and

$$v_2 = \frac{u_2}{1 - (\lambda^+ - \lambda^-)\left[(\lambda^+ + \lambda^-) u_1 - u_2\right]}.$$

A similar argument can be done to determine that

$$u_1 = \frac{v_1}{1 + (\lambda^+ - \lambda^-)\left[(\lambda^+ + \lambda^-) v_1 - v_2\right]}$$

and

$$u_2 = \frac{v_2}{1 + (\lambda^+ - \lambda^-)\left[(\lambda^+ + \lambda^-) v_1 - v_2\right]}.$$

The above reasoning allows us to state that if $\lambda^+ \neq \lambda^-$ the quadratic system

$$\begin{cases} \dot{u}_1 &= (t^- - \lambda^+)\, u_1 - u_2 - p_{M^-}(\lambda^+) u_1^2, \\ \dot{u}_2 &= \left[m^- + (\lambda^+)^2\right] u_1 - 2\lambda^+ u_2 - p_{M^-}(\lambda^+) u_1 u_2 - 1, \end{cases}$$

is equivalent to the linear system

$$\begin{cases} \dot{v}_1 &= (t^- - \lambda^-)\, v_1 - v_2, \\ \dot{v}_2 &= \left[m^- + (\lambda^-)^2\right] v_1 - 2\lambda^- v_2 - 1, \end{cases}$$

for each open half-plane determined by the straight line

$$1 - \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right) u_1 - u_2\right] = 0,$$

and in the same way, the quadratic system

$$\begin{cases} \dot{v}_1 &= (t^- - \lambda^-)\, v_1 - v_2 - p_{M^+}(\lambda^-) v_1^2, \\ \dot{v}_2 &= \left[m^+ + (\lambda^-)^2\right] v_1 - 2\lambda^- v_2 - p_{M^+}(\lambda^-) v_1 v_2 - 1, \end{cases}$$

is equivalent to the linear system

$$\begin{cases} \dot{u}_1 &= (t^+ - \lambda^+)\, u_1 - u_2, \\ \dot{u}_2 &= \left[m^+ + (\lambda^+)^2\right] u_1 - 2\lambda^+ u_2 - 1, \end{cases}$$

for each open half-plane determined by the straight line

$$1 + \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right) v_1 - v_2\right] = 0.$$

Observe that periodic orbits of piecewise quadratic system (3.2) must belong to the open half-plane

$$\Omega^+ = \left\{(u_1, u_2) \in \mathbb{R}^2 : 1 - \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right) u_1 - u_2\right] > 0\right\}, \qquad (3.6)$$

because the invariant cones of system (3.1) located above the plane Π^+ also have to be located above the plane Π^-. Analogously, the periodic orbits of piecewise quadratic system (3.4) must belong to the open half-plane

$$\Omega^- = \left\{(v_1, v_2) \in \mathbb{R}^2 : 1 + \left(\lambda^+ - \lambda^-\right) \left[\left(\lambda^+ + \lambda^-\right) v_1 - v_2\right] > 0\right\}. \tag{3.7}$$

We represent the half-planes Ω^+ and Ω^- in the Fig. 3.1 in the case $\lambda^- - \lambda^+ > 0$ and $\lambda^- + \lambda^+ < 0$. The remaining cases have analogous representations.

Note that the origin belongs to half-planes Ω^+ and Ω^-, and that $\Omega^+ = \Omega^- = \mathbb{R}$ for $\lambda^+ = \lambda^-$.

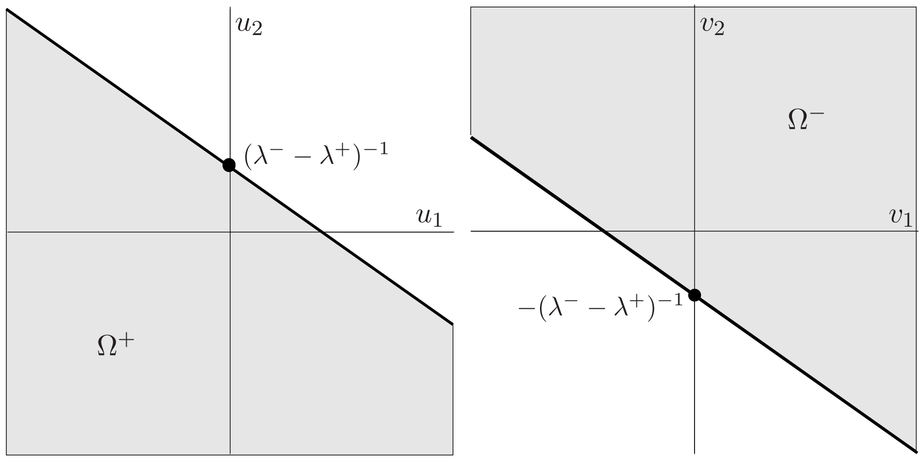


Figure 3.1: The half-planes Ω^+ and Ω^- defined in (3.6) and (3.7) for the case $\lambda^- - \lambda^+ > 0$ and $\lambda^- + \lambda^+ < 0$.

From the above reasoning, we can describe a hybrid planar PWL system, with the proposal to search the periodic orbits of systems (3.2) or (3.4), that is, the invariant cones of system (3.1) above planes Π^- and Π^+.

Proposition 3.6 The invariant cones of system (3.1) located above planes Π^- and Π^+ are in one-to-one correspondence to periodic orbits of the planar hybrid piecewise linear system

$$\begin{cases} \dot{x} = 2(\alpha^- - \lambda^-)x - y \\ \dot{y} = \left[(\alpha^- - \lambda^-)^2 + (\beta^-)^2\right] x - 1 \end{cases} \quad \text{if } x < 0,$$
$$\begin{cases} \dot{x} = 2(\alpha^+ - \lambda^+)x - y \\ \dot{y} = \left[(\alpha^+ - \lambda^+)^2 + (\beta^+)^2\right] x - 1 \end{cases} \quad \text{if } x > 0, \tag{3.8}$$

defined in the region $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \left\{(x,y) \in \mathbb{R}^2 : 1 + \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ - \lambda^-\right)x - y\right] > 0, x \leqslant 0\right\} \tag{3.9}$$

and

$$\Omega_2 = \left\{(x,y) \in \mathbb{R}^2 : 1 - \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^- - \lambda^+\right)x - y\right] > 0, x \geqslant 0\right\}, \tag{3.10}$$

with the reset map

$$\begin{array}{rcl} \mathcal{R} : \Omega \cap \{x = 0\} & \longrightarrow & \Omega \cap \{x = 0\} \\ (0, y) & \longmapsto & (0, \delta(y)), \end{array} \tag{3.11}$$

being

$$\delta(y) = \begin{cases} \dfrac{y}{1 - (\lambda^+ - \lambda^-)y} & \text{if } \ y \leqslant 0, \\ \\ \dfrac{y}{1 + (\lambda^+ - \lambda^-)y} & \text{if } \ y > 0. \end{cases} \tag{3.12}$$

Proof: The following discontinuous change of variables

$$\begin{array}{ll} \begin{cases} v_1 = \dfrac{u_1}{1 - (\lambda^+ - \lambda^-)\left[(\lambda^+ + \lambda^-)u_1 - u_2\right]} \\ v_2 = \dfrac{u_2}{1 - (\lambda^+ - \lambda^-)\left[(\lambda^+ + \lambda^-)u_1 - u_2\right]} \end{cases} & \text{for} \quad u_1 \leqslant 0, \\ \begin{cases} v_1 = u_1, \\ v_2 = u_2, \end{cases} & \text{for} \quad u_1 > 0, \end{array} \tag{3.13}$$

defined in the half-plane Ω^+, where Ω^+ is given in (3.6), transforms system (3.2) into the planar PWL system

$$\begin{array}{ll} \begin{cases} \dot{v}_1 = (t^- - \lambda^-)v_1 - v_2, \\ \dot{v}_2 = \left[m^- + (\lambda^-)^2\right]v_1 - 2\lambda^- v_2 - 1, \end{cases} & \text{if} \quad v_1 < 0, \\ \begin{cases} \dot{v}_1 = (t^+ - \lambda^+)v_1 - v_2, \\ \dot{v}_2 = \left[m^+ + (\lambda^+)^2\right]v_1 - 2\lambda^+ v_2 - 1, \end{cases} & \text{if} \quad v_1 > 0, \end{array} \tag{3.14}$$

and the set Ω^+ into the set $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$, where

$$\tilde{\Omega}_1 = \left\{(v_1, v_2) \in \mathbb{R}^2 : 1 + \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right)v_1 - v_2\right] > 0, v_1 \leqslant 0\right\}$$

and

$$\tilde{\Omega}_2 = \left\{(v_1, v_2) \in \mathbb{R}^2 : 1 - \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ + \lambda^-\right) v_1 - v_2\right] > 0, v_1 \geqslant 0\right\}.$$

Note that system (3.14) needs the reset map

$$\begin{array}{rcl} \tilde{\mathcal{R}} : \tilde{\Omega} \cap \{v_1 = 0\} & \longrightarrow & \tilde{\Omega} \cap \{v_1 = 0\} \\ (0, v_2) & \longmapsto & (0, \tilde{\delta}(v_2)), \end{array}$$

where

$$\tilde{\delta}(v_2) = \begin{cases} \dfrac{v_2}{1 - (\lambda^+ - \lambda^-)v_2} & \text{if } v_2 \leqslant 0, \\ \\ \dfrac{v_2}{1 + (\lambda^+ - \lambda^-)v_2} & \text{if } v_2 > 0, \end{cases}$$

which is a direct consequence of the discontinuous change (3.13). Now, the last change of variables

$$x = v_1, \qquad y = \begin{cases} -2\lambda^- v_1 + v_2 & \text{if} \quad v_1 \leqslant 0, \\ -2\lambda^+ v_1 + v_2 & \text{if} \quad v_1 > 0, \end{cases}$$

transforms system (3.14) into system (3.8), the regions $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ into the regions Ω_1 and Ω_2, given in (3.9) and (3.10), respectively, and the reset map $\tilde{\mathcal{R}}$ into the reset map $\mathcal{R}$. This concludes the proof. □

We represent the sets $\tilde{\Omega}$ and Ω in Fig. 3.2 for the case $\lambda^- - \lambda^+ > 0$ and $\lambda^- + \lambda^+ < 0$. The remaining cases have analogous representations.

It is clear that Proposition 3.6 has a dual result, that we enunciate without proof, for the invariant cones living below planes Π^- and Π^+.

Proposition 3.7 The invariant cones of system (3.1) located below planes Π^- and Π^+ are in one-to-one correspondence to the periodic orbits of the planar hybrid PWL system

$$\begin{aligned} &\begin{cases} \dot{x} = 2(\alpha^- - \lambda^-)x - y \\ \dot{y} = \left[(\alpha^- - \lambda^-)^2 + (\beta^-)^2\right] x + 1 \end{cases} \quad \text{if } x < 0, \\ &\begin{cases} \dot{x} = 2(\alpha^+ - \lambda^+)x - y \\ \dot{y} = \left[(\alpha^+ - \lambda^+)^2 + (\beta^+)^2\right] x + 1 \end{cases} \quad \text{if } x > 0, \end{aligned} \tag{3.15}$$

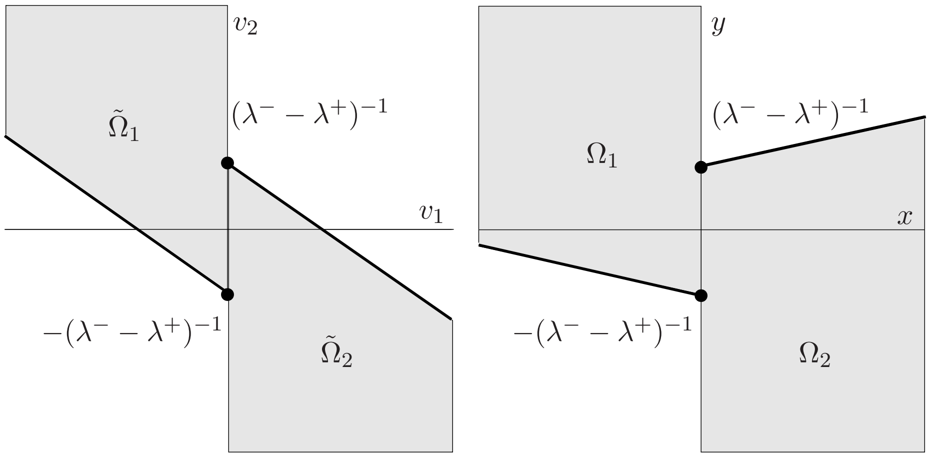


Figure 3.2: The sets $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$ and $\Omega = \Omega_1 \cup \Omega_2$ for $\lambda^- - \lambda^+ > 0$ and $\lambda^- + \lambda^+ < 0$.

defined in the region $\hat{\Omega} = \hat{\Omega}_1 \cup \hat{\Omega}_2$, where

$$\hat{\Omega}_1 = \left\{(x,y) \in \mathbb{R}^2 : 1 - \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^+ - \lambda^-\right)x - y\right] > 0, x \leqslant 0\right\}$$

and

$$\hat{\Omega}_2 = \left\{(x,y) \in \mathbb{R}^2 : 1 + \left(\lambda^+ - \lambda^-\right)\left[\left(\lambda^- - \lambda^+\right)x - y\right] > 0, x \geqslant 0\right\},$$

with the reset map

$$\begin{array}{rcl} \mathcal{R}^{-1} : \Omega \cap \{x=0\} & \longrightarrow & \Omega \cap \{x=0\} \\ (0,y) & \longmapsto & (0, \delta^{-1}(y)), \end{array}$$

where δ is defined in (3.12).

3.3 Statement of main results

To begin this section we present the unperturbed system and its perturbation. As we have written in Sec. 3.1, the dynamical behavior considered in Proposition 3.2 provide us the germ necessary to analyze the existence and bifurcations of two-zonal invariant cones. Note that system (3.1), under conditions of Proposition 3.2, can be written as

$$\dot{\mathbf{x}} = \begin{cases} N^+\mathbf{x} & \text{if} \quad x \geqslant 0, \\ N^-\mathbf{x} & \text{if} \quad x < 0, \end{cases} \tag{3.16}$$

where

$$N^+ = \begin{pmatrix} 3\lambda^- & -1 & 0 \\ 2(\lambda^-)^2 + (\beta^+)^2 & 0 & -1 \\ (\lambda^-)^3 + \lambda^-(\beta^+)^2 & 0 & 0 \end{pmatrix},$$

$$N^- = \begin{pmatrix} 3\lambda^- & -1 & 0 \\ 2(\lambda^-)^2 + (\beta^-)^2 & 0 & -1 \\ (\lambda^-)^3 + \lambda^-(\beta^-)^2 & 0 & 0 \end{pmatrix}, \tag{3.17}$$

with $\lambda^-, \beta^-, \beta^+ \in \mathbb{R}$, $\beta^- > 0$ and $\beta^+ > 0$. This system will be called the unperturbed system.

We ask about invariant cones of the unperturbed system (3.16) that remain when it is perturbed. As the eigenvalues of the coefficient matrices N^- and N^+ are λ^-, $\lambda^- \pm \mathrm{i}\beta^-$ and λ^-, $\lambda^- \pm \mathrm{i}\beta^+$, respectively, to perturb the system it is natural to assume that coefficient matrices of the perturbed system possess a pair of complex conjugate eigenvalues. Furthermore, these eigenvalues must be close to the eigenvalues of the unperturbed system. Hence, we can assume that the eigenvalues of the coefficient matrices of the perturbed system are λ^-, $\alpha^- \pm \mathrm{i}\beta^-$ and λ^+, $\alpha^+ \pm \mathrm{i}\beta^+$, with

$$\alpha^- = \lambda^- + \varepsilon\sigma^-, \alpha^+ = \lambda^- + \varepsilon\sigma^+ \text{ and } \lambda^+ = \lambda^- + \varepsilon\Lambda, \tag{3.18}$$

where $\lambda^-, \sigma^-, \sigma^+, \Lambda, \varepsilon \in \mathbb{R}$ and $|\varepsilon| << 1$.

From system (3.1), assuming that the eigenvalues of matrices M^+ and M^- have the form given in (3.18), we arrive to the perturbed system

$$\dot{\mathbf{x}} = \begin{cases} (N^+ + \varepsilon L^+)\mathbf{x} & \text{if} \quad x \geqslant 0, \\ (N^- + \varepsilon L^-)\mathbf{x} & \text{if} \quad x < 0, \end{cases} \tag{3.19}$$

where N^- and N^+ are given in (3.17), $L^- = \mathbf{b}^-\mathbf{e_1}^T$ and $L^+ = \mathbf{b}^+\mathbf{e_1}^T$, with

$$\mathbf{b}^- = \begin{pmatrix} 2\sigma^- \\ 3\lambda^-\sigma^- + \varepsilon(\sigma^-)^2 \\ 2(\lambda^-)^2\sigma^- + \varepsilon\lambda^-(\sigma^-)^2 \end{pmatrix},$$

$$\mathbf{b}^+ = \begin{pmatrix} 2\sigma^+ + \Lambda \\ 3\lambda^-\sigma^+ + \lambda^-\Lambda + O(\varepsilon) \\ 2(\lambda^-)^2\sigma^+ + \Lambda((\lambda^-)^2 + (\beta^+)^2) + O(\varepsilon) \end{pmatrix}$$

and $\lambda^-, \sigma^-, \sigma^+, \Lambda, \varepsilon \in \mathbb{R}$, $|\varepsilon| \ll 1$.

Note that the invariant plane of the left zone for the perturbed system (3.19) is Π^-, the same as that of the unperturbed system (3.16). Nevertheless, the invariant plane of the right zone, which we will denote by Π^+_ε, has a different expression $\Pi^+_\varepsilon \equiv (\lambda^- + \varepsilon\Lambda)^2 x - (\lambda^- + \varepsilon\Lambda)y + z = 0$.

At this time, we are able to expose the main results of this chapter. The proofs are based on the equivalence between invariant cones in three-dimensional CPWL systems and periodic orbits in planar hybrid PWL systems (which is stated in Sec. 3.2). Thus, two functions whose zeros provide us the invariant cones that persist will be constructed. These functions will be called Melnikov functions. Note that one could also use the method of averaging for manifolds [79], since, as it is proved in [10], this approach can be extended for Lipschitzian systems; but that would require us to work with system (1.20), which is a continuous piecewise cubic system on S^2, rather than a planar piecewise linear system.

The second items of the following theorems will allow us give some results of saddle-node bifurcation of invariant cones of system (3.1).

We only prove Theorem 3.8. Similar reasoning let us prove Theorem 3.9.

Theorem 3.8 Consider the Melnikov function

$$M_1(y_0; \boldsymbol{\mu}) = (\sigma^+ - \Lambda)F(y_0, \beta^+) + \sigma^- G(y_0, \beta^-) - y_0^3\Lambda, \tag{3.20}$$

defined for $y_0 > 0$ and $\boldsymbol{\mu} = (\sigma^-, \sigma^+, \Lambda, \beta^-, \beta^+) \in \mathbb{R}^5$, where

$$F(y_0, a) = \frac{\pi(1 + a^2 y_0^2)}{a^3} + \frac{2y_0}{a^2} - I(y_0, a), \quad G(y_0, a) = \frac{-2y_0}{a^2} + I(y_0, a), \tag{3.21}$$

being

$$I(y_0, a) = \frac{1}{a}\left(\left(\frac{1}{a^2} + y_0^2\right)\sin^{-1}\frac{a y_0}{\sqrt{1 + a^2 y_0^2}} + \frac{y_0}{a}\right) \qquad \text{for } a > 0.$$

Assume that there exist $\bar{y}_0 > 0$ and $\boldsymbol{\mu}_0 \in \mathbb{R}^5$ such that $M_1(\bar{y}_0; \boldsymbol{\mu}_0) = 0$ and denote

$$q_1 = \frac{\partial M_1}{\partial y_0}(\bar{y}_0; \boldsymbol{\mu}_0).$$

Then, the following statements hold.

(a) If $q_1 \neq 0$, then the perturbed system (3.19) has a hyperbolic two-zonal invariant cone located above the planes Π^- and Π_ε^+, for ε different from zero and sufficiently small. Moreover, the invariant cone is repulsive if $\varepsilon \cdot q_1 < 0$ and attractive if $\varepsilon \cdot q_1 > 0$.

(b) If $q_1 = 0$, then there exists a function $\boldsymbol{\mu} = \boldsymbol{\mu}(\varepsilon) = \boldsymbol{\mu}_0 + O(\varepsilon)$ such that system (3.19) has a unique non-hyperbolic semi-attractive two-zonal invariant cone located above the planes Π^- and Π_ε^+, for ε different from zero and sufficiently small.

Proof: For the unperturbed system (3.16), $\alpha^- = \lambda^- = \alpha^+ = \lambda^+$, so applying Proposition 3.6, the invariant cones located above plane $\Pi^- = \Pi^+$ are in one-to-one correspondence to periodic orbits of the planar CPWL system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = (\beta^-)^2 x - 1 \end{cases} \text{if } x < 0, \qquad \begin{cases} \dot{x} = -y \\ \dot{y} = (\beta^+)^2 x - 1 \end{cases} \text{if } x > 0, \tag{3.22}$$

defined in $\mathbb{R}^2$. Note that, as $\lambda^+ = \lambda^-$, the reset map defined in (3.11) is the identity function and the resulting system is a planar CPWL system. Now, let us describe the phase plane of system (3.22). As $\beta^-, \beta^+ > 0$, system (3.22) possesses only one equilibrium point $\bar{x} = (1/(\beta^+)^2, 0)$. Moreover, system (3.22) is invariant under the transformation $(t, y) \to (-t, -y)$. From this invariance and taking into account that the equilibrium point $\bar{x}$ is a linear center of the right zone, we conclude that there exists an unbounded continuum of periodic orbits surrounding the equilibrium point $\bar{x}$ (see Fig. 3.3). In the

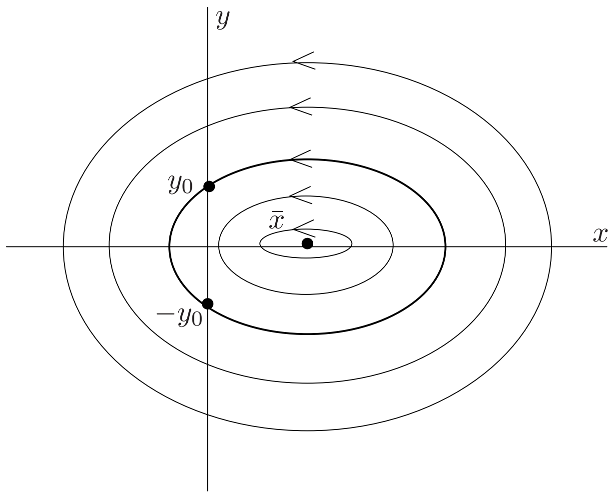


Figure 3.3: Representation of the continuum of periodic orbits of system (3.22).

continuum, there exist periodic orbits contained in the zone $x \geq 0$ and periodic orbits contained in both zones. The orbit passing through $(x, y) = (0, 0)$ is tangent to the vertical axis and corresponds to the invariant cone which is tangent to the separation plane and it is located above the planes Π^- and Π^+. The two-zonal orbits form an unbounded continuum of periodic orbits crossing transversally the separation line $x = 0$ counterclockwise. If we integrate the system, we find out that every two-zonal orbit is formed from two pieces of ellipses, joined with continuity and differentiability at the separation line $x = 0$. This family of periodic orbits can be parameterized as

$$\Gamma_{y_0} \equiv \begin{cases} ((\beta^-)^2 x - 1)^2 + (\beta^-)^2 y^2 = 1 + (\beta^-)^2 y_0^2 & \text{if } x < 0, \\ ((\beta^+)^2 x - 1)^2 + (\beta^+)^2 y^2 = 1 + (\beta^+)^2 y_0^2 & \text{if } x \geq 0, \end{cases}$$

where $y_0 \in (0, +\infty)$ is the intersection point of the periodic orbit with the vertical axis.

On the other hand, applying Proposition 3.6 to system (3.19), we can asserts that invariant cones of this system located above planes Π^- and Π_ε^+ are in one-to-one correspondence to periodic orbits of the planar hybrid PWL system

$$
\begin{cases} \dot{x} = 2\varepsilon\sigma^- x - y \\ \dot{y} = (\varepsilon^2(\sigma^-)^2 + (\beta^-)^2)x - 1 \end{cases} \quad \text{if } x < 0,
$$
$$
\begin{cases} \dot{x} = 2\varepsilon(\sigma^+ - \Lambda)x - y \\ \dot{y} = (\varepsilon^2(\sigma^+ - \Lambda)^2 + (\beta^+)^2)x - 1 \end{cases} \quad \text{if } x > 0, \tag{3.23}
$$

defined into the region $\Omega_\varepsilon = \Omega_{\varepsilon 1} \cup \Omega_{\varepsilon 2}$, where

$$
\Omega_{\varepsilon 1} = \left\{ (x, y) \in \mathbb{R}^2 : 1 + \varepsilon\Lambda\left(\varepsilon\Lambda x - y\right) > 0, x \leqslant 0 \right\}
$$

and

$$
\Omega_{\varepsilon 2} = \left\{ (x, y) \in \mathbb{R}^2 : 1 + \varepsilon\Lambda\left(\varepsilon\Lambda x + y\right) > 0, x \geqslant 0 \right\},
$$

with the reset map

$$
\begin{array}{rccc} \mathcal{R}_\varepsilon : \Omega_\varepsilon \cap \{x = 0\} & \longrightarrow & \Omega_\varepsilon \cap \{x = 0\} \\ (0, y) & \longmapsto & (0, \delta_\varepsilon(y)), \end{array}
$$

being

$$
\delta_\varepsilon(y) = \begin{cases} \dfrac{y}{1 - \varepsilon\Lambda y} & \text{if } \ y \leqslant 0, \\[2ex] \dfrac{y}{1 + \varepsilon\Lambda y} & \text{if } \ y > 0. \end{cases}
$$

Note that system (3.23) is a perturbation of system (3.22).

The previous system (3.23) is a planar hybrid PWL system of the family (2.6). Therefore, the theory that has been developed in Chapter 2 can be applied here. Specifically, the Melnikov function M defined in (2.25) particularized for system (3.23) is given by

$$
M(y_0; \boldsymbol{\mu}) = -\frac{2}{y_0} M_1(y_0; \boldsymbol{\mu}),
$$

where $M_1(y_0; \boldsymbol{\mu}) = M_c(y_0; \boldsymbol{\mu}) - y_0^3\Lambda$, with

$$
M_c(y_0; \boldsymbol{\mu}) = (\sigma^+ - \Lambda)S^+(y_0, \boldsymbol{\mu}) + \sigma^- S^-(y_0, \boldsymbol{\mu}),
$$

being

$$S^{+}(y_0;\boldsymbol{\mu}) = \text{Area}(\text{int}(\Gamma_{y_0} \cap \{(x,y) \in \mathbb{R}^2 : x > 0\})) \tag{3.24}$$

and

$$S^{-}(y_0;\boldsymbol{\mu}) = \text{Area}(\text{int}(\Gamma_{y_0} \cap \{(x,y) \in \mathbb{R}^2 : x < 0\})). \tag{3.25}$$

Finding the explicit formulas of areas (3.24) and (3.25), the function M_1 is given in expression (3.20). The proof conclude by direct application of Theorem 2.8 □

Theorem 3.9 Consider the Melnikov function

$$M_2(y_0;\boldsymbol{\mu}) = (\sigma^+ - \Lambda)G(y_0,\beta^+) + \sigma^- F(y_0,\beta^-) + y_0^3\Lambda, \tag{3.26}$$

defined for $y_0 > 0$ and $\boldsymbol{\mu} = (\sigma^-, \sigma^+, \Lambda, \beta^-, \beta^+) \in \mathbb{R}^5$, where F and G are given in (3.21). Assume that there exist $\bar{y}_0 > 0$ and $\boldsymbol{\mu}_0 \in \mathbb{R}^5$ such that $M_2(\bar{y}_0;\boldsymbol{\mu}_0) = 0$ and denote

$$q_2 = \frac{\partial M_2}{\partial y_0}(\bar{y}_0;\boldsymbol{\mu}_0).$$

Then, the following statements hold.

(a) If $q_2 \neq 0$, then the perturbed system (3.19) has a hyperbolic two-zonal invariant cone located below the planes Π^- and Π^+_ε, for ε different from zero and sufficiently small. Moreover, the invariant cone is repulsive if $\varepsilon \cdot q_2 < 0$ and attractive if $\varepsilon \cdot q_2 > 0$.

(b) If $q_2 = 0$, then there exists a function $\boldsymbol{\mu} = \boldsymbol{\mu}(\varepsilon) = \boldsymbol{\mu}_0 + O(\varepsilon)$ such that system (3.19) has a unique non-hyperbolic semi-attractive two-zonal invariant cone located below the planes Π^- and Π^+_ε, for ε different from zero and sufficiently small.

3.4 Conditions for the existence and saddle-node bifurcation of invariant cones

In this section, by using theorems 3.8 and 3.9, we give specific conditions about the parameters of the perturbed system (3.19), which allow us to guarantee the existence of two-zonal invariant cones. These results agree with those given in Theorem 2 of [25]. Furthermore, by means of the theory developed in this work, we are able to prove the conjecture done in [25] about the existence of a saddle-node bifurcation of invariant cones. In fact, we will give an explicit expression of the function which characterize the saddle-node bifurcation.

In the next section we will see that, although the results given in [21] about the existence and saddle-node bifurcation of invariant cones cannot be applied to system (3.19), we are able to extend the results given there through the theory developed in this work.

Let us begin by proving some results about the existence of invariant cones in system (3.19).

Functions M_1 and M_2 given in (3.20) and (3.26), respectively, are defined for $y_0 > 0$. However, we can extend by continuity the definition to $y_0 = 0$,

$$M_1(0, \boldsymbol{\mu}) = \frac{\pi}{(\beta^+)^3}(\sigma^+ - \Lambda) \quad \text{and} \quad M_2(0, \boldsymbol{\mu}) = \frac{\pi}{(\beta^-)^3}\sigma^-.$$

We will occasionally delete the parameter $\boldsymbol{\mu}$ in the expressions of Melnikov functions and we will denote the derivative with respect to the variable y_0 by a prime.

To start, a result about the existence of exactly two two-zonal invariant cones for the perturbed system (3.19) is stated.

Theorem 3.10 If $\Lambda \cdot \sigma^- < 0$ and $\Lambda(\sigma^+ - \Lambda) > 0$, then the perturbed system (3.19) has exactly two two-zonal invariant cones for $|\varepsilon| \neq 0$ sufficiently small, one of them above the planes Π^- and Π_ε^+ and another below them. Moreover, if $\Lambda \cdot \varepsilon < 0$, the invariant cone located above the planes is attractive and the other one is repulsive. On the contrary, if $\Lambda \cdot \varepsilon > 0$, the invariant cone located above the planes is repulsive and the other one is attractive.

Proof: To prove the existence of invariant cones we first show that functions M_1 and M_2 given in (3.20) and (3.26), respectively, possess simple zeros in $(0, +\infty)$.

The function M_1 satisfies the following properties

$$M_1(0) = \frac{\pi}{(\beta^+)^3}(\sigma^+ - \Lambda) \text{ and } \lim_{y_0 \to +\infty} M_1(y_0) = -(\text{sgn}(\Lambda))(+\infty). \tag{3.27}$$

Therefore, if $\Lambda(\sigma^+ - \Lambda) > 0$, M_1 possesses at least one zero $\bar{y}_0 > 0$. In the same way, the function M_2 satisfies

$$M_2(0) = \frac{\pi}{(\beta^-)^3}\sigma^- \text{ and } \lim_{y_0 \to +\infty} M_2(y_0) = (\text{sgn}(\Lambda))(+\infty),$$

so, if $\sigma^- \cdot \Lambda < 0$, M_2 possesses at least one zero $\bar{y}_0^* > 0$.

Taking into account Remark 3.3, the roots $\bar{y}_0$ and $\bar{y}_0^*$ must be unique and simple. Moreover, it is easy to see that

$$\text{sgn}(M_1'(\bar{y}_0)) = -\text{sgn}(\Lambda) \quad \text{and} \quad \text{sgn}(M_2'(\bar{y}_0^*)) = \text{sgn}(\Lambda),$$

so, the proof concludes by direct application of the first item of theorems 3.8 and 3.9. □

After that, we give two results about the existence of one two-zonal invariant cone above (respectively, below) the planes Π^- and Π_ε^+. We shall only prove the first theorem because the second one can be demonstrated working with the Melnikov function M_2 in the same way as we do with M_1 in the proof of the first one.

Theorem 3.11 If $\Lambda(\sigma^+ - \Lambda) > 0$ and $\Lambda(2(\sigma^- - \sigma^+) - \Lambda) > 0$, then the perturbed system (3.19) has one two-zonal invariant cone above the planes Π^- and Π_ε^+ for $|\varepsilon| \neq 0$ sufficiently small. Moreover, if $\Lambda \cdot \varepsilon < 0$, the invariant cone is attractive and if $\Lambda \cdot \varepsilon > 0$, the invariant cone is repulsive.

Proof: From (3.27), we know that if $\Lambda(\sigma^+ - \Lambda) < 0$, M_1 possesses at least one zero $\bar{y}_0 > 0$. We can now obtain $M_1'(y_0) = M_d(y_0)y_0$, where

$$M_d(y_0) = \\ -3\Lambda y_0 + \frac{2\sigma^-}{\beta^-}\sin^{-1}\frac{(\beta^-)^2 y_0}{\sqrt{1+(\beta^-)^2 y_0^2}} + \frac{2(\sigma^+ - \Lambda)}{\beta^+}\left(\pi - \sin^{-1}\frac{(\beta^+)^2 y_0}{\sqrt{1+(\beta^+)^2 y_0^2}}\right). \tag{3.28}$$

This function satisfies

$$M_d(0) = \frac{2\pi(\sigma^+ - \Lambda)}{\beta^+} \quad \text{and} \quad \lim_{y_0 \to +\infty} M_d(y_0) = -(\text{sgn}(\Lambda))(+\infty),$$

so, under the hypothesis $\Lambda(\sigma^+ - \Lambda) < 0$, M_d possesses at least one zero in $(0, +\infty)$. Moreover, it is easy to see that $M_d'(y_0)$ has at most two strictly positive zeros. Taking into account that $M_d'(0) = 2(\sigma^- - \sigma^+) - \Lambda$ and assuming $\Lambda(2(\sigma^- - \sigma^+) - \Lambda^+) > 0$, it is direct to check that M_d possesses exactly one zero in $(0, +\infty)$ and so, the zero $\bar{y}_0$ is unique and it is simple. The proof concludes by direct application of the first item of Theorem 3.8. □

Theorem 3.12 If $\Lambda \cdot \sigma^- < 0$ and $\Lambda(2(\sigma^- - \sigma^+) - \Lambda) > 0$, then the perturbed system (3.19) has one two-zonal invariant cone below the planes Π^- and Π_ε^+ for $|\varepsilon| \neq 0$ sufficiently small. Moreover, if $\Lambda \cdot \varepsilon > 0$, the invariant cone is attractive and if $\Lambda \cdot \varepsilon < 0$, the invariant cone is repulsive.

Note that, under the hypotheses of theorems 3.10, 3.11 and 3.12, it is impossible to find a saddle-node bifurcation of two-zonal invariant cones, because we know from [25] that the maximum number of two-zonal invariant cones which can appear is two. Nevertheless, if we consider $\beta^- = \beta^+$, we are able to prove the conjecture about the existence of saddle-node bifurcation of two-zonal invariant cones given in [25].

Specifically, the hypotheses in Conjecture 3 of [25] were $t^+ \neq t^-$ and $\lambda^+ \neq \lambda^-$, which in the case of system (3.19) are translated into $\varepsilon(2(\sigma^- - \sigma^+) - \Lambda) \neq 0$ and $\varepsilon\Lambda \neq 0$, hypotheses that are fulfilled in the following two results about the existence of a saddle-node bifurcation of invariant cones.

In the first result, the invariant cones arise above the planes Π^- and Π_ε^+, and in the next one, below them. Only the first theorem is proven because the second one can be demonstrated simply by working with the Melnikov function M_2 in the same way as we do with M_1 in the proof of the first one.

Theorem 3.13 Suppose $\beta^- = \beta^+$, $\Lambda + \sigma^- - \sigma^+ \neq 0$ and $\Lambda(2(\sigma^- - \sigma^+) - \Lambda) > 0$. Let the function $SN_1(\Lambda, \sigma^+, \sigma^-, \beta^+)$ be defined as

$$SN_1(\Lambda, \sigma^+, \sigma^-, \beta^+) = \pi(\Lambda - \sigma^+) + \frac{3}{2}\sqrt{\Lambda(2(\sigma^- - \sigma^+) - \Lambda)} - (\Lambda + \sigma^- - \sigma^+)\sin^{-1}\frac{(\beta^+)^2\tilde{y}_0}{\sqrt{1 + (\beta^+)^2(\tilde{y}_0)^2}}, \tag{3.29}$$

where

$$\tilde{y}_0 = \sqrt{\frac{2(\sigma^- - \sigma^+) - \Lambda}{(\beta^+)^2\Lambda}}. \tag{3.30}$$

If there exist $\bar{\Lambda}, \bar{\sigma}^+, \bar{\sigma}^-, \bar{\beta}^+ \in \mathbb{R}$ for which $SN_1(\bar{\Lambda}, \bar{\sigma}^+, \bar{\sigma}^-, \bar{\beta}^+) = 0$, then there exist functions $\Lambda(\varepsilon) = \bar{\Lambda} + O(\varepsilon)$, $\sigma^+(\varepsilon) = \bar{\sigma}^+ + O(\varepsilon)$ and $\sigma^-(\varepsilon) = \bar{\sigma}^- + O(\varepsilon)$, defined for $|\varepsilon|$ sufficiently small, so that the perturbed system (3.19) with $\Lambda = \Lambda(\varepsilon)$, $\sigma^+ = \sigma^+(\varepsilon)$, $\sigma^- = \sigma^-(\varepsilon)$ and $\beta^- = \beta^+ = \bar{\beta}^+$, has exactly one two-zonal invariant cone above the planes Π^- and Π_ε^+, which is non-hyperbolic and semi-attractive, for $|\varepsilon| \neq 0$ and small enough.

Proof: We are looking for a strictly positive double zero of the Melnikov function M_1, i.e., a solution of the system

$$\begin{cases} M_1(y_0) = 0, \\ M_1'(y_0) = 0. \end{cases} \tag{3.31}$$

This system is equivalent, for $y_0 \neq 0$, to

$$\begin{cases} M_1(y_0) = 0, \\ M_d(y_0) = 0, \end{cases} \tag{3.32}$$

where $M_d(y_0)$ is given in (3.28). Imposing $\beta^- = \beta^+$, we can find the value of $\sin^{-1}((\beta^+)^2 y_0/(1 + (\beta^+)^2 y_0^2)^{-1/2})$ from the first equation of (3.32), provided that $\Lambda + \sigma^+ - \sigma^- \neq 0$. Substituting it in the second equation of (3.32), we arrive to

$$y_0 \left(\frac{2(\sigma^- - \sigma^+) - \Lambda(1 + (\beta^+)^2 y_0^2)}{1 + (\beta^+)^2 y_0^2} \right) = 0.$$

The unique strictly positive solution of the previous equation is given by

$$\tilde{y}_0 = \sqrt{\frac{2(\sigma^- - \sigma^+) - \Lambda}{(\beta^+)^2 \Lambda}},$$

provided that $\Lambda(2(\sigma^- - \sigma^+) - \Lambda) > 0$. Now, substituting $y_0 = \tilde{y}_0$ in the first equation of (3.31), we have that M_1 has a positive double zero when

$$\begin{aligned} &\pi(\Lambda - \sigma^+) + \\ &\frac{3}{2}\sqrt{\Lambda(2(\sigma^- - \sigma^+) - \Lambda)} - (\Lambda + \sigma^- - \sigma^+) \sin^{-1} \frac{(\beta^+)^2 \tilde{y}_0}{\sqrt{1 + (\beta^+)^2 (\tilde{y}_0)^2}} = 0. \end{aligned}$$

The proof concludes by direct application of the second item of Theorem 3.8. □

Theorem 3.14 Suppose $\beta^- = \beta^+$, $\Lambda + \sigma^- - \sigma^+ \neq 0$ and $\Lambda(2(\sigma^- - \sigma^+) - \Lambda) > 0$. Let the function $SN_2(\Lambda, \sigma^+, \sigma^-, \beta^+)$ be defined as

$$\begin{aligned} &SN_2(\Lambda, \sigma^+, \sigma^-, \beta^+) = \\ &\pi\sigma^- + \frac{3}{2}\sqrt{\Lambda(2(\sigma^- - \sigma^+) - \Lambda)} - (\Lambda + \sigma^- - \sigma^+) \sin^{-1} \frac{(\beta^+)^2 \tilde{y}_0}{\sqrt{1 + (\beta^+)^2 (\tilde{y}_0)^2}}, \end{aligned}$$

where $\tilde{y}_0$ is given in (3.30). If there exist $\bar{\Lambda}, \bar{\sigma}^+, \bar{\sigma}^-, \bar{\beta}^+ \in \mathbb{R}$ for which $SN_2(\bar{\Lambda}, \bar{\sigma}^+, \bar{\sigma}^-, \bar{\beta}^+) = 0$, then there exist functions $\Lambda(\varepsilon) = \bar{\Lambda} + O(\varepsilon)$, $\sigma^+(\varepsilon) = \bar{\sigma}^+ + O(\varepsilon)$ and $\sigma^-(\varepsilon) = \bar{\sigma}^- + O(\varepsilon)$, defined for $|\varepsilon|$ sufficiently small, so that the perturbed system (3.19) with $\Lambda = \Lambda(\varepsilon)$, $\sigma^+ = \sigma^+(\varepsilon)$, $\sigma^- = \sigma^-(\varepsilon)$ and $\beta^- = \beta^+ = \bar{\beta}^+$, has exactly one two-zonal invariant cone below the planes Π^- and Π^+_ε, which is non-hyperbolic and semi-attractive, for $|\varepsilon| \neq 0$ and small enough.

To illustrate the behavior of the Melnikov functions described in Theorem 3.13, we represent in Fig. 3.4 the Melnikov function M_1 under the hypotheses of this theorem. Note that this situation corresponds to a saddle-node bifurcation of invariant cones. If we change a little bit the values of the parameters of the perturbed system (3.19), the Melnikov function M_1 can have exactly two simple roots, which give us two two-zonal hyperbolic invariant cones or, in the opposite way, it has not roots, which corresponds to the non-existence of invariant cones of the perturbed system (3.19) in a neighborhood of the invariant cone of the unperturbed system (3.16) corresponding to the periodic orbit $\Gamma_{\tilde{y}_0}$.

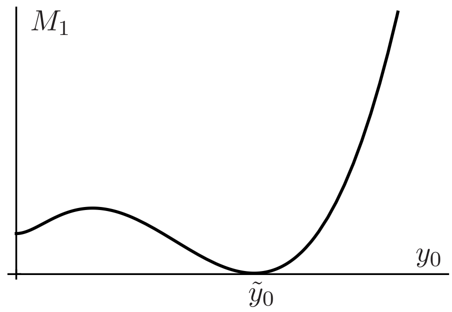


Figure 3.4: The Melnikov function M_1 under the hypotheses of Theorem 3.13 when SN_1 defined in (3.29) vanishes.

3.5 Bifurcations of invariant cones which are tangent to the separation plane

In this last section, we will focus our attention on the existence of invariant cones which arise from the invariant cones of the unperturbed system (3.16) which are tangent to the separation plane

$x = 0$. A similar situation was analyzed in [21]. There, the following parameters were considered,

$$\eta = 3\lambda^+ - \lambda^- - 2\alpha^- \quad \text{and} \quad \tilde{\eta} = 3\lambda^- - \lambda^+ - 2\alpha^+. \tag{3.33}$$

For the perturbed system (3.19), these quantities are

$$\eta = \varepsilon(3\Lambda^- - 2\sigma^-) \quad \text{and} \quad \tilde{\eta} = -\varepsilon(\Lambda + 2\sigma^+). \tag{3.34}$$

It is a necessary condition for the application of the results given in [21], that these parameters were different from zero in system (3.16), which is not fulfilled in our case. Nevertheless, we are able to prove similar results about the existence and bifurcation of invariant cones, by applying the theory developed in this chapter, as we are going to see in detail in the following lines.

Let us begin by analyzing the existence of tangent cones for system (3.16). Note that there are two tangent cones, one on each zone of linearity. We will denote the tangent cone contained in the half-space $\{x \leq 0\}$ as $\mathcal{C}_0^-$ and the tangent cone contained in the half-space $\{x \geq 0\}$ as $\mathcal{C}_0^+$. The invariant cone $\mathcal{C}_0^+$ corresponds to the periodic orbit of planar CPWL unperturbed system (3.22) which is tangent to the separation straight line $x = 0$. A similar comment can be done for the invariant cone $\mathcal{C}_0^-$.

To begin with, we present two results about the existence of one two-zonal invariant cone above (respectively, below) the planes Π^- and Π_ε^+. Only the first theorem will be proven because the second one can be demonstrated working with the Melnikov function M_2.

Theorem 3.15 Suppose $(\sigma^+ - \Lambda)(2\sigma^- - 3\Lambda) < 0$ and $|\sigma^+ - \Lambda|$ is sufficiently small. Then, the perturbed system (3.19) has one two-zonal invariant cone located above the planes Π^- and Π_ε^+, near to $\mathcal{C}_0^+$, for $|\varepsilon| \neq 0$ and small enough. Moreover, if $(\sigma^+ - \Lambda)\varepsilon < 0$, the invariant cone is attractive and if $(\sigma^+ - \Lambda)\varepsilon > 0$, the invariant cone is repulsive.

Proof: Consider the function

$$M_r(y_0) = \frac{M_1(y_0)}{S^+(y_0)}. \tag{3.35}$$

Due to function $S^+(y_0)$ is strictly positive for $y_0 \geq 0$, see (3.24), function M_r possesses the same roots as M_1, with the same multiplicity.

Assume that all the parameters of the system are fixed, except σ^+. We define the following function

$$H(s, \sigma^+) = M_r(s^{1/3}, \sigma^+),$$

where we have assumed that the Melnikov reduced function (3.35) is function of σ^+ too. Then, assuming $2\sigma^- - 3\Lambda \neq 0$, one obtains

$$H(0, \Lambda) = 0 \quad \text{and} \quad \frac{\partial H}{\partial s}(0, \Lambda) = (\beta^+)^3(2\sigma^- - 3\Lambda)/(3\pi) \neq 0.$$

By applying the Implicit Function Theorem we deduce that there exists a function f which is defined in a neighborhood U of $(0, \Lambda)$, such that $H(f(\sigma^+), \sigma^+) = 0$ for all $\sigma^+ \in U$, i.e., the equation $M_r(y_0) = 0$ has one solution $\hat{y}_0 = (f(\sigma^+))^{1/3}$ provided that $|\sigma^+ - \Lambda|$ is small enough. Moreover, it is easy to see that sgn$(f(\sigma^+)) = -$sgn$((\sigma^+ - \Lambda)(2\sigma^- - 3\Lambda))$. So, the solution $\hat{y}_0$ is positive when $(\sigma^+ - \Lambda)(2\sigma^- - 3\Lambda) < 0$. Therefore, there exists a positive simple root $\hat{y}_0 > 0$ of the Melnikov function M_1 near to $y_0 = 0$. On the other hand, sgn$(M_1'(\hat{y}_0))$=sgn$(M_r'(\hat{y}_0))$ = sgn$(2\sigma^- - 3\Lambda) \neq 0$. The proof concludes by direct application of the first item of Theorem 3.8. □

Theorem 3.16 Suppose $\sigma^-(2\sigma^+ + \Lambda) < 0$ and $|\sigma^-|$ is sufficiently small. Then, the perturbed system (3.19) has one two-zonal invariant cone located below the planes Π^- and Π_ε^+, near to $\mathcal{C}_0^-$, for $|\varepsilon| \neq 0$ and small enough. Moreover, if $\sigma^- \cdot \varepsilon < 0$, the invariant cone is attractive and if $\sigma^- \cdot \varepsilon > 0$, the invariant cone is repulsive.

Finally, by adding some more hypotheses to those considered in theorems 3.15 and 3.16, respectively, we establish two results about a saddle-node bifurcation of two-zonal invariant cones. We only prove the first theorem because the second one can be proven just working with the Melnikov function M_2. To do that, we need a previous result about the roots of a polynomial of degree 5. The result is given in Proposition 6 of [43].

Lemma 3.17 Consider the function

$$P(x) = b_0 + b_3x^3 + b_5x^5, \tag{3.36}$$

with $b_5 \neq 0$. The nonnegative solutions of equation

$$P(x) = 0, \tag{3.37}$$

behave as follows.

1. For $b_0 = 0$ the equation has always the zero solution, it has not positive solution if $b_3 \cdot b_5 > 0$ and it has the positive solution $x = \sqrt{-b_3/b_5} > 0$ when $b_3 \cdot b_5 < 0$.

2. For $b_3 = 0$ the equation has not positive solution if $b_0 \cdot b_5 > 0$ and has one positive solution for $b_0 \cdot b_5 < 0$.

3. If $b_0 \cdot b_5 > 0$ and $b_0 \cdot b_3 > 0$ there are not positive solutions.

4. If $b_0 \cdot b_5 < 0$ there is only one positive solution.

5. If $b_0 \cdot b_5 > 0$ and $b_0 \cdot b_3 < 0$, we can define in the parameter plane (b_0, b_3) the expression given by

$$h^*(b_0, b_3) = b_0 + \frac{2}{5} b_3 \left(-\frac{3b_3}{5b_5} \right)^{3/2}, \tag{3.38}$$

so that,

   (a) If $b_0 h^*(b_0, b_3) < 0$, then equation (3.37) has two positive solutions.

   (b) If $b_0 h^*(b_0, b_3) = 0$, then equation (3.37) has only one positive solution, namely $x = \sqrt{-3b_3/5b_5} > 0$.

   (c) If $b_0 h^*(b_0, b_3) > 0$, the equation (3.37) has not positive solutions.

Theorem 3.18 Assume $|\sigma^+ - \Lambda|$ and $|2\sigma^- - 3\Lambda|$ are sufficiently small, $(\sigma^+ - \Lambda)(2\sigma^- - 3\Lambda) < 0$ and $(\sigma^+ - \Lambda)\sigma^- < 0$. Then, there exists a function $SN_{01}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-)$ whose local expression is given by

$$\begin{aligned} &SN_{01}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) = \\ &= \sigma^+ - \Lambda + 2(\beta^+)^3 \frac{2\sigma^- - 3\Lambda}{15\pi} \left(\frac{3(2\sigma^- - 3\Lambda)}{2(\beta^-)^2\sigma^- + 5(\beta^+)^2(2\sigma^- - 3\Lambda)} \right)^{3/2} + \dots \end{aligned}$$

such that the following statements hold.

(a) If $(\sigma^+ - \Lambda)SN_{01}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) < 0$, then system (3.19) has two two-zonal invariant cones above the planes Π^- and Π^+_ε, near to $\mathcal{C}_0^+$, for $|\varepsilon| \neq 0$ and small enough. One of them is attractive and the other one is repulsive.

(b) If there exist $\bar{\Lambda}$, $\bar{\sigma}^+$, $\bar{\sigma}^-$, $\bar{\beta}^+$, $\bar{\beta}^- \in \mathbb{R}$ for which $SN_{01}(\bar{\Lambda}, \bar{\sigma}^+, \bar{\sigma}^-, \bar{\beta}^+, \bar{\beta}^-) = 0$, then there exist functions $\Lambda(\varepsilon) = \bar{\Lambda} + O(\varepsilon)$, $\sigma^+(\varepsilon) = \bar{\sigma}^+ + O(\varepsilon)$ and $\sigma^-(\varepsilon) = \bar{\sigma}^- + O(\varepsilon)$, defined for $|\varepsilon|$ sufficiently small, such that the perturbed system (3.19) with $\Lambda = \Lambda(\varepsilon)$, $\sigma^+ = \sigma^+(\varepsilon)$, $\sigma^- = \sigma^-(\varepsilon)$, $\beta^+ = \bar{\beta}^+$ and $\beta^- = \bar{\beta}^-$, has exactly one two-zonal invariant cone above the planes Π^- and Π^+_ε, near to $\mathcal{C}_0^+$, which is non-hyperbolic and semi-attractive, for $|\varepsilon| \neq 0$ and small enough.

(c) If $(\sigma^+ - \Lambda)SN_{01}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) > 0$, then system (3.19) has no two-zonal invariant cones above the planes Π^- and Π^+_ε, for $|\varepsilon| \neq 0$ and small enough, near to $\mathcal{C}_0^+$.

Proof: As we have noted in the proof of Theorem 3.15, function M_r defined in (3.35) possesses the same roots as M_1 with the same multiplicity. This function can be written in a neighborhood of $y_0 = 0$ into the form

$$M_r(y_0) =$$
$$\sigma^+ - \Lambda + (\beta^+)^3 \left(\frac{2\sigma^- - 3\Lambda}{3\pi} y_0^3 - \frac{2(\beta^-)^2\sigma^- + 5(\beta^+)^2(2\sigma^- - 3\Lambda)}{15\pi} y_0^5 \right) + O(y_0^6).$$

Therefore, if $(\sigma^+ - \Lambda, 2\sigma^- - 3\Lambda) \simeq (0, 0)$, then we can find zeros of M_r for $|y_0|$ sufficiently small finding the roots of

$$Q(y_0) = \sigma^+ - \Lambda + \frac{(\beta^+)^3(2\sigma^- + 3\Lambda)}{3\pi} y_0^3 - \frac{(\beta^+)^3(2(\beta^-)^2\sigma^- + 5(\beta^+)^2(2\sigma^- - 3\Lambda))}{15\pi} y_0^5. \quad (3.39)$$

By applying statement 5 of Lemma 3.17 and taking into account that for the polynomial Q given in (3.39),

$$b_0 = \sigma^+ - \Lambda, \qquad b_3 = \frac{(\beta^+)^3(2\sigma^- - 3\Lambda)}{3\pi}$$

and

$$b_5 = -\frac{(\beta^+)^3(2(\beta^-)^2\sigma^- + 5(\beta^+)^2(2\sigma^- - 3\Lambda))}{15\pi},$$

it follows that for $(\sigma^+ - \Lambda)(2\sigma^- + 3\Lambda) < 0$ and $(\sigma^+ - \Lambda) < 0$, there exits a function given by

$$h(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) =$$
$$= \sigma^+ - \Lambda + 2(\beta^+)^3 \frac{2\sigma^- - 3\Lambda}{15\pi} \left(\frac{3(2\sigma^- - 3\Lambda)}{2(\beta^-)^2\sigma^- + 5(\beta^+)^2(2\sigma^- - 3\Lambda)} \right)^{3/2},$$

such that,

(a) If $(\sigma^+ - \Lambda)h(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) < 0$, then polynomial Q has two positive roots.

(b) If $h(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) = 0$, then polynomial Q has only one positive root.

(c) If $(\sigma^+ - \Lambda)h(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) > 0$, the polynomial Q has not positive roots.

The proof concludes by direct application of Theorem 3.8. □

Theorem 3.19 Assume $|\sigma^-|$ and $|2\sigma^+ + \Lambda|$ are sufficiently small, $\sigma^- \cdot (2\sigma^+ + \Lambda) < 0$ and $(\sigma^+ - \Lambda) \cdot \sigma^- < 0$. Then, there exists a function $SN_{02}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-)$ whose local expression is

$$SN_{02}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) =$$
$$= \sigma^- + 2(\beta^-)^3 \frac{2\sigma^+ + \Lambda}{15\pi} \left(\frac{3(2\sigma^+ + \Lambda)}{5(\beta^-)^2(2\sigma^+ + \Lambda) + 2(\beta^+)^2(\sigma^+ - \Lambda)} \right)^{3/2} + \ldots$$

such that the following statements hold.

(a) If $\sigma^- \cdot SN_{02}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) < 0$, then system (3.19) has two two-zonal invariant cones below the planes Π^- and Π^+_ε, near to $\mathcal{C}_0^-$, for $|\varepsilon| \neq 0$ and small enough. One of them is attractive and the other one is repulsive.

(b) If there exist $\bar{\Lambda}$, $\bar{\sigma}^+$, $\bar{\sigma}^-$, $\bar{\beta}^+$, $\bar{\beta}^- \in \mathbb{R}$ for which $SN_{02}(\bar{\Lambda}, \bar{\sigma}^+, \bar{\sigma}^-, \bar{\beta}^+, \bar{\beta}^-) = 0$, then there exist functions $\Lambda(\varepsilon) = \bar{\Lambda} + O(\varepsilon)$, $\sigma^+(\varepsilon) = \bar{\sigma}^+ + O(\varepsilon)$ and $\sigma^-(\varepsilon) = \bar{\sigma}^- + O(\varepsilon)$, defined for $|\varepsilon|$ sufficiently small, such that the perturbed system (3.19) with $\Lambda = \Lambda(\varepsilon)$, $\sigma^+ = \sigma^+(\varepsilon)$, $\sigma^- = \sigma^-(\varepsilon)$, $\beta^+ = \bar{\beta}^+$ and $\beta^- = \bar{\beta}^-$, has exactly one two-zonal invariant cone below the planes Π^- and Π^+_ε, near to $\mathcal{C}_0^-$, which is non-hyperbolic and semi-attractive, for $|\varepsilon| \neq 0$ and small enough.

(c) If $\sigma^- \cdot SN_{02}(\Lambda, \sigma^+, \sigma^-, \beta^+, \beta^-) > 0$, then system (3.19) has no two-zonal invariant cones below the planes Π^- and Π^+_ε, near to $\mathcal{C}_0^-$, for $|\varepsilon| \neq 0$ and small enough.

It is worth mentioning that theorems 3.15 and 3.16 describe the existence of invariant cones in the perturbed system (3.19). We can say that these periodic orbits arise, under the hypothesis of Theorem 3.15 (respectively, 3.16), from the invariant cone tangent to the separation plane $x = 0$ which is in the half-space $x \geq 0$ (respectively $x \leq 0$). The appearance of this invariant cone is known as the focus-center-limit cycle bifurcation. A generic situation of this bifurcation is described in [21]. However, it is not possible to apply the results given in [21] to our unperturbed system (3.16) because the coefficients η and $\tilde{\eta}$ given in expression (3.34), which characterize the bifurcation, are zero for system (3.16). We can say that theorems 3.15 and 3.16 give conditions for spreading out in one direction the focus-center-limit cycle bifurcation when $\eta = 0$ (respectively, $\tilde{\eta} = 0$), and theorems 3.18 and 3.19 study the degeneration of this bifurcation.

Chapter 4

Periodic Orbits for Perturbations of Three-Dimensional Non-Controllable Continuous Piecewise Linear Systems

In this chapter, we consider the existence of periodic orbits in a class of three-dimensional CPWL systems with two zones.

In order to analyze a family of dynamical systems, it is usual to begin detecting the elements of the family which satisfy some non-generic property (in our case, the lack of controllability). Next, the dynamical behavior of this non-generic system is studied. After that, if it is possible, some systems in the family are described as perturbations of the non-generic system studied, and then, the dynamical behavior of the perturbed systems is analyzed [8, 30, 70, 87, 94].

The non-generic system object of study in this chapter is a non-controllable 2CPWL$_3$ system which possesses a continuum of periodic orbits. As it has been pointed out in the previous chapters, if we perturb a planar differential system having continuum of periodic orbits, we can think about the number and positions of the periodic orbits that persist under the perturbation. To solve this problem in the planar case, we find the Melnikov theory, as we have been studying in the previous chapters.

In the three-dimensional systems object of study, 2CPWL$_3$ systems, we are going to extend the ideas of the Melnikov theory for planar systems to dimension three, by defining a function whose roots will provide us the number and position of the periodic orbits that survive after the perturbation.

The chapter is structured as follows. In a first section, the non-controllable 2CPWL$_3$ system is introduced. Subsequently, in Sec. 4.2, we perform a perturbation which makes the system controllable. After that, in Sec. 4.3 we construct a suitable Melnikov function. The following section is devoted to the analysis of the Melnikov function. Finally, in the last section we state some results about the existence and stability of periodic orbits and we perform a bifurcation analysis.

Some of the main results of this chapter are published in [17].

4.1 The unperturbed system

We consider 2CPWL$_3$ systems. To be able to adapt the ideas of the Melnikov theory, we need a 2CPWL$_3$ system having a continuum of periodic orbits. From the analysis done in [22, 23] the appropriate candidate is a non-controllable observable system of the family (1.7). By imposing the existence of periodic orbits to systems of the form (1.7), (that is, $\alpha = 0$) and doing an appropriate change of variable we obtain the following partially decoupled system

$$\begin{cases} \dot{x} = \lambda^{\nabla} x - y, \\ \dot{y} = z, \\ \dot{z} = 1 - y, \end{cases} \tag{4.1}$$

with $\lambda^+, \lambda^- \in \mathbb{R}$.

Moreover, coefficient matrices in both linear zones share the pair of complex eigenvalues $\pm\mathrm{i}$. The matrix of the zone $x < 0$ possesses the real eigenvalue λ^- and the matrix of the zone $x > 0$ has the real eigenvalue λ^+.

To start, the solution of the linear system

$$\begin{cases} \dot{y} = z, \\ \dot{z} = 1 - y, \end{cases}$$

with initial condition

$$(y(0), z(0)) = (1 + r\cos(\theta_0), r\sin(\theta_0)), \quad \text{being} \quad r \geqslant 0 \quad \text{and} \quad \theta_0 \in [0, 2\pi),$$

is given by

$$y(t) = 1 + r\cos(t - \theta_0), \qquad z(t) = r\sin(t - \theta_0).$$

Therefore, the cylinders of equation

$$(y-1)^2 + z^2 = r^2, \quad \text{with} \quad r \geq 0 \tag{4.2}$$

are invariant manifolds for system (4.1).

Taking $\theta = t - \theta_0$ as a new independent variable, we obtain

$$y(\theta) = 1 + r\cos\theta, \qquad z(\theta) = r\sin\theta. \tag{4.3}$$

The substitution of (4.3) in the first equation of (4.1) drives us to the following non-autonomous one-dimensional reduced equation

$$\frac{dx}{d\theta} = \begin{cases} \lambda^+ x - 1 - r\cos\theta, & \text{if } x \geq 0, \\ \lambda^- x - 1 - r\cos\theta, & \text{if } x < 0. \end{cases} \tag{4.4}$$

This equation collects the dynamical behavior of the three-dimensional system (4.1). It reminds us to a Ricatti equation with periodic coefficients [53].

The analysis done in [23] about the equation (4.4) lets us state the following theorem about system (4.1).

Theorem 4.1 System (4.1) satisfies the following properties.

(a) The periodic orbits, if they exist, have a period of 2π.

(b) If $\lambda^+ \leq 0$ and $\lambda^- \geq 0$, system (4.1) has neither equilibrium points nor periodic orbits.

(c) If $\lambda^- < 0$ and $\lambda^+ \leq 0$ (respectively, $\lambda^- \geq 0$ and $\lambda^+ > 0$), then system (4.1) has a unique equilibrium point $(\bar{x}, \bar{y}, \bar{z}) = (1/\lambda^-, 1, 0)$, (respectively, $(\bar{x}, \bar{y}, \bar{z}) = (1/\lambda^+, 1, 0)$) and an unbounded continuum of periodic orbits.

(d) If $\lambda^- < 0$ y $\lambda^+ > 0$, then system (4.1) has exactly two equilibrium points $(\bar{x}, \bar{y}, \bar{z}) = (1/\lambda^-, 1, 0)$ and $(\bar{x}, \bar{y}, \bar{z}) = (1/\lambda^+, 1, 0)$ and a bounded continuum of periodic orbits. Moreover, the system possesses a heteroclinic orbit of equation $\{(x, 1, 0) : 1/\lambda^- < x < 1/\lambda^+\}$ joining both equilibrium points.

The properties stated in Theorem 4.1 let us draw the bifurcation diagram of Fig. 4.1. The lines $\lambda^+ = 0$ and $\lambda^- = 0$ are bifurcation straight-lines of system (4.1). There, the equilibrium points target to infinity, where they disappear and the structure of the periodic orbits changes.

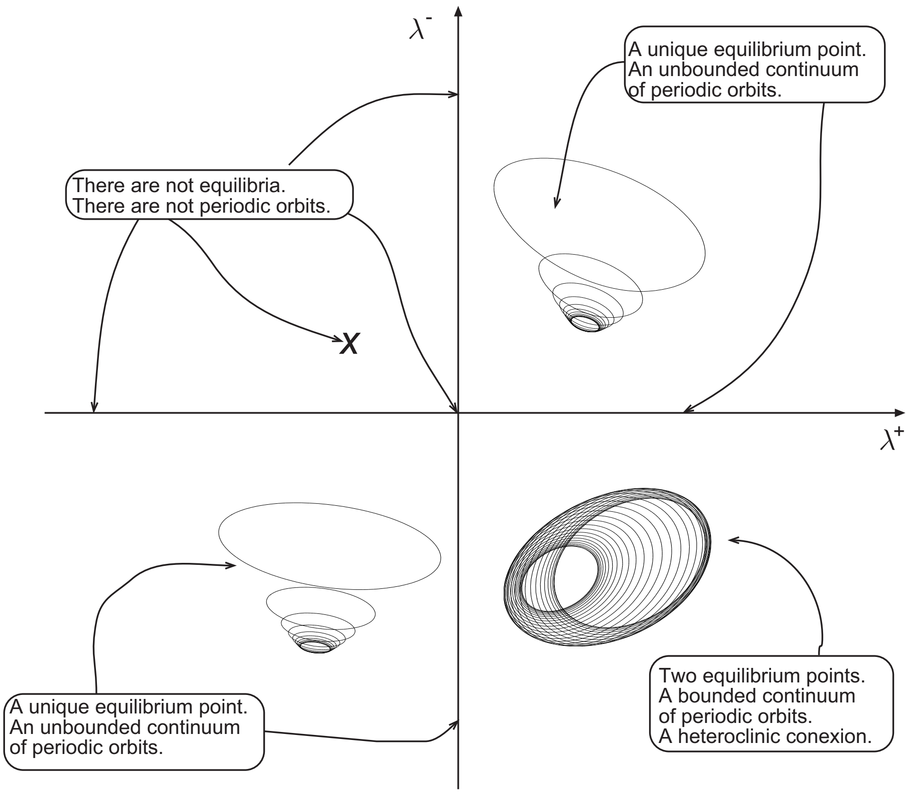
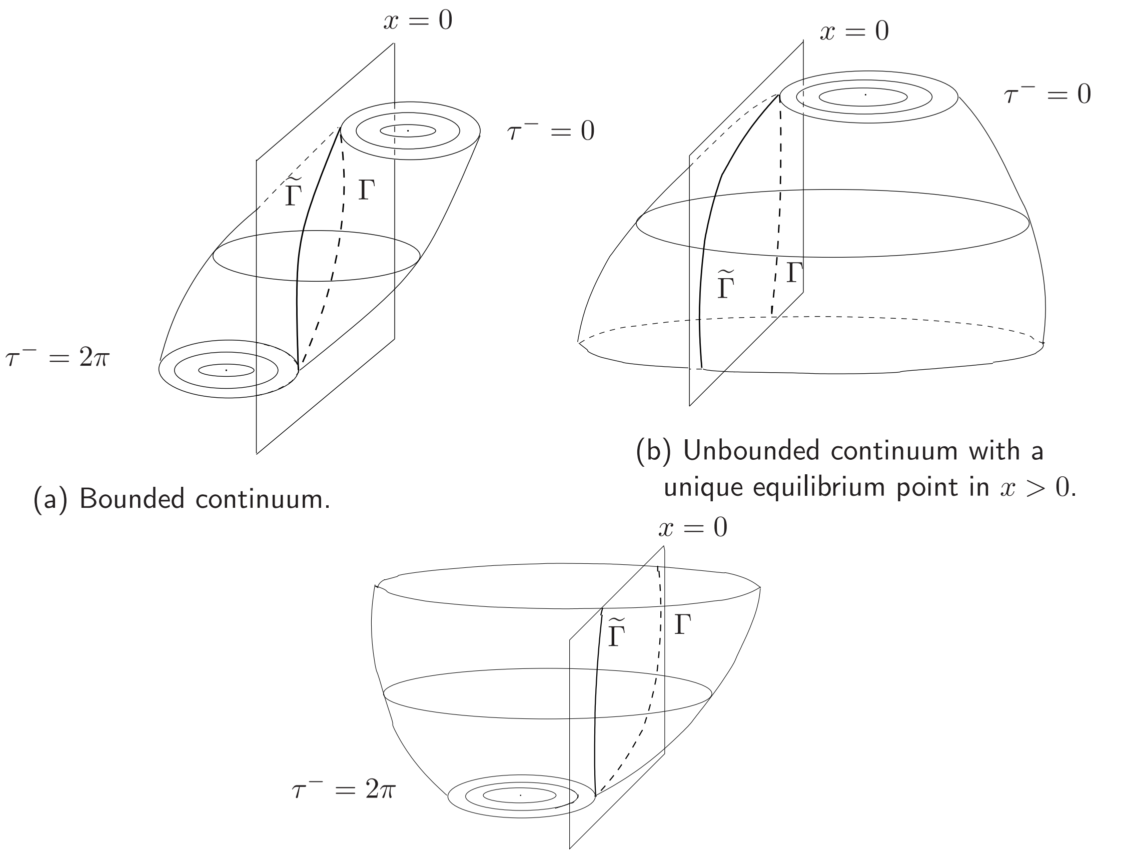


Figure 4.1: Bifurcation diagram of system (4.1).

For the development which will be explored later, it is important to describe some other properties of system (4.1). We show these properties graphically in Fig. 4.2.

We know from Theorem 4.1 that the unperturbed system (4.1) has one invariant manifold foliated by periodic orbits in the cases $\lambda^- < 0$, $\lambda^+ \leq 0$ (resp. $\lambda^- \geq 0$, $\lambda^+ > 0$) and $\lambda^- < 0$, $\lambda^+ > 0$. Consider one of these situations. For the sake of simplicity, we work in the polar coordinates introduced in (4.3). We will focus our attention on the periodic orbits which have points in common with the separation plane. Each one of these periodic orbits, except those that have a non-transversal intersection with the separation plane, intersects this plane at two points, $(0, \theta_0, r_0)$, $(0, \theta_1, r_0)$, where $\theta_0 \in I_0 \subset (0, 2\pi)$, $\theta_1 \in I_1 \subset (0, 2\pi)$, with I_0, I_1 open intervals. As a consequence, the continuum intersects the separation plane in two curves which can be parameterized as

$$\Gamma \equiv r = \hat{r}(\theta_0), \theta_0 \in \bar{I}_0 \text{ and } \widetilde{\Gamma} \equiv r = \widetilde{r}(\theta_1), \theta_1 \in \bar{I}_1$$



(a) Bounded continuum.

(b) Unbounded continuum with a unique equilibrium point in $x > 0$.

(c) Unbounded continuum with a unique equilibrium point in $x < 0$.

Figure 4.2: Continuum of periodic orbits of the unperturbed system and its intersection with the separation plane for cases explained in Theorem 4.1.

with $1 + \hat{r}(\theta_0)\cos\theta_0 \geq 0$ and $1 + \widetilde{r}(\theta_1)\cos\theta_1 \leq 0$, where $\bar{I}_0$ and $\bar{I}_1$ are, respectively, the intervals I_0 and I_1 by adding it some of their endpoints, depending on the properties of the chosen continuum, as it will be explained in a few lines. Therefore, for the unperturbed system, each point of the curve $(\theta_0, \hat{r}(\theta_0))$ with $\theta_0 \in \bar{I}_0$, is a fixed point of the Poincaré map.

Let $\bar{\theta}_0$ be a point in $\bar{I}_0$. The periodic orbit of system (4.1) with initial condition $(0, \bar{\theta}_0, \hat{r}(\bar{\theta}_0))$ has a left half-period $\tau^-(\bar{\theta}_0)$ and a right half-period $\tau^+(\bar{\theta}_0) = 2\pi - \tau^-(\bar{\theta}_0)$ because, as it is stated in Theorem 4.1, every periodic orbit of the unperturbed system has a period of 2π. Moreover, the half-period of each periodic orbit is unique and allows us to parameterize the curve $r = \hat{r}(\theta_0)$, $\theta_0 \in \bar{I}_0$

as

$$\Gamma \equiv (\theta_0(\tau^-), r(\tau^-)), \tag{4.5}$$

with τ^- in a subinterval I of $[0, 2\pi]$. The corresponding periodic orbit of the continuum will be denoted by χ_{τ^-}.

On the one hand, if $\lambda^- < 0$ and $\lambda^+ > 0$, we know from Theorem 4.1 that system (4.1) has two equilibrium points, one in zone $x > 0$ and the other in zone $x < 0$. These equilibria are surrounded by periodic orbits found in a plane, and in the corresponding half-space. The last two of these periodic orbits living in only one zone, have tangential intersection with the separation plane, see Fig. 4.2 (a). The one which surrounds the equilibrium point in $x \leq 0$ has left half-period $\tau^- = 2\pi$ $(\tau^+ = 0)$ and the one which surrounds the equilibrium point in $x > 0$ has left half-period $\tau^- = 0$ $(\tau^+ = 2\pi)$. Thus, the interval of definition where is located the parameter τ^- for the description of the curve Γ is the interval $[0, 2\pi]$, see [17].

On the other hand, if $\lambda^+ > 0$ and $\lambda^- \geq 0$, system (4.1) possesses an unbounded continuum of periodic orbits and a unique equilibrium point in the zone $x > 0$. This point is locally surrounded by periodic orbits living in a plane and are in the zone $x > 0$, see Fig. 4.2 (b). The last one of these one-zonal periodic orbits, touches tangentially the separation plane, and we can say that its left half-period is $\tau^- = 0$ $(\tau^+ = 2\pi)$. Therefore, the parameter which defines the curve Γ is located in an interval of the form $I = [0, \tilde{\tau}^-)$, where $\tilde{\tau}^-$ will be determined in the following result.

Analogously, if $\lambda^- < 0$ and $\lambda^+ \leq 0$, system (4.1) possesses an unbounded continuum of periodic orbits and a unique equilibrium point which is located in the zone $x < 0$. This point is locally surrounded by periodic orbits living in a plane and are in the zone $x < 0$, see Fig. 4.2 (c). The last one of these one-zonal periodic orbits, touches tangentially the separation plane, and we can say that its left half-period is $\tau^- = 2\pi$ $(\tau^+ = 0)$. Therefore, the parameter which defines the curve Γ is located in an interval of the form $I = (\tilde{\tau}^-, 2\pi]$, where $\tilde{\tau}^-$ will be determined in the following result.

Proposition 4.2 Consider the function

$$\begin{aligned} N(\tau^-) = \quad & (\lambda^- - \lambda^+)(e^{\lambda^+(\tau^- - 2\pi) + \lambda^- \tau^-} + 1) - \\ & (\lambda^- - \lambda^+)(e^{\lambda^+(\tau^- - 2\pi)} + e^{\lambda^- \tau^-}) \cos \tau^- - \\ & (1 + \lambda^- \lambda^+)(e^{\lambda^- \tau^-} - e^{\lambda^+(\tau^- - 2\pi)}) \sin \tau^-. \end{aligned} \tag{4.6}$$

If $\lambda^+ \cdot \lambda^- \geq 0$ and $\lambda^+ + \lambda^- \neq 0$ then, there exists a unique $\tilde{\tau}^- \in (0, 2\pi)$ such that

$$N(\tilde{\tau}^-) = 0. \tag{4.7}$$

Moreover, in this case the value $\tilde{\tau}$ allows us to define the interval of definition of the curve Γ.

Proof: We begin by doing $r \to +\infty$ in the reduced equation (4.4). To do that, first we perform the change of variable $X = x/r$. By doing $r \to +\infty$ we obtain the equation

$$\frac{dX}{d\theta} = \lambda^{\nabla} X - \cos\theta,$$

which renaming X in small letter, corresponds to the three-dimensional homogeneous system

$$\begin{cases} \dot{x} = \lambda^{\nabla} x - y, \\ \dot{y} = z, \\ \dot{z} = -y, \end{cases} \tag{4.8}$$

which is the homogeneous system associated to system (4.1). According to [25] this system has a unique two-zonal invariant cone which is foliated by period orbits. These periodic orbits has a period of 2π and the same left half-period $\tilde{\tau}^-$.

From Proposition 11 of [25], $\tilde{\tau}^-$ is determined as the only solution in $(0, 2\pi)$ of the system

$$\begin{cases} \lambda^- + [(\lambda^-)^2 + 1]\dfrac{e^{-\lambda^- s}\sin s}{\varphi_{-\lambda^-}(s)} = \lambda^+ - [(\lambda^+)^2 + 1]\dfrac{e^{\lambda^+(2\pi - s)}\sin(2\pi - s)}{\varphi_{\lambda^+}(2\pi - s)} \\ \lambda^- - [(\lambda^-)^2 + 1]\dfrac{e^{\lambda^- s}\sin s}{\varphi_{\lambda^-}(s)} = \lambda^+ + [(\lambda^+)^2 + 1]\dfrac{e^{-\lambda^+(2\pi - s)}\sin(2\pi - s)}{\varphi_{-\lambda^+}(2\pi - s)}, \end{cases} \tag{4.9}$$

where $\varphi_\omega(s) = 1 - e^{\omega s}(\cos s - \omega \sin s)$ is the Andronov function, see [1].

Both equations of system (4.9) are equivalents. Hence, $\tilde{\tau}^-$ is the only solution in $(0, 2\pi)$ of

$$\lambda^- + [(\lambda^-)^2 + 1]\frac{e^{-\lambda^- s}\sin s}{\varphi_{-\lambda^-}(s)} = \lambda^+ - [(\lambda^+)^2 + 1]\frac{e^{\lambda^+(2\pi - s)}\sin(2\pi - s)}{\varphi_{\lambda^+}(2\pi - s)}. \tag{4.10}$$

It is easy to see that (4.10) is equivalent to $N(s) = 0$ with N given in (4.6).

Therefore $\tilde{\tau}^-$ is given as the only solution in the interval $(0, 2\pi)$ of the equation $N(s) = 0$. □

Remark 4.3 Note that, for the bounded continuum there exists $\hat{r}$ such that there are not periodic orbits with a ratio greater than $\hat{r}$, i.e., it corresponds to the most external periodic orbit of the continuum. This value is given in Theorem 4.19 in [23]. On the corresponding cylinder, there exits a unique half-stable periodic orbit with left half-period

$$\tau_*^- = 2\lambda^+\pi/(\lambda^+ - \lambda^-). \tag{4.11}$$

For the sake of brevity, we will denote the corresponding intersection point of this periodic orbit with the curve Γ given in (4.5) as $(\widetilde{\theta}, \widetilde{r})$. It is remarkable that the corresponding continuum is compact but it is not normally hyperbolic [34].

At this point, we are able to ask about the periodic orbits of the continuum described in Theorem 4.1 which persist after a perturbation.

To analyze that, we translate to a three-dimensional piecewise linear systems family, the ideas of the Melnikov theory for planar systems.

4.2 Construction of the perturbed system

As it was analyzed in Chapter 1, under the observability hypothesis, every 2CPWL$_3$ system can be written in the Liénard's form

$$\dot{\mathbf{x}} = \begin{pmatrix} t^{\triangledown} & -1 & 0 \\ m^{\triangledown} & 0 & -1 \\ d^{\triangledown} & 0 & 0 \end{pmatrix} \mathbf{x} - \mathbf{e}_3. \tag{4.12}$$

From now on, it will be useful to write the Liénard system (4.12) in cylindrical coordinates. That is what we will do in the next proposition.

Proposition 4.4 There is a change of variables that transforms the Liénard system (4.12) into the form

$$\begin{cases} \dot{x} = t^+ x - 1 - r\cos\theta, \\ \dot{r} = ((m^+ - 1)\cos\theta + (t^+ - d^+)\sin\theta)x, & \text{if } x \geq 0, \\ \dot{\theta} = ((t^+ - d^+)\cos\theta - (m^+ - 1)\sin\theta)\dfrac{x}{r} - 1, \end{cases}$$
$$\begin{cases} \dot{x} = t^- x - 1 - r\cos\theta, \\ \dot{r} = ((m^- - 1)\cos\theta + (t^- - d^-)\sin\theta)x, & \text{if } x < 0, \\ \dot{\theta} = ((t^- - d^-)\cos\theta - (m^- - 1)\sin\theta)\dfrac{x}{r} - 1, \end{cases} \tag{4.13}$$

with $r > 0,\ \theta \in [0, 2\pi)$.

Proof: After the change of variables $Z = x - z$ and the additional change $y = 1 + r\cos\theta,\ Z = r\sin\theta,$ with $r \geq 0, \theta \in [0, 2\pi)$, system (4.12) takes the form (4.13). □

Note that the matrices of the unperturbed system (4.1) share the pair of complex conjugate eigenvalues $\pm$i. If we want to perturb system (4.1) it is natural to suppose that coefficient matrices of the perturbed system have a pair of complex conjugate eigenvalues. Then, we assume that the coefficient matrix in the right zone $x > 0$ has the eigenvalues $\lambda^+, \alpha^+ \pm \mathrm{i}\beta^+$ and the coefficient matrix in the left zone $x < 0$ has the eigenvalues $\lambda^-, \alpha^- \pm \mathrm{i}\beta^-$, with $\beta^+ \cdot \beta^- \neq 0$.

Furthermore, these eigenvalues must be near to the spectrum of the coefficient matrices of the unperturbed system (4.1), therefore, we can assume

$$\alpha^- = \varepsilon\Lambda^-, \beta^- = 1, \alpha^+ = \varepsilon\Lambda^+, \beta^+ = 1 + \varepsilon B, \tag{4.14}$$

where ε is sufficiently small and $\Lambda^-, \Lambda^+, B \in \mathbb{R}$. Note that is not restrictive assuming $\beta^- = 1$, because always exists a change of variable, affecting to the temporal variable, so that β^- can be equal to the unity.

From system (4.13) and with the choice made in (4.14) we arrive to the perturbed system

$$\dot{\mathbf{x}} = \begin{cases} \begin{pmatrix} \lambda^- x - rc_\theta - 1 \\ 0 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} 2\Lambda^- x \\ (a^- c_\theta + b^- s_\theta)x \\ (b^- c_\theta - a^- s_\theta)\frac{x}{r} \end{pmatrix} \text{ if } x < 0, \\ \begin{pmatrix} \lambda^+ x - rc_\theta - 1 \\ 0 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} 2\Lambda^+ x \\ (a^+ c_\theta + b^+ s_\theta)x \\ (b^+ c_\theta - a^+ s_\theta)\frac{x}{r} \end{pmatrix} \text{ if } x \geq 0, \end{cases} \tag{4.15}$$

where,

$$\begin{gathered} c_\theta = \cos\theta, s_\theta = \sin\theta, \\ a^- = \varepsilon(\Lambda^-)^2 + 2\Lambda^-\lambda^-, \; b^- = 2\Lambda^- - \varepsilon(\Lambda^-)^2\lambda^-, \\ a^+ = \varepsilon(\Lambda^+)^2 + 2\Lambda^+\lambda^+ + 2B + \varepsilon B^2, \\ b^+ = 2\Lambda^+ - \varepsilon(\Lambda^+)^2\lambda^+ - \varepsilon B^2\lambda^+ - 2B\lambda^+. \end{gathered} \tag{4.16}$$

It is clear that if $\varepsilon = 0$, system (4.15) is the expression in polar coordinates of the unperturbed system (4.1).

4.3 Derivation of the Melnikov function

As it has been done in the previous chapters, we will construct a Melnikov function from a Poincaré map and displacement function on. The definition of a Poincaré map and a displacement function can be done by means of the composition of the Poincaré half-maps, following a development similar to those done in Chapter 1. For this particular case, it is better to define these functions by considering cylindrical coordinates. Thus, to find periodic orbits of system (4.15) we must find roots of the displacement function

$$d(\theta, r, \varepsilon) = P(\theta, r, \varepsilon) - (\theta, r) \tag{4.17}$$

equivalently,

$$d(\theta, r, \varepsilon) = \begin{pmatrix} d_1(\theta, r, \varepsilon) \\ d_2(\theta, r, \varepsilon) \end{pmatrix} = \begin{pmatrix} P_1(\theta, r, \varepsilon) - \theta \\ P_2(\theta, r, \varepsilon) - r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.18}$$

For system (4.1), the Poincaré map can be written into the form

$$P(\theta, r, 0) = \begin{pmatrix} P_1(\theta, r, 0) \\ P_2(\theta, r, 0) \end{pmatrix} = \begin{pmatrix} Q_1(\theta, r) \\ r \end{pmatrix}$$

since cylinders (4.2) are invariants. The function Q_1 is determined by the one-dimensional equation (4.4). Then,

$$d(\theta, r, 0) = \begin{pmatrix} d_1(\theta, r, 0) \\ d_2(\theta, r, 0) \end{pmatrix} = \begin{pmatrix} Q_1(\theta, r) - \theta \\ 0 \end{pmatrix}.$$

Consider I the interval of definition of the curve Γ. Let $\bar{\tau}_0^- \in \text{int}(I)$, i.e. a point of curve Γ, which we denote by $(\bar{\theta}_0, \bar{r}_0)$. This point belongs to the periodic orbit $\chi_{\bar{\tau}_0^-}$. The periodic orbit $\chi_{\bar{\tau}_0^-}$ has two points in common with the separation plane that intersect it transversally.

We have that $d(\bar{\theta}_0, \bar{r}_0, 0) = (0, 0)^T$. The Jacobian matrix of $d(\theta, r, 0)$ with respect to (θ, r) evaluated in $(\bar{\theta}_0, \bar{r}_0)$ is

$$\begin{pmatrix} \dfrac{\partial Q_1}{\partial \theta}(\bar{\theta}_0, \bar{r}_0) - 1 & \dfrac{\partial Q_1}{\partial r}(\bar{\theta}_0, \bar{r}_0) \\ 0 & 0 \end{pmatrix}.$$

This matrix has no full rank and we cannot apply the Implicit Function Theorem. Nevertheless, from [23] it follows that

$$\frac{\partial Q_1}{\partial \theta}(\bar{\theta}_0, \bar{r}_0) = e^{\lambda^- \bar{\tau}_0^- + (2\pi - \bar{\tau}_0^-)\lambda^+}$$

and so, if the periodic orbit is not the most external periodic orbit of the bounded continuum, we deduce (see Remark 4.3)

$$\frac{\partial d_1}{\partial \theta}(\bar{\theta}_0, \bar{r}_0, 0) = e^{\lambda^- \bar{\tau}_0^- + (2\pi - \bar{\tau}_0^-)\lambda^+} - 1 \neq 0 \tag{4.19}$$

and if we apply the Implicit Function Theorem, there exists a function $g_{\bar{\theta}_0}$, defined in a neighborhood V of $(r_0, 0)$, with $g_{\bar{\theta}_0}(\bar{r}_0, 0) = \bar{\theta}_0$ such that $d_1(g_{\bar{\theta}_0}(r, \varepsilon), r, \varepsilon) = 0$, for all $(r, \varepsilon) \in V$. Replacing it in the second equation of (4.18) we arrive to $d_2(g_{\bar{\theta}_0}(r, \varepsilon), r, \varepsilon) = 0$.

Denote

$$\widetilde{d}_2(r, \varepsilon) = d_2(g_{\bar{\theta}_0}(r, \varepsilon), r, \varepsilon). \tag{4.20}$$

Now, to study the periodic orbits that persist in the perturbed system (4.15), we just analyze the equation

$$\widetilde{d}_2(\bar{r}_0, \varepsilon) = 0. \tag{4.21}$$

We have $\widetilde{d}_2(\bar{r}_0, \varepsilon) = \varepsilon \widetilde{D}_2(\bar{r}_0, \varepsilon)$, since $\widetilde{d}_2(\bar{r}_0, 0) = d_2(g_{\bar{\theta}_0}(\bar{r}_0, 0), \bar{r}_0, 0) = 0$. As $\bar{r}_0$ depends on value τ_0^-, we can write that $\widetilde{D}_2$ depends only on $\bar{\tau}_0^-$ and ε.

Therefore, if there exists $\hat{\tau}^-$ such that

$$\widetilde{D}_2(\hat{\tau}^-, 0) = 0, \ \frac{\partial \widetilde{D}_2}{\partial \tau^-}(\hat{\tau}^-, 0) \neq 0, \tag{4.22}$$

by applying the Implicit Function Theorem, we can state that there exist $\varepsilon_0 > 0$ and a function $\tau^- = \tau^-(\varepsilon)$ defined in $(-\varepsilon_0, \varepsilon_0)$ such that $\widetilde{D}_2(\tau^-(\varepsilon), \varepsilon) = 0$, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. So, to find the roots of equation (4.21) when $|\varepsilon| \neq 0$ and sufficiently small, it is enough to find simple zeros of equation $\widetilde{D}_2(\bar{\tau}_0^-, 0) = 0$. In other words, the simple roots of

$$\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0) = 0. \tag{4.23}$$

The roots of the previous equation will give us some of the periodic orbits of the continuum that remain in the perturbed system (4.15).

Next, we establish $\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0)$ through the second component of the derivatives of the Poincaré half-maps.

Lemma 4.5 Let $\bar{\tau}_0^- \in \text{int}(I)$ be a left half-period different from τ_*^- given in (4.11), i.e., a point $(\bar{\theta}_0, \bar{r}_0) = (\theta_0(\bar{\tau}_0^-), r_0(\bar{\tau}_0^-)) \in \Gamma \setminus \{(\widetilde{\theta}, \widetilde{r})\}$, with Γ given in (4.5). The function $\widetilde{d}_2$ defined in (4.20) satisfies

$$\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0) = \frac{\partial P_2^-}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0) + \frac{\partial P_2^+}{\partial \varepsilon}(\bar{\theta}_1, \bar{r}_0, 0), \tag{4.24}$$

where $\bar{\theta}_1 = P_1^-(\bar{\theta}_0, \bar{r}_0, 0)$.

Proof: From the definition of $\widetilde{d}_2$ given in (4.20) we obtain

$$\begin{aligned} \frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0) &= \frac{\partial d_2}{\partial \theta}(g_{\bar{\theta}_0}(\bar{r}_0, 0), \bar{r}_0, 0) \frac{\partial g_{\bar{\theta}_0}}{\partial \varepsilon}(\bar{r}_0, 0) + \frac{\partial d_2}{\partial \varepsilon}(g_{\bar{\theta}_0}(\bar{r}_0, 0), \bar{r}_0, 0) = \\ &= \frac{\partial d_2}{\partial \varepsilon}(g_{\bar{\theta}_0}(\bar{r}_0, 0), \bar{r}_0, 0). \end{aligned}$$

On the other hand $d_2(\bar{\theta}_0, \bar{r}_0, \varepsilon) = P_2(\bar{\theta}_0, \bar{r}_0, \varepsilon) - \bar{r}_0$, and then,

$$\frac{\partial d_2}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, \varepsilon) = \frac{\partial P_2}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, \varepsilon),$$

hence,

$$\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0) = \frac{\partial d_2}{\partial \varepsilon}(g_{\bar{\theta}_0}(\bar{r}_0, 0), \bar{r}_0, 0) = \frac{\partial d_2}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0) = \frac{\partial P_2}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0).$$

Moreover, $P(\bar{\theta}_0, \bar{r}_0, \varepsilon) = P^+(P^-(\bar{\theta}_0, \bar{r}_0, \varepsilon), \varepsilon) = P^+(P_1^-(\bar{\theta}_0, \bar{r}_0, \varepsilon), P_2^-(\bar{\theta}_0, \bar{r}_0, \varepsilon), \varepsilon)$. Therefore, we can write

$$\begin{aligned}
&\frac{\partial P_2}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0) = \\
&\frac{\partial P_2^+}{\partial \theta}(\bar{\theta}_1, \bar{r}_0, 0)\frac{\partial P_1^-}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0) + \frac{\partial P_2^+}{\partial r}(\bar{\theta}_1, \bar{r}_0, 0)\frac{\partial P_2^-}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0) + \frac{\partial P_2^+}{\partial \varepsilon}(\bar{\theta}_1, \bar{r}_0, 0),
\end{aligned}$$

where $\bar{\theta}_1 = P_1^-(\bar{\theta}_0, \bar{r}_0, 0)$. We know that $P_2^+(\theta, r, 0) = r$, thus we obtain (4.24) and the proof is completed. □

In the next proposition we give an integral expression of $\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0)$.

Proposition 4.6 Let $\bar{\tau}_0^- \in \text{int}(I)$ be a left half-period different from τ_*^- given in (4.11), i.e., a point $(\bar{\theta}_0, \bar{r}_0) = (\theta_0(\bar{\tau}_0^-), r_0(\bar{\tau}_0^-)) \in \Gamma \setminus \{(\widetilde{\theta}, \widetilde{r})\}$, with Γ given in (4.5) and denote $\bar{\theta}_1 = P_1^-(\bar{\theta}_0, \bar{r}_0, 0)$. The function $\widetilde{d}_2$ defined in (4.20) satisfies

$$\begin{aligned}
\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0) = \quad & 2\Lambda^- \int_0^{\bar{\tau}_0^-} (\lambda^- \cos(\bar{\theta}_0 - t) + \sin(\bar{\theta}_0 - t))x^-(t)dt + \\
& 2B \int_0^{2\pi - \bar{\tau}_0^-} (\cos(\bar{\theta}_1 - t) - \lambda^+ \sin(\bar{\theta}_1 - t))x^+(t)dt + \\
& 2\Lambda^+ \int_0^{2\pi - \bar{\tau}_0^-} (\lambda^+ \cos(\bar{\theta}_1 - t) + \sin(\bar{\theta}_1 - t))x^+(t)dt,
\end{aligned} \tag{4.25}$$

where $x^-(t)$ is the solution of the initial value problem

$$\left\{ \begin{array}{rcl} \dot{x}^- &=& \lambda^- x^- - \bar{r}_0 \cos(\bar{\theta}_0 - t) - 1, \\ x^-(0) &=& 0, \end{array} \right. \tag{4.26}$$

and $x^+(t)$ is the solution of the initial value problem

$$\left\{\begin{array}{rcl} \dot{x}^+ &=& \lambda^+ x^+ - \bar{r}_0\cos(\bar{\theta}_1 - t) - 1,\\ x^+(0) &=& 0. \end{array}\right. \tag{4.27}$$

Proof: On the one hand, from Lemma 4.5, one obtains

$$\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0) = \frac{\partial P_2^-}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0) + \frac{\partial P_2^+}{\partial \varepsilon}(\bar{\theta}_1, \bar{r}_0, 0).$$

On the other hand, it is easy to see that $P_2^-(\bar{\theta}_0, \bar{r}_0, \varepsilon) = r^-(\bar{\tau}_0^-, \bar{\theta}_0, \bar{r}_0, \varepsilon)$, where $\mathbf{x}^-(t, \bar{\theta}_0, \bar{r}_0, \varepsilon) = (x^-(t, \bar{\theta}_0, \bar{r}_0, \varepsilon), r^-(t, \bar{\theta}_0, \bar{r}_0, \varepsilon), \theta^-(t, \bar{\theta}_0, \bar{r}_0, \varepsilon))$ is the solution of the initial value problem

$$\left\{\begin{array}{l} \begin{pmatrix} \dot{x} \\ \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \lambda^- x - r\cos\theta - 1 \\ 0 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} 2\Lambda^- x \\ (a^-\cos\theta + b^-\sin\theta)x \\ (b^-\cos\theta - a^-\sin\theta)\dfrac{x}{r} \end{pmatrix}, \\ \begin{pmatrix} x(0) \\ r(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{r}_0 \\ \bar{\theta}_0 \end{pmatrix}, \end{array}\right.$$

with a^- and b^- given in (4.16). Thus, $\frac{\partial P_2^-}{\partial \varepsilon}(\bar{\theta}_0, \bar{r}_0, 0) = \frac{\partial r^-}{\partial \varepsilon}(\bar{\tau}_0^-, \bar{\theta}_0, \bar{r}_0, 0)$. Similarly, we can see that $\frac{\partial P_2^+}{\partial \varepsilon}(\bar{\theta}_1, \bar{r}_0, 0) = \frac{\partial r^+}{\partial \varepsilon}(2\pi - \bar{\tau}_0^-, \bar{\theta}_1, \bar{r}_0, 0)$, where $\mathbf{x}^+(t, \bar{\theta}_1, \bar{r}_0, \varepsilon) = (x^+(t, \bar{\theta}_1, \bar{r}_0, \varepsilon), \theta^+(t, \bar{\theta}_1, \bar{r}_0, \varepsilon), r^+(t, \bar{\theta}_1, \bar{r}_0, \varepsilon))$ is the solution of the initial value problem

$$\left\{\begin{array}{l} \begin{pmatrix} \dot{x} \\ \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \lambda^+ x - r\cos\theta - 1 \\ 0 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} 2\Lambda^+ x \\ (a^+\cos\theta + b^+\sin\theta)x \\ (b^+\cos\theta - a^+\sin\theta)\dfrac{x}{r} \end{pmatrix}, \\ \begin{pmatrix} x(0) \\ r(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{r}_0 \\ \bar{\theta}_1 \end{pmatrix}, \end{array}\right.$$

with a^+ and b^+ given in (4.16).

Hence,

$$\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0) = \frac{\partial r^-}{\partial \varepsilon}(\bar{\tau}_0^-, \bar{\theta}_0, \bar{r}_0, 0) + \frac{\partial r^+}{\partial \varepsilon}(2\pi - \bar{\tau}_0^-, \bar{\theta}_1, \bar{r}_0, 0). \tag{4.28}$$

Now, we will find $\frac{\partial r^-}{\partial \varepsilon}(\bar{\tau}_0^-, \bar{\theta}_0, \bar{r}_0, 0)$. Denotes $\frac{\partial r^-}{\partial \varepsilon}(t, \bar{\theta}_0, \bar{r}_0, 0) = r_\varepsilon^-(t, \bar{\theta}_0, \bar{r}_0, 0)$.

If we take the derivative in the left zone of (4.15), i.e., $x < 0$, with respect to ε and we make $\varepsilon = 0$, we deduce that $r_\varepsilon^-(\cdot, \bar{\theta}_0, \bar{r}_0, 0)$ satisfies the initial value problem

$$\begin{cases} \dot{r}_\varepsilon^- = (2\Lambda^-\lambda^- \cos(\bar{\theta}_0 - t) + 2\Lambda^- \sin(\bar{\theta}_0 - t))x^-(t), \\ r_\varepsilon^-(0) = 0, \end{cases}$$

with $x^-(t)$ the solution of the initial value problem (4.26).

Then,

$$r_\varepsilon^-(\bar{\tau}_0^-, \bar{\theta}_0, \bar{r}_0, 0) = 2\Lambda^- \int_0^{\bar{\tau}_0^-} (\lambda^- \cos(\bar{\theta}_0 - t) + \sin(\bar{\theta}_0 - t))x^-(t)dt. \tag{4.29}$$

Denote $\dfrac{\partial r^+}{\partial \varepsilon}(t, \bar{\theta}_1, \bar{r}_0, 0) = r_\varepsilon^+(t, \bar{\theta}_1, \bar{r}_0, 0)$. Reasoning as we did to find $r_\varepsilon^-(\bar{\tau}_0^-, \bar{\theta}_0, \bar{r}_0, 0)$ and taking into account $\bar{\tau}_0^+ = 2\pi - \bar{\tau}_0^-$, we get

$$\begin{aligned} r_\varepsilon^+(2\pi - \bar{\tau}_0^-, \bar{\theta}_1, \bar{r}_0, 0) = \;& 2B \int_0^{2\pi - \bar{\tau}_0^-} (\cos(\bar{\theta}_1 - t) - \lambda^+ \sin(\bar{\theta}_1 - t))x^+(t)dt + \\ & 2\Lambda^+ \int_0^{2\pi - \bar{\tau}_0^-} (\lambda^+ \cos(\bar{\theta}_1 - t) + \sin(\bar{\theta}_1 - t))x^+(t)dt. \end{aligned} \tag{4.30}$$

From (4.29), (4.30) and (4.28), the proof is finished. □

Remark 4.7 Note that the definition of $\widetilde{d}_2$, given in (4.20), involves the function $g_{\bar{\theta}_0}$ that exists if $\bar{\tau}_0^- \neq \tau_*^-$, with τ_*^- given in (4.11), i.e. if the periodic orbit does not correspond to the most external periodic orbit of the bounded continuum. However, the derivative of $\frac{\partial \widetilde{d}_2}{\partial \varepsilon}(\bar{r}_0, 0)$ has an expression that involves the partial derivatives with respect to ε of the Poincare half-maps, that are well defined in every periodic orbit transversal to the separation plane, even in the most external periodic orbit of the bounded continuum. Therefore, we are now able to define the Melnikov function for all $\tau^- \in \text{int}(I)$, as we are going to do in the next definition.

Definition 4.8 Let $\tau^- \in \text{int}(I)$ be a left half-period, i.e., a point (θ_0, r_0) of the curve Γ given in (4.5) and denote $\theta_1 = P_1^-(\theta_0, r_0, 0)$.

We define the Melnikov function of system (4.15) as

$$\begin{aligned} M(\tau^-;\boldsymbol{\mu}) = \; & 2\Lambda^- \int_0^{\tau^-} (\lambda^- \cos(\theta_0 - t) + \sin(\theta_0 - t))x^-(t)dt + \\ & 2B \int_0^{2\pi-\tau^-} (\cos(\theta_1 - t) - \lambda^+ \sin(\theta_1 - t))x^+(t)dt + \\ & 2\Lambda^+ \int_0^{2\pi-\tau^-} (\lambda^+ \cos(\theta_1 - t) + \sin(\theta_1 - t))x^+(t)dt \end{aligned} \tag{4.31}$$

where $\boldsymbol{\mu} = (\lambda^+, \lambda^-, \Lambda^+, \Lambda^-, B)$, $x^-(t)$ is the solution of the initial value problem

$$\begin{cases} \dot{x}^- &= \lambda^- x^- - r_0 \cos(\theta_0 - t) - 1, \\ x^-(0) &= 0, \end{cases} \tag{4.32}$$

and $x^+(t)$ is the solution of the initial value problem

$$\begin{cases} \dot{x}^+ &= \lambda^+ x^+ - r_0 \cos(\theta_1 - t) - 1, \\ x^+(0) &= 0. \end{cases} \tag{4.33}$$

Note that, due to the continuity and differentiability of the Poincaré half-maps, it is clear that the Melnikov function is analytic in $\tau^- \in \text{int}(I)$. Indeed, as we will see later, we can extend the definition up to some of the endpoints of I.

To finish this section, we write the Melnikov function in another way.

Proposition 4.9 Let $\tau^- \in \text{int}(I)$ be a left half-period, i.e. a point (θ_0, r_0) of the curve (4.5). We can express the Melnikov function given in (4.31) as

$$\begin{aligned} & M(\tau^-;\boldsymbol{\mu}) = \\ & 2\Lambda^- \left(\frac{r_0\tau^-}{2} - s_0 c_{\tau^-} + c_0 s_{\tau^-} + s_0 + \frac{r_0}{4} s_{2\tau^-}(c_0^2 - s_0^2) + r_0 c_0 s_0 s_{\tau^-}^2 \right) + \\ & 2B \left(-\frac{r_0}{2} s_{\tau^-}^2 (c_0^2 - s_0^2) + \frac{r_0}{2} c_0 s_0 s_{2\tau^-} - c_0 + c_0 c_{\tau^-} + s_0 s_{\tau^-} \right) + \\ & 2\Lambda^+ \left(r_0\pi - \frac{r_0\tau^-}{2} + s_0 c_{\tau^-} - c_0 s_{\tau^-} - s_0 - \frac{r_0}{4} s_{2\tau^-}(c_0^2 - s_0^2) - r_0 c_0 s_0 s_{\tau^-}^2 \right), \end{aligned} \tag{4.34}$$

where $\boldsymbol{\mu} = (\lambda^+, \lambda^-, \Lambda^+, \Lambda^-, B)$, $c_0 = \cos\theta_0$, $s_0 = \sin\theta_0$, $c_{\tau^-} = \cos\tau^-$, $s_{\tau^-} = \sin\tau^-$ and $s_{2\tau^-} = \sin 2\tau^-$.

Proof: First, we are going to find $r_\varepsilon^-(\tau^-\bar{\theta}_0, \bar{r}_0, 0)$, i.e. the value of

$$r_\varepsilon^-(\tau^-\bar{\theta}_0, \bar{r}_0, 0) = 2\Lambda^- \int_0^{\tau^-} (\lambda^- \cos(\theta_0 - t) + \sin(\theta_0 - t))x^-(t)dt,$$

where $x^-(t)$ is the solution of the initial value problem (4.32).

If we multiply $\dot{x}^-(t) = \lambda^- x^- - r_0 \cos(\theta_0 - t) - 1$ by $\cos(\theta_0 - t)$ and we integrate this equation from 0 to τ^- we have

$$\begin{aligned}&\int_0^{\tau^-} \dot{x}^-(t)\cos(\theta_0 - t)dt = \\ &\lambda^- \int_0^{\tau^-} x^-(t)\cos(\theta_0 - t)dt - r_0 \int_0^{\tau^-} \cos^2(\theta_0 - t)dt - \int_0^{\tau^-} \cos(\theta_0 - t)dt.\end{aligned}$$

Then, by integrating by parts taking into account $x^-(0) = 0$, $x^-(\tau^-) = 0$, we arrive to

$$\begin{aligned}&\int_0^{\tau^-} (\lambda^- \cos(\theta_0 - t) + \sin(\theta_0 - t))x^-(t)dt = \\ &r_0 \int_0^{\tau^-} \cos^2(\theta_0 - t)dt + \int_0^{\tau^-} \cos(\theta_0 - t)dt.\end{aligned} \tag{4.35}$$

Now, we calculate the integrals of the right part to deduce the expression

$$\begin{aligned}&r_\varepsilon^-(\tau^-, \bar{\theta}_0, \bar{r}_0,) = \\ &2\Lambda^- \left(\frac{r_0 \tau^-}{2} - s_0 c_{\tau^-} + c_0 s_{\tau^-} + s_0 + \frac{r_0}{4} s_{2\tau^-}(c_0^2 - s_0^2) + r_0 c_0 s_0 s_{\tau^-}^2 \right),\end{aligned} \tag{4.36}$$

with c_0, s_0 and s_{τ^-} given in (4.34). Reasoning as we did to find $r_\varepsilon^-(\bar{\tau}^-, \bar{\theta}_0, \bar{r}_0, 0)$ given in (4.36) and taking into account that $x^+(0) = 0, x^+(2\pi - \tau^-) = 0$, where $x^+(t)$ is the solution of the initial value problem (4.33) and $\theta_1 = \theta_0 - \tau^-$, yields

$$\begin{aligned}&\int_0^{2\pi - \tau^-} (\cos(\theta_1 - t) - \lambda^+ \sin(\theta_1 - t))x^+(t)dt = \\ &-r_0 \int_0^{2\pi - \tau^-} \cos(\theta_1 - t)\sin(\theta_1 - t)dt - \int_0^{2\pi - \tau^-} \sin(\theta_1 - t)dt,\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}&\int_0^{2\pi-\tau^-}(\lambda^+\cos(\theta_1-t)+\sin(\theta_1-t))x^+(t)dt=\\ &r_0\int_0^{2\pi-\tau^-}\cos^2(\theta_1-t)dt+\int_0^{2\pi-\tau^-}\cos(\theta_1-t)dt.\end{aligned}\tag{4.38}$$

Now, it is enough to compute the integrals of the second terms of (4.37) and (4.38) and taking into account that $\theta_1=\theta_0-t$ to arrive to

$$\begin{aligned}&r_\varepsilon^+(2\pi-\tau^-,\bar\theta_1,\bar r_0,0)=\\ &2B\left(-\frac{r_0}{2}s_{\tau^-}^2(c_0^2-s_0^2)+\frac{r_0}{2}c_0s_0s_{2\tau^-}-c_0+c_0c_{\tau^-}-+s_0s_{\tau^-}\right)+\\ &2\Lambda^+\left(r_0\pi-\frac{r_0\tau^-}{2}+s_0c_{\tau^-}-c_0s_{\tau^-}-s_0-\frac{r_0}{4}s_{2\tau^-}(c_0^2-s_0^2)-r_0c_0s_0s_{\tau^-}^2\right),\end{aligned}\tag{4.39}$$

with c_0, s_0, c_{τ^-} and s_{τ^-} given in (4.34).

As (4.36) and (4.39) hold, the proof is finished. □

In order to summarize the information developed up till now, we state the following result whose proof is straightforward.

Theorem 4.10 Assume that the Melnikov function $M(\cdot,\mu)$ given in (4.31) possesses a simple zero $\hat\tau^-\in\text{int}(I)$. The following properties hold.

(a) If $\lambda^+>0$ and $\lambda^-\geq 0$ (or $\lambda^+\leq 0$ and $\lambda^-<0$), then the perturbed system (4.15) possesses a periodic orbits in a neighborhood of $\chi_{\hat\tau^-}$ for $|\varepsilon|\neq 0$ sufficiently small.

(b) If $\lambda^+>0$ and $\lambda^-<0$ and $\hat\tau^-\neq\tau_*^-$ with τ_*^- given in (4.11), the perturbed system (4.15) has a limit cycle in a neighborhood of $\chi_{\hat\tau^-}$, for $|\varepsilon|\neq 0$ sufficiently small.

To complete this section, let us introduce a new function

$$M_r(\tau^-;\boldsymbol{\mu})=\frac{M(\tau^-;\boldsymbol{\mu})}{r_0(\tau^-,\lambda^-,\lambda^+)}$$

where $\boldsymbol{\mu}=(\lambda^+,\lambda^-,\Lambda^+,\Lambda^-,B)$, which will be called the reduced Melnikov function and has the same zeros and the same multiplicity that the Melnikov function M.

For the sake of brevity, we will occasionally delete the parameter $\boldsymbol{\mu}$ in the Melnikov and the reduced Melnikov function's writing.

The reduced Melnikov function presents more interesting properties than the Melnikov function, because it lets us determine the existence of zeros of the Melnikov function in an easier way. Moreover, if $M(\hat{\tau}^-) = 0$, then it is easy to see that sgn$(M'(\hat{\tau}^-))$ =sgn$(M_r'(\hat{\tau}^-))$. In fact, as

$$M_r'(\tau^-) = \frac{M'(\tau^-)r_0(\tau^-) - M(\tau^-)r_0'(\tau^-)}{r_0^2(\tau^-)},$$

if $M(\hat{\tau}^-) = 0$, it is satisfied that

$$M_r'(\hat{\tau}^-) = \frac{M'(\hat{\tau}^-)}{r_0(\hat{\tau}^-)}$$

and so,

$$\operatorname{sgn}(M'(\hat{\tau}^-)) = \operatorname{sgn}(M_r'(\hat{\tau}^-)). \tag{4.40}$$

4.4 Properties of the Melnikov function

In this section we will study some properties of the Melnikov function defined in (4.31) and the reduced Melnikov function. These properties will be very useful to analyze, in the next section, the existence of periodic orbits, their stabilities and the bifurcations that may appear in the perturbed system (4.15) when the values of the parameters change.

We will analyze some symmetry properties of the Melnikov function. Concretely, we will study the Melnikov function in the case that the system is reversible.

After that, we will perform an analysis of the value of the Melnikov function and its derivatives up to some order in the endpoints of the existence domain.

4.4.1 Reversibility

We want to know how is the behavior of the Melnikov function under some reversibility hypothesis. Concretely, we would like to know what occurs to the Melnikov function if we consider a reversible perturbation and the unperturbed system is reversible too. Let us consider reversible systems under the involution

$$\begin{aligned} &\mathbf{R} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ &(x, y, z)^T \mapsto (-x, y, -z)^T, \end{aligned}$$

that is, they are invariant under the transformation

$$(\mathbf{x}, t) \mapsto (\mathbf{R}(\mathbf{x}), -t).$$

Note that this is the reversibility given in (1.9) that the PWL Michelson system (1.14) introduced in Chapter 1 has.

Coming back to system (4.12), it is reversible under the symmetry $\mathbf{R}$ if and only if

$$t^+ = -t^-, m^+ = m^- \text{ and } d^+ = -d^-.$$

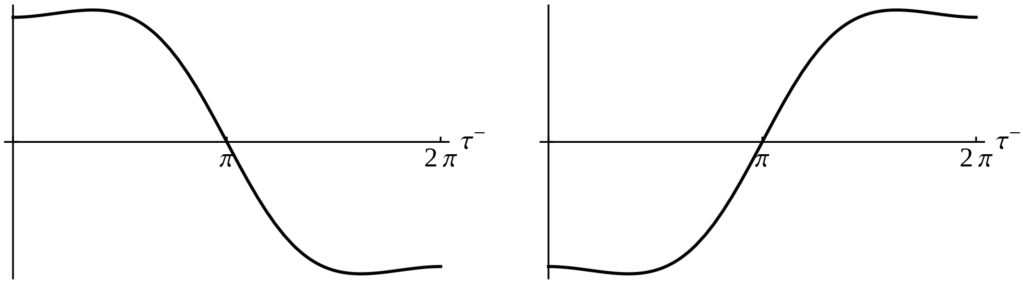
Analogously, the unperturbed system (4.1) is reversible under $\mathbf{R}$ if and only if $\lambda^+ = -\lambda^-$.

We will center our attention now on the case $\lambda^+ = -\lambda^- > 0$ (bounded continuum). We wonder about the behavior if one considers a reversible perturbation, i.e., if in the perturbed system (4.15) $\Lambda^+ = -\Lambda^-$ and $B = 0$. From the expression of the Melnikov function given in (4.34), we can prove the following symmetry property, whose proof is straightforward.

Proposition 4.11 Assume that in the unperturbed system (4.1) we choose $\lambda^+ = -\lambda^- > 0$. If we do a reversible perturbation, i.e. if we consider the perturbed system (4.15) with $\Lambda^+ = -\Lambda^-$ and $B = 0$, then the Melnikov function satisfies the symmetry property $-M(\tau^- + \pi) = M(-\tau^- + \pi)$. Moreover, $M(\pi) = 0$.

In Fig. 4.42 we observe that under the reversibility hypotheses, the symmetry properties studied in Proposition 4.11 hold.

Remark 4.12 Although the Melnikov function for the reversible case vanishes at $\tau_*^- = \pi$, which corresponds to the most external periodic orbit of the bounded continuum in the reversible case (see Remark 4.3), we cannot assure, at the beginning, the persistence of this periodic orbit (see Remark 4.7). However, in this case, it can be proven the persistence of this periodic orbit, due to its transversal intersection with the separation plane and its reversible character.



(a) $\lambda^+ = -\lambda^-$, $\Lambda^+ = -\Lambda^- > 0$. (b) $\lambda^+ = -\lambda^-$, $\Lambda^+ = -\Lambda^- < 0$.

Figure 4.3: Melnikov function M defined in (4.31) in the reversible case.

4.4.2 Analysis at the endpoints of the intervals

In this subsection, we want to extend the value of the reduced Melnikov function and their derivatives up to some order in the endpoints of the definition interval.

Later on, limits at the endpoints of the existence domain of M will be considered laterals. For the sake of brevity, we will omit the corresponding notation. Likewise, we will not use special notation to denote lateral consecutive derivatives of the Melnikov function M and the reduced Melnikov function M_r with respect to τ^- in the endpoints of the existence domain. The proofs of the following results are tedious but direct. In spite of that, to calculate the derivatives of an order greater than three, we have used the symbolic manipulation programs Maple and Mathematica, and we have obtained the same results in both programs.

Proposition 4.13 The following properties hold.

(a) If $\lambda^- < 0$, $\lambda^+ > 0$, or $\lambda^+ > 0$, $\lambda^- \geq 0$, the Melnikov function M in the left endpoint of its definition interval $\tau^- = 0$, behaves

$$\lim_{\tau^- \to 0} M(\tau^-) = 2\Lambda^+ \pi \sqrt{1 + \frac{1}{(\lambda^+)^2}}.$$

(b) If $\lambda^- < 0$, $\lambda^+ > 0$, or $\lambda^+ \leq 0$, $\lambda^- < 0$, the Melnikov function M in the right endpoint of its definition interval $\tau^- = 2\pi$, behaves

$$\lim_{\tau^- \to 2\pi} M(\tau^-) = 2\Lambda^- \pi \sqrt{1 + \frac{1}{(\lambda^-)^2}}.$$

From Proposition 4.13 we can define by continuity the Melnikov function in its corresponding endpoints of definition as

$$\begin{aligned} M(0) &= 2\Lambda^+ \pi \sqrt{1 + \frac{1}{(\lambda^+)^2}}, \\ M(2\pi) &= 2\Lambda^- \pi \sqrt{1 + \frac{1}{(\lambda^-)^2}}. \end{aligned} \tag{4.41}$$

With these values of definition at the endpoints of the domain, we can obtain the successive derivatives of the Melnikov function. However, we can obtain results in an easier way by working with the reduced Melnikov function.

In the next proposition we will analyze the derivatives of the Melnikov reduced function at $\tau^- = 0$ and $\tau^- = 2\pi$.

Proposition 4.14 The following properties are satisfied.

(a) If $\lambda^- < 0,\ \lambda^+ > 0$, or $\lambda^+ > 0,\ \lambda^- \geq 0$, the Melnikov reduced function M_r satisfies the following properties at the left endpoint of its interval of definition.

$$\begin{aligned} &M_r(0) = 2\Lambda^+\pi,\ M_r'(0) = 0,\ M_r''(0) = 0,\ M_r^{iv}(0) = 0, \\ &M_r'''(0) = \frac{B(\lambda^- - \lambda^+) + (\Lambda^- - \Lambda^+)(1 + \lambda^-\lambda^+)}{1 + (\lambda^+)^2}, \\ &M_r^{v}(0) = \frac{B(\lambda^- - \lambda^+)(5(\lambda^+)^2 - 10\lambda^-\lambda^+ - (\lambda^-)^2 - 9)}{3(1 + (\lambda^+)^2)} + \\ &\frac{(\Lambda^+ - \Lambda^-)(3 - 5(\lambda^-)^2 + 19\lambda^-\lambda^+ + (\lambda^-)^3\lambda^+ - 11(\lambda^+)^2 + 4(\lambda^-)^2(\lambda^+)^2 - 5\lambda^-(\lambda^+)^3)}{3(1 + (\lambda^+)^2)}. \end{aligned}$$

(b) If $\lambda^- < 0,\ \lambda^+ > 0$, or $\lambda^+ \leq 0,\ \lambda^- < 0$, the Melnikov reduced function M_r satisfies the following properties at the right endpoint of its interval of definition.

$$M_r(2\pi) = 2\Lambda^-\pi,\ M_r'(2\pi) = 0,\ M_r''(2\pi) = 0,\ M_r^{iv}(2\pi) = 0,$$
$$M_r'''(2\pi) = \frac{B(\lambda^+ - \lambda^-) + (\Lambda^- - \Lambda^+)(1 + \lambda^-\lambda^+)}{1 + (\lambda^-)^2},$$
$$M_r^{v}(2\pi) = \frac{B(\lambda^+ - \lambda^-)(5(\lambda^-)^2 - 10\lambda^-\lambda^+ - (\lambda^+)^2 - 9)}{3(1 + (\lambda^-)^2)} +$$
$$\frac{(\Lambda^+ - \Lambda^-)(3 - 5(\lambda^+)^2 + 19\lambda^-\lambda^+ + (\lambda^+)^3\lambda^- - 11(\lambda^-)^2 + 4(\lambda^-)^2(\lambda^+)^2 - 5\lambda^+(\lambda^-)^3)}{3(1 + (\lambda^-)^2)}.$$

Particularly, when the third derivative is zero, the fifth derivative has a more compact expression.

Proposition 4.15 Under the hypotheses of Proposition 4.14, if $M_r'''(0) = 0$, then $M_r^{v}(0) = 2(\Lambda^- - \Lambda^+)(1 + (\lambda^-)^2)$.

Proposition 4.16 Under the hypotheses of Proposition 4.14, if $M_r'''(2\pi) = 0$, then $M_r^{v}(2\pi) = 2(\Lambda^- - \Lambda^+)(1 + (\lambda^+)^2)$.

To conclude this section, we ask about the behavior of the Melnikov function in a neighborhood of the value $\widetilde{\tau}^-$ defined in Proposition 4.2 when $\lambda^+ > 0$, $\lambda^- \geq 0$ and $\lambda^+ \leq 0$, $\lambda^- < 0$.

From expressions given in (4.35), (4.37) and (4.38) it is deduced that the Melnikov function can be written as

$$M(\tau^-) = 2(\Lambda^- I^-(\tau^-) - BI_1^+(\tau^-) + \Lambda^+ I_2^+(\tau^-))r_0(\tau^-) + C(\tau^-),$$

where

$$
\begin{aligned}
I^-(\tau^-) &= \int_0^{\tau^-} \cos^2(\theta_0 - t)dt,\\
I_1^+(\tau^-) &= \int_0^{2\pi-\tau^-} \cos(\theta_1 - t)\sin(\theta_1 - t)dt,\\
I_2^+(\tau^-) &= \int_0^{2\pi-\tau^-} \cos^2(\theta_1 - t)dt,\\
C(\tau^-) &= 2\left(\Lambda^- \int_0^{\tau^-} \cos(\theta_0 - t)dt + \Lambda^+ \int_0^{2\pi-\tau^-} \cos(\theta_1 - t)dt - B\int_0^{2\pi-\tau^-} \sin(\theta_1 - t)dt\right).
\end{aligned}
$$

It can be proven that $\lim_{\tau^-\to\widetilde{\tau}^-} r_0(\tau^-) = +\infty$ and that, in some cases, the Melnikov function is not bounded. For instance, by substituting $B = 0$ and taking into account that $I^-(\tau^-) > 0$ and $I_2^+(\tau^-) > 0$, we can conclude that $\lim_{\tau^-\to\widetilde{\tau}^-} M(\tau^-) = +\infty$ if $\Lambda^+ > 0$ and $\Lambda^- > 0$ and that $\lim_{\tau^-\to\widetilde{\tau}^-} M(\tau^-) = -\infty$ if $\Lambda^+ < 0$ and $\Lambda^- < 0$.

However, in other cases the function M can be bounded. Therefore, if we choose Λ^-, Λ^+ and B such that

$$\Lambda^- I^-(\tilde{\tau}^-) - BI_1^+(\tilde{\tau}^-) + \Lambda^+ I_2^+(\tilde{\tau}^-) = 0,$$

then $\lim_{\tau^-\to\widetilde{\tau}^-} M(\tau^-)$ is not, at the beginning, determined, and we may have a finite limit.

We can determine this limit, for example, for $\lambda^+ = \lambda^-$ and $\Lambda^+ = -\Lambda^-$. In such a case, it is easy to see that $\widetilde{\tau}^- = \pi$. The Melnikov function has, in that case, the following expression

$$M(\tau^-) = 2\Lambda^+(\pi - \tau^- + \sin\tau^-)\sqrt{\frac{\sec^2(\tau^-/2)(1 + (\lambda^-)^2)}{(\lambda^-)^2}},$$

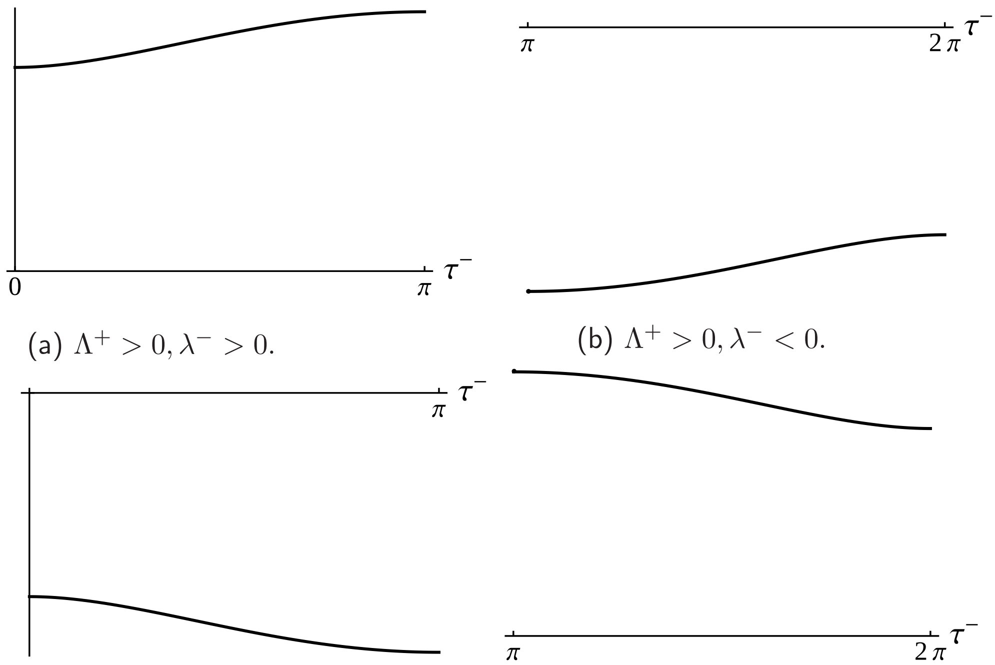
with $\tau^- \in [0, \pi)$ if $\lambda^- > 0$ and $\tau^- \in (\pi, 2\pi]$ if $\lambda^- < 0$. Note that it does not depend on the parameter B. It is easy to see that

$$\lim_{\tau^-\to\pi^-} M(\tau^-) = 8\Lambda^+\sqrt{1 + \frac{1}{(\lambda^-)^2}}$$

$$\lim_{\tau^-\to\pi^+} M(\tau^-) = -8\Lambda^+\sqrt{1 + \frac{1}{(\lambda^-)^2}}$$

hence, in both cases, M is bounded in its definition domain.

In Fig. 4.4 it can be seen the behavior of function M for $\lambda^+ = \lambda^- \neq 0$ and $\Lambda^+ = -\Lambda^- \neq 0$, depending on the signs of λ^+ and Λ^+.



(a) $\Lambda^+ > 0, \lambda^- > 0.$ (b) $\Lambda^+ > 0, \lambda^- < 0.$

(c) $\Lambda^+ < 0, \lambda^- > 0.$ (d) $\Lambda^+ < 0, \lambda^- < 0.$

Figure 4.4: The Melnikov function M in cases $\lambda^+ = \lambda^- \neq 0$ and $\Lambda^+ = -\Lambda^- \neq 0$, depending on the signs of λ^+ and λ^+.

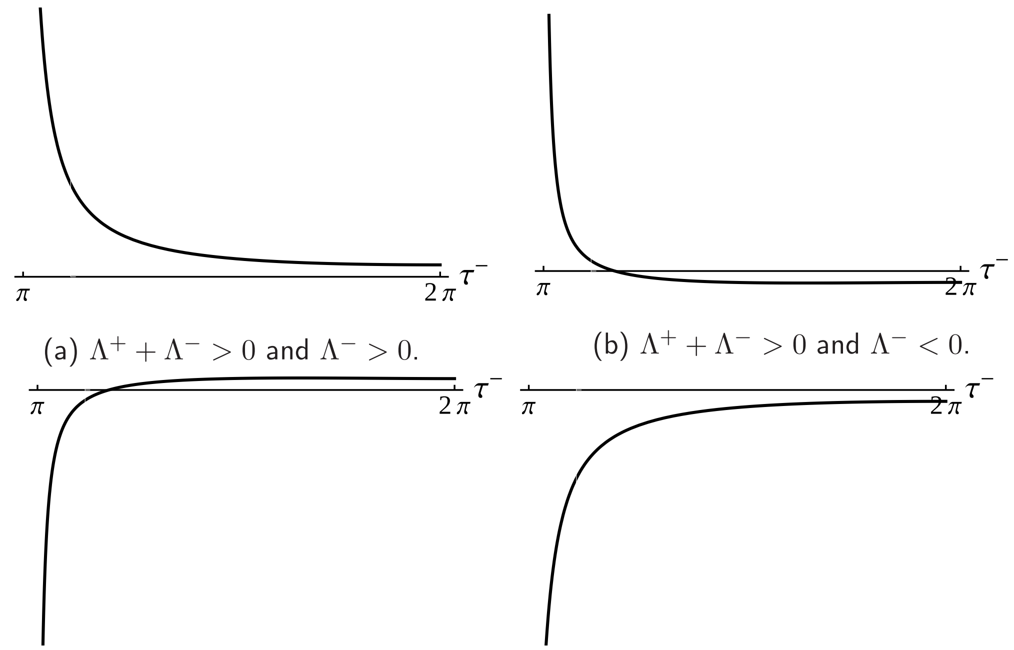
Observe that for the case $\lambda^+ = \lambda^- \neq 0$ and $\Lambda^+ = -\Lambda^- \neq 0$, the Melnikov function M does not vanish in its existence domain.

In fact, it is easy to analyze the case $\lambda^+ = \lambda^- \neq 0$. In this case, the unperturbed system (4.1) is linear and the perturbed system (4.15) is non-controllable, see Chapter 1. Then, its dynamics is basically planar and it is organized on the invariant plane defined by the common real eigenvalue $\lambda^+ = \lambda^-$ and the invariant cylinders given in (4.2). Now, the Melnikov function is simply

$$M(\tau^-) = (2\Lambda^+\pi - (\Lambda^+ - \Lambda^-)\tau^- + (\Lambda^+ - \Lambda^-)\sin\tau^-)\sqrt{\frac{\sec^2(\tau^-/2)(1+(\lambda^-)^2)}{(\lambda^-)^2}}. \qquad (4.42)$$

Note that it does not depend on the parameter B. Their behavior can be easily established and in particular, its zeros in its existence domain. In Fig. 4.5 we can see different cases that may appear.

This behavior let us do a bifurcation analysis of periodic orbits in this case. Such analysis, as we will



(a) $\Lambda^+ + \Lambda^- > 0$ and $\Lambda^- > 0$.

(b) $\Lambda^+ + \Lambda^- > 0$ and $\Lambda^- < 0$.

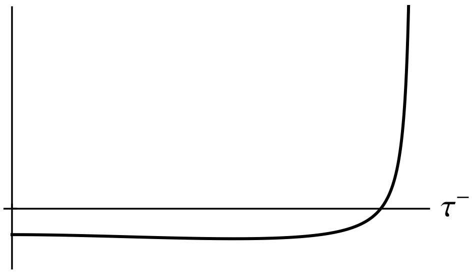
(c) $\Lambda^+ + \Lambda^- < 0$ and $\Lambda^- > 0$.

(d) $\Lambda^+ + \Lambda^- < 0$ and $\Lambda^- < 0$.

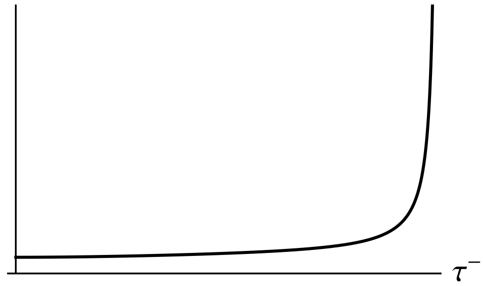
Figure 4.5: Melnikov function M in cases $\lambda^+ = \lambda^- < 0$ and $\Lambda^+ + \Lambda^- \neq 0$. Observe that the Melnikov function M can have only one zero.

see in Sec. 4.5.3, coincides with the analysis done in [13, 41, 44, 85] about the Hopf bifurcation from infinity, for planar piecewise linear systems with two zones.

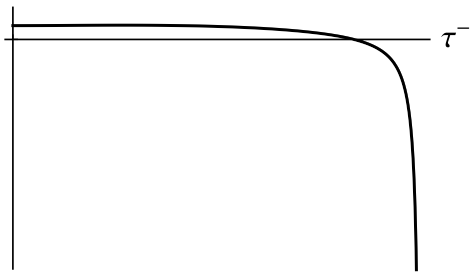
Coming back to the general case, it is possible to give unbounded behaviors of the Melnikov function M, with or without roots. See Fig. 4.6.



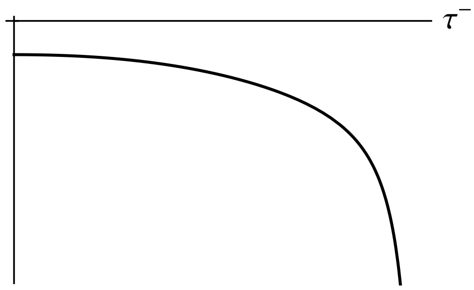
(a) $\Lambda^- = 0.17, B = 0, \Lambda^+ = -0.2,$
$\lambda^- = 0.1, \lambda^+ = 0.2.$



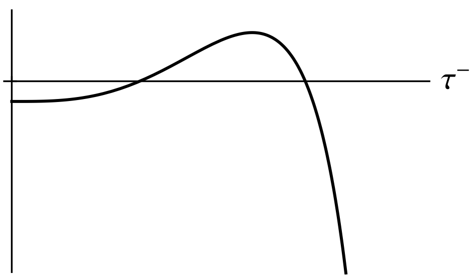
(b) $\Lambda^- = -0.1, B = 0, \Lambda^+ = 0.3,$
$\lambda^- = 0.4, \lambda^+ = 0.5.$



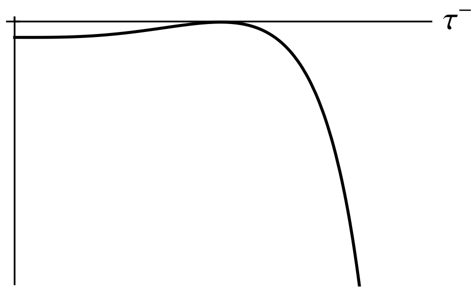
(c) $\Lambda^- = -0.1, B = 10, \Lambda^+ = 0.3,$
$\lambda^- = 0.4, \lambda^+ = 0.5.$



(d) $\Lambda^- = 0.17, B = 6, \Lambda^+ = -0.2,$
$\lambda^- = 0.1, \lambda^+ = 0.2.$



(e) $\Lambda^- = -0.25, B = -5, \Lambda^+ = -0.003,$
$\lambda^- = 0.4, \lambda^+ = 0.5.$



(f) $\Lambda^- = -0.167, B = -3, \Lambda^+ = -0.003,$
$\lambda^- = 0.4, \lambda^+ = 0.5.$

Figure 4.6: Examples of unbounded behaviors of the Melnikov function M in the case $\lambda^+, \lambda^- > 0$ and $\Lambda^+ \neq 0$, $\Lambda^- \neq 0$. Observe that the Melnikon function M can have no, one or two zeros. Thus, a saddle-node bifurcation may appear, see picture (f).

4.5 Existence and stability of periodic orbits. Bifurcation analysis

In this section we are going to state some of the most important results of this chapter.

First, we state some results about the existence of periodic orbits in the perturbed system (4.15) and their stabilities. Later, we will study their possible bifurcations.

All this analysis will be done through the Melnikov and the reduced Melnikov function and we will use their the properties studied in Sec. 4.4.

4.5.1 First results about the existence and stability of periodic orbits

In this subsection, we will analyze the existence of zeros of the Melnikov function and their relation to periodic orbits of the perturbed system (4.15).

Remind that the periodic orbits of the continuum with common points with the separation plane $x = 0$ are denoted by χ_{τ^-}, where τ^- is the left half-period of the orbit.

To begin with, we state an easy result about the existence of periodic orbits for the unperturbed system (4.1)

Theorem 4.17 Assume that $\lambda^+ > 0$ and $\lambda^- < 0$. If $\Lambda^+ \cdot \Lambda^- < 0$, then the Melnikov function M has a root $\hat{\tau}^- \in (0, 2\pi)$. Moreover, if $\hat{\tau}^- \neq \tau_*^-$, where τ_*^- was given in (4.11) and $M'(\hat{\tau}^-) \neq 0$, then the perturbed system (4.15) has a two-zonal limit cycle in a neighborhood of $\chi_{\hat{\tau}^-}$, for $|\varepsilon| \neq 0$ sufficiently small.

Proof: By taking into account that the Melnikov function in the case $\lambda^+ > 0$ and $\lambda^- < 0$ is continuous in $[0, 2\pi]$ and moreover (see (4.41))

$$M(0) = 2\Lambda^+\pi\sqrt{1 + \frac{1}{(\lambda^+)^2}} \quad \text{and} \quad M(2\pi) = 2\Lambda^-\pi\sqrt{1 + \frac{1}{(\lambda^-)^2}},$$

it follows that the Melnikov function M possesses a zero $\hat{\tau}^- \in (0, 2\pi)$. Then, from Theorem 4.10 we know that if this root is simple and $\hat{\tau}^- \neq \tau_*^-$, then the perturbed system (4.1) has a limit cycle in a neighborhood of the periodic orbit $\chi_{\hat{\tau}^-}$ and the proof is finished. □

Remark 4.18 Usually, conditions imposed in Theorem 4.17 will be satisfied, because $\tau^- = \tau_*^-$ would be a solution of $M(\tau^-) = 0$ when Λ^+, Λ^- and B satisfy condition

$$\Lambda^- a + \Lambda^+ b + Bc = 0, \tag{4.43}$$

where

$$\begin{aligned} a &= \int_0^{\tau_*^-} (\lambda^- \cos(\theta_0 - t) + \sin(\theta_0 - t))x^-(t)dt + \\ b &= \int_0^{2\pi-\tau_*^-} (\cos(\theta_1 - t) - \lambda^+ \sin(\theta_1 - t))x^+(t)dt + \\ c &= \int_0^{2\pi-\tau_*^-} (\lambda^+ \cos(\theta_1 - t) + \sin(\theta_1 - t))x^+(t)dt. \end{aligned}$$

This relationship is not generic. It is the equation of a plane in the three-dimensional space Λ^+–Λ^-–B.

For instance, if $\lambda^+ = 0.15, \lambda^- = -0.25, \Lambda^+ = 0.1, \Lambda^- = -0.2$, there exists a unique B which satisfies (4.43), and its approximate value is $B = \bar{B} \simeq 0.33979$. That is, if $B \neq \bar{B}$, then Theorem 4.17 can be applied. Nevertheless, for $B = \bar{B}$ Theorem 4.17 cannot be applied, and in this case we do not know if the periodic orbit exists or not in a neighborhood of $\chi_{\tau_*^-}$. Note that, in the reversible case, it remains, as we have described in Remark 4.12.

Now, we will look for the existence of periodic orbits of the perturbed system (4.15), close to the periodic orbits of the unperturbed system (4.1) which are tangent to the separation plane $x = 0$.

We want to study the behavior near to periodic orbits χ_0 and $\chi_{2\pi}$ of the unperturbed system, when this makes sense.

To prove the existence of limit cycles in the perturbed system close to the periodic orbits χ_0 and $\chi_{2\pi}$ of the unperturbed system, we must study the existence of simple zeros of the Melnikov function defined in (4.31) near to $\tau^- = 0$ and $\tau^- = 2\pi$.

In the following results, we use the parameters

$$\begin{aligned} \sigma^+ &= B(\lambda^- - \lambda^+) + (\Lambda^- - \Lambda^+)(1 + \lambda^-\lambda^+), \\ \sigma^- &= B(\lambda^+ - \lambda^-) + (\Lambda^- - \Lambda^+)(1 + \lambda^-\lambda^+). \end{aligned} \tag{4.44}$$

Firstly, we state the next result about the existence of a periodic orbit of the perturbed system close to the periodic orbit χ_0 of the unperturbed. The following development and reasonings are similar to those used in Section 3.5.

Theorem 4.19 Consider the perturbed system (4.15) in the cases $\lambda^+ > 0,\ \lambda^- < 0$, or $\lambda^+ > 0$, $\lambda^- \geq 0$. Suppose $\Lambda^+ \cdot \sigma^+ < 0$ and $|\Lambda^+|$ is sufficiently small. Then, the perturbed system (4.15) has a two-zonal limit cycle in a neighborhood of $\chi_{\hat{\tau}^-}$, for $|\varepsilon| \neq 0$ and sufficiently small, where $\hat{\tau}^-$ is the only solution of the Melnikov function close to $\tau^- = 0$.

Furthermore, one of the characteristic multipliers of this limit cycle is always strictly greater than 1 and the other one is strictly greater than 1 if $\sigma^+ \cdot \varepsilon > 0$ and strictly less than 1 if $\sigma^+ \cdot \varepsilon < 0$.

Proof: On the one hand, we define the following function

$$F(s, \Lambda^+) = M_r(s^{1/3}, \Lambda^+),$$

where we have assumed that the Melnikov reduced function is function of Λ^+ too. Then,

$$F(0,0) = 0 \text{ and } \frac{\partial F}{\partial s}(0,0) = \sigma^+/(1 + (\lambda^+)^2) \neq 0.$$

By applying the Implicit Function Theorem we deduce that there exists a function f which is defined in a neighborhood U of the origin, such that $F(f(\Lambda^+), \Lambda^+) = 0$ for all $\Lambda^+ \in U$, i.e., the equation $M_r(\tau^-) = 0$ has one solution $\hat{\tau}^- = (f(\Lambda^+))^{1/3}$ provided that $|\Lambda^+|$ is small enough. Moreover, it is easy to see that sgn$(f(\Lambda^+)) = -$sgn$(\Lambda^+ \cdot \sigma^+)$. Therefore, the solution $\hat{\tau}^-$ is positive when $\Lambda^+ \cdot \sigma^+ < 0$. Moreover, sgn$(M_r'(\hat{\tau}^-))$ =sgn$(\sigma^+) \neq 0$ and from (4.40) sgn$(M_r'(\hat{\tau}^-))$ =sgn$(M'(\hat{\tau}^-))$. Then, sgn$(M'(\hat{\tau}^-))$ =sgn$(\sigma^+) \neq 0$.

Hence, there exists a positive simple root $\hat{\tau}^- > 0$ of the Melnikov function M near to $\tau^- = 0$, and we can suppose that it is different from τ_*^- given in (4.11). Furthermore, from (4.40) it is satisfied that

$$\text{sgn}(M'(\hat{\tau}^-)) = \text{sgn}(M_r'(\hat{\tau}^-)),$$

and it is obvious that if $|\Lambda^+| \simeq 0$ and $\sigma^+ \neq 0$, then

$$\text{sgn}(M_r'(\hat{\tau}^-)) = \text{sgn}(\sigma^+) \neq 0,$$

so we get that

$$\text{sgn}(M'(\hat{\tau}^-)) = \text{sgn}(\sigma^+) \neq 0.$$

On the other hand, simple roots of the Melnikov function different from τ_*^- correspond to periodic orbits of the perturbed system (4.15). Then the system has, for $|\varepsilon| \neq 0$ and sufficiently small, a two-zonal limit cycle in a neighborhood of the periodic orbit $\chi_{\hat{\tau}^-}$ of the unperturbed system (4.1), which we denote by $\Upsilon_{\hat{\tau}^-}$ and is near to χ_0. Now, we are going to examine the characteristic multipliers of

$\Upsilon_{\hat{\tau}^-}$.

The characteristic multipliers are the eigenvalues of the Jacobian matrix of the Poincare map evaluated at the periodic orbit. From the analysis done in [23], the characteristic multipliers of $\chi_{\hat{\tau}^-}$ for the unperturbed system (4.1) are $\mu_1 = \exp(\lambda^-\hat{\tau}^- + \lambda^+(2\pi - \hat{\tau}^-))$ and $\mu_2 = 1$. As $\hat{\tau}^- \simeq 0$ and $\lambda^+ > 0$ yields $\mu_1 > 1$. Due to the continuity and differentiability, one characteristic multiplier of the periodic orbit $\Upsilon_{\hat{\tau}^-}$ must be greater than 1 if $|\Lambda^+|$ and ε are different from zero and small enough.

For determining the other characteristic multiplier, we must study the sign of $\frac{\partial d_2}{\partial r}(g_{\theta(\hat{\tau}^-)}(r(\hat{\tau}^-),\varepsilon), r(\hat{\tau}^-),\varepsilon)$. From the analysis developed up to now, we have $d_2(\theta, r, \varepsilon) = \varepsilon D_2(\theta, r, 0) + O(\varepsilon)$, so

$$\frac{\partial d_2}{\partial r}(g_{\theta(\hat{\tau}^-)}(r(\hat{\tau}^-),\varepsilon), r(\hat{\tau}^-),\varepsilon) = \varepsilon\frac{\partial \widetilde{D}_2}{\partial r}(r(\hat{\tau}^-),0) + O(\varepsilon).$$

We know $\widetilde{D}_2(r(\hat{\tau}^-),0) = M(\hat{\tau}^-)$, therefore

$$\frac{\partial \widetilde{D}_2}{\partial r}(r(\hat{\tau}^-),0) = \frac{\frac{\partial M}{\partial \tau^-}(\hat{\tau}^-)}{\frac{\partial r}{\partial \tau^-}(\hat{\tau}^-)},$$

and the denominator is different from zero provided that we are not in the most external periodic orbit of the continuum. Then, if $\frac{\partial M}{\partial \tau^-}(\hat{\tau}^-) \neq 0$, we are able to establish the sign of $\frac{\partial d_2}{\partial r}(g_{\theta(\hat{\tau}^-)}(r(\hat{\tau}^-),\varepsilon), r(\hat{\tau}^-),\varepsilon)$.

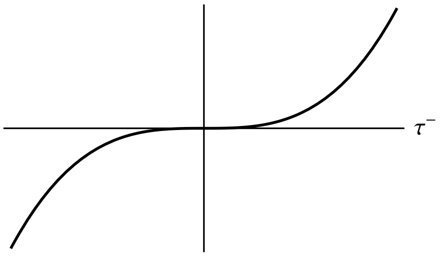
We know that sgn$\left(\frac{\partial M}{\partial \tau^-}(\hat{\tau}^-)\right)$=sgn$(\sigma^+) \neq 0$. Moreover $\frac{\partial r}{\partial \tau^-}(\hat{\tau}^-) > 0$ if we are near to the periodic orbit $\chi_{\hat{\tau}^-}$ which is close to χ_0. As a result, we conclude sgn$\left(\frac{\partial d_2}{\partial r}(g_{\theta(\hat{\tau}^-)}(r(\hat{\tau}^-),\varepsilon), r(\hat{\tau}^-),\varepsilon)\right) =$ sgn$(\varepsilon \cdot \sigma^+)$, and the proof is completed. □

Analogously, we can prove the following result, which provides us the existence of a periodic orbit of the perturbed system close to the periodic orbit $\chi_{2\pi}$ of the unperturbed system, which is tangent to the separation plane.

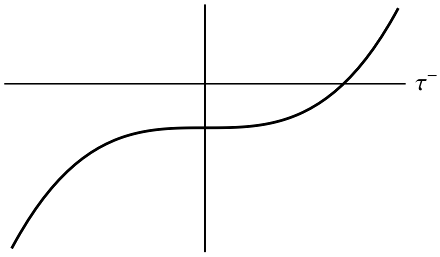
Theorem 4.20 Consider the perturbed system (4.15) in the cases $\lambda^+ > 0$, $\lambda^- < 0$, or $\lambda^+ \leq 0$, $\lambda^- < 0$. Suppose $\Lambda^- \cdot \sigma^- > 0$ and $|\Lambda^-|$ is sufficiently small. Then, the perturbed system (4.15) has a two-zonal limit cycle in a neighborhood of $\chi_{\hat{\tau}^-}$, for $|\varepsilon| \neq 0$ and sufficiently small, where $\hat{\tau}^-$ is the only solution of the Melnikov function close to $\tau^- = 2\pi$.

Furthermore, one of the characteristic multipliers is always strictly less than 1 and the other one is strictly greater than 1 if $\sigma^- \cdot \varepsilon < 0$ and strictly less than 1 if $\sigma^- \cdot \varepsilon > 0$.

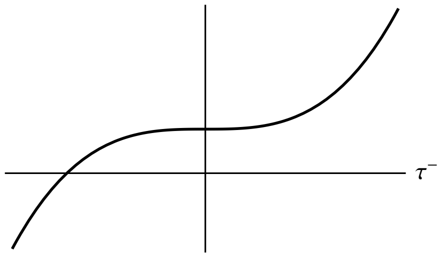
In Fig. 4.7 we have represented the different possibilities for the graphic of the Melnikov function M in a neighborhood of $\tau^- = 0$ for $|\Lambda^+|$ small and $\sigma^+ > 0$.



(a) $\Lambda^+ = 0$.



(b) $\Lambda^+ < 0$.



(c) $\Lambda^+ > 0$.

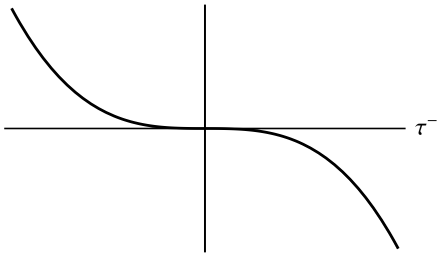
Figure 4.7: Graphic of the Melnikov function M in a neighborhood of $\tau^- = 0$ for $|\Lambda^+|$ sufficiently small and $\sigma^+ > 0$.

In Fig. 4.8 we have represented different possibilities of the graphic of the Melnikov function M in a neighborhood of $\tau^- = 0$ for $|\Lambda^+|$ small and $\sigma^+ < 0$.

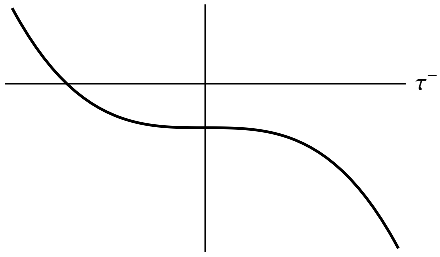
4.5.2 Saddle-node bifurcation of periodic orbits

In the results given in the previous subsection, we assumed that σ^+ and σ^- were different from zero, but we can suppose that they vanish. In this case, we can have up to two zeros of the Melnikov function and a saddle-node bifurcation of periodic orbits can be described. To do that, we will need Lemma 3.17 presented in Chapter 3.

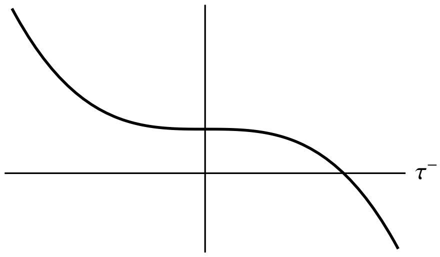
Theorem 4.21 Consider the perturbed system (4.15) in the cases $\lambda^+ > 0$, $\lambda^- < 0$, or $\lambda^+ \leq 0$, $\lambda^- < 0$. Assume $\Lambda^+ \neq \Lambda^-$, $\Lambda^+ \cdot \Lambda^- > 0$, $\Lambda^+ \cdot \sigma^+ < 0$ and $|\Lambda^+|$,$|\sigma^+|$ are sufficiently small. Then,



(a) $\Lambda^+ = 0$.



(b) $\Lambda^+ < 0$.



(c) $\Lambda^+ > 0$.

Figure 4.8: Graphic of the Melnikov function M in a neighborhood of $\tau^- = 0$ for $|\Lambda^+|$ sufficiently small and $\sigma^+ < 0$.

there is a function $SN_0(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, B)$ defined locally by

$$\begin{aligned} &SN_0(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, \sigma^+) = \\ &= \frac{120\Lambda^+\pi}{(1+(\lambda^+)^2)(\Lambda^- - \Lambda^+)} - 24\sqrt{6}\left(-\frac{\sigma^+}{(1+(\lambda^+)^2)(\Lambda^- - \Lambda^+)}\right)^{5/2} + \dots \end{aligned} \tag{4.45}$$

such that the following statements hold.

(a) If $\Lambda^+ \cdot SN_0(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, B) < 0$ and $|\varepsilon| \neq 0$ is sufficiently small, the perturbed system (4.15) has two two-zonal periodic orbits in a neighborhood of χ_0.

(b) If $SN_0(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, B) = 0$, there exist functions $\Lambda^+(\varepsilon)$, $\Lambda^-(\varepsilon)$ and $B(\varepsilon)$, defined for $|\varepsilon|$ sufficiently small, such that the perturbed system (4.15) with $\Lambda^+ = \Lambda^+(\varepsilon)$, $\Lambda^- = \Lambda^-(\varepsilon)$ and $B = B(\varepsilon)$ has exactly one periodic orbit in a neighborhood of χ_0 for $|\varepsilon| \neq 0$ sufficiently small.

(c) If $\Lambda^+ \cdot SN_0(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, B) > 0$ and $|\varepsilon| \neq 0$ is sufficiently small, the perturbed system (4.15) has not periodic orbits in a neighborhood of χ_0.

Proof: The reduced Melnikov function can be written in a neighborhood of $\tau^- = 0$ into the form

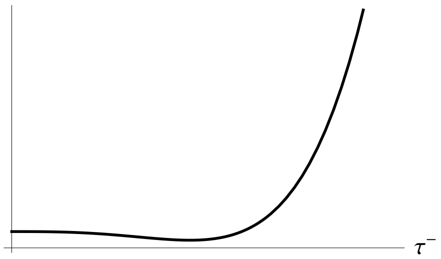
$$M_r(\tau^-) = 2\Lambda^+\pi + \frac{\sigma^+}{6(1+(\lambda^+)^2)}(\tau^-)^3 + \frac{(\Lambda^- - \Lambda^+)(1+(\lambda^+)^2) + O(\sigma^+)}{60}(\tau^-)^5 + O((\tau^-)^6).$$

Therefore, if $(\Lambda^+, \sigma^+) \simeq (0,0)$, then we can find roots of $M_r(\tau^-) = 0$ for $|\tau^-|$ sufficiently small solving

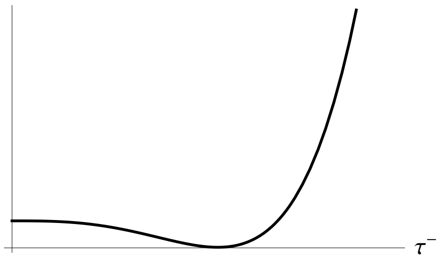
$$2\Lambda^+\pi + \frac{\sigma^+}{6(1+(\lambda^+)^2)}(\tau^-)^3 + \frac{(\Lambda^- - \Lambda^+)(1+(\lambda^+)^2)}{60}(\tau^-)^5 = 0.$$

Finally, by applying Lemma 3.17, the Weierstrass Preparation Theorem and analogous reasoning to those given in the proof of Theorem 1.3 of [8], the conclusion is straightforward. □

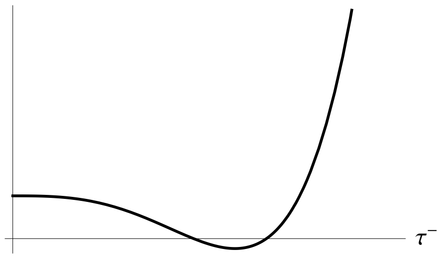
In Fig. 4.9 we have represented different possibilities for the graphic of the Melnikov function M in a neighborhood of $\tau^- = 0$ for $\Lambda^+ \cdot \Lambda^- > 0$, $\Lambda^+ \cdot \sigma^+ < 0$ with $|\Lambda^+|$ and $|\sigma^+|$ small enough.



(a) $\Lambda^+ \cdot SN_0(\Lambda^+, \Lambda^-, B, \lambda^+, \lambda^-) > 0.$



(b) $SN_0(\Lambda^+, \Lambda^-, B, \lambda^+, \lambda^-) = 0.$



(c) $\Lambda^+ \cdot SN_0(\Lambda^+, \Lambda^-, B, \lambda^+, \lambda^-) < 0.$

Figure 4.9: Graphic of the Melnikov function M in a neighborhood of $\tau^- = 0$ for $\Lambda^+ \cdot \Lambda^- > 0$, $\Lambda^+ \cdot \sigma^+ < 0$ with $|\Lambda^+|$ and $|\sigma^+|$ small enough.

An analogous analysis can be done for $\chi_{2\pi}$, when this makes sense.

Theorem 4.22 Consider the perturbed system (4.15) in the cases $\lambda^+ > 0$, $\lambda^- < 0$, or $\lambda^+ \leq 0$, $\lambda^- < 0$. Assume $\Lambda^+ \neq \Lambda^-$, $\Lambda^+ \cdot \Lambda^- > 0$, $\Lambda^- \cdot \sigma^- > 0$ and $|\Lambda^-|$, $|\sigma^-|$ are sufficiently small. Then, there is a function $SN_{2\pi}(\Lambda^+, \Lambda^-, B, \lambda^+, \lambda^-)$ defined locally by

$$SN_{2\pi}(\Lambda^+, \Lambda^-, B, \lambda^+, \lambda^-) = \\ = 2\Lambda^-\pi - \frac{\sigma^-}{15(1+(\lambda^-)^2)} \left(\frac{-6\sigma^-}{(\Lambda^- - \Lambda^+)(1+(\lambda^-)^2)(1+(\lambda^+)^2)} \right)^{3/2} + \ldots \tag{4.46}$$

such that the next statements hold.

(a) If $\Lambda^- \cdot SN_{2\pi}(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, B) > 0$ and $|\varepsilon| \neq 0$ is sufficiently small, the perturbed system (4.15) has two two-zonal periodic orbits in a neighborhood of $\chi_{2\pi}$.

(b) If $SN_{2\pi}(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, B) = 0$, there exist functions $\Lambda^+(\varepsilon)$, $\Lambda^-(\varepsilon)$ and $B(\varepsilon)$ defined for $|\varepsilon|$ sufficiently small, such that the perturbed system (4.15) with $\Lambda^+ = \Lambda^+(\varepsilon)$, $\Lambda^- = \Lambda^-(\varepsilon)$ and $B = B(\varepsilon)$ has exactly one periodic orbit in a neighborhood of $\chi_{2\pi}$ for $|\varepsilon| \neq 0$ sufficiently small.

(c) If $\Lambda^- \cdot SN_{2\pi}(\Lambda^+, \Lambda^-, \lambda^+, \lambda^-, B) < 0$ and $|\varepsilon| \neq 0$ sufficiently small, the perturbed system (4.15) has not periodic orbits in a neighborhood of $\chi_{2\pi}$.

Remark 4.23 It is worth mentioning here that theorems 4.19 and 4.20 describe the existence of periodic orbits in the perturbed system (4.15). We can say that these periodic orbits arise, in the hypothesis of Theorem 4.19 (respectively 4.20), from the periodic orbit tangent to the separation plane $x = 0$ which is in the half-space $x \geq 0$ (respectively $x \leq 0$). The appearance of this periodic orbit is known as the focus-center-limit cycle bifurcation. A generic situation of this bifurcation is described in Theorem 1 of [24]. It is not possible to apply this result to our unperturbed system (4.1) because the coefficient δ of Theorem 1 of [24], which characterize the bifurcation, is zero for our system, namely,

$$\delta = -2(\Lambda^- + B(\lambda^- - \lambda^+) + (\Lambda^- + \Lambda^+)\lambda^-\lambda^+ - \Lambda^+(\lambda^+)^2)\varepsilon + O(\varepsilon^2).$$

We can say that theorems 4.19 and 4.20 give conditions for spreading out in one direction the focus-center-limit cycle bifurcation when $\delta = 0$ and theorems 4.21 and 4.22 study the degeneration of this bifurcation.

4.5.3 Hopf bifurcation from infinity

This subsection is devoted to the study of the Hopf bifurcation from infinity for part of the systems object of study in this chapter. We want to analyze, under the presence of an unbounded continuum, the existence of zeros of the Melnikov function, close to the value $\widetilde{\tau}^-$ which was described in Proposition 4.2.

We will analyze the case $\lambda^+ = \lambda^- \neq 0$. In this case, as we have described in the previous section, the Melnikov function takes the form given in (4.42), the reduced Melnikov function is

$$M_r(\tau^-) = 2\Lambda^+\pi - (\Lambda^+ - \Lambda^-)\tau^- + (\Lambda^+ - \Lambda^-)\sin\tau^- \tag{4.47}$$

and the value $\widetilde{\tau}^- = \pi$.

Before stating the result about the Hopf bifurcation from infinity, we give a result about the existence of periodic orbits.

Theorem 4.24 Assume that $\lambda^+ = \lambda^- \neq 0$. Then, the following conditions are satisfied.

(a) If $\lambda^+ = \lambda^- > 0$ and $\Lambda^+(\Lambda^+ + \Lambda^-) < 0$, then the perturbed system (4.15) possesses a unique limit cycle for $|\varepsilon| \neq 0$ sufficiently small. Moreover, the limit cycle possesses a characteristic multiplier grater that 1 and another greater that 1 if $\varepsilon(\Lambda^+ + \Lambda^-) > 0$ and less that 1 if $\varepsilon(\Lambda^+ + \Lambda^-) < 0$.

(b) If $\lambda^+ = \lambda^- > 0$ and $\Lambda^+(\Lambda^+ + \Lambda^-) > 0$, then the perturbed system (4.15) has not periodic orbits for $|\varepsilon| \neq 0$ sufficiently small.

(c) If $\lambda^+ = \lambda^- < 0$ and $\Lambda^-(\Lambda^+ + \Lambda^-) < 0$, then the perturbed system (4.15) possesses a unique limit cycle for $|\varepsilon| \neq 0$ sufficiently small. Moreover, the limit cycle possesses a characteristic multiplier less that 1 and another less that 1 if $\varepsilon(\Lambda^+ + \Lambda^-) < 0$ and greater than 1 if $\varepsilon(\Lambda^+ + \Lambda^-) > 0$.

(d) If $\lambda^+ = \lambda^- > 0$ and $\Lambda^-(\Lambda^+ + \Lambda^-) > 0$, then the perturbed system (4.15) has not periodic orbits for $|\varepsilon| \neq 0$ sufficiently small.

Proof: It is enough to study the growth of the reduced Melnikov function in its interval of definition and its value at the endpoints of that interval.

By remembering that in this case the reduced Melnikov function has the expression given in (4.47), we have that

$$M_r'(\tau^-) = -(\Lambda^+ - \Lambda^-) + (\Lambda^+ - \Lambda^-)\cos\tau^-$$

does not vanishes in the interval $(0, 2\pi)$, so M_r is strictly increasing in such interval.

Furthermore, $M_r(0) = 2\Lambda^+\pi$, $M_r(\pi) = (\Lambda^+ + \Lambda^-)\pi$, $M_r(2\pi) = 2\Lambda^-\pi$ and taking into account the chosen signs in each item, the proof is direct, just by using a compactness argument for items (b) and (d), and reminding that periodic orbits, if they exist, are in a plane.

Finally, the conclusions about the characteristic multipliers are straightforward, by using similar reasonings to those made in the proof of Theorem 4.19. □

Remark 4.25 The behavior described in Theorem 4.24 agrees with the study made in [13, 41, 44, 85] about periodic orbits of planar piecewise linear systems with two zones. From these works, it can be deduced that a system written in Liénard form

$$\begin{cases} \dot{x} = 2\alpha^+ x - y, \\ \dot{y} = ((\alpha^+)^2 + (\beta^+)^2)x + a \end{cases} \text{if } x \geq 0, \qquad \begin{cases} \dot{x} = 2\alpha^- x - y, \\ \dot{y} = ((\alpha^-)^2 + (\beta^-)^2)x + a \end{cases} \text{if } x < 0, \tag{4.48}$$

with $a \in \{-1, 1\}$, only possesses a limit cycle in the following situations.

(a) The equilibrium point is in the zone $x > 0$ and $\alpha^+ \left(\dfrac{\alpha^+}{\beta^+} + \dfrac{\alpha^-}{\beta^-}\right) < 0$.

(b) The equilibrium point is in the zone $x < 0$ and $\alpha^- \left(\dfrac{\alpha^+}{\beta^+} + \dfrac{\alpha^-}{\beta^-}\right) < 0$.

When $\lambda^+ = \lambda^- \neq 0$, the dynamics of the perturbed system (4.15) is basically planar and on the invariant plane it is organized by a system of the form (4.48), where $\alpha^+ = \varepsilon\Lambda^+$, $\alpha^+ = \varepsilon\Lambda^+$, $\beta^+ = 1 + \varepsilon B$ and $\beta^- = 1$. When $|\varepsilon| \neq 0$ is sufficiently small, it is easy to see that conditions of Theorem 4.24 are not more than a new redaction of the results about the existence of limit cycles for system (4.48) which were described in [13, 41, 44, 85].

Finally, we give a result which collects the appearance of a large periodic orbit, i.e., a result that proves the existence of a Hopf bifurcation from infinity.

Theorem 4.26 Assume that $\lambda^+ = \lambda^- \neq 0$. Then, the following properties hold.

(a) If $\lambda^+ = \lambda^- > 0$, $\Lambda^+(\Lambda^+ + \Lambda^-) < 0$ and $|\Lambda^+ + \Lambda^-|$ is sufficiently small, then the perturbed system (4.15) possesses a unique limit cycle for $|\varepsilon| \neq 0$ small enough with left half-period $\tau^- \simeq \pi$ and less than π. Moreover, this limit cycle is born with infinite amplitude.

(b) If $\lambda^+ = \lambda^- < 0$, $\Lambda^-(\Lambda^+ + \Lambda^-) < 0$ and $|\Lambda^+ + \Lambda^-|$ is sufficiently small, then the perturbed system (4.15) possesses a unique limit cycle for $|\varepsilon| \neq 0$ small enough with left half-period $\tau^- \simeq \pi$ and grater than π. Moreover, this limit cycle is born with infinite amplitude.

Proof: It is enough to take into account that $M_r(\pi) = 0$ when $\Lambda^+ + \Lambda^- = 0$.

Now, if we just develop M_r in a neighborhood of π and apply the reasonings from theorems 4.19 and 4.20 to guarantee the existence of zeros of M_r close to π when $|\Lambda^+ + \Lambda^-|$ is sufficiently small.

Note that the limit cycle is born with infinite amplitude because the zeros that appear are close to π. □

In Fig. 4.10 we show some graphics for illustrating the behavior of the Melnikov function in a vicinity of the appearance of the Hopf bifurcation from infinity.

In this chapter, we have analyzed the existence of periodic orbits in a perturbation of a class of three-dimensional non-controllable CPWL systems via an adaptation of the Melnikov theory. In the next chapter, we use different skills to analyze the existence of periodic orbits in a perturbation of a class of three-dimensional non-observable CPWL systems.

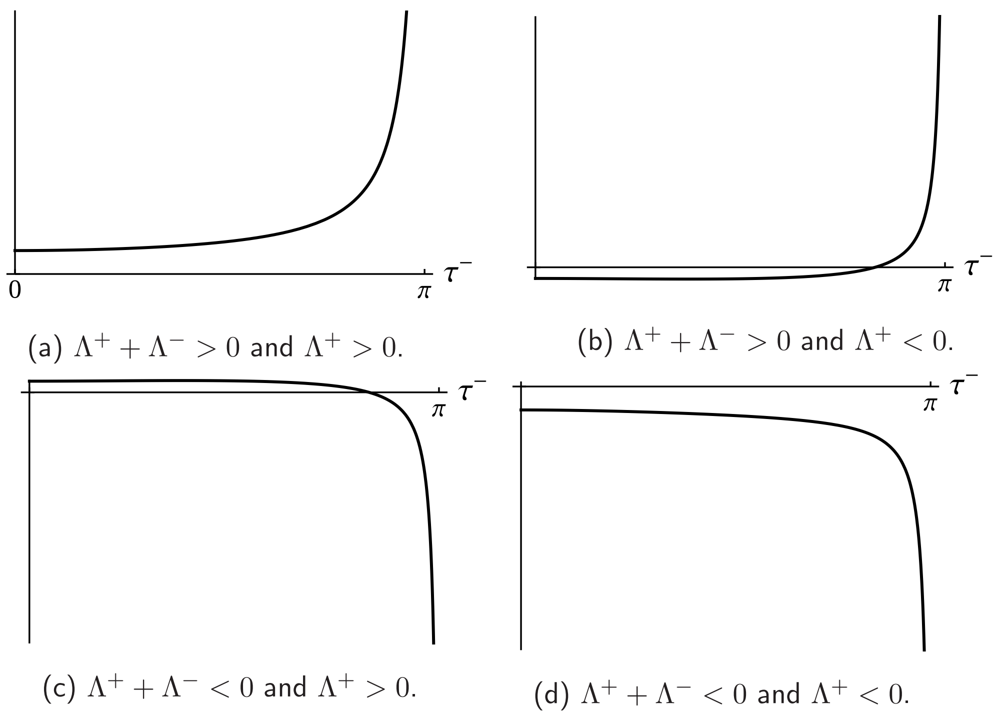


Figure 4.10: Melnikov function M in cases $\lambda^+ = \lambda^- > 0$ and $\Lambda^+ + \Lambda^- \neq 0$.

Chapter 5

Periodic Orbits for Perturbations of Three-Dimensional Non-Observable Continuous Piecewise Linear Systems.

At it was pointed out in the last section of the introductory chapter, the existence of invariant cones is an important matter for analyzing the stability of homogeneous $2CPWL_3$ systems.

The works about invariant cones that we can find in the current literature [21, 25, 65, 66] assume observability hypothesis. In Chapter 3 of this work, we have extended the results obtained in [21] and we have proven the conjecture about the existence of saddle-node bifurcation of invariant cones that was stated in [25]. At the beginning of this chapter, we deal with the analysis of invariant cones in non-observable homogeneous $2CPWL_3$ systems.

In [26] the existence of an observable $2CPWL_3$ system having an invariant cone foliated by periodic orbits is proven. However, it was not possible to characterize this property. In this chapter, we consider the problem in non-observable PWL systems, and we are able to find explicitly a system having an invariant cone foliated by periodic orbits. Once we have this non-generic situation, one can wonder about the number and position of the periodic orbits that remain if one perturbs the system, by making it observable and non-homogeneous.

To analyze the existence of periodic orbits in dynamical systems, apart from the Menikov theory, we find in the literature the averaging method [87, 94]. The main idea of this method is to relate the periodic solutions of a system to the equilibria of an autonomous one, the averaged differential system. In [9] the authors analyze from the point of view of the averaging method, the persistence of periodic orbits of an invariant manifold of periodic solutions. The method of averaging has been generalized to continuous non-smooth dynamical systems [10, 11, 13, 64]. However, in this Chapter we have opted to tackle the problem from another point of view. Specifically, we write the system in the appropriate form to be able to use the techniques of Chapter 14 of [31] where non-autonomous perturbations of a linear system with periodic solutions were studied.

The chapter is organized as follows. In a first section we remember the normal form of non-observable 2CPWL$_3$ systems introduced in the first chapter of this work. In a second section, we analyze the existence of invariant cones in homogeneous non-observable 2CPWL$_3$ systems. Sec. 5.3 is devoted to the obtention of an homogeneous non-observable 2CPWL$_3$ system having an invariant cone foliated by periodic orbit, the unperturbed system. In Sec. 5.4, this non-generic situation is perturbed, by making the system observable and non-homogeneous. The last section of this chapter is devoted to the analysis of the persistence of the periodic orbits of the continuum after the perturbation.

5.1 Non-observable 2CPWL$_3$ systems

In the introductory Chapter 1 we have discussed that every 2CPWL$_3$ system can be written into the form (1.4), namely,

$$\dot{\mathbf{x}} = A^{\triangledown}\mathbf{x} + \mathbf{b} = \left(\begin{array}{c|c} a_{11}^{\triangledown} & A_{12} \\ \hline A_{21}^{\triangledown} & A_{22} \end{array}\right)\mathbf{x} + \mathbf{b}, \tag{5.1}$$

where $\mathbf{b} = (b_1, b_2, b_3)^T \in \mathbb{R}^3$, $a_{11}^+, a_{11}^- \in \mathbb{R}$, $A_{12} \in \mathcal{M}_{1\times 2}(\mathbb{R})$, $A_{21}^+, A_{21}^- \in \mathcal{M}_{2\times 1}(\mathbb{R})$ and $A_{22} \in \mathcal{M}_2(\mathbb{R})$.

Furthermore, if system (5.1) is non-observable and $A_{12} \neq (0,0)^T$, then there exist a linear change of variables which transforms it into the form (1.6), i.e.,

$$\left\{\begin{array}{l} \dot{x} = b_{11}^{\triangledown}x - y, \\ \dot{y} = b_{21}^{\triangledown}x + b_{22}y + b_2, \\ \dot{z} = b_{31}^{\triangledown}x + b_{32}y + b_{33}z + b_3. \end{array}\right. \tag{5.2}$$

In looking for invariant cones, we are interested in homogeneous systems. Under the hypothesis of homogeneity, the system is

$$\left\{\begin{array}{l} \dot{x} = b_{11}^{\triangledown}x - y, \\ \dot{y} = b_{21}^{\triangledown}x + b_{22}y, \\ \dot{z} = b_{31}^{\triangledown}x + b_{32}y + b_{33}z. \end{array}\right. \tag{5.3}$$

In the following proposition, system (5.3) will be further simplified.

Proposition 5.1 There exists a linear change of variable which transforms system (5.3) into

$$\begin{cases} \dot{x} = c_{11}^{\bigtriangledown}x - y, \\ \dot{y} = c_{21}^{\bigtriangledown}x, \\ \dot{z} = c_{31}^{\bigtriangledown}x + c_{33}z, \end{cases} \tag{5.4}$$

where

$$c_{11}^{\pm} = b_{11}^{\pm} + b_{22}, \quad c_{21}^{\pm} = b_{21}^{\pm}, \quad c_{31}^{\pm} = b_{32}b_{11}^{\pm} + b_{31}^{\pm} - b_{32}b_{33}, \quad c_{33} = b_{33}. \tag{5.5}$$

Proof: By performing the change of variable

$$Z = b_{32}x + z$$

and just renaming the variable Z as z, it is easy to see that system (5.3) is transformed into

$$\begin{cases} \dot{x} = b_{11}^{\bigtriangledown}x - y, \\ \dot{y} = b_{21}^{\bigtriangledown}x + b_{22}y, \\ \dot{z} = (b_{32}b_{11}^{\bigtriangledown} + b_{31}^{\bigtriangledown} - b_{32}b_{33})x + b_{33}z. \end{cases} \tag{5.6}$$

Now, by changing

$$Y = b_{22}x + y$$

and renaming the variable Y as y, system (5.7) is transformed into

$$\begin{cases} \dot{x} = (b_{11}^{\bigtriangledown} + b_{22})x - y, \\ \dot{y} = b_{21}^{\bigtriangledown}x, \\ \dot{z} = (b_{32}b_{11}^{\bigtriangledown} + b_{31}^{\bigtriangledown} - b_{32}b_{33})x + b_{33}z \end{cases} \tag{5.7}$$

and the conclusion follows. □

5.2 Invariant cones in non-observable 2CPWL$_3$ systems

In this section we analyze the existence of invariant cones in non-observable 2CPWL$_3$ systems, which we have just seen that can be written into the form (5.4). In looking for invariant cones, by analogy with the observable case, we will assume the existence of complex eigenvalues in both zones of linearity. Then, system (5.4) can be expressed in terms of the eigenvalues of the coefficient matrices, namely, $\alpha^{\pm}\pm i\beta^{\pm}$ and λ, as

$$\left\{\begin{array}{l} \dot{x} = 2\alpha^{\triangledown}x - y, \\ \dot{y} = ((\alpha^{\triangledown})^2 + (\beta^{\triangledown})^2)x, \\ \dot{z} = c_{31}^{\triangledown}x + \lambda z, \end{array}\right. \tag{5.8}$$

where

$$\alpha^{\pm} = \frac{1}{2}c_{11}^{\pm}, \quad \beta^{\pm} = \frac{1}{2}\sqrt{4c_{21}^{\pm} - (c_{11}^{\pm})^2} \quad \text{and} \quad \lambda = c_{33}. \tag{5.9}$$

Note that λ is a shared real eigenvalue of both coefficient matrices due to the lack of the observability. Moreover, we assume in the rest of this chapter that the system cannot be fully decoupled in any of both zones of linearity, i.e., $c_{31}^{+} \cdot c_{31}^{-} \neq 0$.

Later on, it will be useful to introduce the following notation

$$\gamma^{\pm} = \frac{\alpha^{\pm} - \lambda}{\beta^{\pm}} \quad \text{and} \quad \delta^{\pm} = \frac{(\alpha^{\pm} - \lambda)^2 + (\beta^{\pm})^2}{c_{31}^{\pm}} \neq 0. \tag{5.10}$$

Before proceeding analyzing the two-zonal invariant cones of system (5.8), we will show some properties of planar invariant surfaces for this system and their relative position with respect to the two-zonal invariant cones. The following study is an adaptation to non-observable systems of the study done in [25] for observable systems. We begin by introducing some concepts.

Let A be a real matrix with eigenvalues λ and $\alpha \pm i\beta$, where $\beta > 0$. The invariant plane containing the origin for the system $\dot{\mathbf{x}} = A\mathbf{x}$ will be referred to as the focal plane of the matrix A. That focal plane is also a planar invariant cone for the system $\dot{\mathbf{x}} = A\mathbf{x}$.

Lemma 5.2 If the real matrix M has the form

$$M = \begin{pmatrix} 2\alpha & -1 & 0 \\ (\alpha^2 + \beta^2) & 0 & 0 \\ c_{31} & 0 & \lambda \end{pmatrix}$$

with $\beta > 0$ and $c_{31} \neq 0$, then its focal plane is given by $\Pi_F \equiv -\lambda x + y - \delta z = 0$, where

$$\delta = \frac{(\alpha - \lambda)^2 + \beta^2}{c_{31}}.$$

Proof: It is obvious that the eigenvalues of matrix M are given by λ and $\alpha \pm i\beta$. It is well known that if $\mathbf{v}$ is an eigenvector of M^T associated to the real eigenvalue λ, then $\mathbf{v}$ is orthogonal to the focal plane of M. Here, it is easy to see that $\mathbf{v} = (-\lambda, 1, -\delta)^T$ is an eigenvector of M^T associated to λ, and the conclusion follows. □

By using Lemma 5.2 under the assumption $c_{31}^+ \cdot c_{31}^- \neq 0$, coefficient matrices of both zones of linearity of system (5.8) have, respectively, the focal planes

$$\Pi_F^+ \equiv -\lambda x + y - \delta^+ z = 0 \quad \text{and} \quad \Pi_F^- \equiv -\lambda x + y - \delta^- z = 0, \tag{5.11}$$

where $\delta^\pm$ are given in (5.10).

Next, the focal half-planes of system (5.8) are introduced.

Definition 5.3 The half-plane

$$\Pi_{HF}^- \equiv \Pi_F^- \cap \{x \leq 0\}, \tag{5.12}$$

where Π_F^- is given in (5.11) is said to be the left focal half-plane of system (5.8) and the half-plane

$$\Pi_{HF}^+ \equiv \Pi_F^+ \cap \{x \geq 0\}, \tag{5.13}$$

where Π_F^+ is given in (5.11) is said to be the right focal half-plane of system (5.8).

Note that, when $\delta^+ = \delta^-$, the union of both focal half-planes constitutes a planar two-zonal invariant cone of system (5.8).

Now, the relative position of the focal half-planes and the two-zonal invariant cones, if any, for non-observable CPWL systems will be established. Remember that, we say that a cone is above (respectively, below) a plane if for every point (x_1, y_1, z_1) not at the origin and belonging to the cone, there exists another point (x_1, y_1, z_2) belonging to the plane such that $z_1 > z_2$ (respectively, $z_1 < z_2$).

Lemma 5.4 If system (5.8) has a non-planar invariant cone $\mathcal{C}$, then $\mathcal{C}$ is either above or below both focal half-planes Π_{HF}^+ and Π_{HF}^-.

Proof: If $\delta^- = \delta^+$, then as indicated before, system (5.8) has one invariant plane and the conclusion is straightforward.

Next, we will consider the case $\delta^+ \neq \delta^-$. The focal half-planes intersect the separation plane $x = 0$ at the lines

$$r^+ \equiv \{x = 0, \delta^+ z - y = 0\} \quad \text{and} \quad r^- \equiv \{x = 0, \delta^- z - y = 0\}.$$

At the plane $x = 0$, from the first equation of system (5.8) we have $\dot{x}|_{x=0} = -y$, so the flow enters into the region $x < 0$ for $y > 0$ and enters into the region $x > 0$ for $y < 0$.

Now, if the half straight-line $r_0 \equiv \{z = (1/\delta_0)y, y \geq 0\}$, where $\delta_0 \in \mathbb{R} \setminus \{0\}$ generates a two-zonal invariant cone $\mathcal{C}$, then the line r_0 is transformed into the half straight-line $r_1 = P^-(r_0) \equiv \{x = 0, z = (1/\delta_1)y, y \leq 0\}$, by means of the flow in the region $x < 0$, and we obtain $r_0 = P^+(r_1)$ by means of the flow in the region $x > 0$.

Assume that $\delta^+ > \delta^-$ and the invariant cone $\mathcal{C}$ is above the half-plane Π^-_{HF}, i.e. $\delta^- > \delta_0$ (see Fig. 5.1). If $\delta_1 < \delta^+$, then $P^+(r_1)$ must be below the straight line r^+ (see Fig. 5.1 again). If $\delta_1 = \delta^+$, then $r_1 \equiv r^+ \cap \{y \leq 0\}$ and $P^+(r_1) \equiv r^+ \cap \{y \geq 0\}$. In both cases, $P^+(r_1) \neq r_0$ and r_0 does not generate a two-zonal invariant cone. Consequently, we must have $\delta_1 > \delta^+$, i.e., the invariant cone is also above Π^+_{HF}.

The remaining cases can be analogously proven. □

In the following result we will prove that the non-planar two-zonal invariant cones of system (5.8), if any, do not share points (apart from the origin) with the whole focal planes of the coefficient matrices.

Proposition 5.5 If system (5.8) has a non-planar two-zonal invariant cone $\mathcal{C}$, then $\mathcal{C}$ is either above or below both whole focal planes Π^+_F and Π^-_F.

Proof: If $\delta^+ = \delta^-$, the proof is obvious.

When $\delta^+ \neq \delta^-$, performing in system (5.8) the change

$$Z = -\lambda x + y - \delta^+ z,$$

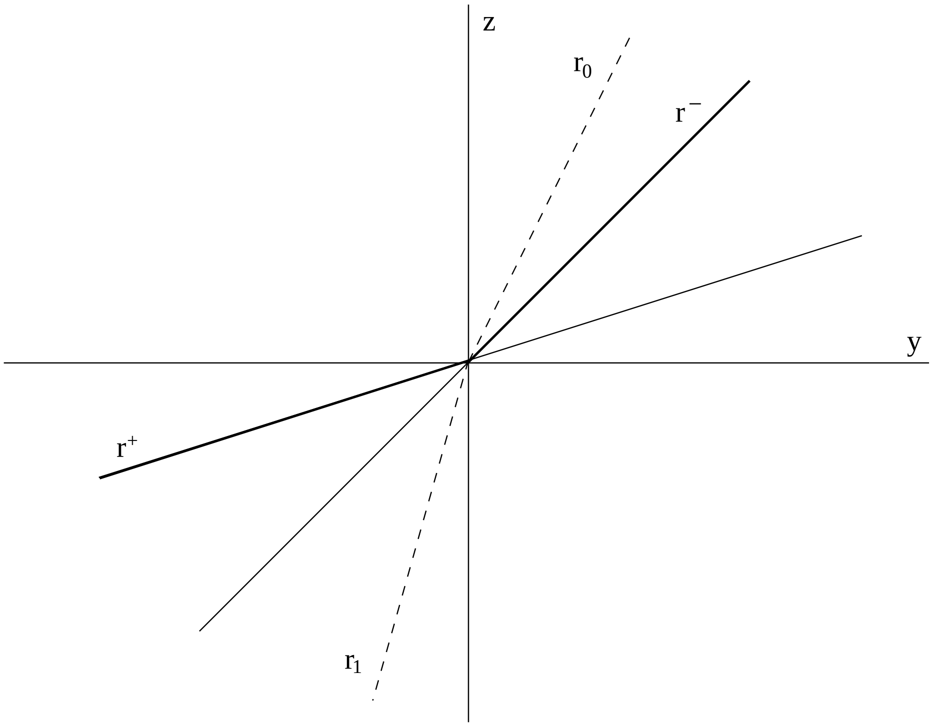


Figure 5.1: The straight lines r^+ and r^- and the half straight-lines r_0 and r_1 when $\delta^+ > \delta^-$ and $\delta_1 < \delta^+$.

one gets

$$\left\{\begin{array}{l}\dot{x} = 2\alpha^- x - y,\\ \dot{y} = ((\alpha^-)^2 + (\beta^-)^2)x,\\ \dot{Z} = (\delta^- - \delta^+)c_{31}^- x + \lambda Z.\end{array}\right. \quad \text{if } x < 0, \quad \left\{\begin{array}{l}\dot{x} = 2\alpha^+ x - y,\\ \dot{y} = ((\alpha^+)^2 + (\beta^+)^2)x,\\ \dot{Z} = \lambda Z.\end{array}\right. \quad \text{if } x > 0. \tag{5.14}$$

Moreover, the right focal half-plane of (5.14) is $\{Z = 0\} \cap \{x \geq 0\}$.

If system (5.8) has a two-zonal invariant cone $\mathcal{C}$, then system (5.14) has a two-zonal invariant cone $\tilde{\mathcal{C}}$. This cone $\tilde{\mathcal{C}}$ does not intersect the half-plane $\{Z = 0\} \cap \{x \geq 0\}$, because the cone $\mathcal{C}$ does not cut the right focal half-plane Π_{HF}^+ of (5.8), see Lemma 5.4. Therefore, the cone $\tilde{\mathcal{C}}$ only may cut the plane $Z = 0$ for $x < 0$.

Due to $\delta^+ \neq \delta^-$, the derivative $\dot{z}$ on the half-plane $\{Z = 0\} \cap \{x < 0\}$, is given by $\dot{Z} = (\delta^- - \delta^+)c_{31}^- x \neq 0$, i.e., this derivative does not change its sign on this half-plane, so the invariant cone $\tilde{\mathcal{C}}$ cannot cut the cited half-plane and the two-zonal invariant cone $\mathcal{C}$ does not intersect with the plane Π_F^+.

Analogously, one can prove that the non-planar two-zonal invariant cone $\mathcal{C}$, cannot intersect the plane Π_F^-.

Finally, it can be deduced, by considering again Lemma 5.4, that the non-planar two-zonal invariant cone is either above or below both focal planes. □

Now, we are able to analyze the existence of invariant cones in system (5.8), as we perform in the following proposition.

Proposition 5.6 Define the values

$$\eta = \delta^-\delta^+\left(1 - e^{(\gamma^-+\gamma^+)\pi}\right), \tag{5.15}$$

$$\mu = \delta^+(1 + e^{\gamma^-\pi}) - \delta^- e^{\gamma^-\pi}(1 + e^{\gamma^+\pi}) \tag{5.16}$$

and

$$\rho = e^{-(\gamma^++\gamma^-)\pi}, \tag{5.17}$$

where $\gamma^\pm$ and $\delta^\pm$ are given in (5.10). The following properties hold.

(a) If $\mu \neq 0$, system (5.8) possesses a unique two-zonal invariant cone such that $v_0^c := \eta/\mu$ provides the slope of the half straight-line intersection between the invariant cone and the separation plane $x = 0$ which is localized in the half-plane $y > 0$. Moreover, the invariant cone is attractive when $\rho < 1$ and repulsive when $\rho > 1$.

(b) If $\eta \neq 0$ and $\mu = 0$, system (5.8) does not possess any two-zonal invariant cone.

Proof: For the sake of clarity, let us denote $T^\triangledown = 2\alpha^\triangledown$ and $D^\triangledown = (\alpha^\triangledown)^2 + (\beta^\triangledown)^2$. Thus, system (5.8) can be expressed as

$$\begin{cases} \dot{x} = T^\triangledown x - y, \\ \dot{y} = D^\triangledown x, \\ \dot{z} = c_{31}^\triangledown x + \lambda z. \end{cases} \tag{5.18}$$

We search for an invariant cone of system (5.18). Consider a half straight-line contained on the separation plane and passing through the origin. This line has a slope of $v_0 = y_0/z_0$, where $z_0 \neq 0$.

Note that we assume $z_0 \neq 0$ because if $z_0 = 0$, then the half straight-line is locus in the axis z, i.e., $y = 0$. Then, due to $\dot{x}|_{x=0} = -y$, it will be for the points in the half straight-line that $\dot{x} = 0$, and the flow cannot cross the separation plane $x = 0$. Therefore, the existence of invariant cones is not possible.

Assume that $y_0 > 0$. Then, first the system of the left zone is applied. Due to the homogeneity, the line is mapped into another half straight-line of the same class with a slope of $v_1 = y_1/z_1$ with $z_1 \neq 0$. After that, the system of the right zone is applied and the half straight-line is transformed into another half straight-line of the same type with a slope of $v_2 = y_2/z_2$ with $z_2 \neq 0$. The existence of a two-zonal invariant cone is obviously characterized by the equation

$$v_0 = v_2. \tag{5.19}$$

From Proposition 5.5 it is known that, the non-planar two-zonal invariant cones, if any, are located above or below both focal planes Π_F^+ and Π_F^-, whose expressions are given in (5.11).

Consider the following change of variable in the left zone

$$Z = -\lambda x + y - \delta^- z,$$

which takes system (5.18) in the left zone to

$$\begin{cases} \dot{x} = T^- x - y, \\ \dot{y} = D^- x, \\ \dot{Z} = \lambda Z. \end{cases} \tag{5.20}$$

In looking for invariant cones above or below the left focal plane, we consider when $Z \neq 0$ the following last change of variable

$$U = x/Z, \quad V = y/Z,$$

which transform system (5.20) into

$$\begin{cases} \dot{U} = (T^- - \lambda)U - V, \\ \dot{V} = D^- U - \lambda V, \\ \dot{Z} = \lambda Z. \end{cases} \tag{5.21}$$

Note that,

$$V = \frac{y}{Z} = \frac{y}{-\lambda x + y - \delta^- z} = \frac{y/z}{-\lambda x/z + y/z - \delta^-}$$

when $z \neq 0$. In looking for invariant cones, it is enough to consider the planar subsystem in the variables U and V. Coefficient matrices of this system have the pair of complex conjugated eigenvalues $(\alpha^- - \lambda) \pm \mathrm{i}\beta^-$. Take a point such that $U = 0$, $(0, V_0)$ with $V_0 > 0$, that is,

$$V_0 = \frac{y_0/z_0}{y_0/z_0 - \delta^-} = \frac{v_0}{v_0 - \delta^-},$$

where $v_0 = y_0/z_0$. First, the system of the left zone is applied. It is easy to see by integrating system (5.21) that the following intersection point with the line $U = 0$ is given by $(0, V_1)$ with

$$V_1 = -e^{\pi\gamma^-} V_0, \tag{5.22}$$

where γ^- is given in (5.10). Due to V_1 is located in the line $U = 0$, it could be written as

$$V_1 = \frac{y_1/z_1}{y_1/z_1 - \delta^-} = \frac{v_1}{v_1 - \delta^-}, \tag{5.23}$$

and then, equation (5.22) is translated into

$$\frac{v_1}{v_1 - \delta^-} = -e^{\gamma^- \pi} \frac{v_0}{v_0 - \delta^-}. \tag{5.24}$$

Now, by reasoning analogously with the system of the right zone, it is clear that the following intersection point with the line $U = 0$ is given by a point $(0, V_2)$ with

$$V_2 = \frac{v_2}{v_2 - \delta^+} = -e^{\gamma^+ \pi} \frac{v_1}{v_1 - \delta^+}. \tag{5.25}$$

From expressions (5.24) and (5.25) it can be obtained v_2 as a function of v_0, that is,

$$v_2 = h(v_0) = \frac{\delta^+ \delta^- e^{(\gamma^- + \gamma^+)\pi} v_0}{\delta^-(1 + e^{\gamma^+ \pi}) e^{\gamma^- \pi} v_0 + \delta^+(\delta^- - (1 + e^{\gamma^- \pi}) v_0)}.$$

From equation (5.19), we know that there exists an invariant cone when $v_0 = v_2$, i.e., if function h

has a fix point, that is, when

$$v_0 = v_0^c := \frac{\eta}{\mu} = \frac{\delta^-\delta^+\left(1 - e^{(\gamma^-+\gamma^+)\pi}\right)}{\delta^+(1+e^{\gamma^-\pi}) - \delta^- e^{\gamma^-\pi}(1+e^{\gamma^+\pi})}. \tag{5.26}$$

If $\mu \neq 0$, it is obvious that v_0^c provides the slope of the half straight-line intersection between the invariant cone and the separation plane $x = 0$, which is localized in the half-plane $y > 0$.

If $\mu = 0$ and $\eta \neq 0$ the half straight-line is locus in the axis z, i.e., $y = 0$. Then, due to $\dot{x}|_{x=0} = -y$, it is fulfilled that $\dot{x} = 0$ for the points in the half straight-line, and the flow cannot cross the separation plane $x = 0$. Therefore, the existence of invariant cones is not possible.

Finally, the derivative of function h with respect to v_0 evaluated in v_0^c results the value ρ given in (5.17) and the proof is finished. □

It is worth noting that, from Proposition 5.6, the impossibility of having two two-zonal invariant cones in the non-observable system (5.8), follows. Therefore, it is not possible to get saddle-node bifurcation of two-zonal invariant cones, and the dynamics is simpler than the observable case, where the saddle-node bifurcation of invariant cones occurs, as it has been proven in Chapter 3 of this work.

On the other hand, when $\delta^- = \delta^+ = \delta$, the slope v_0^c given in (5.26) becomes δ, and the condition for the existence of a two-zonal invariant cone, $\mu \neq 0$ with μ given in (5.16), is satisfied provided that $\gamma^+ + \gamma^- \neq 0$. In this non-generic case, focal planes of both zones of linearity coincide and become

$$-\lambda x + y - \delta z = 0.$$

Thus, the invariant cone is this invariant plane. Therefore, in looking for non-planar two-zonal invariant cones, condition $\delta^+ \neq \delta^-$ must be satisfied.

5.3 Non-observable 2CPWL$_3$ system with an invariant cone foliated by periodic orbits: the unperturbed system

Among the non-observable systems of the family (5.8), we want to obtain one having an invariant cone foliated by periodic orbits. To get it, by following the ideas of the Van der Pol and Duffing canonical forms [13], we simplify the expression of the family (5.8), by performing a linear change of variables that linearize the third equation of the system.

Proposition 5.7 Consider system (5.8). If one of the following statements holds

(a) $c_{31}^+ = c_{31}^-$

(b) $c_{31}^+ \neq c_{31}^-$, $\alpha^+ = \alpha^-$ and $\beta^+ \neq \beta^-$,

(c) $c_{31}^+ \neq c_{31}^-$, $\alpha^+ \neq \alpha^-$, and $\delta^+ c_{31}^+ \neq \delta^- c_{31}^-$,

then there exists a linear change of variables which transforms system (5.8) into

$$\begin{cases} \dot{x} = 2\alpha^{\triangledown} x - y, \\ \dot{y} = ((\alpha^{\triangledown})^2 + (\beta^{\triangledown})^2)x, \\ \dot{z} = kx + \lambda z, \end{cases} \tag{5.27}$$

where

$$k = \begin{cases} c_{31} & \text{if} \quad c_{31} := c_{31}^+ = c_{31}^-, \\ \dfrac{c_{31}^- c_{31}^+(\delta^+ - \delta^-)}{(\beta^+)^2 - (\beta^-)^2} & \text{if} \quad c_{31}^+ \neq c_{31}^-, \quad \alpha^+ = \alpha^- \quad \text{and} \quad \beta^+ \neq \beta^-, \\ \dfrac{c_{31}^- c_{31}^+(\delta^- - \delta^+)}{c_{31}^+ - c_{31}^-} & \text{if} \quad c_{31}^+ \neq c_{31}^-, \quad \alpha^+ \neq \alpha^- \quad \text{and} \quad \delta^+ c_{31}^+ \neq \delta^- c_{31}^-. \end{cases} \tag{5.28}$$

Proof: Take into account that if $c_{31}^+ = c_{31}^-$, the system is already in the desired form (5.27) with $k = c_{31} := c_{31}^- = c_{31}^+$. Let us consider the remaining cases, where it will be assumed that $c_{31}^+ \neq c_{31}^-$.

First, we consider the case $\alpha^+ \neq \alpha^-$ and $\delta^+ c_{31}^+ \neq \delta^- c_{31}^-$. By performing the following change of variables

$$Z = \frac{c_{31}^+ - c_{31}^-}{2(\alpha^- - \alpha^+)} x + z$$

and renaming the variable Z in small letter again, it is easy to see that system (5.4) is transformed into

$$\begin{cases} \dot{x} = 2\alpha^{\triangledown} x - y, \\ \dot{y} = ((\alpha^{\triangledown})^2 + (\beta^{\triangledown})^2)x, \\ \dot{z} = d_{31}x + d_{32}y + \lambda z, \end{cases} \tag{5.29}$$

where

$$d_{31} = \frac{2\alpha^+ c_{31}^- - 2\alpha^- c_{31}^+ + \lambda(c_{31}^+ - c_{31}^-)}{2(\alpha^+ - \alpha^-)} \quad \text{and} \quad d_{32} = \frac{c_{31}^- - c_{31}^+}{2(\alpha^- - \alpha^+)}. \tag{5.30}$$

Finally, the change

$$Z = (d_{32}c - \lambda)x + y + cz,$$

where

$$c = \frac{\delta^- c_{31}^- - \delta^+ c_{31}^+}{c_{31}^+ - c_{31}^-} \neq 0, \tag{5.31}$$

takes the system to the promised form (5.27), just renaming the variable Z in small letter.

To finish with, we must consider the case $\alpha := \alpha^+ = \alpha^-$ and $\beta^+ \neq \beta^-$, i.e., system (5.8) is given by

$$\begin{cases} \dot{x} = 2\alpha x - y, \\ \dot{y} = (\alpha^2 + (\beta^\nabla)^2)x, \\ \dot{z} = c_{31}^\nabla x + \lambda z. \end{cases} \tag{5.32}$$

The change of variable

$$Z = -A\lambda x + Ay + z,$$

where $A = (c_{31}^- - c_{31}^+)/((\beta^+)^2 - (\beta^-)^2))$, allows us to transform system (5.32) into (5.27) with $\alpha^+ = \alpha^- = \alpha$, by concluding the proof. □

If ones assumes that $k \neq 0$, where k is given in (5.28), then system (5.27) can be written as

$$\begin{cases} \dot{x} = 2\alpha^\nabla x - y, \\ \dot{y} = ((\alpha^\nabla)^2 + (\beta^\nabla)^2)x, \\ \dot{z} = x + \lambda z. \end{cases} \tag{5.33}$$

Note that if $k = 0$, then system (5.27) is decoupled and variable z evolves independently of variables x and y.

System (5.33) possesses, under some hypotheses, a two-zonal invariant cone foliated by periodic orbits, as we detail in the following proposition.

Proposition 5.8 System (5.33) when $\alpha^+/\beta^+ + \alpha^-/\beta^- = 0$ and $\lambda \neq 0$ possesses a two-zonal invariant cone foliated by periodic orbits with a period of $\mathcal{T} = \pi/\beta^- + \pi/\beta^+$. Moreover

$$\tilde{v}_0^c := \frac{\delta^-\delta^+(1 - e^{-\lambda(1/\beta^- + 1/\beta^+)\pi})}{\delta^+ + e^{\gamma^-\pi}(\delta^+ - \delta^-) - \delta^- e^{-\lambda(1/\beta^- + 1/\beta^+)\pi}} \neq 0 \tag{5.34}$$

where γ^- and $\delta^\pm$ are given in (5.10), provides the slope of the half straight-line intersection between the invariant cone and the separation plane $x = 0$ which is localized in the half-plane $y > 0$. Furthermore, the invariant cone is attractive when $\lambda < 0$ and repulsive when $\lambda > 0$.

Proof: On the one hand, consider the planar system

$$\begin{cases} \dot{x} = 2\alpha^\nabla x - y, \\ \dot{y} = ((\alpha^\nabla)^2 + (\beta^\nabla)^2)x. \end{cases} \tag{5.35}$$

Take a point $(0, y_0)$ with $y_0 < 0$. First, the system of the right zone is applied. It is easy to see that the following intersection point of the orbit passing through this point with the separation line $x = 0$ is given by $(0, -e^{\pi\alpha^+/\beta^+}y_0)$. After that, the system of the left zone is applied and the following intersection point with the separation line is given by $(0, e^{\pi(\alpha^+/\beta^+ + \alpha^-/\beta^-)}y_0)$. Therefore, system (5.35) possesses a continuum of periodic orbits with a period of $\mathcal{T} = \pi/\beta^- + \pi/\beta^+$ when

$$\alpha^+/\beta^+ + \alpha^-/\beta^- = 0. \tag{5.36}$$

Under condition (5.36) and by assuming $\lambda \neq 0$, it is easy to see that statement (a) of Proposition 5.6 holds, so the system possesses a two-zonal invariant cone and

$$\tilde{v}_0^c := \frac{\delta^-\delta^+(1 - e^{-\lambda(1/\beta^- + 1/\beta^+)\pi})}{\delta^+ + e^{\gamma^-\pi}(\delta^+ - \delta^-) - \delta^- e^{-\lambda(1/\beta^- + 1/\beta^+)\pi}} \neq 0$$

provides the slope of the half straight-line intersection between the invariant cone and the separation plane $x = 0$ which is localized in the half-plane $y > 0$. The attractiveness of the invariant cone depends on the parameter ρ given in (5.17), which under the hypotheses results $e^{\lambda(1/\beta^+ + 1/\beta^-)\pi}$. Taking into account that $e^{\lambda(1/\beta^+ + 1/\beta^-)\pi} > 1$ if $\lambda > 0$ and $e^{\lambda(1/\beta^+ + 1/\beta^-)\pi} < 1$ if $\lambda < 0$, the attractiveness follows.

Finally, the conjunction of the existence of the two-zonal invariant cone together with the existence of the continuum of periodic orbits, forces the two-zonal invariant cone to be foliated by periodic orbits.

□

Note that system (5.35) in the particular case $\alpha^- = -\alpha^-$ and $\beta^+ = \beta^- = 1$, has been studied in Chapter 2, see (2.36), where it was proven that it has a continuum of periodic orbits. In that case condition (5.36) is clearly fulfilled.

From now on, system (5.33) under the hypothesis of Proposition 5.8 is called the unperturbed system.

5.4 Construction of the perturbed system

Under the hypothesis of Proposition 5.7 and $k \neq 0$, with k given in (5.28), system (5.8) becomes system (5.33). In Proposition 5.8 of the previous Sec. 5.3, we have set the hypothesis for the existence of a cone foliated of periodic orbits in system (5.33), from now on, the unperturbed system. In this section, our aim is the form of the perturbation of this system, to study in the next section the conditions for the persistence of periodic orbits of the continuum in the perturbed system.

We are going to perturb system (5.8) and we proceed by doing the same change of variables that we have done to system (5.8) in Proposition 5.7, if it is possible, to the perturbation of system (5.8). The perturbation intends to be the most generic possible. First, for the persistence of isolated period orbits, we need to make the system non-homogeneous. To achieve this non-homogeneity we add the independent term εc_2 in the second equation of system (5.8). Any other independent term can be annihilated by means of linear changes of variable. On the other hand, we add a term which breaks the non-observability hypothesis. This term will be εz and it is added in the second equation.

Specifically, let us consider the following perturbation of system (5.8), which becomes observable and non-homogeneous,

$$\left\{\begin{array}{l} \dot{x} = 2\alpha^{\nabla} x - y, \\ \dot{y} = ((\alpha^{\nabla})^2 + (\beta^{\nabla})^2)x + \varepsilon(z + c_2), \\ \dot{z} = c_{31}^{\nabla} x + \lambda z, \end{array}\right. \tag{5.37}$$

where $|\varepsilon| \ll 1$ and $c_2 \in \mathbb{R}$.

Under hypotheses of Proposition 5.7, it is possible proceeding by performing the same changes of variables done in the proof of case of Proposition 5.7 to system (5.37). In particular, we analyze most generic case, that is, when statement (c) is fulfilled, i.e.,

$$c_{31}^{+} \neq c_{31}^{-} \quad , \quad \alpha^{+} \neq \alpha^{-}, \quad \text{and} \quad \delta^{+} c_{31}^{+} \neq \delta^{-} c_{31}^{-}.$$

The remaining cases can be analyzed analogously. Under the hypotheses of statement (c) of

Proposition 5.7, system (5.37) can be transformed into

$$\begin{cases} \dot{x} = 2\alpha^{\triangledown} x - y, \\ \dot{y} = ((\alpha^{\triangledown})^2 + (\beta^{\triangledown})^2)x + \varepsilon \left(\dfrac{\lambda x}{c} - \dfrac{y}{c} + \dfrac{z}{c} + c_2 \right), \\ \dot{z} = kx + \lambda z + \varepsilon \left(\dfrac{\lambda x}{c} - \dfrac{y}{c} + \dfrac{z}{c} + c_2 \right), \end{cases} \tag{5.38}$$

where c is given in (5.31) and k is given by the third expression in (5.28).

In order to reduce the size of the independent term in the third equation, under the assumption $\lambda \neq 0$, we do one more change of variable

$$Z = z - \frac{\varepsilon c_2}{\lambda},$$

and renaming the variable Z in small letter, the system is written as

$$\begin{cases} \dot{x} = 2\alpha^{\triangledown} x - y, \\ \dot{y} = ((\alpha^{\triangledown})^2 + (\beta^{\triangledown})^2)x + \dfrac{\varepsilon}{c} \left(\lambda x - y + z - \dfrac{\varepsilon c_2}{\lambda} \right) + \varepsilon c_2, \\ \dot{z} = kx + \lambda z + \dfrac{\varepsilon}{c} \left(\lambda x - y + z - \dfrac{\varepsilon c_2}{\lambda} \right). \end{cases} \tag{5.39}$$

Note that, if $k \neq 0$, it is possible to rescale the system as

$$\begin{cases} \dot{x} = 2\alpha^{\triangledown} x - y, \\ \dot{y} = ((\alpha^{\triangledown})^2 + (\beta^{\triangledown})^2)x + \dfrac{\varepsilon}{c} \left(\lambda x - y + kz - \dfrac{\varepsilon c_2}{\lambda} \right) + \varepsilon c_2, \\ \dot{z} = x + \lambda z + \dfrac{\varepsilon}{kc} \left(\lambda x - y + kz - \dfrac{\varepsilon c_2}{\lambda} \right), \end{cases} \tag{5.40}$$

or equivalently,

$$\begin{cases} \dot{x} = 2\alpha^{\triangledown} x - y, \\ \dot{y} = ((\alpha^{\triangledown})^2 + (\beta^{\triangledown})^2)x + \varepsilon \left(\tilde{N}(x,y,z,\varepsilon) + c_2 \right), \\ \dot{z} = x + \lambda z + \dfrac{\varepsilon}{k} \tilde{N}(x,y,z,\varepsilon), \end{cases} \tag{5.41}$$

where

$$\tilde{N}(x,y,z,\varepsilon) = \frac{1}{c} \left(\lambda x - y + kz - \frac{\varepsilon c_2}{\lambda} \right).$$

Note that when $\varepsilon = 0$, system (5.41) is system (5.33) and under the hypothesis of Proposition 5.8 system (5.33) is the unperturbed system. Thus, system (5.41) for $\varepsilon \neq 0$ and under the hypothesis of Proposition 5.8 will be called the perturbed system. Now, our objective is the analysis of the periodic orbits of the perturbed system which come from the periodic orbits of the continuum of the unperturbed system.

5.5 The persistence of periodic orbits in the perturbed system

In this section, we analyze the periodic orbits of the continuum of the unperturbed system (system (5.33) under the hypotheses of Proposition 5.8), that persist in the perturbed system (5.41).

To study the existence, uniqueness and asymptotic stability of T-periodic orbits of the perturbed system, we are going to use the ideas from Chapter 14 of [31]. To apply these ideas, it is necessary to write the system in the form,

$$\frac{d\mathbf{x}}{ds} = A\mathbf{x} + \varepsilon\mathbf{f}(s, \mathbf{x}, \varepsilon), \tag{5.42}$$

where $A \in \mathcal{M}_2(\mathbb{R})$, $\mathbf{x} \in \mathbb{R}^2$, $0 < \varepsilon \ll 1$ and $\mathbf{f} \in C^0(\mathbb{R} \times \mathbb{R}^2 \times [0, 1), \mathbb{R}^2)$ is T-periodic in the first variable.

In looking for a system of the form (5.42), we perform some changes of variables to system (5.41).

In a first step, the idea is doing a change of variables to piecewise defined generalized polar coordinates and obtaining a system that if the third variable vanishes, then it represents the invariant cone.

Remember that the period of the orbits of the continuum of periodic orbits of system (5.33) under the hypotheses of Proposition 5.8 is given by $\mathcal{T} = \pi/\beta^- + \pi/\beta^+$.

We perform a piecewise change for variables x and y, taking into account the periodic orbits of system (5.33). Concretely, in $x < 0$, we change the variables x, y by s, y_0, and in $x > 0$ by y_0, r where y_0 is the positive intersection of the corresponding periodic orbit with the axis $y = 0$, s is the time that takes the orbit to arrive to the point (x, y) from the initial condition $(0, y_0)$ and $r = s - \pi/\beta^- - \pi/\beta^+$. Note that $s \in (0, \pi/\beta^-)$ for $x < 0$ and $s \in (\pi/\beta^-, \mathcal{T})$ for $x > 0$, see Fig. 5.2.

On the other hand, we change variable z by $\tilde{z}$, a new variable which measures the distance between the third component z of the point $\mathbf{p} = (x, y, z)$ and the invariant cone of system (5.33), see Fig. 5.3.

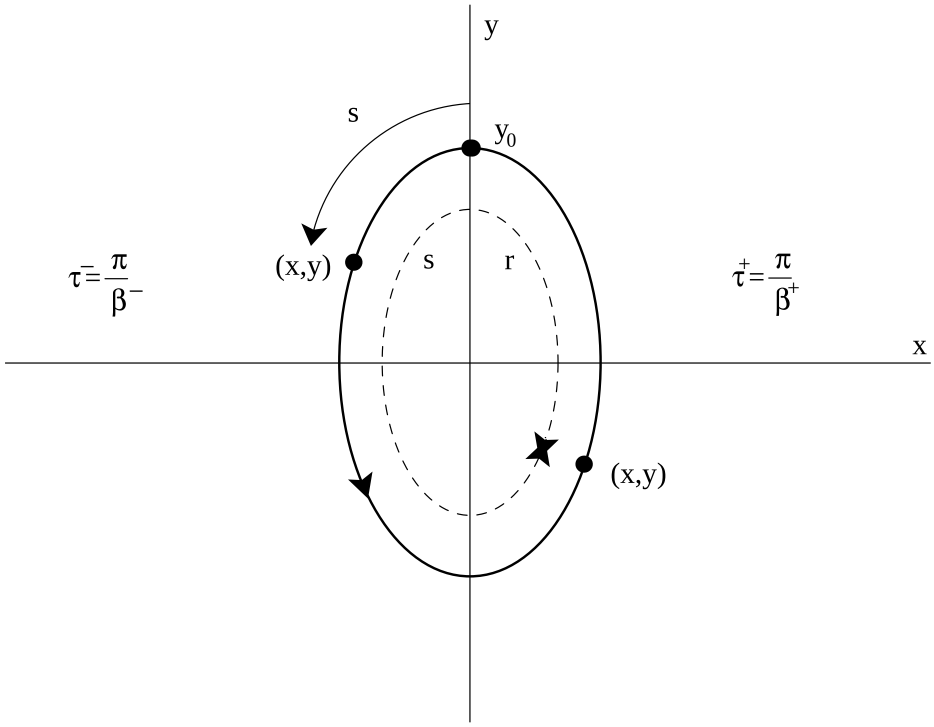


Figure 5.2: Change of the variables (x, y) to piecewise defined generalized polar coordinates (s, y_0).

Let us denote

$$\phi(s;(x_0,y_0,z_0)) = (\phi_1(s;(x_0,y_0,z_0)), \phi_2(s;(x_0,y_0,z_0)), \phi_3(s;(x_0,y_0,z_0)))$$

the solution of system (5.33) with initial condition $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$. With the considerations above, we perform the following piecewise change of variables,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} \left(\begin{array}{cc|c} \multicolumn{2}{c|}{e^{\tilde{A}^- s}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \begin{pmatrix} 0 \\ y_0 \\ \tilde{z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \phi_3(s;(0, y_0, y_0/\tilde{v}_0^c)) \end{pmatrix} & \text{if } x < 0, \\ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{e^{\tilde{A}^+ (s-\mathcal{T})}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \begin{pmatrix} 0 \\ y_0 \\ \tilde{z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \phi_3(s-\mathcal{T}, (0, y_0, y_0/\tilde{v}_0^c)) \end{pmatrix} & \text{if } x > 0, \end{cases} \tag{5.43}$$

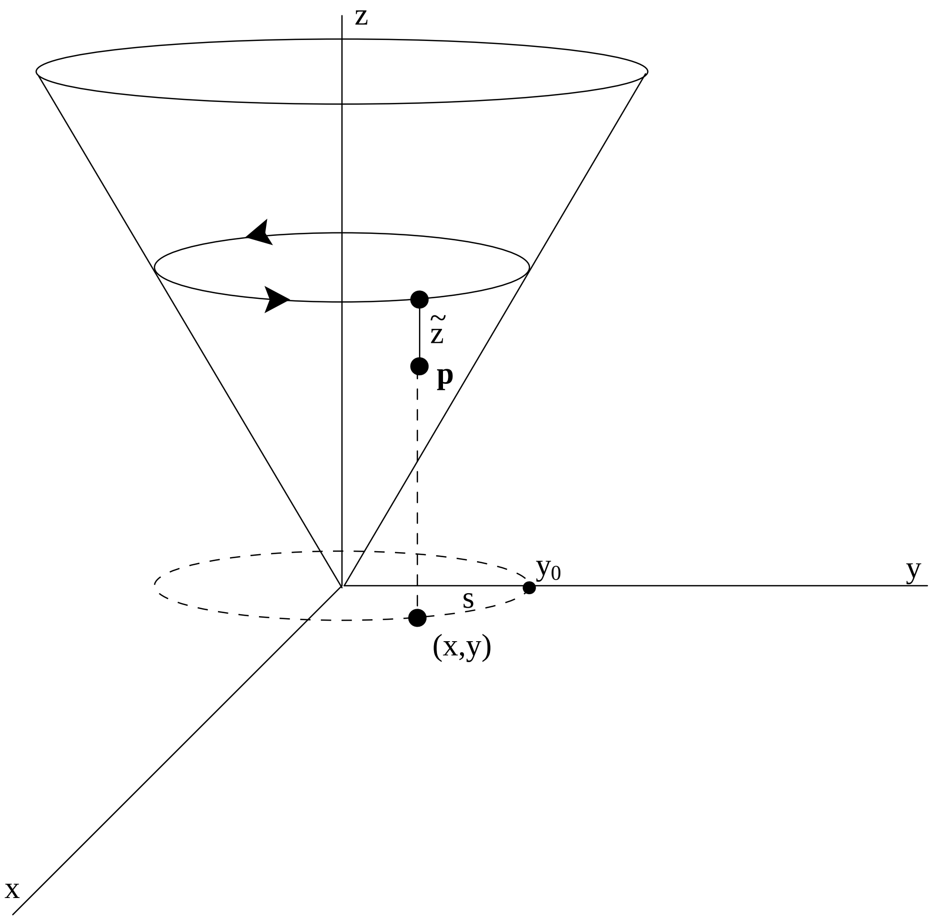


Figure 5.3: Change of the variables (x, y, z) to generalized polar coordinates $(s, y_0, \tilde{z})$.

where $y_0 > 0$, with $\tilde{v}_0^c$ given in (5.34) and

$$\tilde{A}^{\pm} = \begin{pmatrix} 2\alpha^{\pm} & -1 \\ (\alpha^{\pm})^2 + (\beta^{\pm})^2 & 0 \end{pmatrix}.$$

Note that $(0, y_0, y_0/\tilde{v}_0^c)$ is the point located at the intersection between the separation plane and the cone such that $y_0 > 0$.

It is easy to see that the explicit expression of the solution $\phi_3(s;(0,y_0,y_0/\tilde{v}_0^c))$ is given by

$$\phi_3(s;(0,y_0,y_0/\tilde{v}_0^c)) = \begin{cases} \phi_3^-(s,y_0,\tilde{v}_0^c) & \text{if } s\in(0,\pi/\beta^-), \\ \phi_3^+(s-\mathcal{T},y_0,\tilde{v}_0^c) & \text{if } s\in(\pi/\beta^-,\mathcal{T}), \end{cases}$$

where

$$\phi_3^\pm(s,y_0,\tilde{v}_0^c) = y_0 e^{\lambda s}\frac{\beta^\pm(\delta^\pm/\tilde{v}_0^c - 1) + e^{(\alpha^\pm-\lambda)s}(\beta^\pm\cos(\beta^\pm s) + (\lambda-\alpha^\pm)\sin(\beta^\pm s))}{\beta^\pm\delta^\pm}.$$

After the change of variables (5.43), function $\tilde{N}$ is rewritten in the left zone as

$$\tilde{N}_p^-(s,y_0,\tilde{z},\varepsilon) = \frac{1}{c}\left[e^{\alpha^- s}\left((\lambda+\alpha^-)\sin(\beta^- s) - \beta^-\cos(\beta^- s)\right)\frac{y_0}{\beta^-} + k\left(\tilde{z} + \phi_3^-(s,y_0,\tilde{v}_0^c)\right) - \frac{\varepsilon c_2}{\lambda}\right] \tag{5.44}$$

with $s\in(0,\pi/\beta^-)$ and in the right zone as

$$\tilde{N}_p^+(s,y_0,\tilde{z},\varepsilon) = \frac{1}{c}\left[e^{\alpha^+ S}\left((\lambda+\alpha^+)\sin(\beta^+ S) - \beta^+\cos(\beta^+ S)\right)\frac{y_0}{\beta^+} + k\left(\tilde{z} + \phi_3^+(S,y_0,\tilde{v}_0^c)\right) - \frac{\varepsilon c_2}{\lambda}\right], \tag{5.45}$$

where $S = s-\mathcal{T}$ and $s\in(\pi/\beta^-,\mathcal{T})$.

The change of variables (5.43) transforms system (5.41) into

$$\begin{cases} \dot{s} = 1 - \varepsilon M_1^\nabla(s,y_0,\tilde{z},\varepsilon), \\ \dot{y}_0 = \varepsilon M_2^\nabla(s,y_0,\tilde{z},\varepsilon), \\ \dot{\tilde{z}} = \lambda\tilde{z} + \varepsilon M_3^\nabla(s,y_0,\tilde{z},\varepsilon), \end{cases} \tag{5.46}$$

where

$$M_1^\nabla(s,y_0,\tilde{z},\varepsilon) = \begin{cases} \dfrac{w_1^-}{y_0}(\tilde{N}_p^-(s,y_0,\tilde{z},\varepsilon)+c_2), & \text{if } s\in(0,\pi/\beta^-), \\ \dfrac{w_1^+}{y_0}(\tilde{N}_p^+(s,y_0,\tilde{z},\varepsilon)+c_2), & \text{if } s\in(\pi/\beta^-,\mathcal{T}), \end{cases}$$

$$M_2^\nabla(s,y_0,\tilde{z},\varepsilon) = \begin{cases} w_2^-(\tilde{N}_p^-(s,y_0,\tilde{z},\varepsilon)+c_2), & \text{if } s\in(0,\pi/\beta^-), \\ w_2^+(\tilde{N}_p^+(s,y_0,\tilde{z},\varepsilon)+c_2), & \text{if } s\in(\pi/\beta^-,\mathcal{T}), \end{cases} \tag{5.47}$$

and

$$M_3^{\bigtriangledown}(s,y_0,\tilde{z},\varepsilon)=\begin{cases}\dfrac{d\phi_3^-}{ds}(s,y_0,\tilde{v}_0^c)(\tilde{N}_p^-(s,y_0,\tilde{z},\varepsilon)+c_2)+\dfrac{\tilde{N}_p^-(s,y_0,\tilde{z},\varepsilon)}{k}), & \text{if } s\in(0,\pi/\beta^-),\\ \dfrac{d\phi_3^+}{ds}(s,y_0,\tilde{v}_0^c)(\tilde{N}_p^+(s,y_0,\tilde{z},\varepsilon)+c_2)+\dfrac{\tilde{N}_p^+(s,y_0,\tilde{z},\varepsilon)}{k}, & \text{if } s\in(\pi/\beta^-,\mathcal{T}),\end{cases} \tag{5.48}$$

being

$$w_1^{\pm}=\begin{cases} w_1^-:=\dfrac{e^{-\alpha^- s\sin(\beta^- s)}}{\beta^-} & \text{if } s\in(0,\pi/\beta^-),\\ w_1^+:=\dfrac{e^{-\alpha^+(s-\mathcal{T})\sin(\beta^+(s-\mathcal{T}))}}{\beta^+} & \text{if } s\in(\pi/\beta^-,\mathcal{T})\end{cases}$$

and

$$w_2^{\pm}=\begin{cases} w_2^-:=\dfrac{e^{-\alpha^- s}(\beta^-\cos(\beta^- s)+\alpha^-\sin(\beta^- s))}{\beta^-} & \text{if } s\in(0,\pi/\beta^-),\\ w_2^+:=\dfrac{e^{-\alpha^+(s-\mathcal{T})}(\beta^+\cos(\beta^+(s-\mathcal{T}))+\alpha^+\sin(\beta^+(s-\mathcal{T})))}{\beta^+} & \text{if } s\in(\pi/\beta^-,\mathcal{T}).\end{cases}$$

System (5.46) can be transformed into a planar system of the form (5.42), where the role of the temporal variable will be captured by the variable s. Thus, from system (5.46),

$$\begin{cases} y_0'=\dfrac{dy_0}{ds}=\varepsilon\dfrac{M_2^{\bigtriangledown}(s,y_0,\tilde{z},\varepsilon)}{1-\varepsilon M_1^{\bigtriangledown}(s,y_0,\tilde{z},\varepsilon)},\\ \tilde{z}'=\dfrac{d\tilde{z}}{ds}=\dfrac{\lambda\tilde{z}+\varepsilon M_3^{\bigtriangledown}(s,y_0,\tilde{z},0)}{1-\varepsilon M_1^{\bigtriangledown}(s,y_0,\tilde{z},\varepsilon)}.\end{cases} \tag{5.49}$$

The previous system can be rewritten as,

$$\begin{cases} y_0'=\varepsilon M_2^{\bigtriangledown}(s,y_0,\tilde{z},0)+O(\varepsilon^2),\\ \tilde{z}'=\lambda\tilde{z}+\varepsilon M_3^{\bigtriangledown}(s,y_0,\tilde{z},0)+O(\varepsilon^2).\end{cases} \tag{5.50}$$

This is a system of the desired form

$$\mathbf{x}'=A\mathbf{x}+\varepsilon\mathbf{f}(s,\mathbf{x},\varepsilon), \tag{5.51}$$

where $\mathbf{x} = (y_0, \tilde{z})^T$,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix},$$

and

$$\mathbf{f}(s, \mathbf{x}, \varepsilon) = \begin{pmatrix} M_2^{\nabla}(s, \mathbf{x}, 0) + O(\varepsilon^2) \\ M_3^{\nabla}(s, \mathbf{x}, 0) + O(\varepsilon^2) \end{pmatrix}.$$

Now, we look for periodic orbits of the perturbed system (5.51).

Consider the solution $\psi = \psi(s, \mathbf{x}^*, \varepsilon)$, where $\psi(0, \mathbf{x}^*, \varepsilon) = \mathbf{x}^*$, which exists for s in some interval containing $0 \leq s \leq \mathcal{T}$, and it is continuous in ε for ε sufficiently close to $\varepsilon = 0$. From (5.51), using the variation-of-constants formula, we obtain

$$\psi(s, \mathbf{x}^*, \varepsilon) = e^{sA}\mathbf{x}^* + \varepsilon \int_0^s e^{(s-r)A}\mathbf{f}(r, \psi(r, \mathbf{x}^*, \varepsilon), \varepsilon) dr. \tag{5.52}$$

It follows directly from (5.52), that a necessary and sufficient condition for ψ to be periodic of period $\mathcal{T}$ is that

$$(e^{\mathcal{T}A} - I)\mathbf{x}^* + \varepsilon \int_0^{\mathcal{T}} e^{(\mathcal{T}-r)A}\mathbf{f}(r, \psi(r, \mathbf{x}^*, \varepsilon), \varepsilon) dr = (0, 0)^T, \tag{5.53}$$

or equivalently, if we denote $\mathbf{x}^* = (y_0^*, \tilde{z}^*)^T$,

$$\begin{pmatrix} 0 & 0 \\ 0 & e^{\lambda\mathcal{T}} - 1 \end{pmatrix} \begin{pmatrix} y_0^* \\ \tilde{z}^* \end{pmatrix} + \varepsilon \int_0^{\mathcal{T}} \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda(\mathcal{T}-r)} \end{pmatrix} \begin{pmatrix} M_2^{\nabla}(r, \mathbf{x}^*, 0) + O(\varepsilon^2) \\ M_3^{\nabla}(r, \mathbf{x}^*, 0) + O(\varepsilon^2) \end{pmatrix} dr = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.54}$$

This is a system of two equations for the two unknowns consisting of the components y_0^* and $\tilde{z}^*$ of $\mathbf{x}^*$, namely,

$$\left\{ \begin{aligned} \varepsilon \int_0^{\mathcal{T}} \left(M_2^{\nabla}(r, \mathbf{x}^*, 0) + O(\varepsilon^2) \right) dr &= 0, \\ (e^{\lambda\mathcal{T}} - 1)\tilde{z}^* + \varepsilon \int_0^{\mathcal{T}} e^{\lambda(\mathcal{T}-r)} \left(M_3^{\nabla}(r, \mathbf{x}^*, 0) + O(\varepsilon^2) \right) dr &= 0. \end{aligned} \right. \tag{5.55}$$

It is obvious that, if the following equation holds,

$$\int_0^{\mathcal{T}} \left(M_2^{\nabla}(r, \mathbf{x}^*, 0) + O(\varepsilon^2) \right) dr = 0, \tag{5.56}$$

the first equation of system (5.55) is fulfilled. Then, let us replace system (5.55) by

$$\left\{\begin{array}{r} \displaystyle\int_0^{\mathcal{T}} \left(M_2^{\bigtriangledown}(r,\mathbf{x}^*,0)+O(\varepsilon^2)\right) dr = 0, \\ \displaystyle(e^{\lambda\mathcal{T}}-1)\tilde{z}^* + \varepsilon\int_0^{\mathcal{T}} e^{\lambda(\mathcal{T}-r)}\left(M_3^{\bigtriangledown}(r,\mathbf{x}^*,0)+O(\varepsilon^2)\right) dr = 0. \end{array}\right. \tag{5.57}$$

Note that he second equation of system (5.57) is fulfilled for $\varepsilon = 0$ and $\tilde{z}^* = 0$. On the other hand, for $\varepsilon = 0$ and $\tilde{z}^* = 0$, the first equation of system (5.57) is equivalent to

$$I(y_0^*(0)) = \int_0^{\mathcal{T}} M_2^{\bigtriangledown}(r, y_0^*(0), 0, 0)dr = 0. \tag{5.58}$$

For the sake of brevity, let us denote $y_0^*(0) = y_0$. Due to the piecewise definition of function $M_2^{\bigtriangledown}$, the computation of integral (5.58) must be split into two pieces, in the following form,

$$I(y_0) = I_1(y_0) + I_2(y_0) = \int_0^{\pi/\beta^-} M_2^-(s, y_0, 0, 0)ds + \int_{\pi/\beta^-}^{\mathcal{T}} M_2^+(s, y_0, 0, 0)ds. \tag{5.59}$$

The direct calculation of integrals I_1 and I_2 by taking into account condition (5.36) allows us to write

$$I(y_0) = a + by_0, \tag{5.60}$$

where

$$a = 2c_2\frac{\alpha^-}{\beta^-}\frac{\left(1+e^{\frac{\alpha^-}{\beta^-}\pi}\right)\left(\frac{1}{\beta^-}+\frac{1}{\beta^+}\right)}{\left(\frac{\alpha^-}{\beta^-}\right)^2+1} \tag{5.61}$$

and

$$\begin{aligned} b = \;& c^{-1}\left(\frac{\pi}{2(\beta^-)^3}((\alpha^-)^2-(\beta^-)^2-\alpha^-\lambda)+\frac{kJ^-}{(\beta^-)^2\delta^-}+\right. \\ & \left.\frac{\pi}{2(\beta^+)^3}((\alpha^+)^2-(\beta^+)^2-\alpha^+\lambda)+\frac{kJ^+}{(\beta^+)^2\delta^+}\right), \end{aligned} \tag{5.62}$$

being

$$J^- = \frac{(\beta^-)^2(2\alpha^- - \lambda)(\delta^-/\tilde{v}_0^c - 1)(1+e^{-\gamma^-\pi})}{\delta^-} + \frac{\left(1-e^{\frac{-\lambda}{\beta^-}\pi}\right)(\beta^-)^2(2(\beta^-)^2+\lambda^2-2\alpha^-(\lambda-\alpha^-))}{\lambda(4(\beta^-)^2+\lambda^2)},$$

$$J^+ = \frac{-(\beta^+)^2(2\alpha^+ - \lambda)(\delta^+/\tilde{v}_0^c - 1)(1+e^{\gamma^+\pi})}{\delta^+} + \frac{\left(e^{\frac{\lambda}{\beta^+}\pi} - 1\right)(\beta^+)^2(2(\beta^+)^2 + \lambda^2 - 2\alpha^+(\lambda - \alpha^+))}{\lambda(4(\beta^+)^2 + \lambda^2)},$$

with $\gamma^{\pm}$ and $\delta^{\pm}$ given in (5.10), c given in (5.31), k given in (5.28) and $\tilde{v}_0^c$ given in (5.34).

Therefore, equation (5.58) has solution for $y_0^*(0) = -a/b := y_0^c$, provided that $b \neq 0$, where a and b are given in (5.61) and (5.62), respectively.

It is obvious that,

$$p(s) = \psi(s, (y_0^c, 0), 0) \tag{5.63}$$

is a periodic solution with period $\mathcal{T}$ of system (5.51) when $\varepsilon = 0$. It is worth noting that this orbit corresponds, when it is possible to undo the change of variables, to that periodic orbit of the continuum of the unperturbed system (5.33) whose intersection with the separation plane $\{x = 0\}$ is given by y_0^c.

In the following theorem we prove the existence of a periodic solution of system (5.51) for $|\varepsilon| \neq 0$ sufficiently small and consequently, the existence of a periodic orbit of system (5.41).

Theorem 5.9 Consider the perturbed system (5.41) with $\alpha^+/\beta^+ + \alpha^-/\beta^- = 0$, $\lambda \neq 0$ and $k \neq 0$ where k is given in (5.28). If $a \cdot b < 0$, where a and b are given in (5.61) and (5.62), and $|\varepsilon| \neq 0$ is sufficiently small, then there exists a periodic orbit of system (5.41) with a period of $\mathcal{T} + O(\varepsilon)$, coming from the orbit of the continuum whose positive intersection with the separation plane $\{x = 0\}$ is given by $y_0^c = -a/b$.

Proof: First we prove that under the assumptions $\lambda \neq 0$ and $b \neq 0$ where b is given in (5.62), there exists a unique periodic solution $q = q(s, \varepsilon)$ of system (5.51) with period $\mathcal{T}$, which is continuous in (s, ε) for all s and $|\varepsilon|$ sufficiently small, and which for $\varepsilon = 0$ reduces to $q(s, 0) = p(s)$ where $p(s)$ is given in (5.63). To obtain this, it will be shown that, for sufficiently small $|\varepsilon|$, system (5.57) has a unique $\mathbf{x}^* = \mathbf{x}^*(\varepsilon)$, continuous in ε and with $\mathbf{x}^*(0) = (0, y_0^c)^T$. From this, it follows directly that $q(s, \varepsilon) = \psi(s, \mathbf{x}^*(\varepsilon), \varepsilon)$ is the desired solution.

For $\varepsilon = 0$, the second equation of system (5.57) is a homogeneous linear equation whose only solution is $\tilde{z}^*(0) = 0$. Moreover, the first equation of system (5.57) for $\varepsilon = 0$ is equation (5.58). It has been computed that this equation has a solution $y_0^*(0) = y_0^c$ when $b \neq 0$, with b given in (5.62).

Let us rewrite system (5.57) as

$$\begin{cases} F_1(\varepsilon, y_0^*, \tilde{z}^*) = 0, \\ F_2(\varepsilon, y_0^*, \tilde{z}^*) = 0. \end{cases} \tag{5.64}$$

Function F_1 is linear in the variables $(y_0^*, \tilde{z}^*)$ due to the linear character of function M_2^{∇} given in (5.47) and function F_2 is quadratic in the variables $(y_0^*, \tilde{z}^*)$ due to the quadratic character of function M_3^{∇} given in (5.48). Thus, to apply the Implicit Function Theorem, it is necessary to prove that the jacobian of the left side of system (5.64) with respect to y_0^* and $\tilde{z}^*$ evaluated at $\varepsilon = 0$ and $(y_0^*, \tilde{z}^*) = (y_0^c, 0)$ does not vanish, i.e. $\det(J(0, y_0^c, 0)) \neq 0$, where

$$J(\varepsilon, y_0^c, \tilde{z}^*) = \begin{pmatrix} \dfrac{\partial F_1}{\partial y_0^*}(\varepsilon, y_0^*, \tilde{z}^*) & \dfrac{\partial F_1}{\partial \tilde{z}^*}(\varepsilon, y_0^*, \tilde{z}^*) \\ \dfrac{\partial F_2}{\partial y_0^*}(\varepsilon, y_0^*, \tilde{z}^*) & \dfrac{\partial F_2}{\partial \tilde{z}^*}(\varepsilon, y_0^*, \tilde{z}^*) \end{pmatrix}. \tag{5.65}$$

It is easy to see that

$$\frac{\partial F_2}{\partial y_0^*}(0, y_0^c, 0) = 0 \quad \text{and} \quad \frac{\partial F_2}{\partial \tilde{z}^*}(0, y_0^c, 0) = e^{\lambda \mathcal{T}} - 1,$$

and

$$\frac{\partial F_1}{\partial y_0^*}(0, y_0^c, 0) = b.$$

By hypothesis $b \neq 0$ and $\lambda \neq 0$, and so,

$$J(0, y_0^c, 0) = b\left(e^{\lambda \mathcal{T}} - 1\right) \neq 0.$$

Therefore, by the Implicit Function Theorem, system (5.57) has a unique solution $\mathbf{x}^* = \mathbf{x}^*(\varepsilon)$ for sufficiently small $|\varepsilon|$, which is continuous in ε, and such that $\mathbf{x}^*(0) = (y_0^c, 0)$. Thus, there exists a unique periodic solution $q = q(s, \varepsilon)$ of system (5.51) with period $\mathcal{T}$, which is continuous in (s, ε) for all s and $|\varepsilon|$ sufficiently small, and which for $\varepsilon = 0$ reduces to $q(s, 0) = p(s)$ where $p(s)$ is given in (5.63).

Simply undoing the change of variables (5.43), which is possible provided that $y_0^c > 0$ (i.e., $a \cdot b < 0$) and taking into account the first equation of system (5.46), the conclusion follows. □

Note that Theorem 5.9 could be stated, under hypothesis (c) of Proposition 5.7, for system

(5.37).

In this chapter, we have analyzed the existence of invariant cones in non-observable PWL systems. Among the non-observable systems having an invariant cone, we have found a specific system with an invariant cone foliated by periodic orbits. Finally, after a perturbation which made the system observable and non-homogeneous, we have proven the persistence of one periodic orbit of the continuum. Note that, from the continua of periodic orbits analyzed in Chapter 4, we have proven that in some cases two periodic orbits remain after the perturbation. However, in this chapter, we have only captured the persistence of one periodic orbit of the continuum.

Chapter 6

Transversal Tangency in PWL Michelson System

In the previous chapters of this work, we have analyzed the existence of periodic orbits in planar and three-dimensional piecewise linear systems. We have studied periodic orbits with transversal intersection to the separation boundary and, in some cases, periodic orbits where the contact between the orbit and the separation boundary is tangential, but the orbit does not cross the separation boundary. In this chapter, we will analyze the behavior of a three-dimensional piecewise linear version of the Michelson system (1.14) around a two-zonal reversible periodic orbit with two intersection points with the separation plane which crosses it by one of these points tangentially. Specifically, we will see that this tangency fosters the emergence of reversible periodic orbits with four intersection points with the separation plane.

In the last section of this chapter we will see numerically that the orbit with transversal tangency to the separation plane is the starting-point for the appearance of the noose bifurcation. The results of this last section appeared previously in [46] and we add it to this chapter for the sake of completeness. We will analyze numerically the different bifurcations that the family of reversible periodic orbits with two and four transversal intersection points with the separation plane experiments, by comparing the behavior to that of the smooth Michelson system.

The chapter is outlined as follows. In a first section we remember the PWL version of the Michelson system which was introduced in Chapter 1. Sec. 6.2 is focused on the problem of the existence of reversible periodic orbits with two points of intersection with the separation plane. Subsequently, Sec. 6.3 states the problem of the existence of reversible periodic orbits with four points of intersection with the separation plane. This problem consist of the so-called closing equations and some inequalities. Sec. 6.4 is centered on the proof of the existence of solution for the closing equations set out in Sec. 6.3. After that, Sec. 6.5 is focused on the analysis of the inequalities set out in Sec. 6.3. Finally, Sec. 6.6 is devoted to performing some numerical analysis and to studying some bifurcations of periodic orbits that appear on the system, centering our attention on the existence of the noose bifurcation. In this section, it will be pointed out the similitude between the piecewise linear version of the Michelson system and the Michelson differentiable system.

The results of this chapter are gathered in [14].

6.1 The piecewise linear version of the Michelson system

The original Michelson [39, 58, 67, 77, 96] system is given by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = \rho^2 - y - \dfrac{1}{2}x^2, \end{cases} \tag{6.1}$$

where the parameter ρ is strictly positive and the dot denotes the derivative with respect to the temporal variable t.

As we have seen in the introductory chapter, specifically at the end of Section 1.1.1, just performing a linear change of variables, followed by the change of function $x^2 \to |x|$, the following piecewise linear version of the Michelson system is gotten

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = 1 - y - \lambda(1+\lambda^2)|x|, \end{cases} \tag{6.2}$$

with the parameter λ strictly positive.

Let us begin with the properties of system (6.2). The Michelson system (6.1) and the piecewise linear version (6.2) are volume-preserving and time-reversible with respect to the involution (1.9), $\mathbf{R}(x, y, z) = (-x, y, -z)$.

Continuous system (6.2) is given by two linear systems separated by the plane $\Sigma \equiv \{x = 0\}$, called the separation plane, and it can be written in matrix form as

$$\dot{\mathbf{x}} = \begin{cases} A^+\mathbf{x} + \mathbf{e}_3 & \text{if } x \geq 0, \\ A^-\mathbf{x} + \mathbf{e}_3 & \text{if } x \leq 0, \end{cases}$$

with $\mathbf{x} = (x, y, z)^T$, $\mathbf{e}_3 = (0, 0, 1)^T$,

$$A^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda(1+\lambda^2) & -1 & 0 \end{pmatrix} \text{ and } A^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda(1+\lambda^2) & -1 & 0 \end{pmatrix}. \tag{6.3}$$

In the half-space $\{x < 0\}$, the system has exactly one saddle-focus equilibrium point $\mathbf{p}^- = (-1/(\lambda(1+\lambda^2)), 0, 0)^T$, since the eigenvalues of matrix A^- are λ, $\alpha \pm \mathrm{i}\beta$, with

$$\alpha = -\frac{\lambda}{2}, \quad \beta = \frac{\sqrt{4+3\lambda^2}}{2}. \tag{6.4}$$

By the reversibility with respect to $\mathbf{R}$, there exists exactly one saddle-focus equilibrium $\mathbf{p}^+ = -\mathbf{p}-$ in the half-space $\{x > 0\}$, whose eigenvalues are given by $-\lambda$ and $-\alpha \pm \mathrm{i}\,\beta$.

On the other hand, taking into consideration that system (6.2) is formed by two linear systems separated by the plane Σ, it is also interesting to understand the behavior of the flow crossing this plane. From the first equation of the system, it is clear that an orbit which intersects the plane Σ at a point $(0, y_0, z_0)$ with $y_0 > 0$, crosses transversally the separation plane from $\{x < 0\}$ to $\{x > 0\}$. Analogously, when $y_0 < 0$ the orbit crosses transversally the separation plane from $\{x > 0\}$ to $\{x < 0\}$. In the case $y_0 = 0$, the local shape of the orbit depends on the sign of z_0. For $z_0 > 0$ the orbit is locally contained in $\{x \geq 0\}$, for $z_0 < 0$ the orbit is locally contained in $\{x \leq 0\}$ and for $z_0 = 0$ the orbit crosses the plane Σ tangentially from $\{x < 0\}$ to $\{x > 0\}$. The $z-$axis is called the tangency line of the system. More details about this behavior can be found in [19, 20, 72].

Taking into account that both linear systems correspond to saddle-focus equilibria, it is direct to check that a periodic orbit must visit both half-spaces, $\{x > 0\}$ and $\{x < 0\}$. Hence, the periodic orbits of system (6.2) have to intersect the separation plane at least at two points. Moreover, in this chapter we focus our attention on reversible periodic orbits, i.e., periodic orbits which are invariant with respect to the involution $\mathbf{R}$.

6.2 Reversible Periodic Orbits with two intersection points with the separation plane

In this section, we are going to introduce a set of conditions to characterize RP2-orbits (Reversible Periodic orbits with 2 points of intersection with the separation plane), as it was done in [15].

It is well known that a periodic orbit is reversible if and only if it intersects twice the set of fixed points of involution $\mathbf{R}$, $\mathrm{Fix}(\mathbf{R})$, which in this case corresponds to the y-axis.

For every point $\mathbf{p} = (x_0, y_0, z_0)^T \in \mathbb{R}^3$, we denote by $\mathbf{x}(t;\lambda,\mathbf{p}) = (x(t;\lambda,\mathbf{p}), y(t;\lambda,\mathbf{p}), z(t;\lambda,\mathbf{p}))^T$ the solution of system (6.2) with parameter λ and initial condition $\mathbf{x}(0;\lambda,\mathbf{p}) = \mathbf{p}$. Assume that there exist three real values $\hat{t}_1 > 0$, $\hat{\lambda} > 0$ and $\hat{y}_0$ such that

$$\hat{\mathbf{p}}_1 := \mathbf{x}(\hat{t}_1; \hat{\lambda}, \hat{\mathbf{p}}_0) \in \mathrm{Fix}(\mathbf{R}), \tag{6.5}$$

$$\hat{y}_0 < 0, \tag{6.6}$$

$$x(s; \hat{\lambda}, \hat{\mathbf{p}}_0) < 0 \quad \text{for all} \quad s \in (0, \hat{t}_1), \tag{6.7}$$

where $\hat{\mathbf{p}}_0 = (0, \hat{y}_0, 0)$. Then, under hypotheses (6.5)–(6.7), system (6.2) has for $\lambda = \hat{\lambda}$ an RP2-orbit whose period is $2\hat{t}_1$ (half-period $\hat{t}_1$), and whose points of intersection with the separation plane are $\hat{\mathbf{p}}_0$ and $\hat{\mathbf{p}}_1 = (0, \hat{y}_1, 0)$. In Fig. 6.1, a schematic drawing of an RP2-orbit is shown.

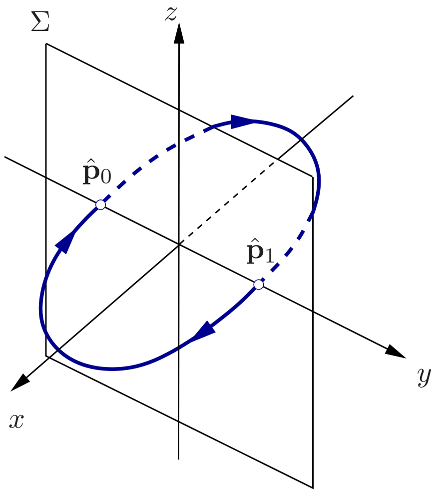


Figure 6.1: Qualitative picture of a reversible periodic orbit of system (6.2) which has exactly two intersections with the plane Σ.

Note that, from hypotheses (6.5)–(6.7) and the properties of the flow crossing the separation plane, the inequality $\hat{y}_1 \geq 0$ must be fulfilled.

In the case $\hat{y}_1 = 0$, the RP2-orbit crosses the plane $\{x = 0\}$ at $\hat{\mathbf{p}}_1 = (0, 0, 0)$ tangentially. The existence of this periodic orbit was established in the second statement of Theorem 1 of [15]. This result is the starting-point of this chapter, and is written subsequently.

Proposition 6.1 There exists a value $\lambda = \lambda_C > 0$ such that system (6.2) has a RP2-orbit with periods less than 4π which crosses the separation plane through the origin tangentially.

The half-period of the RP2-orbit that crosses the plane Σ tangentially, is denoted by t_C. In Lemma 4 of [15] it has been proven that this half-period is contained in an interval that depends on the value λ_C, concretely,

$$t_C \in \left(\frac{3\pi}{\sqrt{3\lambda_C^2 + 4}}, \frac{4\pi}{\sqrt{3\lambda_C^2 + 4}} \right). \tag{6.8}$$

Numerical computations allow us to obtain the values of $t_C \simeq 5.2434$ and $\lambda_C \simeq 0.5851$. The RP2-orbit that crosses the separation plane tangentially for $\lambda = \lambda_C$ is drawn in Fig. 6.2.

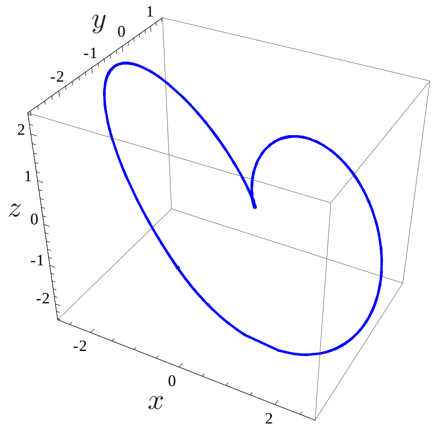


Figure 6.2: RP2-orbit of system (6.2) which crosses the separation plane tangentially.

It is natural to think that beyond RP2-orbit, RP4-orbits (Reversible Periodic orbits with 4 points of intersection with the separation plane) appear, because it is known that close to an orbit crossing the separation plane with a cubic tangency, orbits which cross the separation plane by three points transversally, appear [68]. Moreover, numerical simulations support this fact. The rest of the chapter is focused on the proof by hand of the appearance of RP4-orbits.

6.3 Reversible Periodic Orbits with four intersection points with the separation plane

This section is devoted to introducing a set of conditions to characterize RP4-orbits.

Assume that there exist four real values $\hat{t}_1 > 0$, $\hat{t}_2 > 0$, $\hat{\lambda} > 0$ and $\hat{y}_0$ such that

$$\hat{\mathbf{p}}_1 := \mathbf{x}(\hat{t}_1; \hat{\lambda}, \hat{\mathbf{p}}_0) \in \Sigma, \quad (6.9)$$

$$\hat{\mathbf{p}}_2 := \mathbf{x}(\hat{t}_2; \hat{\lambda}, \hat{\mathbf{p}}_1) \in \mathrm{Fix}(\mathbf{R}), \quad (6.10)$$

$$\hat{y}_0 < 0, \quad (6.11)$$

$$x(s; \hat{\lambda}, \hat{\mathbf{p}}_0) < 0 \text{ for all } s \in (0, \hat{t}_1), \quad (6.12)$$

$$x(s; \hat{\lambda}, \hat{\mathbf{p}}_1) > 0 \text{ for all } s \in (0, \hat{t}_2), \quad (6.13)$$

where $\hat{\mathbf{p}}_0 = (0, \hat{y}_0, 0)$. Then, under hypotheses (6.9)–(6.13), provided that $\hat{\mathbf{p}}_0 \neq \hat{\mathbf{p}}_2$, system (6.2) has for $\lambda = \hat{\lambda}$ an RP4-orbit, whose half-period is $\hat{t} = \hat{t}_1 + \hat{t}_2$ and whose intersections with the separation plane are $\hat{\mathbf{p}}_0$, $\hat{\mathbf{p}}_1$, $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{p}}_3 = \mathbf{R}(\hat{\mathbf{p}}_1)$. In Fig. 6.3, a schematic picture of an RP4-orbit is shown. The points $\hat{\mathbf{p}}_0$, $\hat{\mathbf{p}}_1$, $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{p}}_3$ are also represented.

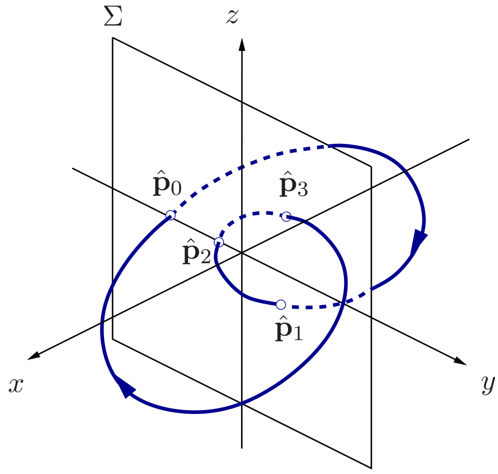


Figure 6.3: Qualitalive picture of a reversible periodic orbit of system (6.2) which exactly four intersections with the plane Σ.

Solutions of conditions (6.9)–(6.13) may correspond to RP4-orbits or RP2-orbits, as we characterize in the following remark.

Remark 6.2 Suppose that a triple $(t_1, t_2, \lambda) = (\hat{t}_1, \hat{t}_2, \hat{\lambda})$, with $\hat{t}_1 > 0$, $\hat{t}_2 \geq 0$ and $\hat{\lambda} > 0$ verifies conditions (6.9)–(6.13). Then, the following statements hold.

(a) If $\hat{\mathbf{p}}_0 \neq \hat{\mathbf{p}}_2$ and $\hat{t}_2 \neq 0$, system (6.2) has for $\lambda = \hat{\lambda}$ an RP4-orbit.

(b) If $\hat{\mathbf{p}}_0 = \hat{\mathbf{p}}_2$, then $\hat{t}_1 = \hat{t}_2 \neq 0$ and system (6.2) has for $\lambda = \hat{\lambda}$ an RP2-orbit.

(c) If $\hat{t}_2 = 0$, system (6.2) has for $\lambda = \hat{\lambda}$ an RP2-orbit. In particular, if $(\hat{t}_1, \hat{t}_2, \hat{\lambda}) = (t_C, 0, \lambda_C)$, then system (6.2) has for $\lambda = \lambda_C$ an RP2-orbit that crosses the plane Σ tangentially through the origin (see Proposition 6.1).

Note that, from hypotheses (6.9)–(6.13) and the properties of the flow crossing the separation plane, the inequalities

$$\hat{y}_1 = y(\hat{t}_1; \hat{\lambda}, \hat{\mathbf{p}}_0) \geq 0 \tag{6.14}$$

and

$$\hat{y}_2 = y(\hat{t}_2; \hat{\lambda}, \hat{\mathbf{p}}_1) \leq 0 \tag{6.15}$$

must be satisfied. For the points satisfying conditions (6.9)–(6.13) corresponding to RP4-orbits, these inequalities must be strict.

Inequalities (6.14) and (6.15) will help us to prove inequalities given in (6.11)–(6.13).

6.4 Analysis of the closing equations

This section is focused on the proof of the existence of solution of (6.9) and (6.10), from now on, the closing equations. We are going to see that the system of the closing equations is a system of six equations and seven unknowns, namely, $(t_1, t_2, y_0, y_1, z_1, y_2, \lambda)$. The first step will be its reduction to a system of two equations and three unknowns, namely, (t_1, t_2, λ). After that, we will check that the values $(t_C, 0, \lambda_C)$ corresponding to the RP2-orbit that crosses the separation plane tangentially satisfy the closing equations. Nevertheless, in looking for solutions which correspond to RP4-orbits, we need solutions with $t_2 \neq 0$. Therefore, the idea is the application of the Implicit Function Theorem to get solutions $(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))$ starting from $(\tilde{t}_1(0), 0, \tilde{\lambda}(0)) = (t_C, 0, \lambda_C)$ such that $t_2 \neq 0$. However, we will see that it is impossible to apply directly the Implicit Function Theorem, and it will be necessary a previous desingularization of the system. Thus, we will get solutions of the closing equations with $t_2 \neq 0$.

We begin with the reduction of the system of six equations and seven unknowns to a system of two equations and three unknowns.

Let $(t_1, t_2, y_0, y_1, z_1, y_2, \lambda) = (\hat{t}_1, \hat{t}_2, \hat{y}_0, \hat{y}_1, \hat{z}_1, \hat{y}_2, \hat{\lambda})$ be a solution of the system given by (6.9) and (6.10), i.e.,

$$\mathbf{p}_1 := \mathbf{x}(t_1; \lambda, \mathbf{p}_0) \in \Sigma, \tag{6.16}$$

$$\mathbf{p}_2 := \mathbf{x}(t_2; \lambda, \mathbf{p}_1) \in \mathrm{Fix}(\mathbf{R}), \tag{6.17}$$

with $y_0 < 0$, where $\mathbf{p}_0 = (0, y_0, 0)$. According to conditions (6.11)–(6.13), the expression of $\mathbf{x}$ in equation (6.16) (respectively, in equation (6.17)) can be obtained by integrating the linear system in the half-space $\{x < 0\}$ (respectively, in the half-space $\{x > 0\}$). So, it is convenient to introduce the following notation.

Let $\mathbf{x}^-(t; \lambda, \mathbf{p})$ (respectively, $\mathbf{x}^+(t; \lambda, \mathbf{p})$) be the solution of the linear system $\dot{\mathbf{x}} = A^-\mathbf{x} + \mathbf{e}_3$ (respectively, $\dot{\mathbf{x}} = A^+\mathbf{x} + \mathbf{e}_3$) with parameter λ and initial condition $\mathbf{x}(0; \lambda, \mathbf{p}) = \mathbf{p}$, where the matrices A^- and A^+ are given in (6.3).

The system formed by (6.16) and (6.17) is a system with six equations and seven unknowns,

$$x^-(t_1; \lambda, (0, y_0, 0)) = 0, \tag{6.18}$$

$$y^-(t_1; \lambda, (0, y_0, 0)) = y_1, \tag{6.19}$$

$$z^-(t_1; \lambda, (0, y_0, 0)) = z_1, \tag{6.20}$$

$$x^+(t_2; \lambda, (0, y_1, z_1)) = 0, \tag{6.21}$$

$$y^+(t_2; \lambda, (0, y_1, z_1)) = y_2, \tag{6.22}$$

$$z^+(t_2; \lambda, (0, y_1, z_1)) = 0. \tag{6.23}$$

However, after some manipulations, it can be transformed into a system with two equations and three unknowns.

Let us begin with the analysis of equations (6.18)–(6.20), which can be written explicitly in the following form,

$$\begin{aligned}2\beta\left[(1+\lambda^2)(1+\lambda^2 y_0)e^{t_1\lambda}-(1+3\lambda^2)\right]e^{\frac{t_1}{2}\lambda}-2\lambda^2\beta\left[(1+\lambda^2)y_0-2\right]\cos(\beta t_1)\\+\lambda\left[(1+\lambda^2)(2+3\lambda^2)y_0-2\right]\sin(\beta t_1)=0\end{aligned} \tag{6.24}$$

$$\begin{aligned}\frac{e^{t_1\lambda}}{2\beta(1+3\lambda^2)}\Bigg[2\beta(1+\lambda^2 y_0)+e^{-\frac{3t_1}{2}\lambda}\cdot\\ \cdot\left[2\beta\left((1+2\lambda^2)y_0-1\right)\cos\left(\beta t_1\right)+\lambda\left(y_0-3\right)\sin\left(\beta t_1\right)\right]\Bigg]=y_1,\end{aligned} \tag{6.25}$$

$$\begin{aligned}\frac{e^{t_1\lambda}}{2\beta(1+3\lambda^2)}\Bigg[2\beta\lambda(1+\lambda^2 y_0)-2\beta\left(1+\lambda^2 y_0\right)\lambda e^{-\frac{3t_1}{2}\lambda}\cos\left(\beta t_1\right)\\-\Big((2+6\lambda^2+3\lambda^4)y_0-(2+3\lambda^2)\Big)e^{-\frac{3t_1}{2}\lambda}\sin\left(\beta t_1\right)\Bigg]=z_1.\end{aligned} \tag{6.26}$$

From equation (6.24), the value y_0 can be expressed as a function of t_1 and λ. We denote this function by $y_0 = Y_0(t_1,\lambda)$ and it is given by

$$Y_0(t_1,\lambda)=\frac{2\beta\left[1+3\lambda^2-(1+\lambda^2)e^{\lambda t_1}\right]e^{\frac{t_1}{2}\lambda}-4\lambda^2\beta\cos(\beta t_1)+2\lambda\sin(\beta t_1)}{\lambda(1+\lambda^2)\left[2\lambda\beta\left(e^{\frac{3t_1}{2}\lambda}-\cos(\beta t_1)\right)+(2+3\lambda^2)\sin(\beta t_1)\right]},$$

provided that the denominator does not vanish, where β was given in (6.4). Denote the denominator as

$$D(t_1,\lambda)=\lambda(1+\lambda^2)\left[2\lambda\beta\left(e^{\frac{3\lambda t_1}{2}}-\cos(\beta t_1)\right)+(2+3\lambda^2)\sin(\beta t_1)\right]. \tag{6.27}$$

Functions $D(t_1,\lambda)$ and $Y_0(t_1,\lambda)$ were studied in [15]. Concretely, it was proven that they verify $D(t_1,\lambda)>0$ and $Y_0(t_1,\lambda)<0$ for every (t_1,λ) such that it corresponds to an RP2-orbit, including the orbit which crosses the separation plane tangentially, thus

$$D(t_C,\lambda_C)>0 \tag{6.28}$$

and

$$Y_0(t_C,\lambda_C)<0. \tag{6.29}$$

By substituting $y_0 = Y_0(t_1,\lambda)$ in equation (6.25), it can be obtained an explicit expression of y_1 in terms of t_1 and λ, which is called $Y_1(t_1,\lambda)$. Similarly, from equation (6.26) it can be obtained an explicit expression of z_1 in terms of t_1 and λ, which is called $Z_1(t_1, \lambda)$. More concretely, we define

$$\begin{aligned} X_1(t_1, \lambda) &= x^-(t_1; \lambda, (0, Y_0(t_1, \lambda), 0)), \\ Y_1(t_1, \lambda) &= y^-(t_1; \lambda, (0, Y_0(t_1, \lambda), 0)), \\ Z_1(t_1, \lambda) &= z^-(t_1; \lambda, (0, Y_0(t_1, \lambda), 0)). \end{aligned} \tag{6.30}$$

With this notation, $X_1(t_1, \lambda) = 0$,

$$Y_1(t_1, \lambda) = \frac{M(t_1, \lambda)}{D(t_1, \lambda)} \tag{6.31}$$

and

$$Z_1(t_1, \lambda) = \frac{Q(t_1, \lambda)}{D(t_1, \lambda)},$$

where

$$\begin{aligned} M(t_1, \lambda) &= 2\beta(e^{\lambda t_1} - 1)\left[\lambda^2(1 + e^{\lambda t_1})e^{-\frac{\lambda t_1}{2}} - (1 + 2\lambda^2)\cos(\beta t_1)\right] + \lambda(1 + e^{\lambda t_1})\sin(\beta t_1), \\ Q(t_1, \lambda) &= 2\lambda^3\beta\left[e^{-\frac{\lambda t_1}{2}}(1 + e^{2\lambda t_1}) - (1 + e^{\lambda t_1})\cos(\beta t_1)\right] + (e^{\lambda t_1} - 1)(2 + 6\lambda^2 + 3\lambda^4)\sin(\beta t_1) \end{aligned}$$

and $D(t_1, \lambda)$ is given in (6.27).

Thus, if $D(t_1, \lambda)$ does not vanish, the system of equations (6.18)–(6.23) is reduced to

$$\begin{cases} X_2(t_1, t_2, \lambda) = 0, \\ Z_2(t_1, t_2, \lambda) = 0, \end{cases} \tag{6.32}$$

where

$$\begin{cases} X_2(t_1, t_2, \lambda) = x^+(t_2; \lambda, (0, Y_1(t_1, \lambda), Z_1(t_1, \lambda)), \\ Z_2(t_1, t_2, \lambda) = z^+(t_2; \lambda, (0, Y_1(t_1, \lambda), Z_1(t_1, \lambda))). \end{cases} \tag{6.33}$$

Moreover, the notation

$$Y_2(t_1, t_2, \lambda) = y^+(t_2; \lambda, (0, Y_1(t_1, \lambda), Z_1(t_1, \lambda))), \tag{6.34}$$

will be useful later on.

Therefore, it is clear that when (t_1, t_2, λ) is a solution of system (6.32) such that $D(t_1, \lambda)$ does not vanish, then $(t_1, t_2, \lambda, Y_0(t_1, \lambda), Y_1(t_1, \lambda), Z_1(t_1, \lambda), Y_2(t_1, t_2, \lambda))$ is a solution of (6.18)–(6.23). Reciprocally, if $(t_1, t_2, \lambda, y_0, y_1, z_1, y_2)$ is a solution of (6.18)–(6.23) such that $D(t_1, \lambda)$ does not vanish, then (t_1, t_2, λ) is a solution of (6.32). We can say, roughly speaking, that (6.9) and (6.10) is equivalent to (6.32). Note that the solutions of this system correspond to periodic orbits when inequalities (6.11)–(6.13) are satisfied.

Once we have reduced the system of the closing equations to (6.32), we are going to center our attention on the analytic proof of its existence of solution.

It is important to remind that taking into account inequalities (6.28) and (6.29), the solution $(t_1, t_2, \lambda) = (t_C, 0, \lambda_C)$ of system (6.32) corresponds to the RP2-orbit which crosses the separation plane tangentially at the origin, see Remark 6.2. That lead us to study the system (6.32) in a neighborhood of $(t_C, 0, \lambda_C)$. The idea is to apply the Implicit Function Theorem to system (6.32) in a neighborhood of the point $(\tilde{t}_1(0), 0, \tilde{\lambda}(0)) = (t_C, 0, \lambda_C)$ to get solutions $(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))$ with $t_2 \neq 0$. However, it is impossible to apply directly the Implicit Function Theorem and a previous desingularization of the system is required.

Let us write the first equation of system (6.32) through its Taylor series expansion in a neighborhood of $t_2 = 0$,

$$X_2(t_1, t_2, \lambda) = X_2(t_1, 0, \lambda) + \frac{\partial X_2}{\partial t_2}(t_1, 0, \lambda)t_2 + H(t_1, t_2, \lambda)t_2^2 = 0. \tag{6.35}$$

From the first equation of system (6.33) and taking into account equation (6.21), it follows that $X_2(t_1, 0, \lambda) = x^+(0; \lambda, (0, Y_1(t_1, \lambda), Z_1(t_1, \lambda))) = 0$. Moreover, by taking into consideration the first equation of system (6.2), $\dot{x} = y$, it follows that

$$\frac{\partial X_2}{\partial t_2}(t_1, 0, \lambda) = \frac{\partial x^+}{\partial t_2}(0; \lambda, (0, Y_1(t_1, \lambda), Z_1(t_1, \lambda))) = Y_1(t_1, \lambda),$$

so, equation (6.35) becomes into

$$t_2 \left[Y_1(t_1, \lambda) + H(t_1, t_2, \lambda)t_2 \right] = 0, \tag{6.36}$$

which is always zero for $t_2 = 0$. Therefore, it is not possible to apply directly the Implicit Function Theorem to equation (6.36). We define

$$\tilde{X}_2(t_1, t_2, \lambda) = \frac{1}{t_2} X_2(t_1, t_2, \lambda) = Y_1(t_1, \lambda) + H(t_1, t_2, \lambda)t_2,$$

for $t_2 \neq 0$. We are going to be more specific with the last expression because it will be useful later on. Taking into account that solutions of system (6.2) satisfy $\dot{y} = z$, we can write

$H(t_1,t_2,\lambda)=\frac{1}{2}Z_1(t_1,\lambda)+\tilde{H}(t_1,t_2,\lambda)t_2$, and then,

$$\tilde{X}_2(t_1,t_2,\lambda) \quad = Y_1(t_1,\lambda)+\frac{1}{2}Z_1(t_1,\lambda)t_2+\tilde{H}(t_1,t_2,\lambda)t_2^2. \tag{6.37}$$

Hence, solutions of system (6.32) with $t_2 \neq 0$ correspond to solutions of

$$\begin{cases} \tilde{X}_2(t_1,t_2,\lambda)=0, \\ Z_2(t_1,t_2,\lambda)=0. \end{cases}$$

Note that despite $t_2=0$, the triple $(t_1,t_2,\lambda)=(t_C,0,\lambda_C)$ is also a solution of this system because $Y_1(t_C,\lambda_C)=0$ for the RP2-orbit which crosses the separation plane tangentially.

After the desingularization, we can use the Implicit Function Theorem to get solutions $(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))$ with $t_2\neq 0$ and small enough, starting from the tangency, that is, $(\tilde{t}_1(0),0,\tilde{\lambda}(0))=(t_C,0,\lambda_C)$. Specifically, it must be proven that $\det(J(t_C,0,\lambda_C))\neq 0$, where

$$J(t_1,t_2,\lambda)=\begin{pmatrix} \dfrac{\partial \tilde{X}_2}{\partial t_1}(t_1,t_2,\lambda) & \dfrac{\partial \tilde{X}_2}{\partial \lambda}(t_1,t_2,\lambda) \\ & \\ \dfrac{\partial Z_2}{\partial t_1}(t_1,t_2,\lambda) & \dfrac{\partial Z_2}{\partial \lambda}(t_1,t_2,\lambda) \end{pmatrix}. \tag{6.38}$$

As a previous step in the proof of this condition, the following two lemmas are stated.

For the sake of brevity, we are going to denote

$$\mathbf{q}=(0,Y_0(t_1,\lambda),0) \quad \text{and} \quad \mathbf{q}_C=(0,Y_0(t_C,\lambda_C),0). \tag{6.39}$$

Lemma 6.3 The function $x^-(t_1;\lambda,\mathbf{q})$ satisfies the following property

$$\frac{\partial x^-}{\partial y_0}(t_C;\lambda_C,\mathbf{q}_C)=\lambda_C(1+\lambda_C^2)(Y_0(t_C,\lambda_C))^2,$$

where $\mathbf{q}=(0,Y_0(t_1,\lambda),0)$ and $\mathbf{q}_C=(0,Y_0(t_C,\lambda_C),0)$ are given in (6.39).

Proof: The function

$$\frac{\partial x^-}{\partial y_0}(t;\lambda_C,\mathbf{q}_C)$$

for every $0 \leq t \leq t_C$ is the first component of the solution of system

$$\dot{\mathbf{w}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_C(1+\lambda_C^2) & -1 & 0 \end{pmatrix} \mathbf{w} \tag{6.40}$$

which satisfies the initial condition $\mathbf{w}(0) = (0,1,0)^T$.

Consider the functions

$$\begin{aligned} \mathbf{v}_1(t) &= \dot{\mathbf{x}}(t;\lambda_C,\mathbf{q}_C) = \left(y^-, z^-, 1-y^- + \lambda_C(1+\lambda_C^2)x^-\right)^T\Big|_{(t;\lambda_C,\mathbf{q}_C)}, \\ \mathbf{v}_2(t) &= \dot{\mathbf{v}}_1(t) = \left(z^-, 1-y^- + \lambda_C(1+\lambda_C^2)x^-, -z^- + \lambda_C(1+\lambda_C^2)y^-\right)^T\Big|_{(t;\lambda_C,\mathbf{q}_C)}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{v}_3(t) &= \dot{\mathbf{v}}_2(t) \\ &= \left(1-y^- + \lambda_C(1+\lambda_C^2)x^-, -z^- + \lambda_C(1+\lambda_C^2)y^-, -1+y^- + \lambda_C(1+\lambda_C^2)(z^- - x^-)\right)^T\Big|_{(t;\lambda_C,\mathbf{q}_C)}. \end{aligned}$$

Functions $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form a fundamental system of solutions of system (6.40) because

$$\det(\mathbf{v}_1(t_C)|\mathbf{v}_2(t_C)|\mathbf{v}_3(t_C)) = \det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = -1 \neq 0.$$

Therefore, the function $\mathbf{w}(t)$ can be written as $\mathbf{w}(t) = \alpha_1\mathbf{v}_1(t) + \alpha_2\mathbf{v}_2(t) + \alpha_3\mathbf{v}_3(t)$, where $(\alpha_1,\alpha_2,\alpha_3)^T$ satisfies

$$\alpha_1\mathbf{v}_1(0) + \alpha_2\mathbf{v}_2(0) + \alpha_3\mathbf{v}_3(0) = (0,1,0)^T.$$

That is, $(\alpha_1,\alpha_2,\alpha_3)^T$ is the unique solution of system

$$\begin{pmatrix} Y_0(t_C,\lambda_C) & 0 & 1-Y_0(t_C,\lambda_C) \\ 0 & 1-Y_0(t_C,\lambda_C) & \lambda_C(1+\lambda_C^2)Y_0(t_C,\lambda_C) \\ 1-Y_0(t_C,\lambda_C) & \lambda_C(1+\lambda_C^2)Y_0(t_C,\lambda_C) & Y_0(t_C,\lambda_C)-1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus, taking into account that equality $\det(\mathbf{v}_1(t_C)|\mathbf{v}_2(t_C)|\mathbf{v}_3(t_C)) = \det(\mathbf{v}_1(0)|\mathbf{v}_2(0)|\mathbf{v}_3(0))$ holds, by solving the system we obtain $\alpha_1 = \lambda_C(1+\lambda_C^2)Y_0(t_C,\lambda_C)(Y_0(t_C,\lambda_C)-1)$, $\alpha_2 = 1 - Y_0(t_C,\lambda_C)$ and $\alpha_3 = \lambda_C(1+\lambda_C^2)(Y_0(t_C,\lambda_C))^2$. Then,

$$\mathbf{w}(t_C) = \alpha_1\mathbf{v}_1(t_C) + \alpha_2\mathbf{v}_2(t_C) + \alpha_3\mathbf{v}_3(t_C) = (\alpha_3, \alpha_2, \alpha_1 - \alpha_3)^T$$

and the conclusion is direct. □

Lemma 6.4 The determinant of the jacobian matrix (6.38) evaluated at $(t_1,t_2,\lambda) = (t_C,0,\lambda_C)$ can be written as

$$\det(J(t_C,0,\lambda_C)) = -\frac{\partial Y_1}{\partial \lambda}(t_C,\lambda_C).$$

Proof: We are going to prove that

$$\frac{\partial \tilde{X}_2}{\partial t_1}(t_C,0,\lambda_C) = 0 \tag{6.41}$$

and

$$\frac{\partial Z_2}{\partial t_1}(t_C,0,\lambda_C) = 1. \tag{6.42}$$

Then, the determinant of the jacobian matrix (6.38) is reduced to

$$\det(J(t_C,0,\lambda_C)) = -\frac{\partial \tilde{X}_2}{\partial \lambda}(t_C,0,\lambda_C)$$

and by using (6.37), we get

$$\frac{\partial \tilde{X}_2}{\partial \lambda}(t_C,0,\lambda_C) = \frac{\partial Y_1}{\partial \lambda}(t_C,\lambda_C), \tag{6.43}$$

and the conclusion follows.

Let us begin with the achievement of relation (6.41). From expression (6.37), one obtains

$$\frac{\partial \tilde{X}_2}{\partial t_1}(t_1,0,\lambda) = \frac{\partial Y_1}{\partial t_1}(t_1,\lambda).$$

Taking into account the equations of system (6.2), it follows that

$$\begin{aligned} \frac{\partial Y_1}{\partial t_1}(t_1,\lambda) = & \frac{\partial}{\partial t_1}\left(y^-(t_1;\lambda,\mathbf{q})\right) \\ = & Z_1(t_1,\lambda) + \frac{\partial y^-}{\partial y_0}(t_1;\lambda,\mathbf{q})\frac{\partial Y_0}{\partial t_1}(t_1,\lambda). \end{aligned} \tag{6.44}$$

On the other hand, taking into consideration the first equation of system (6.2), $\dot{x} = y$, one obtains that

$$\begin{aligned} \frac{\partial X_1}{\partial t_1}(t_1,\lambda) = & \frac{\partial}{\partial t_1}\left(x^-(t_1;\lambda,\mathbf{q})\right) \\ = & Y_1(t_1,\lambda) + \frac{\partial x^-}{\partial y_0}(t_1;\lambda,\mathbf{q})\frac{\partial Y_0}{\partial t_1}(t_1,\lambda). \end{aligned}$$

Since $X_1(t_1,\lambda)$ is identically zero, it follows that $\dfrac{\partial X_1}{\partial t_1}(t_1,\lambda)$ is indentically zero too. Therefore, from the last equality it is satisfied that

$$Y_1(t_1,\lambda) + \frac{\partial x^-}{\partial y_0}(t_1;\lambda,\mathbf{q})\frac{\partial Y_0}{\partial t_1}(t_1,\lambda) = 0. \tag{6.45}$$

Remember that $Y_1(t_C,\lambda_C) = 0$. Moreover $Y_0(t_C,\lambda_C) < 0$, thus $\frac{\partial x^-}{\partial y_0}(t_C;\lambda_C,\mathbf{q}_C)) \neq 0$ (see Lemma 6.3). Therefore, by evaluating equation (6.45) at $(t_1,\lambda) = (t_C,\lambda_C)$ it follows that

$$\frac{\partial Y_0}{\partial t_1}(t_C,\lambda_C) = 0. \tag{6.46}$$

From this equality, by evaluating equation (6.44) at $(t_1,\lambda) = (t_C,\lambda_C)$, and taking into account that $Z_1(t_C,\lambda_C) = 0$, we obtain

$$\frac{\partial Y_1}{\partial t_1}(t_C,\lambda_C) = 0, \tag{6.47}$$

and so, relation (6.41) holds.

Now, we proceed to prove equality (6.42). Let us compute the following Taylor series expansion

$$Z_2(t_1,t_2,\lambda) = Z_2(t_1,0,\lambda) + \frac{\partial Z_2}{\partial t_2}(t_1,0,\lambda)t_2 + \bar{H}(t_1,t_2,\lambda)t_2^2.$$

The second equation of system (6.33) evaluated in $t_2 = 0$ is

$$Z_2(t_1,0,\lambda) = z^+(0;\lambda,(0,Y_1(t_1,\lambda),Z_1(t_1,\lambda))) = Z_1(t_1,\lambda).$$

Therefore, it is obvious that

$$\frac{\partial Z_2}{\partial t_1}(t_C, 0, \lambda_C) = \frac{\partial Z_1}{\partial t_1}(t_C, \lambda_C). \tag{6.48}$$

Taking this into consideration and that solutions of system (6.2) satisfy $\dot{z} = 1 - y + \lambda(1+\lambda^2)x$ in the half-space $\{x < 0\}$, we obtain that

$$\begin{aligned}\frac{\partial Z_1}{\partial t_1}(t_1, \lambda) = & \frac{\partial}{\partial t_1}\left[z^-(t_1; \lambda, \mathbf{q})\right] \\ = & 1 - Y_1(t_1, \lambda) + \lambda(1+\lambda^2)X_1(t_1, \lambda) + \frac{\partial z^-}{\partial y_0}(t_1; \lambda, \mathbf{q})\frac{\partial Y_0}{\partial t_1}(t_1, \lambda).\end{aligned}$$

By evaluating (6.49) at $(t_1, \lambda) = (t_C, \lambda_C)$, taking into account expression (6.46) and that $X_1(t_C, \lambda_C) = Y_1(t_C, \lambda_C) = 0$, it follows that

$$\frac{\partial Z_1}{\partial t_1}(t_C, \lambda_C) = 1, \tag{6.49}$$

and then, from equalities (6.48) and (6.49), expression (6.42) holds and the proof is completed. □

Note that, to apply the Implicit Function Theorem, the problem is now reduced to prove that

$$\frac{\partial Y_1}{\partial \lambda}(t_C, \lambda_C) \neq 0. \tag{6.50}$$

From the explicit expression of $Y_1(t_1, \lambda)$ given in (6.31), by taking derivative with respect to λ we obtain

$$\frac{\partial Y_1}{\partial \lambda}(t_1, \lambda) = \frac{\frac{\partial M}{\partial \lambda}(t_1, \lambda)\, D(t_1, \lambda) - M(t_1, \lambda)\frac{\partial D}{\partial \lambda}(t_1, \lambda)}{(D(t_1, \lambda))^2}.$$

Remind that $Y_1(t_C, \lambda_C) = 0$, and so, $M(t_C, \lambda_C) = 0$. Then, the derivative of function $Y_1(t_1, \lambda)$ with respect to λ at $(t_1, \lambda) = (t_C, \lambda_C)$ is

$$\frac{\partial Y_1}{\partial \lambda}(t_C, \lambda_C) = \frac{\partial M}{\partial \lambda}(t_C, \lambda_C)\frac{1}{D(t_C, \lambda_C)}. \tag{6.51}$$

Hence, condition (6.50) is equivalent to

$$\frac{\partial M}{\partial \lambda}(t_C, \lambda_C) \neq 0.$$

In the following proposition, we prove the existence of solution of the closing equations of the

RP4-orbits, just by checking that the previous condition holds.

Proposition 6.5 There exist a value $\tilde{\varepsilon} > 0$ and two analytic functions

$$\tilde{t}_1 = \tilde{t}_1(t_2), \quad \tilde{\lambda} = \tilde{\lambda}(t_2) \tag{6.52}$$

defined for $|t_2| < \tilde{\varepsilon}$, such that $\tilde{t}_1(0) = t_C$, $\tilde{\lambda}(0) = \lambda_C$ and $(\tilde{t}_1, t_2, \tilde{\lambda})$ is solution of system (6.32). Furthermore,

$$(\tilde{t}_1, t_2, \tilde{\lambda}, Y_0(\tilde{t}_1, \tilde{\lambda}), Y_1(\tilde{t}_1, \tilde{\lambda}), Z_1(\tilde{t}_1, \tilde{\lambda}), Y_2(\tilde{t}_1, t_2, \tilde{\lambda}))$$

is solution of equations (6.18)–(6.23).

Proof: As it has been explained before, we must prove that

$$\frac{\partial M}{\partial \lambda}(t_C, \lambda_C) \neq 0.$$

More suitable coordinates are chosen by replacing t_1 with the new variable $\tau = \beta t_1$, where β was given in (6.4). By doing this change of coordinates we obtain

$$\tilde{M}(\tau, \lambda) = M(\tau/\beta, \lambda) = \tilde{A}(\tau, \lambda) + \tilde{B}(\tau, \lambda)\cos\tau + \tilde{C}(\tau, \lambda)\sin\tau, \tag{6.53}$$

with

$$\begin{aligned}
\tilde{A}(\tau, \lambda) &= 2\beta\lambda^2(e^{\frac{2\lambda\tau}{\beta}} - 1)e^{-\frac{\lambda\tau}{2\beta}}, \\
\tilde{B}(\tau, \lambda) &= -2\beta(1 + 2\lambda^2)(e^{\frac{\lambda\tau}{\beta}} - 1), \\
\tilde{C}(\tau, \lambda) &= \lambda(1 + e^{\frac{\lambda\tau}{\beta}}).
\end{aligned}$$

For $\lambda \neq 0$, we consider the function

$$S(\tau, \lambda) := -\frac{M(\tau, \lambda)}{\tilde{A}(\tau, \lambda)} = \frac{\cos\tau}{\tilde{X}(\tau, \lambda)} + \frac{\sin\tau}{\tilde{Y}(\tau, \lambda)} - 1,$$

being $\tilde{X}(\tau, \lambda) = -\tilde{A}(\tau, \lambda)/\tilde{B}(\tau, \lambda)$ and $\tilde{Y}(\tau, \lambda) = -\tilde{A}(\tau, \lambda)/\tilde{C}(\tau, \lambda)$ for every $\tau > 0$.

Let us denote $\tau_C = \beta_C t_C$, where $\beta_C := \sqrt{4 + 3\lambda_C^2}/2$. From the bound for t_C given in (6.8), it follows that $\tau_C \in (3\pi/2, 2\pi)$. For that reason, we restrict the study to this interval.

Fix a value $\tau \in (3\pi/2, 2\pi)$. In order to analyze the derivative of function $S(\tau, \lambda)$ with respect to

λ, we are going to compute the derivative of functions $\tilde{X}(\tau,\lambda)$ and $\tilde{Y}(\tau,\lambda)$ with respect to λ. The derivative of function $\tilde{X}$ with respect to λ is

$$\frac{\partial \tilde{X}}{\partial \lambda}(\tau,\lambda) = \frac{\lambda\left(\lambda\tau(e^{\frac{\lambda\tau}{\beta}}-1)(1+2\lambda^2)+4\beta^3(1+e^{\frac{\lambda\tau}{\beta}})\right)e^{-\frac{\lambda\tau}{2\beta}}}{2\beta^3(1+2\lambda^2)^2},$$

which is strictly positive, for $\lambda > 0$ and $\tau > 0$.

On the other hand, the derivative of function $\tilde{Y}$ with respect to λ is

$$\frac{\partial \tilde{Y}}{\partial \lambda}(\tau,\lambda) = \frac{\left(\beta(1-e^{\frac{\lambda\tau}{\beta}})(2+3\lambda^2)-\lambda\tau(1+e^{\frac{\lambda\tau}{\beta}})\right)e^{-\frac{\lambda\tau}{2\beta}}}{\beta^2},$$

which is strictly negative, for $\lambda > 0$ and $\tau > 0$.

Therefore, the derivative of function $S(\tau,\lambda)$ with respect to λ,

$$\frac{\partial S}{\partial \lambda}(\tau,\lambda) = -\frac{\partial \tilde{X}}{\partial \lambda}(\tau,\lambda)\frac{\cos\tau}{(\tilde{X}(\tau,\lambda))^2} - \frac{\partial \tilde{Y}}{\partial \lambda}(\tau,\lambda)\frac{\sin\tau}{(\tilde{Y}(\tau,\lambda))^2},$$

is strictly negative, for every $\tau \in (3\pi/2, 2\pi)$.

Taking into account that $\tilde{A}(\tau,\lambda)$ is strictly positive, for every $\lambda > 0$ and $\tau > 0$, and that $S(\tau_C,\lambda_C) = M(\tau_C,\lambda_C) = 0$, it follows that

$$\frac{\partial \tilde{M}}{\partial \lambda}(\tau_C,\lambda_C) = -A(\tau_C,\lambda_C)\frac{\partial S}{\partial \lambda}(\tau_C,\lambda_C) > 0.$$

Therefore, it is concluded that

$$\frac{\partial M}{\partial \lambda}(t_C,\lambda_C) > 0. \tag{6.54}$$

Furthermore, from inequality (6.28) we know that $D(t_C,\lambda_C) > 0$ and then, by continuity with respect to t_2, there exists $\tilde{\varepsilon} > 0$ and functions $\tilde{t}_1(t_2)$, and $\tilde{\lambda}(t_2)$ such that $D(\tilde{t}_1(t_2),\tilde{\lambda}(t_2)) > 0$ for $|t_2| < \tilde{\varepsilon}$. Then, by taking into account that solutions of system (6.32) such that $D(t_1,\lambda)$ does not vanish, correspond to solutions of system (6.18)–(6.23), the proof is finished. □

Note that, from expression (6.51) and taking into account inequalities (6.28) and (6.54), it follows that

$$\frac{\partial Y_1}{\partial \lambda}(t_C,\lambda_C) > 0. \tag{6.55}$$

6.5 Verification of the inequalities. Statement of the main result

This section is focused on the analysis of the inequalities that the solutions of system (6.18)–(6.23) must satisfy to correspond to an RP4-orbit.

To check whether the solution $(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))$ of system (6.18)–(6.23) given in Proposition 6.5 corresponds to an RP4-orbit, inequalities (6.11)–(6.13) must be fulfilled. This inequalities are translated into

$$Y_0(\tilde{t}_1(t_2), \tilde{\lambda}(t_2)) < 0, \tag{6.56}$$

$$x^-(s, \tilde{\lambda}(t_2), (0, Y_0(\tilde{t}_1(t_2), 0))) < 0 \text{ for all } s \in (0, \tilde{t}_1(t_2)) \tag{6.57}$$

$$x^+(\tilde{t}_1(t_2), s, \tilde{\lambda}(t_2), (0, Y_0(\tilde{t}_1(t_2), 0))) > 0 \text{ for all } s \in (0, t_2), \tag{6.58}$$

for $0 \leq t_2 < \tilde{\varepsilon}$, with $\tilde{\varepsilon}$ given in Proposition 6.5.

It is known that $Y_0(t_C, \lambda_C) < 0$, see (6.29). Hence, due to the continuity of the solutions with respect to t_2, it is obvious that inequality (6.56) holds for $0 \leq t_2 < \tilde{\varepsilon}$.

As a previous step in the proof of (6.57) and (6.58), we are going to focus our attention on proving conditions (6.14) and (6.15), which using the notation introduced in definitions (6.30) and (6.33) can be written as

$$\left\{ \begin{array}{l} Y_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2)) \geq 0, \\ Y_2(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2)) \leq 0, \end{array} \right. \tag{6.59}$$

for $0 \leq t_2 < \tilde{\varepsilon}$.

Once these inequalities will be proven, we will do the proof of inequalities (6.57) and (6.58).

To begin with the proof of inequalities (6.59), we obtain in the following lemma an approximation of functions $\tilde{t}_1 = \tilde{t}_1(t_2)$ and $\tilde{\lambda} = \tilde{\lambda}(t_2)$ up to the first non-zero term after the constant term.

Lemma 6.6 The solution functions of the closing equations given in Proposition 6.5 satisfy

$$\left\{ \begin{array}{l} \tilde{t}_1(t_2) = t_C - t_2 + O(t_2^2), \\ \tilde{\lambda}(t_2) = \lambda_C - \dfrac{1}{6} \left(\dfrac{\partial Y_1}{\partial \lambda}(t_C, \lambda_C) \right)^{-1} t_2^2 + O(t_2^3), \end{array} \right. \tag{6.60}$$

where $|t_2| < \tilde{\varepsilon}$.

Proof: It is clear that

$$\tilde{\lambda}(0) = \lambda_C \quad \text{and} \quad \tilde{t}_1(0) = t_C. \tag{6.61}$$

From now on, let us denote by a prime the derivatives with respect to t_2 of functions $\tilde{t}_1$ and $\tilde{\lambda}$. From Proposition 6.5, $(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))$ satisfies the system of equations (6.32), that is,

$$\begin{cases} \tilde{X}_2(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2)) = 0, \\ Z_2(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2)) = 0, \end{cases} \tag{6.62}$$

for $|t_2| < \tilde{\varepsilon}$.

Taking derivatives with respect to t_2, and evaluating it in $t_2 = 0$, it is easy to see that $\tilde{\lambda}'(0)$ and $\tilde{t}_1'(0)$ must satisfy the system of equations

$$\begin{pmatrix} \dfrac{\partial \tilde{X}_2}{\partial t_1}(t_C, 0, \lambda_C) & \dfrac{\partial \tilde{X}_2}{\partial \lambda}(t_C, 0, \lambda_C) \\ \dfrac{\partial Z_2}{\partial t_1}(t_C, 0, \lambda_C) & \dfrac{\partial Z_2}{\partial \lambda}(t_C, 0, \lambda_C) \end{pmatrix} \begin{pmatrix} \tilde{t}_1'(0) \\ \tilde{\lambda}'(0) \end{pmatrix} = - \begin{pmatrix} \dfrac{\partial \tilde{X}_2}{\partial t_2}(t_C, 0, \lambda_C) \\ \dfrac{\partial Z_2}{\partial t_2}(t_C, 0, \lambda_C) \end{pmatrix}.$$

From equalities (6.41)-(6.43), this is equivalent to

$$\begin{pmatrix} 0 & \dfrac{\partial Y_1}{\partial \lambda}(t_C, \lambda_C) \\ 1 & \dfrac{\partial Z_2}{\partial \lambda}(t_C, 0, \lambda_C) \end{pmatrix} \begin{pmatrix} \tilde{t}_1'(0) \\ \tilde{\lambda}'(0) \end{pmatrix} = - \begin{pmatrix} \dfrac{\partial \tilde{X}_2}{\partial t_2}(t_C, 0, \lambda_C) \\ \dfrac{\partial Z_2}{\partial t_2}(t_C, 0, \lambda_C) \end{pmatrix}. \tag{6.63}$$

Moreover, from definition (6.37), it is easy to check that

$$\frac{\partial \tilde{X}_2}{\partial t_2}(t_C, 0, \lambda_C) = 0.$$

On the other hand, remember that the Taylor series expansion of function $Z_2(t_1, t_2, \lambda)$ is

$$Z_2(t_1, t_2, \lambda) = Z_1(t_1, \lambda) + \frac{\partial Z_2}{\partial t_2}(t_1, 0, \lambda) t_2 + \bar{H}(t_1, t_2, \lambda) t_2^2.$$

By taking this into account and that solutions of system (6.2) satisfy $\dot{z} = 1 - y - \lambda(1 + \lambda^2)x$ in the half-space $\{x > 0\}$, the Taylor series expansion can be written as

$$Z_2(t_1, t_2, \lambda) = Z_1(t_1, \lambda) + (1 - Y_2(t_1, 0, \lambda) - \lambda(1 + \lambda^2) X_2(t_1, 0, \lambda)) t_2 + \bar{H}(t_1, t_2, \lambda) t_2^2.$$

Simply remembering definitions given in (6.33) and (6.34) and evaluating them in $t_2 = 0$, this can be rewritten as

$$\begin{aligned} Z_2(t_1, t_2, \lambda) = &\ Z_1(t_1, \lambda) + (1 - Y_1(t_1, \lambda) - \lambda(1+\lambda^2)X_1(t_1, \lambda))t_2 \\ + &\ \bar{H}(t_1, t_2, \lambda)t_2^2, \end{aligned}$$

and from this development, it can be computed for $(t_1, t_2, \lambda) = (t_C, 0, \lambda_C)$ that

$$\frac{\partial Z_2}{\partial t_2}(t_C, 0, \lambda_C) = 1 - Y_1(t_C, \lambda_C) - \lambda_C(1+\lambda_C^2)X_1(t_C, \lambda_C) = 1.$$

Thus, system (6.63) becomes

$$\begin{pmatrix} 0 & \dfrac{\partial Y_1}{\partial \lambda}(t_C, \lambda_C) \\ 1 & \dfrac{\partial Z_2}{\partial \lambda}(t_C, 0, \lambda_C) \end{pmatrix} \begin{pmatrix} \tilde{t}_1'(0) \\ \tilde{\lambda}'(0) \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Taking into account that condition (6.55) holds, this system has a unique solution which is given by

$$\tilde{\lambda}'(0) = 0 \quad \text{and} \quad \tilde{t}_1'(0) = -1. \tag{6.64}$$

Therefore, we have the following first approximation of the functions $\tilde{t}_1$ and $\tilde{\lambda}$,

$$\begin{cases} \tilde{t}_1(t_2) = t_C - t_2 + O(t_2^2), \\ \tilde{\lambda}(t_2) = \lambda_C + O(t_2^2). \end{cases}$$

We have found a non-zero term after the constant one for the function $\tilde{t}_1$, but not for $\tilde{\lambda}$. Thus, the following objective is to compute $\tilde{\lambda}''(0)$.

The derivative of $\tilde{X}_2$ with respect to t_2 is

$$\begin{aligned} \frac{d}{dt_2}\left[\tilde{X}_2(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))\right] = &\ \frac{\partial \tilde{X}_2}{\partial t_1}(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))\tilde{t}_1'(t_2) + \frac{\partial \tilde{X}_2}{\partial t_2}(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2)) \\ + &\ \frac{\partial \tilde{X}_2}{\partial \lambda}(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))\tilde{\lambda}'(t_2). \end{aligned}$$

If we calculate the derivative of this expression with respect to t_2 and we evaluate it in $t_2 = 0$, by taking into account equalities (6.61) and (6.64), one arrives to

$$\begin{aligned}\left[\frac{d^2}{dt_2^2}\left[\tilde{X}_2(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Bigg|_{t_2=0} =& -\left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_1}(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Bigg|_{t_2=0} + \frac{\partial \tilde{X}_2}{\partial t_1}(t_C,0,\lambda_C)\tilde{t}_1''(0)\\ &+\left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_2}(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Bigg|_{t_2=0} + \frac{\partial \tilde{X}_2}{\partial \lambda}(t_C,0,\lambda_C)\tilde{\lambda}''(0).\end{aligned} \tag{6.65}$$

From the first equation of system (6.62), it can be affirmed that

$$\left[\frac{d^2}{dt_2^2}\left[\tilde{X}_2(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Bigg|_{t_2=0} = 0,$$

which from equation (6.65) and taking into consideration (6.41), is translated into

$$\begin{aligned}&-\left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_1}(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Bigg|_{t_2=0} + \left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_2}(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Bigg|_{t_2=0}\\ &+\frac{\partial \tilde{X}_2}{\partial \lambda}(t_C,0,\lambda_C)\tilde{\lambda}''(0) = 0.\end{aligned}$$

From this equation, by taking into account expression (6.43) and inequality (6.55), it is possible to obtain that

$$\tilde{\lambda}''(0) = \frac{\left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_1}(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right] - \frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_2}(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Bigg|_{t_2=0}}{\frac{\partial Y_1}{\partial \lambda}(t_C,\lambda_C)}. \tag{6.66}$$

For the sake of brevity, we only specify the computation of the first addend of the numerator of expression (6.66). The second one can be computed analogously.

To begin with, remember the following expression of function $\tilde{X}_2$ (see (6.37))

$$\tilde{X}_2(t_1,t_2,\lambda) \quad = Y_1(t_1,\lambda) + \frac{1}{2}Z_1(t_1,\lambda)t_2 + \tilde{H}(t_1,t_2,\lambda)t_2^2.$$

From this, and taking into consideration (6.44) and (6.49), the derivative of $\tilde{X}_2$ with respect to t_1

is given by

$$\begin{aligned}&\frac{\partial \tilde{X}_2}{\partial t_1}(t_1,t_2,\lambda) = Z_1(t_1,\lambda) + \frac{\partial y^-}{\partial y_0}(t_1;\lambda,\mathbf{q})\frac{\partial Y_0}{\partial t_1}(t_1,\lambda)\\&+\frac{1}{2}\left(1 - Y_1(t_1,\lambda) + \lambda(1+\lambda^2)X_1(t_1,\lambda) + \frac{\partial z^-}{\partial y_0}(t_1;\lambda,\mathbf{q})\frac{\partial Y_0}{\partial t_1}(t_1,\lambda)\right)t_2\\&+\frac{\partial \tilde{H}}{\partial t_1}(t_1,t_2,\lambda)t_2^2,\end{aligned}$$

where $\mathbf{q}$ was defined in (6.39).

Now, taking into account (6.64), if one substitutes $(t_1,\lambda) = (\tilde{t}_1(t_2),\tilde{\lambda}(t_2))$ in the last equality, takes the derivative of this expression with respect to t_2 and evaluates it in $t_2 = 0$, it follows that

$$\begin{aligned}\left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_1}(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))\right]\right]\Big|_{t_2=0} &= -\frac{\partial Z_1}{\partial t_1}(t_C,\lambda_C)\\&+\left[\frac{d}{dt_2}\left[\frac{\partial y^-}{\partial y_0}(\tilde{t}_1(t_2);\tilde{\lambda}(t_2),\tilde{\mathbf{q}})\right]\frac{\partial Y_0}{\partial t_1}(\tilde{t}_1(t_2),\tilde{\lambda}(t_2))\right]\Big|_{t_2=0}\\&+\left[\frac{\partial y^-}{\partial y_0}(\tilde{t}_1(t_2);\tilde{\lambda}(t_2),\tilde{\mathbf{q}})\frac{d}{dt_2}\left[\frac{\partial Y_0}{\partial t_1}(\tilde{t}_1(t_2),\tilde{\lambda}(t_2))\right]\right]\Big|_{t_2=0}\\&+\frac{1}{2}\Big(1 - Y_1(t_C,\lambda_C) + \lambda_C(1+\lambda_C^2)X_1(t_C,\lambda_C)\\&+\frac{\partial z^-}{\partial y_0}(t_C;\lambda_C,\mathbf{q}_C)\frac{\partial Y_0}{\partial t_1}(t_C,\lambda_C)\Big),\end{aligned}\tag{6.67}$$

where

$$\tilde{\mathbf{q}} = (0, Y_0(\tilde{t}_1(t_2),\tilde{\lambda}(t_2)),0)\tag{6.68}$$

and $\mathbf{q}_C$ was defined in (6.39).

Let us compute the previous expression. By taking into consideration equalities (6.64), it is easy to see that

$$\left[\frac{d}{dt_2}\left[\frac{\partial Y_0}{\partial t_1}(\tilde{t}_1(t_2),\tilde{\lambda}(t_2))\right]\right]\Big|_{t_2=0} = -\frac{\partial^2 Y_0}{\partial t_1^2}(t_C,\lambda_C).\tag{6.69}$$

If we take the derivative with respect to t_1 in expression (6.45) and evaluate it in $(t_1,\lambda) = (t_C,\lambda_C)$, it follows that

$$\frac{\partial Y_1}{\partial t_1}(t_C,\lambda_C)+\left[\frac{\partial}{\partial t_1}\left[\frac{\partial x^-}{\partial y_0}(t_1;\lambda,\mathbf{q})\right]\right]\Big|_{(t_C,\lambda_C,\mathbf{q}_C)}\frac{\partial Y_0}{\partial t_1}(t_C,\lambda_C)+\frac{\partial x^-}{\partial y_0}(t_C;\lambda_C,\mathbf{q}_C)\frac{\partial^2 Y_0}{\partial t_1^2}(t_C,\lambda_C) = 0.$$

From this expression and taking into account Lemma 6.3 and equalities (6.46) and (6.47), it follows that

$$\frac{\partial^2 Y_0}{\partial t_1^2}(t_C, \lambda_C) = 0. \tag{6.70}$$

Therefore, from equality (6.69), one obtains

$$\left[\frac{d}{dt_2}\left[\frac{\partial Y_0}{\partial t_1}(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))\right]\right]\Big|_{t_2=0} = 0.$$

From relation (6.67), by taking into consideration that $X_1(t_C, \lambda_C) = Y_1(t_C, \lambda_C) = 0$, expressions (6.46), (6.49) and the previous equality, one concludes that

$$\left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_1}(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))\right]\right]\Big|_{t_2=0} = -\frac{1}{2}.$$

Analogously, it can be proven that

$$\left[\frac{d}{dt_2}\left[\frac{\partial \tilde{X}_2}{\partial t_2}(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))\right]\right]\Big|_{t_2=0} = -\frac{1}{6}.$$

By substituting these two last equalities in expression (6.66), it follows that

$$\tilde{\lambda}''(0) = -\left(3\frac{\partial Y_1}{\partial \lambda}(t_C, \lambda_C)\right)^{-1} \tag{6.71}$$

and the proof is completed. □

Through the approximation of functions $\tilde{\lambda}$ and $\tilde{t}_1$ got in Lemma 6.6, we are going to prove several inequalities. Two of them are given in (6.59) and one more that will be useful in the proof of inequality (6.57).

Lemma 6.7 Let $\tilde{t}_1(t_2)$ and $\tilde{\lambda}(t_2)$ be the functions defined in Proposition 6.5. The Taylor series expansion up to the first non-zero term of functions $Y_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))$, $Y_2(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2))$ and

$Z_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))$, is given by

$$\begin{cases} Y_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2)) = \dfrac{1}{3}t_2^2 + O(t_2^3), \\ Y_2(\tilde{t}_1(t_2), t_2, \tilde{\lambda}(t_2)) = -\dfrac{1}{6}t_2^2 + O(t_2^3), \\ Z_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2)) = -t_2 + O(t_2^2). \end{cases}$$

Proof: The proofs of the three equalities are similar, so we only prove the first one.

By the definition given in (6.30), the equality $Y_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2)) = y^-(\tilde{t}_1(t_2); \tilde{\lambda}(t_2), \tilde{\mathbf{q}})$, where $\tilde{\mathbf{q}}$ was given in (6.68), holds.

We proceed to compute the Taylor series expansion of function $Y_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))$ up to the first non-zero term. The constant term of the expansion is given by

$$Y_1(\tilde{t}_1(0), \tilde{\lambda}(0)) = Y_1(t_C, \lambda_C) = 0. \tag{6.72}$$

Consider now the derivative with respect to t_2,

$$\begin{aligned} &\frac{d}{dt_2}\left[Y_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))\right] = \frac{d}{dt_2}\left[y^-(\tilde{t}_1(t_2); \tilde{\lambda}(t_2), \tilde{\mathbf{q}})\right] \\ &= z^-(\tilde{t}_1(t_2); \tilde{\lambda}(t_2), \tilde{\mathbf{q}})\tilde{t}_1'(t_2) + \frac{\partial y^-}{\partial \lambda}(\tilde{t}_1(t_2); \tilde{\lambda}(t_2), \tilde{\mathbf{q}})\tilde{\lambda}'(t_2) \\ &+ \frac{\partial y^-}{\partial y_0}(\tilde{t}_1(t_2); \tilde{\lambda}(t_2), \tilde{\mathbf{q}})\left(\frac{\partial Y_0}{\partial t_1}(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))\tilde{t}_1'(t_2) + \frac{\partial Y_0}{\partial \lambda}(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))\tilde{\lambda}'(t_2)\right). \end{aligned} \tag{6.73}$$

By evaluating the derivative given in (6.73) for $t_2 = 0$ using the notation $\mathbf{q}_C$ introduced in (6.39), and taking into account (6.64), it is clear that

$$\begin{aligned} &\frac{d}{dt_2}\left[Y_1(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))\right]\Bigg|_{t_2=0} \\ &= \left[-z^-(\tilde{t}_1(t_2); \tilde{\lambda}(t_2), \tilde{\mathbf{q}}) - \frac{\partial y^-}{\partial y_0}(\tilde{t}_1(t_2); \tilde{\lambda}(t_2), \tilde{\mathbf{q}})\frac{\partial Y_0}{\partial t_1}(\tilde{t}_1(t_2), \tilde{\lambda}(t_2))\right]\Bigg|_{(t_C;\lambda_C,\mathbf{q}_C)} = 0, \end{aligned} \tag{6.74}$$

due to $z(t_C; \lambda_C, \mathbf{q}_C) = 0$ and relationship (6.46).

For the sake of brevity, sometimes we delete the dependence with respect to the arguments in the following expressions.

By taking the derivative with respect to t_2 of expression (6.73) and particularizing it in $t_2 = 0$, it follows that

$$\begin{aligned}&\frac{d^2}{dt_2^2}\left[y^-(\tilde{t}_1(t_2);\tilde{\lambda}(t_2),\tilde{\mathbf{q}})\right]\Big|_{t_2=0}\\&=\left[1-\frac{\partial z^-}{\partial y_0}\frac{\partial Y_0}{\partial t_1}+\frac{\partial y^-}{\partial\lambda}\tilde{\lambda}''(0)-\frac{\partial y^-}{\partial y_0}\frac{d}{dt_2}\left(\frac{\partial Y_0}{\partial t_1}\right)+\frac{\partial y^-}{\partial y_0}\frac{\partial Y_0}{\partial\lambda}\tilde{\lambda}''(0)\right]\Big|_{(t_C;\lambda_C,\mathbf{q}_C)}=\frac{2}{3},\end{aligned}\tag{6.75}$$

due to equations (6.46), (6.69)–(6.71) and the equality

$$\left(\frac{\partial y}{\partial y_0}\frac{\partial Y_0}{\partial\lambda}+\frac{\partial y}{\partial\lambda}\right)\Big|_{(t_C;\lambda_C,\mathbf{q}_C)}=\frac{\partial Y_1}{\partial\lambda}(t_C,\lambda_C).$$

From equalities (6.72), (6.74) and (6.75), we conclude that

$$y^-(\tilde{t}_1(t_2);\tilde{\lambda}(t_2),\tilde{\mathbf{q}})=\frac{1}{3}t_2^2+O(t_2^3)$$

and the proof is finished. □

In Lemma 6.7 we have computed the Taylor series expansion up to the first non-zero term, of functions $Y_1(\tilde{t}_1(t_2),\tilde{\lambda}(t_2))$, $Y_2(\tilde{t}_1(t_2),t_2,\tilde{\lambda}(t_2))$ and $Z_1(\tilde{t}_1(t_2),\tilde{\lambda}(t_2))$. It is a remarkably fact that, although the first terms of the Taylor series expansion of functions $\tilde{t}_2(t_2)$ and $\tilde{\lambda}(t_2)$ depends on the values t_C and λ_C (see (6.60)), the approximations given in Lemma 6.7 do not depend on these values.

At this point, we are going to center our attention on proving the inequalities given in (6.57) and (6.58).

On the one hand, from the second inequality in (6.59) and taking into account that $0\le t_2<\tilde{\varepsilon}$, when $\tilde{\varepsilon}$ is sufficiently small, (6.58) follows.

On the other hand, to finish this section, we prove inequality (6.57) in the following proposition.

Proposition 6.8 Consider functions $\tilde{t}_1(t_2)$ and $\tilde{\lambda}(t_2)$, given in Proposition 6.5, and $\tilde{\mathbf{q}}=(0,Y_0(\tilde{t}_1(t_2),\tilde{\lambda}(t_2)),0)$ given in (6.68). For every $t_2>0$ and sufficiently small, it is satisfied that $x^-(s;\tilde{\lambda}(t_2),\tilde{\mathbf{q}})<0$, for all $s\in(0,\tilde{t}_1(t_2))$.

Proof: For the sake of brevity, we are going to remove the argument of functions $\tilde{t}_1(t_2)$ and $\tilde{\lambda}(t_2)$.

From expressions (6.8) and the first equation of system (6.60), it follows that $\tilde{t}_1=t_C-t_2+$

$O(t_2^2) < 2\pi/\beta_C$, for $t_2 > 0$ and sufficiently small, where $\beta_C = \sqrt{4+3\lambda_C^2}/2$. Moreover, from the second equation of system (6.60), and inequality (6.55), we conclude that $\tilde{\lambda} < \lambda_C$ and so, $\sqrt{4+3\tilde{\lambda}^2}/2 = \tilde{\beta} < \beta_C$. Therefore, $2\pi/\beta_C < 2\pi/\tilde{\beta}$ and $\tilde{t}_1 < 2\pi/\tilde{\beta}$.

Now, we are going to see that, under the hypotheses, if there exists a value $s_x \in (0, \tilde{t}_1)$ such that $x^-(s_x; \tilde{\lambda}, \tilde{\mathbf{q}}) \geq 0$, then the value $\tilde{t}_1$ must be greater than $2\pi/\tilde{\beta}$.

Remember that the first two equations of system (6.2) are $\dot{x} = y$, $\dot{y} = z$, and that $Y_0(\tilde{t}_1, \tilde{\lambda}) < 0$ for $t_2 > 0$ and sufficiently small. Then, function $x^-(s; \tilde{\lambda}, \tilde{\mathbf{q}})$ satisfies $x^-(0; \tilde{\lambda}, \tilde{\mathbf{q}}) = x^-(\tilde{t}_1; \tilde{\lambda}, \tilde{\mathbf{q}}) = 0$, $x^-(s_x; \tilde{\lambda}, \tilde{\mathbf{q}}) \geq 0$ and $\dot{x}(0; \tilde{\lambda}, \tilde{\mathbf{q}}) = y^-(0; \tilde{\lambda}, \tilde{\mathbf{q}}) = Y_0(\tilde{t}_1, \tilde{\lambda}) < 0$.

Furthermore, from Lemma 6.7 one obtains that $y^-(\tilde{t}_1; \tilde{\lambda}, \tilde{\mathbf{q}}) = Y_1(\tilde{t}_1, \tilde{\lambda}) > 0$, for $t_2 > 0$ and sufficiently small. Therefore, there exist three values $0 < s_{y1} < s_{y2} < s_{y3} < \tilde{t}_1$ such that $y^-(s_{y1}; \tilde{\lambda}, \tilde{\mathbf{q}}) < 0$, $y^-(s_{y2}; \tilde{\lambda}, \tilde{\mathbf{q}}) > 0$ and $y^-(s_{y3}; \tilde{\lambda}, \tilde{\mathbf{q}}) < 0$.

It is clear now that there exist two values $0 < s_{z1} < s_{z2} < \tilde{t}_1$, where $z^-(s_{z1}; \tilde{\lambda}, \tilde{\mathbf{q}}) = z^-(s_{z2}; \tilde{\lambda}, \tilde{\mathbf{q}}) = 0$, $y^-(s_{z1}; \tilde{\lambda}, \tilde{\mathbf{q}}) > 0$ and $y^-(s_{z2}; \tilde{\lambda}, \tilde{\mathbf{q}}) < 0$. Moreover, from the hypotheses, it holds that $z^-(0; \tilde{\lambda}, \tilde{\mathbf{q}}) = 0$ and from Lemma 6.7 we get that $z^-(\tilde{t}_1; \tilde{\lambda}, \tilde{\mathbf{q}}) = Z_1(\tilde{t}_1, \tilde{\lambda}) < 0$, for $t_2 > 0$ and sufficiently small.

Consider function $V(s) = -\lambda y^-(s; \tilde{\lambda}, \tilde{\mathbf{q}}) + z^-(s; \tilde{\lambda}, \tilde{\mathbf{q}})$. Since $\tilde{\lambda} > 0$, from the previous reasoning it follows that $V(0) > 0$, $V(s_{z1}) < 0$, $V(s_{z2}) > 0$ and $V(\tilde{t}_1) < 0$. Therefore, function V vanishes at three values $0 < s_{v1} < s_{v2} < s_{v3} < \tilde{t}_1$. But, it is easy to see that the expression of function $V(s)$ is given by

$$V(s) = e^{\tilde{\lambda} s}(h_1 \cos(\tilde{\beta} s) + h_2 \sin(\tilde{\beta} s)),$$

where h_1 and h_2 do not depend on s. Thus, $s_{v3} - s_{v1} \geq 2\pi/\tilde{\beta}$ and since $\tilde{t}_1 > s_{v3} - s_{v1}$, the result is proven. □

The development done up to now allows us to state the main result of this chapter, about the existence of RP4-orbits in system (6.2).

Theorem 6.9 Let λ_C be the defined value in Proposition 6.1. Then, there exists $\varepsilon > 0$ such that system (6.2) possesses, for every $\lambda \in (\lambda_C - \varepsilon, \lambda_C)$, an RP4-orbit with period less than 4π.

6.6 Noose bifurcation and numerical analysis

The numerical analysis included in this section appeared before in [46]. We has decided to include it for the sake of completeness.

Roughly speaking, we are going to see that the existence of the crossing tangency forces the appearance of a small extra loop which grows while the period increases. This loop continues growing until it collides with the large loop. In Fig. 6.4, the projections onto the (x, y)-plane of three RP4-orbits are shown. The first projection corresponds to an RP4-orbit close to the RP2-orbit that crosses the separation plane tangentially. The second one corresponds to an RP4-orbit where the small loop is growing. The third projection corresponds to an RP4-orbit obtained close to the collision of the small loop with the big one.

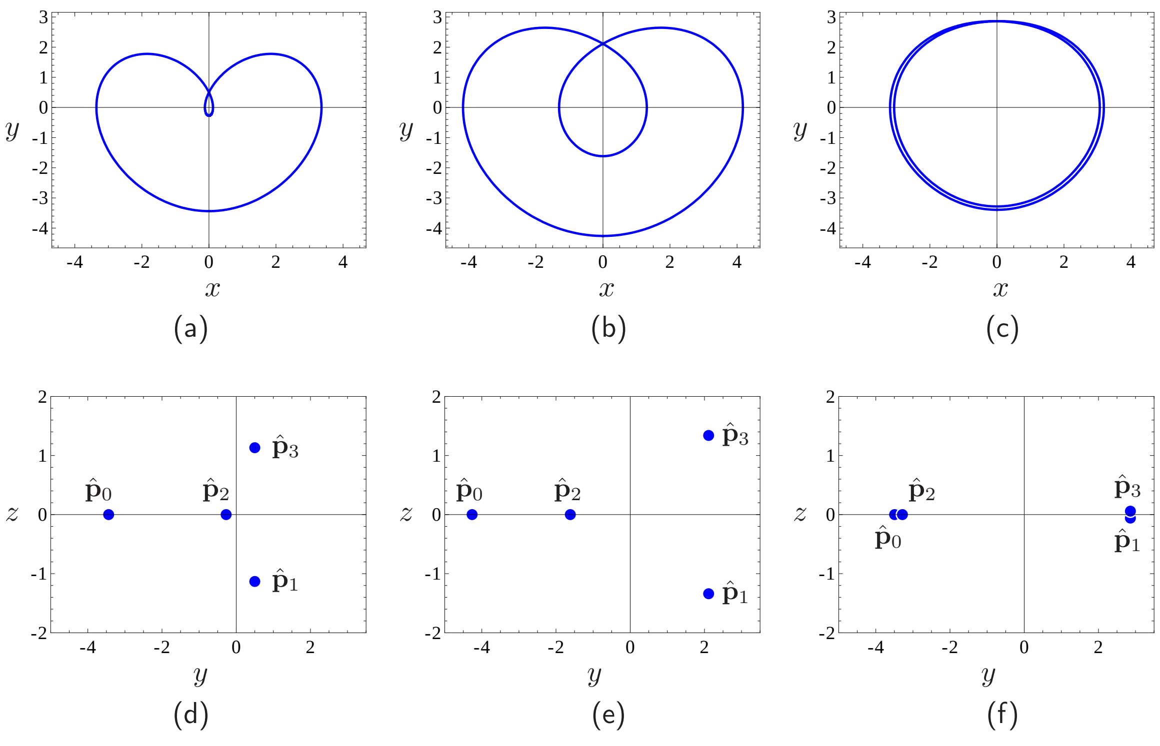


Figure 6.4: (a)–(c) Projections into the plane (x, y) of three RP4-orbits. (d)–(f) Intersection points of these periodic orbits with the separation plane.

In the analysis of reversible periodic orbits by using transversal sections to the flow, it is usual to use the half-period as a natural variable. From now on, this half-period will be used in bifurcation diagrams. For instance, in Fig. 6.5 the family of periodic orbits involved in the noose structure is shown in a λ versus the half-period (that is, $t = t_1 + t_2$) bifurcation diagram. Concretely, the dashed curve corresponds to RP2-orbits (it has been obtained previously in [15]) and the solid one to RP4-orbits. Observe that point $\bar{\mathbf{p}}_{\mathbf{t}} \simeq (5.2434, 0.5851)$ separates these two kinds of periodic orbits and corresponds the RP2-orbit that crosses the separation plane at the origin tangentially. To determine the curve of RP4-orbits a continuation algorithm based on the pseudo arc-length method [48, 57] has been applied. Other points are shown. Point $\overline{\mathbf{sn}} \simeq (3.7237, 0.8481)$ corresponds to a saddle-node bifurcation of periodic orbits whose existence has been proven in Proposition 5 of [15]. Point $\bar{\mathbf{p}}_{\mathbf{d1}} \simeq (3.1669, 0.4259)$ corresponds to a period-doubling bifurcation of period orbits whose existence has been checked numerically. Finally, the thin solid line joins $\bar{\mathbf{p}}_{\mathbf{d1}}$ with $\bar{\mathbf{p}}_{\mathbf{f}} \simeq (6.3337, 0.4259)$ and the noose-shaped curve is closed.

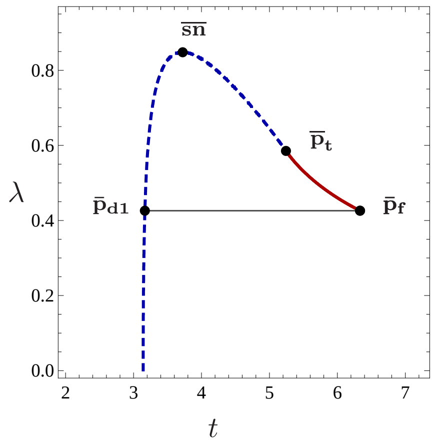


Figure 6.5: Noose structure in the piecewise linear version of the Michelson system (6.2). The solid curve that begins at $\bar{\mathbf{p}}_{\mathbf{t}} \simeq (5.2434, 0.5851)$ and ends at point $\bar{\mathbf{p}}_{\mathbf{f}} \simeq (6.3337, 0.4259)$ corresponds to RP4-orbits. The dashed curve corresponds to RP2-orbits. The points $\overline{\mathbf{sn}} \simeq (3.7237, 0.8481)$ and $\bar{\mathbf{p}}_{\mathbf{d1}} \simeq (3.1669, 0.4259)$ are also shown. The point $\overline{\mathbf{sn}}$ (respectively, $\bar{\mathbf{p}}_{\mathbf{d1}}$) corresponds to a saddle-node (respectively, period-doubling) bifurcation. The thin solid line joins the periodic orbits involved in the period-doubling bifurcation.

To visualize the noose structure of reversible periodic orbits, it is convenient to represent the curves of RP2-orbits and RP4-orbits, which are shown in Fig. 6.5, in the three-dimensional space (t_1, t_2, λ).

Remember that for $t_2 = 0$ solutions of system (6.32) are also solutions of system (6.5), that is, solutions that correspond to RP2-orbits are a particular case of solutions which correspond to

RP4-orbits. Therefore, the curve of RP2-orbits shown in Fig. 6.5 can be represented in the plane $\{t_2 = 0\}$, see Fig. 6.6. Note that, by the reversibility with respect to $\mathbf{R}$, the symmetrical curve with respect to the plane $\mathrm{T} = \{(t_1, t_2, \lambda) \in \mathbb{R}^3 : \; t_1 - t_2 = 0\}$ lies in the plane $\{t_1 = 0\}$, and it is also a curve of RP2-orbits. Moreover, if $t_1 = t_2$, solutions of system (6.32) correspond also to solutions of system (6.5). In this case, the curve of RP2-orbits is located in the plane T. Therefore, the curve of RP2-orbits shown in Fig. 6.5 can be seen as three different curves in the three-dimensional space (t_1, t_2, λ).

In Fig. 6.6, the solid curve corresponds to RP4-orbits. This solid curve joins the two points $\mathbf{p_{t1}} = (t_C, 0, \lambda_C)$ and $\mathbf{p_{t2}} = (0, t_C, \lambda_C)$, and it passes through the point $\mathbf{p_{d1}} \simeq (3.1669, 3.1669, 0.4259)$. By the reversibility with respect to the involution $\mathbf{R}$, this curve of RP4-orbits is symmetrical with respect to the plane T. Moreover, the point $\mathbf{p_{d1}}$ belongs to the curve of RP2-orbits located into the plane T. Therefore, there exists a period-doubling bifurcation at this point and a closed noose can be formed.

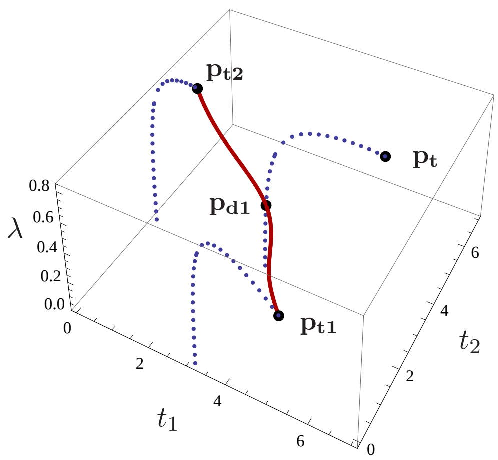


Figure 6.6: Curves of reversible periodic orbits in the three-dimensional space (t_1, t_2, λ). The dashed curves correspond to RP2-orbits. The three curves finish at the tangency point. The solid one corresponds to RP4-orbits. It joins the points $\mathbf{p_{t1}} \simeq (5.2435, 0, 0.5851)$ and $\mathbf{p_{t2}} \simeq (0, 5.2435, 0.5851)$ through the point $\mathbf{p_{d1}} \simeq (3.1669, 3.1669, 0.4259)$.

Now, we analyze the stability and bifurcations of the reversible periodic orbits that are involved in the noose bifurcation. The characteristic (Floquet) multipliers of an RP2-orbit that corresponds to a solution (t_1, λ) of condition (6.5) and without tangency points with the separation plane, are determined by the eigenvalues of the matrix (see [24])

$$\mathcal{M}_1 = \exp\left(A^+ t_1\right) \cdot \exp\left(A^- t_1\right).$$

Analogously, the characteristic multipliers of an RP4-orbit associated with a solution (t_1, t_2, λ) of system (6.32) and without tangency points with the separation plane, are characterized by the eigenvalues of the matrix

$$\mathcal{M}_2 = \exp\left(A^+ t_1\right) \cdot \exp\left(A^- t_2\right) \cdot \exp\left(A^+ t_2\right) \cdot \exp\left(A^- t_1\right).$$

For the particular case $t_2 = 0$, it is obvious that $\mathcal{M}_2 = \mathcal{M}_1$.

Note that, although these results are not valid when a periodic orbit has a tangency with the separation plane, the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ are always defined for all the values of t_1 and t_2. In particular, at the tangency point $\bar{\mathbf{p}}_{\mathbf{t}}$, the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ coincide and their eigenvalues are $\mu_1 \simeq 1$, $\mu_2 \simeq 55.2870$ and $\mu_3 \simeq 0.0181$. On the other hand, the logarithm of module ($\lg|\mu|$) and the principal argument ($\arg \mu$) of the eigenvalues of the matrices $\mathcal{M}_2$ and $\mathcal{M}_2$ are continuous functions of the parameter λ.

Let us observe that one of the characteristic multipliers of reversible periodic orbits of system (6.2) is always 1. This characteristic multiplier is called the trivial Floquet multiplier. The others two eigenvalues are inverse of one another. This is because the periodic orbits are reversible or the system is divergence-free.

In Fig. 6.7, a schematic picture of the logarithm of the module and the principal argument of the characteristic multipliers, versus the half-period, is shown. Concretely, the dashed curve corresponds to these functions when the periodic solution is an RP2-orbit, while the solid one corresponds to these functions for RP4-orbits. From this figure, some conclusions about the stabilities and bifurcations of periodic solutions can be deduced.

From the value $t = \pi$ to the value $t = \mathbf{A} \simeq 3.1669$, the characteristic multipliers have module equal to one, while the arguments go from zero to $\pm\pi$. The RP2-orbit that borns at $\lambda = 0$ is initially elliptic.

For $t = \mathbf{A}$, both multipliers are equal to -1 and they become negative after this value. Therefore, at $t = \mathbf{A}$ there exists a period-doubling bifurcation and the RP2-orbits become Möbius type. This character lasts until $t = \mathbf{B} \simeq 3.6843$, where there exists another period-doubling bifurcation.

From $t = \mathbf{B}$ to $t = \mathbf{C} \simeq 3.7237$, the RP2-orbits are elliptic. At $t = \mathbf{C}$, the characteristic multipliers are equal to one and they become real and positive after this value. Therefore, there exists a saddle-node bifurcation at $t = \mathbf{C}$, (see [15]) and the periodic orbits are hyperbolic until they arrive to $t = \mathbf{D} \simeq 5.5084$.

At $t = \mathbf{D}$ there exists a pitchfork bifurcation, because the characteristic multipliers are equal to one and the arguments of these multipliers are zero. From $t = \mathbf{D}$ to $t = \mathbf{E} \simeq 5.5696$, the RP4-orbits are elliptic, because the characteristic multipliers have module equal to one and its argument is either π or $-\pi$. For $t = \mathbf{E}$, both multipliers are -1 and they become negative after this value. Therefore, at $t = \mathbf{E}$ there exists a period-doubling bifurcation and the RP4-orbits become Möbius type until $t = \mathbf{F} \simeq 6.1260$, where there exists another period doubling bifurcation.

From $t = \mathbf{F}$ to $t = \mathbf{G} \simeq 6.3338$, the RP4-orbits are elliptic. Finally, at $t = \mathbf{G}$ all characteristic multipliers are equal to one. It is the terminal point of the curve of RP4-orbits which closes the noose.

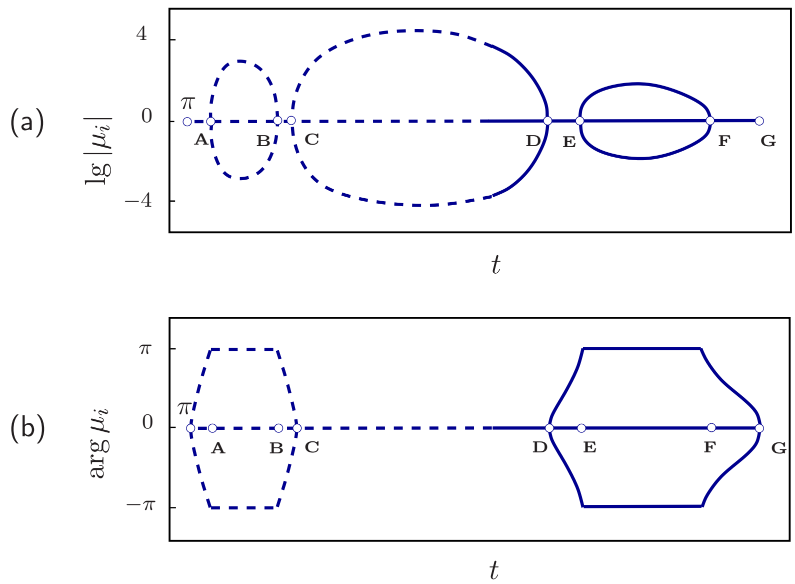


Figure 6.7: Schematic picture of the logarithm of the module and the principal argument of the characteristic multipliers of the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ versus the half-period. The dashed curve corresponds to the logarithm of the module (a) and principal argument (b) of the multipliers of matrix $\mathcal{M}_1$, while the solid one stands for the multipliers of matrix $\mathcal{M}_2$.

In Fig. 6.8, a schematic picture of the noose bifurcation of system (6.2) together with the structure of periodic orbits bifurcations that appears, are shown. Concretely, the dashed line corresponds to RP2-orbits, while the solid one stands for RP4-orbits. On this curve of periodic orbits, there exists four period-doubling bifurcations (at points $\bar{\mathbf{p}}_{\mathbf{d1}}$, $\bar{\mathbf{p}}_{\mathbf{d2}}$, $\bar{\mathbf{p}}_{\mathbf{d3}}$ and $\bar{\mathbf{p}}_{\mathbf{d4}}$, which correspond to $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{F}$, respectively), a saddle-node bifurcation (at point $\overline{\mathbf{sn}}$, that corresponds to $\mathbf{C}$) and a pitchfork bifurcation (at point $\bar{\mathbf{p}}_{\mathbf{b}}$, which corresponds to $\mathbf{D}$). The point $\bar{\mathbf{p}}_{\mathbf{f}}$ corresponds to $\mathbf{G}$, which allows to close the noose. The character of the periodic orbits that appear in each zone are also indicated in this figure.

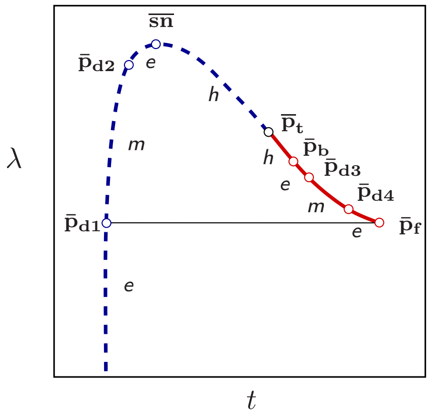


Figure 6.8: Schematic picture of the noose bifurcation in the piecewise linear version of the Michelson system (6.2). The dashed curve corresponds to RP2-orbits, which ends at the tangency point $\bar{\mathbf{p}}_{\mathbf{t}}$. The solid one corresponds to RP4-orbits. This curve begins at this tangency point and finishes at point $\bar{\mathbf{p}}_{\mathbf{f}}$. The period-doubling bifurcations ($\bar{\mathbf{p}}_{\mathbf{d1}}$, $\bar{\mathbf{p}}_{\mathbf{d2}}$, $\bar{\mathbf{p}}_{\mathbf{d3}}$ and $\bar{\mathbf{p}}_{\mathbf{d4}}$), the pitchfork bifurcation ($\bar{\mathbf{p}}_{\mathbf{b}}$) and the saddle-node bifurcation ($\overline{\mathbf{sn}}$) which have been described in the text are shown. The types of periodic orbit are also indicated, where *e*, *h* and *m* stand for elliptic, hyperbolic and Möbius, respectively. The thin solid line does not correspond to periodic orbits. This line connects the points that correspond to periodic orbits involved in the period-doubling bifurcation.

Let us observe that the structure of bifurcations of reversible periodic orbits involved in this noose bifurcation is identical to the original Michelson system [39, 58]. This structure concerns not only the bifurcations of reversible periodic orbits, but also their character (hyperbolic, elliptic or Möbius).

From the analysis done in this chapter for the piecewise version of the Michelson system, we can conclude that the existence of the orbit tangent to the separation plane plays an essential role in the appearance of the small loop that finally ends by closing the noose structure. Thus, we think that in the differentiable Michelson system, an analogous behavior should appear. That is, it should be a periodic orbit tangent to the plane $\{x = 0\}$, such that it forces the appearance of a small loop in the orbit, being the tangent orbit the germ for the closing of the noose bifurcation that exits.

Conclusions and Open Problems

To conclude with this thesis, it is convenient to write a brief summary about the considered problems, the obtained results, the developed methods and the open problems that arise from this work.

Along the whole work we have analyzed the existence of periodic orbits and invariant sets in piecewise-smooth systems, by means of different techniques. Thus, after an introductory chapter, where we introduced the concepts about piecewise linear systems that would be used in the work, in Chapter 2, we have generalized the Melnikov theory to hybrid and discontinuous piecewise-smooth systems. The principal results of this chapter are written in [16].

In Chapter 3, we have analyzed the existence of invariant cones in a class of observable $2CPWL_3$ systems by applying the Melnikov theory developed in Chapter 2 to some related planar hybrid systems. The main result of this chapter is the proof of the existence of a saddle-node bifurcation of invariant cones, as it was conjectured in [25]. This and other interesting results obtained in the chapter are published in [18].

After that, we centered our attention in Chapter 4 in an adaptation of the Melnikov theory to a three-dimensional CPWL system, by using the ideas of the Melnikov theory for planar systems to perturbations of an appropriate class of non-controllable $2CPWL_3$ systems. Part of the results obtained in this chapter are published in [17].

Subsequently, in Chapter 5, first we have studied the existence of invariant cones in non-observable $2CPWL_3$ systems, by using different techniques from that used for observable systems in Chapter 3. After that, we have found a system with an invariant cone foliated by periodic orbits and we have applied an adaptation of the method of Chapter 14 of [31] to catch the periodic orbits that remain after a perturbation of the system.

Finally, in Chapter 6, we have analyzed periodic orbits in a piecewise-smooth version of the well-known Michelson system, and we have concluded that the existence of an orbit tangent to the separation plane plays an essential role in the appearance of the small loop that finally ends by closing the noose structure. The results of this chapter are gathered in [14].

With respect to open problems that have arisen from this work, we should emphasize some of them.

First, the focus-center-limit cycle bifurcation and its degeneration, the saddle-node bifurcation, for the class of hybrid systems analyzed in Chapter 2 can be studied.

With respect to the invariant cones obtained in Chapter 3 as well as 5, it has been analyzed the stability of the invariant surface, but it is an open problem the study of the stability on the cone.

On the other hand, the Hopf bifurcation from infinity obtained in Chapter 4 is a partial result, that we would like to describe completely. Moreover, although the analysis in this chapter is restricted to a piecewise linear system with two zones, it would be interesting to extend it to perturbations of non-controllable systems with more zones of linearity.

The analysis done in Chapter 5 could be generalized to the perturbation of a system (observable or not) having an invariant cone foliated by periodic orbits. We think that under generic hypotheses, only one periodic orbit of the continuum persists, and it should correspond to the unique solution of a linear equation analogous to $I(y_0) = 0$ with $I(y_0)$ given in (5.60).

In Chapter 6, we have stated first necessary conditions for the existence of the noose bifurcation in the PWL Michelson system. The open problem to finish with the research done in Chapter 6, is the proof by hand of the existence of the noose bifurcation in the piecewise version of the Michelson system. After that, we think that this study may be adapted to reversible, divergence-free, piecewise linear systems of more dimensions.

Finally, it is possible to think about the adaptation of the methods and techniques that have been developed in this work to analyze periodic orbits and invariant sets in Filippov systems. For instance, periodic orbits in the so-called Teixeira singularity may be analyzed by adapting the Melnikov theory. A first approximation to the dynamics of the Teixeira singularity has been done in [37].

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