# Theoretical progress on infinite graphs and their average degree: applicability to the European Road Transport Network

M. Cera
Department of Applied Mathematics I
University of Seville, Spain
E-mail: mcera@us.es

E.M. Fedriani
Department of Economics, Quantitative Methods and Economic History
Universidad Pablo de Olavide, ES-41013 Seville, Spain
E-mail: efedmar@upo.es

***Abstract*—There are many problems in Graph Theory for finite graphs relating the number of vertices and the number of edges and, therefore, related to the average degree for finite graphs. However, when dealing with real-life problems involving networks, it is often useful to model the situation by using infinite graphs, which can represent extendable systems. In this paper, we will generalize the concept of average degree for infinite graphs in a family of graphs that we call *average-measurable*. Besides, this new definition allows the generalization of the universal formulae for evaluation of percolation thresholds.**

***Keywords*—*Infinite graph, average degree, complete graph, road transport network, percolation.***

## I. INTRODUCTION

Graph Theory is a tool with many uses in different fields of human knowledge. It is particularly related to networks. We, as authors who deal with them, have a keen interest in road transport networks, since we have been asked to solve a problem of the location of parking areas for dangerous goods in Europe.

However, when trying to solve real problems, scientists may need to develop existing theory beyond the point it has reached so far. In our case, when modeling a road transport network by using a graph, vertices (or nodes) usually correspond to crossroads, while edges represent road sections (where we may be interested in placing a specific service for drivers; see, for example, [13]). In such a situation, the possibility of minimizing the number of roads is relevant. Such roads are usually represented by edges in the models. It is also essential to consider some characteristics which may be related to the presence of certain complete graphs (i.e., graphs including all the possible edges between their vertices). For example, the presence of $K_3$ implies the existence of a cycle which will be, in all likelihood, not be necessary to guarantee communication and transport. The presence of a $K_4$ might affect the future development of the network since it goes against outerplanarity. The presence of $K_5$ prevents the network from being a design without intersections. Finally, road networks tend to expand over time. In Graph Theory this characteristic is usually translated into an infinite graph.

Furthermore, the number of edges (and, eventually, new services) can be limited locally by considering the number of edges incident in each vertex (i.e., their vertex degree). So, this is a good reason to study average degrees in graphs modeling real road transport networks.

On the basis of the above, this paper aims to relate the concepts of average degree and infinite graph. In fact, the average degree has already been used to deal with economic matters (see [11], [14], for example), and it had been defined only for finite graphs. For this reason its properties cannot be used in gradually increasing models.

In another context, percolation models are infinite random graph models for phase transitions and critical phenomena. Percolation is also a key-concept when dealing with hazardous issues such as those concerning dangerous goods transportation. The determination of the critical probability (or percolation threshold) for the percolation model is one of the most interesting problems in this area (see [15], [16], for example). The exact percolation threshold is known only for a few arbitrary trees and infinite 2-dimensional periodic graphs. It is therefore interesting to find bounds for percolation thresholds using *universal formulae* based on features of the infinite model graph. Most of the formulae cited in Physics and Engineering literature are based on the average degree of the infinite graphs (see [17], [18]). The average degree of infinite graphs is only well defined for infinite 2-dimensional periodic graphs. The results that we present in this paper generalize the notion of average degree for a new family of infinite graphs allowing this universal formulae to be used for other infinite graphs.

Theoretically, there exist many problems in Graph Theory involving the relationship between the number of vertices and edges of a graph (see [4], [5], [12], for example), i.e., the average degree. In some cases, many of these problems could be posed for infinite graphs.

Notations and terminologies not explicitly given here can be found in [6], [10].

## II. Average-measurable graphs

In this section, we will define the average degree for a family of infinite graphs that we will call *average-measurable*. We will start from a sequence of finite graphs, and the average degree for infinite graphs will inherit the properties of the average degree for the finite case (see, for example, [1], [17], [19]). We will prove that trees are examples of average-measurable graphs. On the other hand, we will introduce another family of graphs, called *semi-regular* graphs, that will be average-measurable. To achieve these goals we will need some prior notations and definitions.

*Definition 2.1:* Let $G$ be an infinite, locally finite graph and let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of finite subgraphs of $G$. We say that $\{G_n\}_{n\in\mathbb{N}}$ is an *increasing concentric sequence (ICS)* of $G$ if the three following assertions are verified:

- $G_n \subset G_{n+1}$ for all $n$.
- $\partial G_n \cap \partial G_{n+1} = \emptyset$ for all $n$.
- $\bigcup_{n\in\mathbb{N}} G_n = G$,

where $\partial G_n = \langle V(G_n) - V(G_{n-1})\rangle_G$ denotes the *boundary* of $G_n$.

We note that, given an infinite graph $G$ and $v \in G$, it is always possible to find an ICS. In fact, if we consider the subgraphs $G_n(v) = \langle\{u \in V(G) \ : \ d(u,v) \leq n\}\rangle_G$, where $d(u,v)$ denotes the distance between the vertices $u$ and $v$, it is easy to prove that the sequence $\{G_n(v)\}_{n\in\mathbb{N}}$ verifies the conditions described above to be an ICS of $G$.

*Definition 2.2:* Given an infinite, locally finite graph $G$, we define the *inferior-average degree* of $G$ as

$$\underline{d}_\infty(G) = \inf\{\liminf_{n\to+\infty} d(G_n(v)) \ : \ v \in G\},$$

where $d(G_n)$ is average degree of each finite graph $G_n(v)$.

On the other hand, we define the *superior-average degree* of $G$ as

$$\overline{d}_\infty(G) = \sup\{\limsup_{n\to+\infty} d(G_n(v)) \ : \ v \in G\}.$$

*Definition 2.3:* Let $G$ be an infinite, locally finite graph. $G$ is said to be *average-measurable* if $\underline{d}_\infty(G) = \overline{d}_\infty(G) < +\infty$. Besides, in this case, we define the *average degree* of $G$ as $d_\infty(G) = \underline{d}_\infty(G) = \overline{d}_\infty(G)$.

*Example 2.4: Let $H$ be the tree showed in Figure 1, where the vertex $u$ is a root and the degree of the vertices of each level equals its predecessor plus one.*

*If we consider the ICS $\{H_n(u)\}_{n\in\mathbb{N}}$, it is easy to check that*

$$|V(H_n(u))| = \frac{2! + \cdots + (n+2)!}{2}.$$

*On the other hand, taking into account that each finite subgraph $H_n(u)$ of $H$ is a tree,*

$$|E(H_n(u))| = |V(H_n(u))| - 1$$

*and, therefore,*

$$\lim_{n\to+\infty} 2\frac{|E(H_n(u))|}{|V(H_n(u))|} = 2\lim_{n\to+\infty} 1 - \frac{1}{|V(H_n(u))|} = 2.$$

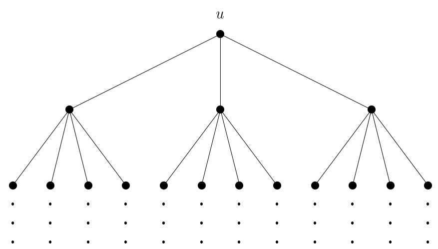


Fig. 1. Tree $H$ with an increasing degree.

*But this property proved for the vertex $u$ is, in fact, true for every vertex of $H$. Moreover, this property is true for every tree.*

*Proposition 2.5: Every infinite, locally finite tree $T$ is average-measurable and $d_\infty(T) = 2$.*

The following example shows a non-average-measurable graph.

*Example 2.6: Let us consider the graph $G$ (see Figure 2), obtained from the definition process of the graph $H$ in the previous example, and verifying*

$$\partial G_1(v) = K_3$$

*and*

$$|E(\partial G_n(v))| = n|V(G_n(v))| \text{ for } n \geq 2.$$

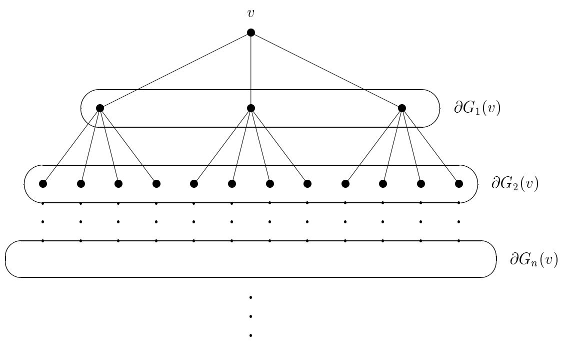


Fig. 2. Graph $G$ obtained from $H$.

*By induction, it is immediate that*

$$n|V(G_n(v))| \leq \binom{|V(\partial G_n(v))|}{2},$$

*and, therefore, it is possible to construct such a graph $G$.*

*On the other hand, from the construction of $G$,*

$$|V(G_n(v))| = \frac{2! + \cdots + (n+2)!}{2}$$

*and*

$$|E(G_n(v))| = |V(G_n(v))| - 1 + 3 + 2|V(G_2(v))| + \cdots + n|V(G_n(v))|.$$

*Now, let $M$ be the graph designed in such a way that $V(M) = V(H) \cup V(G) \cup \{w\}$ and $E(M) = E(H) \cup E(G) \cup \{(u,w),(v,w)\}$ (see Figure 3).*

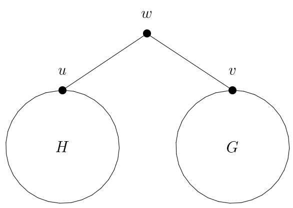


Fig. 3. Graph $M$ obtained from $H$ and $G$.

*We are about to study the sequences $\{M_n(u)\}_{n\in\mathbb{N}}$ and $\{M_n(v)\}_{n\in\mathbb{N}}$. By definition of $M_n(u)$, for $n \geq 2$:*

$$|V(M_n(u))| = |V(H_n(u))| + |V(G_{n-2}(v))| + 1$$

*and*

$$|E(M_n(u))| = |E(H_n(u))| + |E(G_{n-2}(v))| + 2.$$

*To determine the limit of the sequence $\{d(M_n(u))\}_{n\geq 2}$, it is sufficient to analyze the quotient $\dfrac{|E(M_n(u))|}{|V(M_n(u))|}$. By applying the well known Stolz Theorem for sequences:*

$$\frac{|E(M_{n+1}(u))| - |E(M_n(u))|}{|V(M_{n+1}(u))| - |V(M_n(u))|} =$$

$$\frac{\frac{(n+3)!}{2} + \frac{(n+1)!}{2} + (n-1)\cdot\frac{2! + \cdots + (n+1)!}{2}}{\frac{(n+3)!}{2} + \frac{(n+1)!}{2}} =$$

$$1 + \frac{(n-1)\cdot\frac{2! + \cdots + (n+1)!}{2}}{\frac{(n+3)!}{2} + \frac{(n+1)!}{2}}.$$

*If we apply again Stolz Theorem, it holds that*

$$\lim_{n\to+\infty} \frac{(n-1)(2! + \cdots + (n+1)!)}{(n+3)! + (n+1)!} = 0.$$

*So,*

$$\lim_{n\to+\infty} 2\frac{|E(M_n(u))|}{|V(M_n(u))|} = 2$$

*and, therefore, $\underline{d}_\infty(M) \leq 2$.*

*Now we analyze what happens with the sequence $\{M_n(v)\}_{n\in\mathbb{N}}$.*

$$|V(M_n(v))| = |V(H_{n-2}(u))| + |V(G_n(v))| + 1$$

*and*

$$|E(M_n(v))| = |E(H_{n-2}(u))| + |E(G_n(v))| + 2.$$

*We apply Stolz Theorem to compute the limit of the average degree of each subgraph $M_n(v)$ :*

$$\frac{|E(M_{n+1}(v))| - |E(M_n(v))|}{|V(M_{n+1}(v))| - |V(M_n(v))|} = 1 + \frac{(n+1)\cdot\frac{2! + \cdots + (n+3)!}{2}}{\frac{(n+3)!}{2} + \frac{(n+1)!}{2}}.$$

*By applying again Stolz Theorem:*

$$\lim_{n\to+\infty} \frac{(n+1)(2! + \cdots + (n+3)!)}{(n+3)! + (n+1)!} = +\infty.$$

*Hence,*

$$\lim_{n\to+\infty} 2\cdot\frac{|E(M_n(u))|}{|V(M_n(u))|} = +\infty$$

*and, therefore, $\overline{d}_\infty(M) = +\infty$. Thus,*

$$\underline{d}_\infty(M) \leq 2 < \overline{d}_\infty(M) = +\infty,$$

*and $M$ is non-average-measurable.*

## A. Semi-regular graphs

Next we will define a family of infinite graphs which are average-measurable when their maximal degree is bounded.

*Definition 2.7:* Let $G$ be an infinite, locally finite graph. The ICS $\{G_n\}_{n\in\mathbb{N}}$ verifies the so-called *boundary condition* when

$$\lim_{n\to+\infty} \frac{|V(\partial G_n)|}{|V(G_n)|} = 0.$$

*Remark 2.8:* The boundary condition

$$\lim_{n\to+\infty} \frac{|V(\partial G_n)|}{|V(G_n)|} = 0$$

is equivalent to

$$\lim_{n\to+\infty} \frac{|V(G_{n-1})|}{|V(G_n)|} = 1,$$

since

$$\frac{|V(\partial G_n)|}{|V(G_n)|} = \frac{|V(G_n)| - |V(G_{n-1})|}{|V(G_n)|} = 1 - \frac{|V(G_{n-1})|}{|V(G_n)|}.$$

*Definition 2.9:* Let $G$ be an infinite, locally finite graph. $G$ is said to be *semi-regular* if there exists a vertex $v \in G$ such that the ICS $\{G_n(v)\}_{n\in\mathbb{N}}$ verifies the boundary condition.

The following result shows that the previous definition does not depend of the chosen vertex, i.e., if there exists a vertex $v$ for which $\{G_n(v)\}_{n\in\mathbb{N}}$ verifies the boundary condition, then it is verified for all $v$.

*Lemma 2.10:* Let $G$ be an infinite, locally finite graph. If $G$ is semi-regular, then the sequence $\{G_n(v)\}_{n\in\mathbb{N}}$ verifies the boundary condition for all $v \in G$, i.e.,

$$\lim_{n\to+\infty} \frac{|V(\partial G_n(v))|}{|V(G_n(v))|} = 0.$$

*Proof:* Let $G$ be a semi-regular graph and $v_0$ be a vertex such that the ICS $\{G_n(v_0)\}_{n\in\mathbb{N}}$ verifies the boundary condition. Given $v \in G$, we denote by $r$ the distance between $v$ and $v_0$ ($r = d(v, v_0)$). Taking into account Note 2.8, it is sufficient to prove the equality

$$\lim_{n\to+\infty} \frac{|V(G_{n-1}(v))|}{|V(G_n(v))|} = 1$$

to show that the boundary condition is verified.

So, we note that, for $n \geq r+1$ :

$$G_{n-r-1}(v_0) \subseteq G_{n-1}(v) \text{ and } G_n(v) \subseteq G_{n+r}(v_0).$$

Hence,

$$\frac{|V(G_{n-1}(v))|}{|V(G_n(v))|} \geq \frac{|V(G_{n-r-1}(v_0))|}{|V(G_{n+r}(v_0))|}.$$

Furthermore,

$$\frac{|V(G_{n-r-1}(v_0))|}{|V(G_{n+r}(v_0))|} =$$

$$\frac{|V(G_{n-r-1}(v_0))|}{|V(G_{n-r}(v_0))|} \frac{|V(G_{n-r}(v_0))|}{|V(G_{n-r+1}(v_0))|} \cdots \frac{|V(G_{n+r-1}(v_0))|}{|V(G_{n+r}(v_0))|},$$

but

$$\frac{|V(G_{n-r-1}(v_0))|}{|V(G_{n-r}(v_0))|} \xrightarrow[n\to+\infty]{} 1$$

$$\vdots$$

$$\frac{|V(G_{n+r-1}(v_0))|}{|V(G_{n+r}(v_0))|} \xrightarrow[n\to+\infty]{} 1.$$

Finally,

$$1 \geq \lim_{n\to+\infty} \frac{|V(G_{n-1}(v))|}{|V(G_n(v))|} \geq \lim_{n\to+\infty} \frac{|V(G_{n-r-1}(v_0))|}{|V(G_{n+r}(v_0))|} = 1.$$

Thus, $\lim_{n\to+\infty} \frac{|V(G_{n-1}(v))|}{|V(G_n(v))|} = 1$, and the result follows. ∎

*Remark 2.11:* By reasoning as in the previous Lemma, we have that

$$\lim_{n\to+\infty} \frac{|V(G_n(v))|}{|V(G_{n+k}(v))|} = 1$$

for all positive integer $k$.

The previous Lemma may be generalized to other sequences. Actually, given a finite $G_0 \subset G$, we consider the ICS $\{G_n(G_0)\}_{n\geq 0}$ as the sequence defined as follows:

$$G_n(G_0) = \langle \{u \in V(G) \ : \ d(u, G_0) \leq n\} \rangle_G.$$

*Proposition 2.12:* Let $G$ be an infinite, locally finite graph. Then, $G$ is semi-regular if and only if there exists a finite subgraph $G_0 \subset G$ verifying the boundary condition, that is to say, $\lim_{n\to+\infty} \frac{|V(\partial G_n(G_0))|}{|V(G_n(G_0))|} = 0$. Besides, if the aforementioned assertion is true, then the boundary condition is verified for every finite subgraph $G_0 \subset G$.

*Proof:* Let $G$ be an infinite, semi-regular graph, and $G_0 \subset G$ be a finite subgraph. We proving that $\lim_{n\to+\infty} \frac{|V(G_{n-1}(G_0))|}{|V(G_n(G_0))|} = 1$. For this purpose, let us consider $v_0 \in V(G_0)$ and denote by $r$ the diameter of $G_0$ ($r = diam(G_0)$). We have that

$$G_{n-1}(v_0) \subseteq G_{n-1}(G_0) \text{ and } G_n(G_0) \subseteq G_{n+r}(v_0)$$

for all $n$. So

$$\frac{|V(G_{n-1}(G_0))|}{|V(G_n(G_0))|} \geq \frac{|V(G_{n-1}(v_0))|}{|V(G_{n+r}(v_0))|}.$$

Bearing in mind the inequality $\frac{|V(G_{n-1}(G_0))|}{|V(G_n(G_0))|} \leq 1$ and by applying Note 2.11:

$$1 \geq \lim_{n\to+\infty} \frac{|V(G_{n-1}(G_0))|}{|V(G_n(G_0))|} \geq \lim_{n\to+\infty} \frac{|V(G_{n-1}(v_0))|}{|V(G_{n+r}(v_0))|} = 1.$$

Therefore, $\lim_{n\to+\infty} \frac{|V(G_{n-1}(G_0))|}{|V(G_n(G_0))|} = 1$.

In order to prove the opposite implication, we suppose that there exists a $G_0$ such that the sequence $\{G_n(G_0)\}$ verifies the boundary condition. Considering that $v_0 \in V(G_0)$ and $r = diam(G_0)$,

$$G_{n-r-1}(G_0) \subseteq G_{n-1}(v_0)$$

and

$$G_n(v_0) \subseteq G_n(G_0).$$

Hence,

$$\frac{|V(G_{n-1}(v_0))|}{|V(G_n(v_0))|} \geq \frac{|V(G_{n-r-1}(G_0))|}{|V(G_n(v_0))|}.$$

By reasoning as in Note 2.11, it is verified that

$$\lim_{n\to+\infty} \frac{|V(G_{n-r-1}(G_0))|}{|V(G_n(G_0))|} = 1,$$

so

$$\lim_{n\to+\infty} \frac{|V(G_{n-1}(v_0))|}{|V(G_n(v_0))|} = 1.$$

∎

Now, we are about to prove that every semi-regular graph with bounded maximal degree is average-measurable.

*Theorem 2.13:* Let $G$ be an infinite, locally finite graph with $\Delta(G) < +\infty$. If $G$ is semi-regular, then $G$ is average-measurable.

*Proof:* Let $G$ be an infinite graph with $\Delta(G) = \Delta < +\infty$. Given $v \in G$, we consider the sequence $G_n = G_n(v)$, for $n \in \mathbb{N}$. Since $\Delta(G) < +\infty$, we know that $d(G_n) \leq 2\Delta$. To prove that $\{d(G_n)\}$ is convergent, we will see that, in fact, it is a Cauchy sequence, that is, $\lim_{n\to+\infty} |d(G_{n+1}) - d(G_n)| = 0$. Now, we consider the sequence $\{s_n\}$ defined as follows:

$$s_n = \left| \frac{|E(G_{n+1})|}{|V(G_{n+1})|} - \frac{|E(G_n)|}{|V(G_n)|} \right| =$$

$$\left| \frac{|E(G_{n+1})| \cdot |V(G_n)| - |E(G_n)| \cdot |V(G_{n+1})|}{|V(G_n)| \cdot |V(G_{n+1})|} \right|.$$

If we denote

$$E(\partial G_n, \partial G_{n+1}) =$$

$$\{(w_n, w_{n+1}) \in E(G) \ : \ w_n \in \partial G_n, \ w_{n+1} \in \partial G_{n+1}\},$$

then (see Figure 4)

$$|E(G_{n+1})| = |E(G_n)| + |E(\partial G_n, \partial G_{n+1})| + |E(\partial G_{n+1})|.$$

So,

$$s_n =$$

$$\left| \frac{|V(G_n)| \left( |E(G_n)| + |E(\partial G_n, \partial G_{n+1})| + |E(\partial G_{n+1})| \right) - |E(G_n)| \cdot |V(G_{n+1})|}{|V(G_n)| \cdot |V(G_{n+1})|} \right| \leq$$

$$\frac{|E(G_n)| \left| |V(G_n)| - |V(G_{n+1})| \right|}{|V(G_n)| \cdot |V(G_{n+1})|} + \frac{|E(\partial G_n, \partial G_{n+1})|}{|V(G_{n+1})|} + \frac{|E(\partial G_{n+1})|}{|V(G_{n+1})|}.$$

However,

$$\frac{|E(G_n)| \, \left||V(G_n)| - |V(G_{n+1})|\right|}{|V(G_n)| \cdot |V(G_{n+1})|} \leq \Delta \frac{|V(\partial G_{n+1})|}{|V(G_{n+1})|},$$

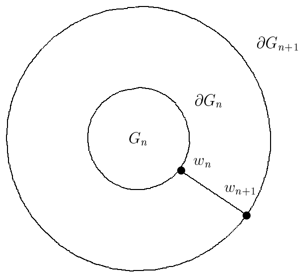


Fig. 4. Edge decomposition $E(G_{n+1})$.

$$\frac{|E(\partial G_n, \partial G_{n+1})|}{|V(G_{n+1})|} \le \frac{\Delta |V(\partial G_n)|}{|V(G_{n+1})|} \le \Delta \frac{|V(\partial G_n)|}{|V(G_n)|},$$

and

$$\frac{|E(\partial G_{n+1})|}{|V(G_{n+1})|} \le \Delta \frac{|V(\partial G_{n+1})|}{|V(G_{n+1})|}.$$

Hence,

$$s_n \le \Delta \frac{|V(\partial G_{n+1})|}{|V(G_{n+1})|} + \Delta \frac{|V(\partial G_n)|}{|V(G_n)|} + \Delta \frac{|V(\partial G_{n+1})|}{|V(G_{n+1})|}.$$

As $\lim\limits_{n\to+\infty} \frac{|V(\partial G_n)|}{|V(G_n|} = 0$, then $\lim\limits_{n\to+\infty} s_n = 0$ and, therefore, the sequence $\{d(G_n)\}_{n\in\mathbb{N}}$ is convergent.

Finally, to reach $\underline{d}_\infty(G) = \overline{d}_\infty(G)$, we are going to prove that, for all $u \in V(G)$, $\lim\limits_{n\to+\infty} \frac{|E(G_n(u))|}{|V(G_n(u)|} = \lim\limits_{n\to+\infty} \frac{|E(G_n(v))|}{|V(G_n(v)|} = t$. For that, we consider $u \in V(G)$ and $r = d_G(u, v)$. So,

$$\begin{aligned} t_n &= \left| \tfrac{|E(G_n(u))|}{|V(G_n(u))|} - \tfrac{|E(G_n(v)|}{|V(G_n(v)|} \right| \\ &= \left| \tfrac{|E(G_n(u))|\cdot|V(G_n(v))| - |E(G_n(v))|\cdot|V(G_n(u))|}{|V(G_n(u))|\cdot|V(G_n(v))|} \right|. \end{aligned}$$

Besides, taking into account that

$$G_{n-r}(v) \subseteq G_n(u) \subseteq G_{n+r}(v)$$

for $n \ge r$, we have that

$$\begin{aligned} t_n &\le \left| \tfrac{|E(G_{n+r}(v))|\cdot|V(G_n(v))| - |E(G_n(v))|\cdot|V(G_{n-r}(v))|}{|V(G_{n-r}(v))|\cdot|V(G_n(v))|} \right| \\ &= \left| \tfrac{|E(G_{n+r}(v))|}{|V(G_{n-r}(v))|} - \tfrac{|E(G_n(v))|}{|V(G_n(v))|} \right|. \end{aligned}$$

Since

$$\lim_{n\to+\infty} \frac{|V(G_n(v))|}{|V(G_{n+1}(v))|} = 1$$

and

$$\frac{|E(G_{n+r}(v))|}{|V(G_{n-r}(v))|} = \frac{|E(G_{n+r}(v))|}{|V(G_{n+r}(v))|} \frac{|V(G_{n+r}(v))|}{|V(G_{n+r-1}(v))|} \cdots \frac{|V(G_{n-r+1}(v))|}{|V(G_{n-r}(v))|},$$

it holds:

$$\lim_{n\to+\infty} \frac{|E(G_{n+r}(v))|}{|V(G_{n-r}(v))|} = t$$

and, therefore, $\lim\limits_{n\to+\infty} t_n = 0$. ■

On the one hand, we note that the condition $\Delta(G) < +\infty$ is necessary in this Theorem, as we can see with the graph $G$ from Figure 5: $G$ is semi-regular but not average-measurable.

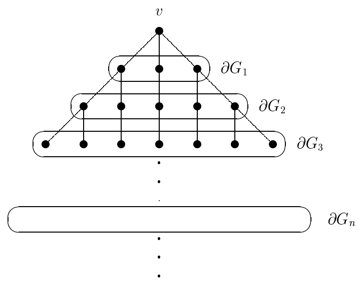


Fig. 5. Semi-regular graph with $\Delta(G) = +\infty$, but non-average-measurable.

This graph is defined in such a way that the subgraph $\partial G_n$ is the complete graph of size $2n+1$, for $n \ge 1$. So,

$$|V(G_n(v))| = 1 + 3 + 5 + \cdots + (2n+1)$$

and

$$|E(G_n(v))| =$$

$$3 + 5 + \cdots + (2n+1) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \cdots + \begin{pmatrix} 2n+1 \\ 2 \end{pmatrix}.$$

$G$ is semi-regular because

$$\lim_{n\to+\infty} \frac{|V(\partial G_n(v))|}{|V(G_n(v))|} = \lim_{n\to+\infty} \frac{2n+1}{1+3+5+\cdots+(2n+1)} = 0.$$

In order to get the value of the limit $\frac{|E(G_n(v))|}{|V(G_n(v))|}$, we apply Stolz Theorem:

$$\frac{|E(G_{n+1}(v))| - |E(G_n(v))|}{|V(G_{n+1}(v))| - |V(G_n(v))|} = \frac{2n+3+\begin{pmatrix} 2n+3 \\ 2 \end{pmatrix}}{2n+3} = +\infty.$$

Thus, $\overline{d}_\infty(G) = +\infty$ and $G$ is not average-measurable.

On the other hand, we consider the graph $H$ from Example 2.4 to find an average-measurable graph which is not semi-regular. Actually, since $H$ is a tree, this graph is average-measurable; however, it is not semi-regular:

$$\frac{|V(\partial H_n(u))|}{|V(H_n(u))|} = \frac{|V(H_n(u))| - |V(H_{n-1}(u))|}{|V(H_n(u))|} =$$

$$\frac{(n+2)!}{2! + \cdots + (n+2)!}.$$

By applying Stolz Theorem:

$$\lim_{n\to+\infty} \frac{(n+3)! - (n+2)!}{(n+3)!} = \lim_{n\to+\infty} \frac{n+2}{n+3} = 1 \ne 0$$

and, therefore, the sequence $\{H_n(u)\}$ does not verify the boundary condition and, then, $H$ is not semi-regular.

### B. Periodic graphs

Now, we will see that the periodic graphs are semi-regular. These graphs are very useful because they are frequent and easily computed. We may find examples of periodic graphs in tiling and patterns [9], Cayley diagrams [3], [8], and they even appear as the resultant graphs of solving linear systems [2].

We recall some prior results about periodic graphs. We denote by $\mathbf{C}$ the unit square $[0,1]\times[0,1]\subset\mathbb{R}^2$ and we define a cellular graph as the graph verifying $V(G)\subset\mathbf{C}$ with no isolated vertices. So, given a cellular graph $G$, we define the 2-dimensional periodic graph ($\mathcal{M}_G$) as the graph obtained from $G$ as follows:

$$V(\mathcal{M}_G)=\{\tau_{(m,n)}(v)\ :\ v\in V(G)\ \text{and}\ (m,n)\in\mathbb{Z}^2\};$$

$$E(\mathcal{M}_G)=\{(\tau_{(m,n)}(u),\tau_{(m,n)}(v))\ :\ (u,v)\in E(G)\ \text{and}\ (m,n)\in\mathbb{Z}^2\},$$

where $\tau_{(m,n)}$ denotes the translation of vector $(m,n)$ in the plane.

If $\mathcal{M}_G$ is a 2-periodic graph generated by the cellular graph $G$, then we define the 8-neighbors of $G$ as the subgraphs $\tau_{(i,j)}(G)$ of $\mathcal{M}_G$ such that $i\in\{-1,0,1\}$ and $j\in\{-1,0,1\}$, with $(i,j)\neq(0,0)$.

Given a cellular graph $G$ and the 2-periodic graph generated by $G$, $\mathcal{M}_G$, we define the $n$-square of center $G$ and radius $n$ ($\prod_n G$) as the subgraph of $\mathcal{M}_G$:

$$\prod_n G=\{\tau_{(i,j)}(G)\ :\ (i,j)\in\mathbb{Z}^2,\ \max\{|i|,|j|\}\leq n\}.$$

We are proving that, in fact, the 2-periodic graphs are semi-regular and average-measurable (since they have bounded maximal degree).

*Proposition 2.14:* Every infinite, periodic, connected graph $\mathcal{M}_G$ generated by the cellular graph $G$ is semi-regular and average-measurable.

*Proof:* Let $G$ be a cellular graph and $\mathcal{M}=\mathcal{M}_G$ be the 2-periodic graph generated from $G$. From Proposition 2.12, it is sufficient to prove that

$$\lim_{n\to+\infty}\frac{|V(\partial\mathcal{M}_n(G))|}{|V(\mathcal{M}_n(G))|}=0$$

to show that $\mathcal{M}$ is semi-regular. We recall that

$$\mathcal{M}_n(G)=\langle u\in V(\mathcal{M})\ :\ d(u,G)_{\mathcal{M}}\leq n\rangle_{\mathcal{M}}.$$

Let us consider $d=\max\{d(u,G)\ :\ u\in G_i,\ 1\leq i\leq 8\}$, where $G_i$ are the 8-neighbors of $G$. Firstly, for all $n\geq d$, it is verified that (see Figure 6):

$$\prod_{\lfloor\frac{n}{d}\rfloor}G\subseteq\mathcal{M}_n(G)\subseteq\prod_n G,$$

because if $v\in\prod_{\lfloor\frac{n}{d}\rfloor}G$, then (see Figure 7):

$$d(u,G)\leq d+d(u,\prod_{\lfloor\frac{n}{d}-1\rfloor}G)\leq\cdots\leq d\left\lfloor\frac{n}{d}\right\rfloor\leq n.$$

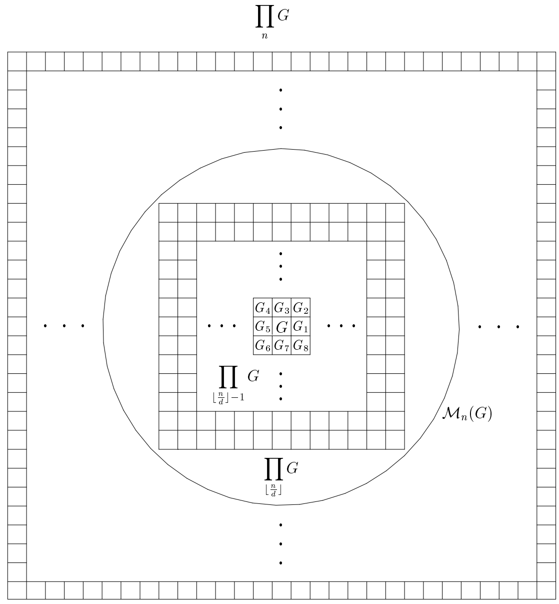


Fig. 6. Chain of inclusions for $\mathcal{M}_n(G)$.

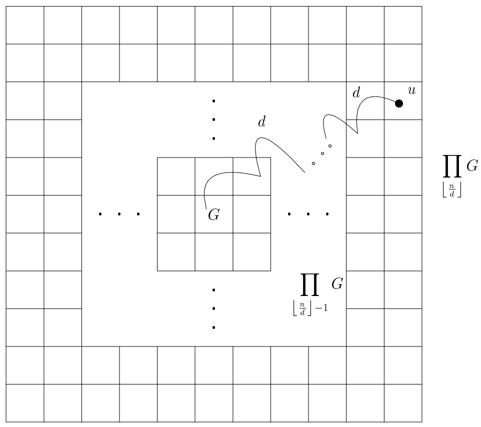


Fig. 7. $\prod_{\lfloor\frac{n}{d}\rfloor}G\subseteq\mathcal{M}_n(G)$.

So,

$$\frac{|V(\partial\mathcal{M}_n(G))|}{|V(\mathcal{M}_n(G))|}\leq\frac{|V(\partial\mathcal{M}_n(G))|}{|V(\prod_{\lfloor\frac{n}{d}\rfloor}G)|}\leq\frac{|V(\partial\mathcal{M}_n(G))|}{\left(2\left\lfloor\frac{n}{d}\right\rfloor+1\right)^2|V(G)|}.$$

On the other hand, for all $n\geq 1$, let us consider $k(n)=|V(\partial\mathcal{M}_n(G))|$. As

$$V(\mathcal{M}_n(G))=\bigcup_{i=0}^{n}V(\partial\mathcal{M}_i(G))\subseteq V(\prod_n G),$$

then:

$$s_n=k(1)+k(2)+\cdots+k(n)\leq(2n+1)^2|V(G)|.$$

Let us suppose that $\limsup_{n\to+\infty}\frac{k(n)}{n^2} = l$, with $l > 0$. By applying Stolz Theorem to the quotient $\frac{s_n}{(2n+1)^2}$, we get that

$$\frac{s_{n+1}-s_n}{(2n+3)^2-(2n+1)^2} = \frac{k(n+1)}{8n+9} = \frac{k(n+1)}{(n+1)^2}\frac{(n+1)^2}{8n+9}$$

and, therefore,

$$\limsup_{n\to+\infty}\frac{s_n}{(2n+1)^2|V(G)|} =$$

$$\frac{1}{|V(G)|}\limsup_{n\to+\infty}\frac{k(n+1)}{(n+1)^2}\frac{(n+1)^2}{8n+9} = +\infty.$$

But this is not possible, because we were supposing that

$$\frac{s_n}{(2n+1)^2|V(G)|} \leq 1.$$

Consequently, $\lim_{n\to+\infty}\frac{k(n)}{n^2} = 0$ and, therefore,

$$\lim_{n\to+\infty}\frac{|V(\partial\mathcal{M}_n(G))|}{\left(2\left\lfloor\frac{n}{d}\right\rfloor+1\right)^2|V(G)|} \leq$$

$$\frac{1}{|V(G)|}\lim_{n\to+\infty}\frac{k(n)}{n^2}\frac{n^2}{\left(2\left(\frac{n}{d}-1\right)+1\right)^2} = 0.$$

Thus,

$$\lim_{n\to+\infty}\frac{|V(\partial\mathcal{M}_n(G))|}{|V(\mathcal{M}_n(G))|} = 0$$

and $\mathcal{M}$ is semi-regular. Besides, since $\mathcal{M}$ is a 2-periodic graph, the assertion $\Delta(\mathcal{M}) < +\infty$ is verified, and (by applying Theorem 2.13) $\mathcal{M}$ is average-measurable. ■

Now, we present an illustration of 2-periodic graph.

*Example 2.15: We consider the 2-periodic graph $\mathcal{M}$ generated by the cellular graph $G$ as in Figure 8.*

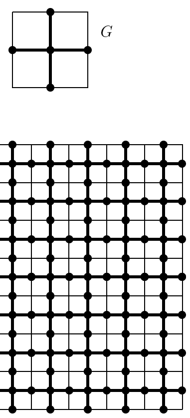


Fig. 8. Periodic graph with average degree $\frac{8}{3}$.

*Since $\mathcal{M}$ is connected, by applying the previous result, we deduce that this graph is average-measurable. Let us consider the sequence $\{\mathcal{M}_n(G)\}$. Since the sequence $\{d(\mathcal{M}_n(G))\}$ is convergent, we know that*

$$\lim_{n\to+\infty} d(\mathcal{M}_n(G)) = \lim_{n\to+\infty} d(\mathcal{M}_{2n}(G)).$$

*On the other hand,*

$$|V(\mathcal{M}_{2n}(G))| = 5 + 16 + 6(4 + 8 + \cdots + 2^n)$$

*and*

$$|E(\mathcal{M}_{2n}(G))| = 4 + 16 + 8(4 + 8 + \cdots + 2^n)$$

*and, therefore, by applying Stolz Theorem:*

$$\lim_{n\to+\infty} d(\mathcal{M}_{2n}(G)) = \frac{8}{3}.$$

Another periodic graph is the 1-dimensional case. This graph $\mathcal{M}_G^1$ is generated by a cellular graph $G$ and horizontal translations $G_i$ of the graph $G$ (see Figure 9). Here we denote by $v_i$ the translated vertex of $v$ in $G_i$, for all integer $i$ and all $v \in V(G)$.

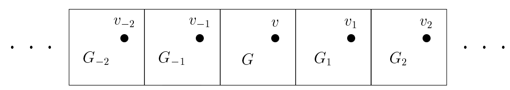


Fig. 9. 1-dimensional periodic graph generated from $G$.

Now, we are able to formulate the final result.

*Proposition 2.16:* Let $G$ be a finite cellular graph. If $\mathcal{M}_G^1$ is connected, then it is semi-regular and, therefore, average-measurable.

*Proof:* Let $G$ be a finite graph, being $M = \mathcal{M}_G^1$ connected. Let us consider the sequence $\{M_n(G)\}_{n\in\mathbb{N}}$ and $d = diam(G)$. We want to study the cardinal of $V(\partial M_n(G))$. To do so, we get $v_k \in V(G_k)$ such that $d(v_k, G)_M = n$, with $k = \min\{i > 0 \ : \ d(v_i, G) = n\}$. So, for all $i \geq k+d+1$, we get $d(v_i, G) \geq n+1$, because if we suppose that $d(v_i, G) \leq n$, as $d(v_i, G_k) \geq i - k$ (see Figure 10), there would be a $w_k \in V(G_k)$ such that $d(v_i, w_k) \geq i - k$ and, therefore:

$$d(w_k, G) \leq d(v_i, G) - d(v_i, w_k) \leq n - i + k \leq n - d - 1.$$

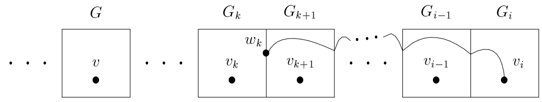


Fig. 10. Path from $v_i$ to $G_k$ in $M$.

However,

$$d(v_k, G) \leq d(v_k, w_k) + d(w_k, G) \leq d + n - d - 1 = n - 1,$$

and this is not possible. Following an analogous reasoning for $k < 0$ :

$$|V(\partial M_n(G))| \leq 2(d+1)|V(G)|.$$

Then,

$$\lim_{n\to+\infty} \frac{|V(\partial\mathcal{M}_n(G))|}{|V(\mathcal{M}_n(G))|} = 0,$$

because

$$\lim_{n\to+\infty} |V(\mathcal{M}_n(G))| = +\infty,$$

and the result follows. ■

## III. Conclusion

In this paper we have extended a definition of average degree $d_\infty(G)$ for a family of infinite graphs which we call average-measurable (see Figure 11). For instance, trees, periodic graphs, and maximal degree bounded semi-regular graphs are average-mesurable graphs.

This generalization of average degree may be useful to model complex, increasing networks, and it extends the use of the universal formulae introduced by J.C. Wierman and D.P. Naor [17] to determine percolation thresholds.

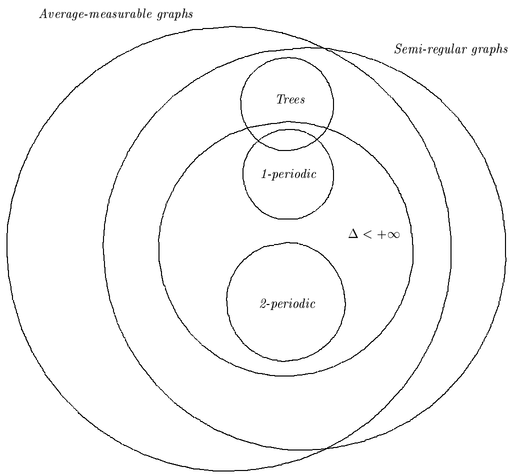


Fig. 11. Relationships among average-measurable graphs.


## Acknowledgments.

The authors would like to thank the European Union ERDF for financial support via the project “Localización de áreas de estacionamiento para mercancías peligrosas en Andalucía y la Red Transeuropea” of the “Programa Operativo FEDER de Andalucía 2007-2013”. We also thank the Public Works Agency and the Regional Ministry of Public Works and Housing of the Regional Government of Andalusia.


## References


[1] P. Barooah and J.P. Hespanha, “Estimation from relative measurements: Electrical analogy and large graphs”, *IEEE Trans. Signal Processing*, vol. 56, no. 6, pp. 2181-2193, 2008.

[2] M. Bauderon, “On System of Equations Defining Infinite Graphs”, in *Graph-theoretic concepts in computer science* (Amsterdam, 1988), Lecture Notes in Comp. Sci., 344, Springer, Berlin, 1989, pp. 54–73.

[3] A. Cayley, “The theory of groups, graphical representation”, *Mathematical Papers, Cambridge*, vol. 10, pp. 26–28, 1895.

[4] M. Cera, A. Diánez and A. Márquez, “The Size of a Graph Without Topological Complete Subgraphs”, *SIAM Journal on Discrete Mathematics*, vol. 13, no. 3 pp. 295–301, 2000.

[5] M. Cera, A. Diánez and A. Márquez, “Extremal Graphs without Topological Complete Subgraphs”, *SIAM Journal on Discrete Mathematics*, vol. 18, no. 2 pp. 288–396, 2004.

[6] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics, vol. 173, Springer-Verlag, 2000.

[7] G.A. Dirac, “In abstrakten graphen vorhandene vollständige 4-graphen und ihre unterteilungen”, *Math. Nachr.*, vol. 22, pp. 61–85, 1960.

[8] R. Frucht, “Herstellung von graphen mit vorgegebener abstrakten gruppe”, *Compositio Math.*, vol. 6, pp. 239–250, 1938.

[9] B. Grünbaum and G.C. Shephard, *Tiling and Patterns*, Freeman, New York, 1987.

[10] W. Mader, “Topological minors in graphs of minimum degree $n$”, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, vol. 49, pp. 199–211, 1998.

[11] E.M. Fedriani, N.Mínguez and A. Martín, “Estabilidad de los indicadores topológicos de pobreza”, *Rect@*, Actas_13, Issue 1, Record No. 39, 2005.

[12] G. Memmi and Y. Raillard, “Some new results about the $(d, k)$ graph problem”, *IEEE Trans. Computers*, vol. 31, no. 8, pp. 784–791, 1982.

[13] W. Peng, G. Dong, K. Yang, and J. Su, “A random road networkmodel and its effect on topological characteristics of mobile delay-tolerant networks”, *IEEE Trans. Mobile Computing*, vol. 13, no. 12, pp. 2706–2718, 2013.

[14] E. Ruiz, M. Hernández, and E.M. Fedriani, “The development of mining heritage tourism: a systemic approach”, in*Tourism Development: Economics, Management and Strategy*. Nova Science Publishers, Inc. Hauppauge, New York, 2008, pp. 121-143.

[15] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, Taylor and Francis, London, 1992.

[16] M. Sahimi, *Applications of Percolation Theory*, Taylor and Francis, London, 1994.

[17] J.C. Wierman and D.P. Naor, “Criteria for evaluation of universal formulas for percolation thresholds”, Physical Review E, vol. 71, 036143, 2005.

[18] J.C. Wierman, D.P. Naor, and R. Cheng, “Improved site percolation threshold universal formula for two-dimensional matching lattices” Physical Review E, vol. 72, 066116, 2005.

[19] A.H. Zemanian, “Infinite electrical networks: a reprise”, *IEEE Trans. Circuits ans Systems*, vol. 35, no. 11, pp. 1346–1358, 1988.