Rough Shapley functions for games with a priori unions

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ABSTRACT

A family of allocation rules for cooperative games with a priori unions is introduced in this paper. These allocation rules, which will be called rough Shapley values, are extensions of the well-known Shapley value for classical cooperative games. The family of rough Shapley values, which is constructed by using rough sets to describe different cooperative options, includes several of the extensions of the Shapley value found in the literature. We prove that the rough Shapley values are the allocation rules for games with a priori unions that satisfy some reasonable properties.

KEYWORDS

game theory; cooperative games; rough sets; Shapley value; a priori unions

1. Introduction

The theory of cooperative games with transferable utility studies situations in which a group of agents cooperate to obtain some profit. One of the main goals of this theory is to provide fair allocation rules for distributing the profit among the agents. The Shapley value [17] is the most commonly used allocation rule. But the use of the Shapley value requires symmetry in the relationships between the agents. However, there are often special alliances or incompatibilities which condition the way in which the agents interact. These situations require specific models. The model on which this article is based is that of cooperative games with a priori unions, introduced by Owen [13] in 1977, which is focused on the study of cooperative situations where there are prior group alliances between agents. Such alliances are described by partitions in the agent set. Owen's model has been generalized using other types of structures, such as coalition configurations [1], proximity relations [5] or colored graphs [12]. Owen obtained and characterized a value for games with a priori unions. This value extends the Shapley value in the sense that in the particular case that there are no previous alliances, it is equal to the Shapley value. Since 1977, multiple extensions of the Shapley value have been obtained for games with a priori unions (see, for instance, [2,3,11]). Besides, modifications of the Shapley value for a priori unions were given by Brink and Dietz [4] and by Gonçalves-Dosantos and Alonso-Meijide [8].

Regarding Owen's model for cooperative games with a priori unions, the reasons for the formation of prior groups can be various, such as interests related to the game itself, external interests, family or friendship relationships or geographical proximity. It is at

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this point that the motivation for the present paper arises. It seems reasonable that the nature of the previous alliances will influence how players from different coalitions cooperate. Thus, a group may act coercively (this is the case of the situations studied in [9]), preventing its members from cooperating unless the entire group participates, or it may be indifferent, or it could even encourage its members to cooperate with other agents, as long as the whole group benefits from such cooperation. These different behaviors should affect the allocation rule. In order to formally describe the different possible behaviors of the prior groups, we propose to use rough sets, introduced by Pawlak [14] in 1982. In each set of agents (coalition) we will consider the partition induced by the prior groups, and then we will use some coefficients to describe the behavior of the sets in this partition. In game theory, rough sets have been used mostly in non-cooperative games (see, for instance, [10]). In cooperative games, Polkowski [15] used them to estimate the worths of certain coalitions in games with incomplete information. In this paper we use rough sets to obtain a family of values for cooperative games with a priori unions. These values are extensions of the Shapley value. The method that we use is constructive and it allows a parametric description of the values in the family, similarly to how Radzik and Driessen [18] describe a family of solutions for standard cooperative games. Besides, we introduce a new approach to obtain solutions for cooperative games with a priori unions, in the sense that we do not use a model based on a two-step negotiation.

This paper is organized as follows. In Section 2 we recall some preliminaries concerning cooperative games and games with a priori unions. In Section 3 we present an axiomatic description of a family of extensions of the Shapley value for games with a priori unions. In Section 4, another family of extensions of the Shapley value is described. In this case the description is not axiomatic. Instead, this family is obtained by applying the Shapley value to a parametric modification of the underlying game taking into account the a priori unions. Finally in Section 5 we show the equivalence between the families obtained in Sections 3 and 4.

2. Games with a priori unions

Let N be a finite set with $|N| = n$. A *cooperative game* on N is defined by a characteristic function $v : 2^N \to \mathbb{R}$ where 2^N is the set of all subsets of N and $v(\emptyset) = 0$. The elements of N are called *players*, the subsets $S \subseteq N$ *coalitions* and $v(S)$ is the *worth* of S, that is, the collective payment that the players in S would obtain if the cooperate. We denote by G^N the family of cooperative games with set of players N. The set of games G^N is a real vector space with dimension $2^n - 1$ if we consider the operations $v + w$ and av with $v, w \in G^N$ and $a \in \mathbb{R}$ given by $(v+w)(S) = v(S)+w(S)$ and $(av)(S) = av(S)$ for every $S \subseteq N$. If $T \in 2^N \setminus \{\emptyset\}$ the *unanimity game* of T is the game $u_T \in G^N$ defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Unanimity games form a basis of G^N, that is, every game $v \in G^N$ can be written as

$$v = \sum_{\{T \subseteq N : T \neq \emptyset\}} \Delta_T^v u_T, \tag{1}$$

where the numbers Δ_T^v are called the Harsanyi coefficients of v. Player $i \in N$ is said to be *null* in $v \in G^N$ if $v(S) = v(S \setminus \{i\})$ for every $S \subseteq N$. If θ is a permutation of N then we denote $\theta i = \theta(i)$ for each $i \in N$ and $\theta S = \{\theta i : i \in S\}$ for each $S \subseteq N$. If $v \in G^N$ then the *permuted game* is the game $\theta v \in G^N$ with $\theta v(\theta S) = v(S)$

for every $S \subseteq N$. A *payoff vector* for a game $v \in G^N$ is any vector $x \in \mathbb{R}^N$. The component x_i is interpreted as the payment of player $i \in N$. A *value* on G^N is a function $f^N : G^N \to \mathbb{R}^N$, which assigns a payoff vector $f^N(v)$ to each game $v \in G^N$. The best-known value for cooperative games is the *Shapley value* [17], ϕ^N. For each game $v \in G^N$ the payoff of a player $i \in N$ is defined by

$$\phi_i^N(v) = \sum_{\{S \subseteq N : i \in S\}} \gamma_s^n \left[v(S) - v(S \setminus \{i\}) \right], \text{ with } \gamma_s^n = \frac{(s-1)!(n-s)!}{n!}. \quad (2)$$

where $s = |S|$. The Shapley value satisfies the following properties: (S1) *efficiency*: if $v \in G^N$, then $\sum_{i \in N} \phi_i^N(v) = v(N)$; (S2) *linearity*: if $v, w \in G^N$ and $a, b \in \mathbb{R}$, then $\phi^N(av + bw) = a\phi^N(v) + b\phi^N(w)$; (S3) *null player property*: if $i \in N$ is a null player for $v \in G^N$, then $\phi_i^N(v) = 0$; (S4) *symmetry*: if θ is a permutation of N and $v \in G^N$, then $\phi_{\theta i}^N(\theta v) = \phi_i^N(v)$ for every $i \in N$. Notice that symmetry implies *equal treatment* of symmetric players, that is, if $i, j \in N$, $v \in G^N$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$, then $\phi_i(v) = \phi_j(v)$. These properties characterize the Shapley value, that is, ϕ is the unique value on G^N that satisfies linearity, efficiency, null player property and symmetry. A *solution function* for cooperative games is a mapping f that assigns a value f^N on G^N for each finite set N. The *Shapley function* is the solution function ϕ that assigns to each finite set of players N the Shapley value on N. If $v \in G^N$ and $N' \subset N$, then the game $v \in G^{N'}$ is the characteristic function v restricted to $2^{N'}$. The Shapley function satisfies this property: (S5) if N is a finite set of players and $i \in N$ is a null player in $v \in G^N$, then $\phi_j^N(v) = \phi_j^{N \setminus \{i\}}(v)$ for all $j \in N \setminus \{i\}$.

Owen [13] introduced games with a priori unions. Starting from the idea that in a cooperative situation some players are more likely to act together than others, he considered prior alliances in the set of players. An *a priori unions structure* or *partition* is a family $\mathcal{P}$ of disjoint nonempty sets of players, namely a partition of a certain set $N(\mathcal{P}) = \bigcup_{A \in \mathcal{P}} A$. In our case $N = N(\mathcal{P})$. Each element $A \in \mathcal{P}$ is called a *union* or *group* and it is interpreted as a set of players with common interests. A partition $\mathcal{P}$ is *individualist* if $|A| = 1$ for all $A \in \mathcal{P}$, and it is *globalist* if $|\mathcal{P}| = 1$. A *coalitional solution function* is a mapping f that assigns a value $f^{\mathcal{P}}$ on G^N to each partition $\mathcal{P}$. In particular, all solution functions are coalitional solution functions which do not take into account the partition, that is, $f^{\mathcal{P}}(v) = f^{\mathcal{P}'}(v)$ if $N(\mathcal{P}) = N(\mathcal{P}') = N$. Owen [13] described a two-step method to obtain a coalitional solution function, which turned out to be an extension of the Shapley function. Let $\mathcal{P} = \{A_1, ..., A_m\}$ be a partition and $v \in G^N$. Let us denote $M = \{1, ..., m\}$. The *quotient game* $v_{\mathcal{P}} \in G^M$ for v is given by $v_{\mathcal{P}}(Q) = v\left(N(\mathcal{P}^Q)\right)$ for all $Q \subseteq M$ where $\mathcal{P}^Q = \{A_q : q \in Q\}$. For each $S \subseteq A \in \mathcal{P}$ we define the restricted partition $\mathcal{P}_S = (\mathcal{P} \setminus \{A\}) \cup \{S\}$. A game $w_q \in G^{A_q}$ for each $q \in M$ is now introduced as $w_q(S) = \phi_q^M(v_{\mathcal{P}_S})$ for all $S \subseteq A_q$. Finally, the *Owen function* is defined as the coalitional solution function ψ where for every partition $\mathcal{P}$, $v \in G^N$ and $i \in N$,

$$\psi_i^{\mathcal{P}}(v) = \phi_i^{A_q}(w_q), \quad (3)$$

where A_q is the group in $\mathcal{P}$ such that $i \in A_q$. In both globalist and individualist partitions the Owen function and the Shapley function coincide, that is, $\psi^{\mathcal{P}}(v) = \phi^N(v)$ for all $v \in G^N$. The Owen value $\psi^{\mathcal{P}}$ satisfies (S1-S3) but not (S4). The Owen function has two main criticisms: it satisfies the null player (thus denying a reasonable payoff to any null player that is part of a coalition with at least one non-null player) and the equal treatment to the individualist partition and the globalist partition (it does not differentiate between these two extreme cases).

3. Coalitional Shapley functions

Several extensions of the Shapley value for games with a priori unions have been studied in the literature [2–4,8,11]. Not all of them retain the essence of the Shapley value. For example, extensions have been defined that are not linear or that provide only group payoffs (not individual ones). With the idea of describing a family of coalitional solution functions, we propose an axiomatic definition that ensures that we maintain fundamental properties of the Shapley value. In addition, we have searched for axioms that are satisfied by the majority of extensions of the Shapley value for games with a priori unions introduced in the literature.

We propose the following axioms for a coalitional solution function f.

Efficiency. If $\mathcal{P}$ is a partition and $v \in G^N$, then $\sum_{i \in N} f_i^{\mathcal{P}}(v) = v(N)$.

Linearity. If $\mathcal{P}$ is a partition, $a, b \in \mathbb{R}$ and $v, w \in G^N$, then $f^{\mathcal{P}}(av + bw) = af^{\mathcal{P}}(v) + bf^{\mathcal{P}}(w)$.

Let $\mathcal{P}$ be a partition. A permutation θ over N is *$\mathcal{P}$-compatible* if $\theta A \in \mathcal{P}$ for all $A \in \mathcal{P}$.

Compatible symmetry. If θ is a permutation $\mathcal{P}$-compatible over N for a partition $\mathcal{P}$, then $f_{\theta i}^{\mathcal{P}}(\theta v) = f_i^{\mathcal{P}}(v)$ for all $v \in G^N$.

Observe that compatible symmetry implies equal treatment of symmetric players $i, j \in N$ in v if i, j are in the same group, in the sense that $f_i^{\mathcal{P}}(v) = f_j^{\mathcal{P}}(v)$ (this is called intracoalitional symmetry in [7]). Let $\mathcal{P}$ be a partition. The set $A \in \mathcal{P}$ is said to be a *null group* for $v \in G^N$ if any $i \in A$ is a null player for game v. The following axiom is a slight modification of the property of independence of null coalitions introduced in [7].

Null group property. Let $\mathcal{P}$ be a partition. If $A \in \mathcal{P}$ is a null group for $v \in G^N$ then

$$f_i^{\mathcal{P}}(v) = \begin{cases} 0, & \text{if } i \in A, \\ f_i^{\mathcal{P} \setminus A}(v), & \text{otherwise.} \end{cases}$$

If we take an individualistic partition then the axioms above correspond to the classical axioms (S1-S4) of the Shapley value plus condition (S5).

Definition 3.1. A coalitional solution function is said to be a coalitional Shapley function if it satisfies efficiency, linearity, compatible symmetry and null group property.

The Shapley function and the Owen function are examples of coalitional Shapley functions. Other examples are the values introduced in [11] and [3]. The proportional Shapley value introduced in [2] is not a coalitional Shapley function since it is not linear. Later we will show that there are more coalitional Shapley functions.

In the following section we present a constructive method that allows to obtain a family of coalitional Shapley functions. With this method we will achieve the following:

- A parametric description of the family, similar to the description obtained by Radzik and Driessen [18] for a family of solutions for standard cooperative games.
- A new approach to obtain solutions for games with a priori unions, which is not based on a two-step negotiation.
- A way of modeling the attitude of the groups in the negotiation.

4. Rough Shapley functions

In the approach proposed by Owen [13] to obtain a solution for games with a priori unions, two assumptions stand out: 1) in order to determine the collective payoff of a prior coalition, all the prior coalitions must act en bloc, and 2) in order to determine the payoff of each player in a prior coalition, the rest of the prior coalitions must act en bloc. Therefore, if the partition of N is $\mathcal{P} = \{A_1, ..., A_m\}$, the only coalitions that matter in Owen's model are those of the form $A_{i_1} \cup \ldots \cup A_{i_{k-1}} \cup E$ where $E \subseteq A_k$. The rest of the coalitions are totally ignored. This, in addition to entailing loss of information, seems somewhat arbitrary, since it presupposes certain attitudes of the players towards coalition formation. Our goal is to introduce a broader model that allows to consider a range of attitudes of the players, and in which Owen's model is a particular case. Thus, in this paper, in order to obtain a coalitional Shapley value, we will have to somehow describe how partitioning affects coalition gains.

4.1. Rough versions of a game with a priori unions

We propose to use rough set theory [14] in the context of cooperative games with a priori unions. Let $\mathcal{P}$ be a partition and $N = N(\mathcal{P})$. If $S \subseteq N$ then the *interior* and the *closure* of S in $\mathcal{P}$ are, respectively,

$$\mathring{S} = \bigcup_{\{A\in\mathcal{P}:A\subseteq S\}} A \quad \text{and} \quad \bar{S} = \bigcup_{\{A\in\mathcal{P}:A\cap S\neq\emptyset\}} A. \tag{4}$$

Coalition S is said to be *crisp* if $\mathring{S} = \bar{S}$, that is, if it is the union of groups in the partition. Any non-crisp coalition is called a *rough coalition*.

Suppose that the players in a coalition S intend to cooperate. If S is crisp, then it is clear all the players in S can cooperate without any restrictions and the players in $N \setminus S$ will play no role. If S is not crisp, then the situation is different, since the players in S that are not in $\mathring{S}$ do not have complete freedom to cooperate within S, since they are in prior groups which are not contained in S, and, therefore, the players in these groups could impede their cooperation. In this case, following rough set theory, we will use the lower approximation (i.e., the interior) and the upper approximation (i.e., the closure) of S, and we will use some coefficients to express the payment achievable by coalition S (taking into account the a priori union structure) as a linear combination of the payments (ignoring the a priori union structure) of the coalitions in the interval $[\mathring{S}, \bar{S}] = \{R : \mathring{S} \subseteq R \subseteq \bar{S}\}$. The coefficients of this linear combination will be denoted α^S_R for each $R \in [\mathring{S}, \bar{S}]$. The set of all coefficients will be $\alpha = \left\{ \left(\alpha^S_R\right)_{R\in[\mathring{S},\bar{S}]} : S \subseteq N \right\}$. In this way we can consider a new game v^α as

$$v^\alpha(S) = \sum_{R\in[\mathring{S},\bar{S}]} \alpha^S_R v(R) \tag{5}$$

for each $S \subseteq N$. It is clear that this construction is too general, and that we need to require conditions on the coefficients α_R^S.

Definition 4.1. Let $\mathcal{P}$ be a partition of a finite set N. A rough coefficient for $\mathcal{P}$ is a set of real numbers $\alpha = \left\{ (\alpha_R^S)_{R \in [\mathring{S}, \bar{S}]} : S \subseteq N \right\}$ that, for any $A \in \mathcal{P}$, satisfies the following three conditions:

1) Sharpness: $\alpha_A^A = 1$.
2) Invariance: $\alpha_{R \cup A}^{S \cup A} = \alpha_R^S$ for all $S \subseteq N \setminus A$ and $R \in [\mathring{S}, \bar{S}]$.
3) Completeness: if $S \subseteq N \setminus A$, $R \in [\mathring{S}, \bar{S}]$ and $S \subsetneq T \subsetneq S \cup A$, then

$$\sum_{L \in [\emptyset, A]} \alpha_{R \cup L}^T = \alpha_R^S.$$

The family of rough coefficients for $\mathcal{P}$ is denoted by $RC^{\mathcal{P}}$. For each $\alpha \in RC^{\mathcal{P}}$ and $v \in G^N$, the game v^α defined by (5) will be called α-rough version of v .

For any partition, the family of rough coefficients is nonempty, since we can take $\alpha_S^S = 1$ for all $S \subseteq N$ and the rest of numbers equal to zero. This rough coefficient is called the *crisp coefficient* and it is denoted by δ. It is clear that $v^\delta = v$ for any cooperative game $v \in G^N$. Moreover if $\mathcal{P}$ is individualistic then the only rough coefficient is the crisp coefficient and hence the only rough version of the game is itself.

Our aim now is to justify the conditions in Definition 4.1. The following proposition shows that the α-rough versions of cooperative games are the games v^α which do not modify either the payments of crisp coalitions or the character of null players which are contained in null groups.

Proposition 4.2. *Let $\mathcal{P}$ be a partition of a finite set N. The numbers $\alpha = \left\{ (\alpha_R^S)_{R \in [\mathring{S}, \bar{S}]} : S \subseteq N \right\}$ form a rough coefficient if and only if, for every cooperative game $v \in G^N$, the following conditions hold:*

1) $v^\alpha(S) = v(S)$ for every crisp coalition $S \subseteq N$,
2) if $i \in A \in \mathcal{P}$ and all players in A are null players in v, then i is null in v^α.

Proof. Firstly we will take a rough coefficient α and prove that the conditions above are satisfied.

1) If S is crisp then S is a union of groups, that is, $S = A_1 \cup \ldots \cup A_k$ with $A_1, \ldots, A_k \in \mathcal{P}$. By sharpness, $\alpha_{A_1}^{A_1} = 1$. By invariance, it follows that $\alpha_{A_1 \cup A_2}^{A_1 \cup A_2} = 1$. Successively applying the invariance property we obtain that $\alpha_S^S = 1$. Since $[\mathring{S}, \bar{S}] = \{S\}$, we have $v^\alpha(S) = \alpha_S^S v(S) = v(S)$.
2) Suppose that $A \in \mathcal{P}$ and all players in A are null players in v. If $T = \{i_1, ..., i_p\} \subseteq A$, then, for all $R \subseteq N$,

$$v(R) = v(R \cup \{i_1\}) = v(R \cup \{i_1, i_2\}) = \cdots = v(R \cup \{i_1, ..., i_{p-1}\}) = v(R \cup T). \quad (6)$$

Let $S, T \subseteq N$ be such that $S \subseteq N \setminus A$ and $T \subseteq A$. We will prove that

$$v^\alpha(S \cup T) = v^\alpha(S). \quad (7)$$

We can suppose that T is nonempty. If $T = A$ then $A \subseteq S \mathring{\cup} A$ and hence we have $[S \mathring{\cup} A, S \bar{\cup} A] = \{R \cup A : R \in [\mathring{S}, \bar{S}]\}$. For each $R \in [\mathring{S}, \bar{S}]$ we have $\alpha_{R \cup A}^{S \cup A} = \alpha_R^S$ by the invariance property of α, and also $v(R \cup A) = v(R)$ by (6). Therefore,

$$v^\alpha(S \cup A) = \sum_{R \in [\mathring{S}, \bar{S}]} \alpha_{R \cup A}^{S \cup A} v(R \cup A) = \sum_{R \in [\mathring{S}, \bar{S}]} \alpha_R^S v(R) = v^\alpha(S).$$

Now we consider $T \neq A$. In this case, we obtain the interval

$$[S \mathring{\cup} T, S \bar{\cup} T] = \{R \cup L : R \in [\mathring{S}, \bar{S}],\ L \in [\emptyset, A]\}.$$

By completeness, each $R \in [\mathring{S}, \bar{S}]$ satisfies $\sum_{L \in [\emptyset, A]} \alpha_{R \cup L}^{S \cup T} = \alpha_R^S$. We also have that $v(R \cup L) = v(R)$ by (6). Therefore,

$$\begin{aligned} v^\alpha(S \cup T) &= \sum_{R \in [\mathring{S}, \bar{S}]} \sum_{L \in [\emptyset, A]} \alpha_{R \cup L}^{S \cup T} v(R \cup L) = \sum_{R \in [\mathring{S}, \bar{S}]} v(R) \sum_{L \in [\emptyset, A]} \alpha_{R \cup L}^{S \cup T} \\ &= \sum_{R \in [\mathring{S}, \bar{S}]} v(R) \alpha_R^S = v^\alpha(S). \end{aligned}$$

Finally we take $i \in A$ and we prove that i is a null player in v^α. Let $S \subseteq N \setminus \{i\}$. If we apply (7) to the pair $(S \setminus A, (S \cap A) \cup \{i\})$ and also to the pair $(S \setminus A, S \cap A)$, we obtain

$$v^\alpha(S \cup \{i\}) = v^\alpha(S \setminus A) = v^\alpha(S).$$

Now let $\alpha = \left\{ \left(\alpha_R^S \right)_{R \in [\mathring{S}, \bar{S}]} : S \subseteq N \right\}$ be such that conditions 1 and 2 described in the proposition are satisfied. Let us prove that α is a rough coefficient.

Take $A \in \mathcal{P}$.

Since A is crisp, we have, by condition 1, that $u_A^\alpha(A) = u_A(A) = 1$. Besides, by definition, $u_A^\alpha(A) = \alpha_A^A u_A(A) = \alpha_A^A$. We conclude that α satisfies sharpness.

Now consider $S \subseteq N \setminus A$ and $\emptyset \neq S' \subseteq A$. We aim to check that if $R \in [\mathring{S}, \bar{S}]$, then

$$\sum_{L \in [\mathring{S}', \bar{S}']} \alpha_{R \cup L}^{S \cup S'} = \alpha_R^S. \tag{8}$$

We will proceed by induction on $d(R) = |\bar{S}| - |R|$. We will use the unanimity game u_R. Notice that all players in A are null players in u_R.

Base case. If $d(R) = 0$ (i.e., $R = \bar{S}$) we take the game $u_{\bar{S}}$. By condition 2 all players in S' are null players in $u_{\bar{S}}^\alpha$. From (6) and definition (5) we obtain

$$\sum_{L \in [\mathring{S}', \bar{S}']} \alpha_{\bar{S} \cup L}^{S \cup S'} = u_{\bar{S}}^\alpha(S \cup S') = u_{\bar{S}}^\alpha(S) = \alpha_{\bar{S}}^S.$$

Inductive step. Suppose that (8) is true for $d(R) < k$. Let us check that then it is true for $d(R) = k$. If we calculate $u_R^\alpha(S \cup S')$ and $u_R^\alpha(S)$ and take into account (6) and condition 2, we obtain

$$\sum_{R'\in[R,\bar{S}]}\sum_{L\in[\mathring{S}',\bar{S}']}\alpha^{S\cup S'}_{R'\cup L}=u^{\alpha}_R(S\cup S')=u^{\alpha}_R(S)=\sum_{R'\in[R,\bar{S}]}\alpha^{S}_{R'}. \tag{9}$$

For each $R'\in[R,\bar{S}]\setminus\{R\}$ we have that $d(R')<k$ and we can apply the induction hypothesis. Therefore, $\sum_{L\in[\mathring{S}',\bar{S}']}\alpha^{S\cup S'}_{R'\cup L}=\alpha^{S}_{R'}$. From this and (9) we conclude that

$$\sum_{L\in[\mathring{S}',\bar{S}']}\alpha^{S\cup S'}_{R\cup L}=\alpha^{S}_{R}.$$

We have proved (8). Now, if we apply this equality to $S'=A$ then $[\mathring{S}',\bar{S}']=\{A\}$ and we obtain invariance. And if we apply (8) to $S'\in[\emptyset,A]\setminus\{\emptyset,A\}$ then $[\mathring{S}',\bar{S}']=[\emptyset,A]$ and, by taking $T=S\cup S'$, we obtain completeness.

□

4.2. The magic cube of a rough coefficient

In this section we will identify a rough coefficient with a magic cube. Then we will solve the system of equations that determines a rough coefficient. This will give us information about the dimension of the family of rough coefficients.

Let $N=\{1,...,n\}$ and $\mathcal{P}=\{A_1,...,A_m\}$. We assume that we have a coalition S as $S=S_1\cup\cdots\cup S_m$ where $S_k\subsetneq A_k$ is a nonempty subset in each group[1]. The completeness condition says

$$\sum_{R\subseteq A_k}\alpha^{S}_{\bigcup_{p=1,p\neq k}^{m}T_p\cup R}=\alpha^{S\setminus S_k}_{\bigcup_{p=1,p\neq k}^{m}T_p}$$

for any $k=1,...,m$ and $T_p\subseteq A_p$ with $p=1,...,m$, $p\neq k$. In a similar way, using again the completeness property,

$$\sum_{R\subseteq A_k}\alpha^{S\setminus\bigcup_{q\in Q}S_q}_{\bigcup_{q\in Q}T_q\cup R}=\alpha^{S\setminus\bigcup_{q\in Q\cup k}S_q}_{\bigcup_{q\in Q}T_q}$$

for all nonempty $Q\subseteq\{1,...,m\}$, with $k\notin Q$ and with cardinality $|Q|<m-1$. If $|Q|=m-1$, namely $Q=\{1,..,m\}\setminus\{k\}$, then we actually have

$$\sum_{R\subseteq A_k}\alpha^{S_k}_{\emptyset\cup R}=\alpha^{\emptyset}_{\emptyset}=1.$$

Thus we obtain a *magic cube* structure. Figure 1 shows this idea with three groups. We use the notation $\alpha^{S}_{R_1R_2R_3}$ to represent $\alpha^{S}_{R_1\cup R_2\cup R_3}$ with $R_k\subseteq A_k$ for $k=1,2,3$.

We aim to calculate the dimension of the set of rough coefficients. We will solve the linear system formed by the conditions of the definition of rough coefficient for a given coalition S. Invariance implies again that we can assume $S=S_1\cup\cdots\cup S_m$ with $\emptyset\neq S_k\subsetneq A_k$ for every $k=1,...,m$. Let $|A_k|=a_k$ and consider $a_1\leq\cdots\leq a_m$. Our unknowns are the coefficients α^{S}_{R} with $R\subseteq N$. For this goal, we will describe the system

[1] If $S_k=A_k$ for some k then the invariance condition implies that this group can be removed.

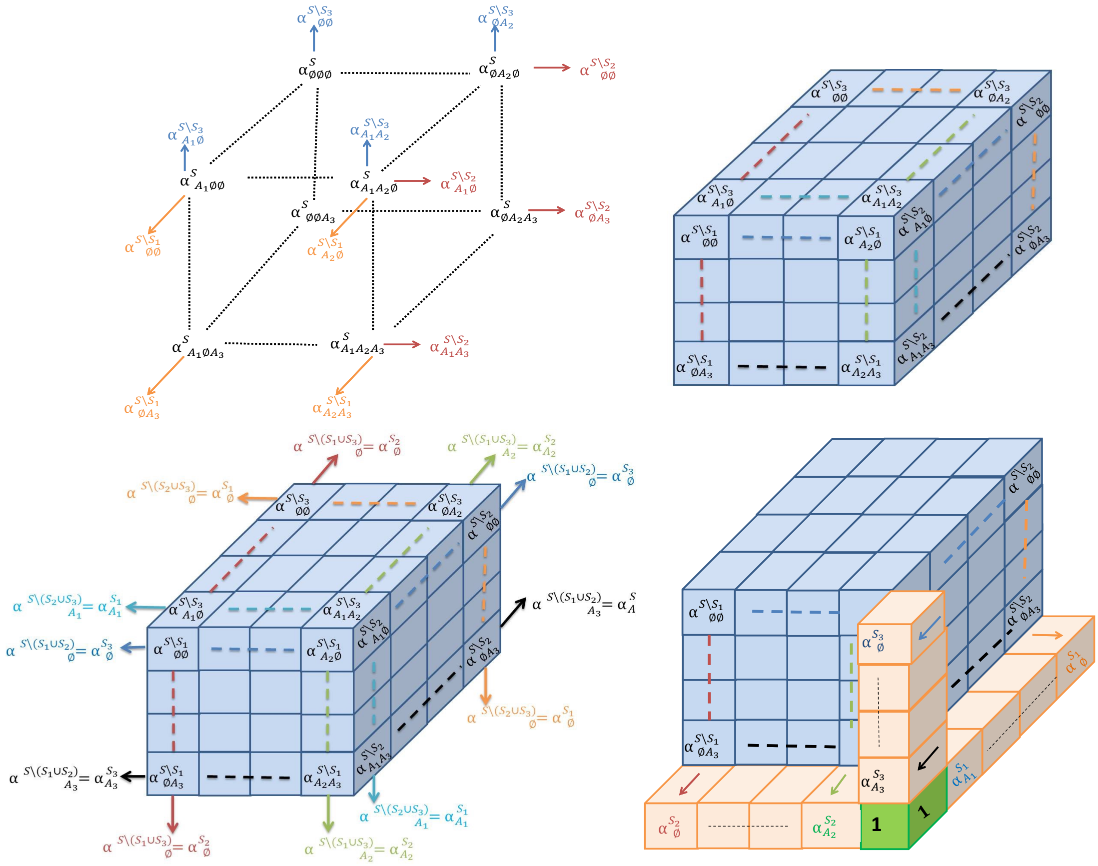

Figure 1. 3D rough magic cube.

as a matrix using the following orders for unknowns and equations. We consider each family 2^{A_k} ordered by the size and the labels. For example, if $A_1 = \{1,2,3\}$ then we take the order in 2^{A_1}: $\{\emptyset, 1, 2, 3, 12, 13, 23, A_1\}$. The unknowns are ordered as follows. First we set a subset of A_1, then one of A_2, and so on until we reach A_m where we traverse all subsets of it using the above internal order in 2^{A_m}. We repeat the process by changing the element fixed in A_{m-1}, later with A_{m-2} an so on until A_1. Let's look at the idea in the three-group example. In this case, we go through the faces in the cube that face us (those of the direction A_1) from the bottom towards us; in each face, from the left column to the right; and, in each column, from top to bottom:

$$\alpha^S_{\emptyset\emptyset\emptyset}, ..., \alpha^S_{\emptyset\emptyset A_3}, ..., \alpha^S_{\emptyset A_2 \emptyset}, ..., \alpha^S_{\emptyset A_2 A_3}, ..., \alpha^S_{A_1 \emptyset\emptyset}, ..., \alpha^S_{A_1 \emptyset A_3}, ..., \alpha^S_{A_1 A_2 \emptyset}, ..., \alpha^S_{A_1 A_2 A_3}.$$

The number of unknowns is $2^{\sum_{k=1}^m a_k}$. We order the equations transversely to the unknowns. First we put the equations running through A_1 with the other groups with fixed coalitions (again taking into account the internal order of the groups). Then the ones that run through A_2 and so on. For example, in the figure the first equations are

$$
\begin{array}{l|l|l}
\alpha^S_{\emptyset\emptyset\emptyset} + \cdots + \alpha^S_{A_1\emptyset\emptyset} = \alpha^{S\setminus S_1}_{\emptyset\emptyset} & \alpha^S_{\emptyset\emptyset\emptyset} + \cdots + \alpha^S_{\emptyset A_2\emptyset} = \alpha^{S\setminus S_2}_{\emptyset\emptyset} & \alpha^S_{\emptyset\emptyset\emptyset} + \cdots + \alpha^S_{\emptyset\emptyset A_3} = \alpha^{S\setminus S_3}_{\emptyset\emptyset} \\
\vdots & \vdots & \vdots \\
\alpha^S_{\emptyset\emptyset A_3} + \cdots + \alpha^S_{A_1\emptyset A_3} = \alpha^{S\setminus S_1}_{\emptyset A_3} & \alpha^S_{\emptyset\emptyset A_3} + \cdots + \alpha^S_{\emptyset A_2 A_3} = \alpha^{S\setminus S_2}_{\emptyset A_3} & \alpha^S_{\emptyset A_2\emptyset} + \cdots + \alpha^S_{\emptyset A_2 A_3} = \alpha^{S\setminus S_3}_{\emptyset A_2} \\
\vdots & \vdots & \vdots \\
\alpha^S_{\emptyset A_2\emptyset} + \cdots + \alpha^S_{A_1 A_2\emptyset} = \alpha^{S\setminus S_1}_{A_2\emptyset} & \alpha^S_{A_1\emptyset\emptyset} + \cdots + \alpha^S_{A_1 A_2\emptyset} = \alpha^{S\setminus S_2}_{A_1\emptyset} & \alpha^S_{A_1\emptyset\emptyset} + \cdots + \alpha^S_{A_1\emptyset A_3} = \alpha^{S\setminus S_3}_{A_1\emptyset} \\
\vdots & \vdots & \vdots \\
\alpha^S_{\emptyset A_2 A_3} + \cdots + \alpha^S_{A_1 A_2 A_3} = \alpha^{S\setminus S_1}_{A_2 A_3} & \alpha^S_{A_1\emptyset A_3} + \cdots + \alpha^S_{A_1 A_2 A_3} = \alpha^{S\setminus S_2}_{A_1 A_3} & \alpha^S_{A_1 A_2\emptyset} + \cdots + \alpha^S_{A_1 A_2 A_3} = \alpha^{S\setminus S_3}_{A_1 A_2}
\end{array}
$$

The number of equations is $\sum_{k=1}^m 2^{\sum_{p=1,p\neq k}^m a_p}$. For each $k = 1, ..., m-1$ we consider the identity matrix I_k of size $2^{a_{k+1}+\cdots+a_m}$ (particularly $I_m = [1]$). Now, for each $k = 1, ..., m$ we define a sequence of blocks:

$$
H_k^0 = [I_k \cdots_{2^{a_k}} I_k]\,, \quad H_k^p = \begin{bmatrix} H_k^{p-1} & & \\ & \ddots_{2^{a_{k-p}}} & \\ & & H_k^{p-1} \end{bmatrix}
$$

for $p = 1, ..., k-1$. The extended matrix of the system of equations is described in m zones as follows

$$
D = \left[\begin{array}{c||c}
H_1^0 & \begin{array}{c} \alpha^{S\setminus S_1}_{\emptyset\cdots\emptyset} \\ \vdots \\ \alpha^{S\setminus S_1}_{A_2\cdots A_m} \end{array} \\ \hline
\vdots & \vdots \\ \hline
H_k^{k-1} & \begin{array}{c} \alpha^{S\setminus S_k}_{\emptyset\cdots{}_{-k}\emptyset} \\ \vdots \\ \alpha^{S\setminus S_k}_{A_1\cdots{}_{-k}A_m} \end{array} \\ \hline
\vdots & \vdots \\ \hline
H_m^{m-1} & \begin{array}{c} \alpha^{S\setminus S_m}_{\emptyset\cdots{}_{-m}\emptyset} \\ \vdots \\ \alpha^{S\setminus S_m}_{A_1\cdots A_{m-1}} \end{array}
\end{array}\right].
$$

Doing a zoom to see the identities of the first block we have

$$
D = \left[\begin{array}{c|c|c|c||c}
 & & & & \alpha^{S\setminus S_1}_{\emptyset\cdots\emptyset} \\
\boxed{\boxed{I_1}} & I_1 & \cdots & I_1 & \vdots \\
 & & & & \alpha^{S\setminus S_1}_{A_2\cdots A_m} \\
\hline
 & & & & \alpha^{S\setminus S_2}_{\emptyset\cdots\emptyset} \\
\boxed{H_2^0} & 0 & \cdots & 0 & \vdots \\
 & & & & \alpha^{S\setminus S_2}_{\emptyset A_3\cdots A_m} \\
---- & ---- & ---- & ---- & ------ \\
0 & H_2^0 & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ---- & ------ \\
 & & & & \vdots \\
0 & 0 & \cdots & H_2^0 & \alpha^{S\setminus S_2}_{A_1 A_3\cdots A_m} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
 & & & & \alpha^{S\setminus S_k}_{\emptyset\cdots\emptyset} \\
\boxed{H_k^{k-2}} & 0 & \cdots & 0 & \vdots \\
 & & & & \alpha^{S\setminus S_k}_{\emptyset A_2\cdots_{-k} A_m} \\
---- & ---- & ---- & ---- & ------ \\
0 & H_k^{k-2} & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ---- & ------ \\
 & & & & \vdots \\
0 & 0 & \cdots & H_k^{k-2} & \alpha^{S\setminus S_k}_{A_1\cdots_{-k} A_m} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
 & & & & \alpha^{S\setminus S_m}_{\emptyset\cdots\emptyset} \\
\boxed{H_m^{m-2}} & 0 & \cdots & 0 & \vdots \\
 & & & & \alpha^{S\setminus S_m}_{\emptyset A_2\cdots A_{m-1}} \\
---- & ---- & ---- & ---- & ------ \\
0 & H_m^{m-2} & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ---- & ------ \\
 & & & & \vdots \\
0 & 0 & \cdots & H_m^{m-2} & \alpha^{S\setminus S_m}_{A_1\cdots A_{m-1}}
\end{array}\right]
$$

Do $\text{rows}_k = \text{rows}_k - H_k^{k-2}\,\text{rows}_1$,

$$
D = \left[\begin{array}{c|c|c|c||c}
\boxed{\boxed{I_1}} & \cdots & \cdots & I_1 & \begin{matrix}\alpha^{S\setminus S_1}_{\emptyset\cdots\emptyset}\\ \vdots \\ \alpha^{S\setminus S_1}_{A_2\cdots A_m}\end{matrix} \\
\hline
0 & -H_2^0 & -H_2^0 & -H_2^0 & \begin{matrix}\alpha^{S\setminus S_2}_{\emptyset\cdots\emptyset} - \sum_{R\subseteq A_2}\alpha^{S\setminus S_1}_{R\emptyset\cdots\emptyset}\\ \vdots \\ \alpha^{S\setminus S_2}_{\emptyset A_3\cdots A_m} - \sum_{R\subseteq A_2}\alpha^{S\setminus S_1}_{RA_3\cdots A_m}\end{matrix} \\
\hdashline
0 & H_2^0 & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\hdashline
0 & 0 & \cdots & H_2^0 & \begin{matrix}\vdots \\ \alpha^{S\setminus S_2}_{A_1A_3\cdots A_m}\end{matrix} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & -H_k^{k-2} & -H_k^{k-2} & -H_k^{k-2} & \begin{matrix}\alpha^{S\setminus S_k}_{\emptyset\cdots\emptyset} - \sum_{R\subseteq A_k}\alpha^{S\setminus S_1}_{\emptyset\cdots R_{(k)}\cdots\emptyset}\\ \vdots \\ \alpha^{S\setminus S_k}_{\emptyset A_2\cdots{}_{-k}A_m} - \sum_{R\subseteq A_k}\alpha^{S\setminus S_1}_{A_2\cdots R_{(k)}\cdots A_m}\end{matrix} \\
\hdashline
0 & H_k^{k-2} & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\hdashline
0 & 0 & \cdots & H_k^{k-2} & \begin{matrix}\vdots \\ \alpha^{S\setminus S_k}_{A_1\cdots{}_{-k}A_m}\end{matrix} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & -H_m^{m-2} & -H_m^{m-2} & -H_m^{m-2} & \begin{matrix}\alpha^{S\setminus S_m}_{\emptyset\cdots\emptyset} - \sum_{R\subseteq A_m}\alpha^{S\setminus S_1}_{\emptyset\cdots\emptyset R}\\ \vdots \\ \alpha^{S\setminus S_m}_{\emptyset A_2\cdots A_{m-1}} - \sum_{R\subseteq A_m}\alpha^{S\setminus S_1}_{A_2\cdots A_{m-1}R}\end{matrix} \\
\hdashline
0 & H_m^{m-2} & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\hdashline
0 & 0 & \cdots & H_m^{m-2} & \begin{matrix}\vdots \\ \alpha^{S\setminus S_m}_{A_1\cdots A_{m-1}}\end{matrix}
\end{array}\right].
$$

We solve $2^{a_2+\cdots+a_m}$ unknowns, those with the form $\alpha^S_{\emptyset*\cdots*}$, in the first zone of the matrix. The k-zone is transformed by the following operations

$$\left[\begin{array}{c|c|c||c}
 & & & \alpha^{S\setminus S_k}_{\emptyset\cdots\emptyset} - \alpha^{S\setminus\{S_1,S_k\}}_{\emptyset\cdots\emptyset} \\
-H_k^{k-2} & \cdots & -H_k^{k-2} & \vdots \\
 & & & \alpha^{S\setminus S_k}_{\emptyset A_2\cdots{}_{-k}A_m} - \alpha^{S\setminus\{S_1,S_k\}}_{A_2\cdots{}_{-k}\cdots A_m} \\
---- & ---- & ---- & ------------- \\
H_k^{k-2} & \cdots & 0 & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ------------- \\
 & & & \vdots \\
0 & \cdots & H_k^{k-2} & \alpha^{S\setminus S_k}_{A_1\cdots{}_{-k}A_m}
\end{array}\right] \longrightarrow$$

$$\left[\begin{array}{c|c|c||c}
 & & & \sum_{R\subseteq A_1}\alpha^{S\setminus S_k}_{R\emptyset\cdots\emptyset} - \alpha^{S\setminus\{S_1,S_k\}}_{\emptyset\cdots\emptyset} \\
0 & \cdots & 0 & \vdots \\
 & & & \sum_{R\subseteq A_1}\alpha^{S\setminus S_k}_{RA_2\cdots{}_{-k}A_m} - \alpha^{S\setminus\{S_1,S_k\}}_{A_2\cdots{}_{-k}\cdots A_m} \\
---- & ---- & ---- & -------------------- \\
H_k^{k-2} & \cdots & 0 & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & -------------------- \\
 & & & \vdots \\
0 & \cdots & H_k^{k-2} & \alpha^{S\setminus S_k}_{A_1\cdots{}_{-k}A_m}
\end{array}\right] \longrightarrow$$

$$\left[\begin{array}{c|c|c||c}
0 & \cdots & 0 & 0 \\
---- & ---- & ---- & ------ \\
 & & & \alpha^{S\setminus S_k}_{\{1\}\emptyset\cdots\emptyset} \\
H_k^{k-2} & \cdots & 0 & \vdots \\
---- & ---- & ---- & ------ \\
\vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ------ \\
 & & & \vdots \\
0 & \cdots & H_k^{k-2} & \alpha^{S\setminus S_k}_{A_1\cdots{}_{-k}A_m}
\end{array}\right]$$

The systems is reduced to

$$
D = \left[\begin{array}{c|c|c||c}
H_2^0 & \cdots & 0 & \alpha^{S\setminus S_2}_{\{1\}\emptyset\cdots\emptyset} \\
 & & & \vdots \\
---- & ---- & ---- & ------ \\
\vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ------ \\
 & & & \vdots \\
0 & \cdots & H_2^0 & \alpha^{S\setminus S_2}_{A_1 A_3 \cdots A_m} \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
H_k^{k-2} & \cdots & 0 & \alpha^{S\setminus S_k}_{\{1\}\emptyset\cdots\emptyset} \\
 & & & \vdots \\
---- & ---- & ---- & ------ \\
\vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ------ \\
 & & & \vdots \\
0 & \cdots & H_k^{k-2} & \alpha^{S\setminus S_k}_{A_1 \cdots_{-k} A_m} \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
H_m^{m-2} & \cdots & 0 & \alpha^{S\setminus S_m}_{\{1\}\emptyset\cdots\emptyset} \\
 & & & \vdots \\
---- & ---- & ---- & ------ \\
\vdots & \ddots & \vdots & \vdots \\
---- & ---- & ---- & ------ \\
 & & & \vdots \\
0 & \cdots & H_m^{m-2} & \alpha^{S\setminus S_m}_{A_1 \cdots A_{m-1}}
\end{array}\right].
$$

If we repeat the process m times we get to solve a set of unknowns in the form $\alpha^S_{*\cdots\emptyset_{(q)}*\cdots*}$ with $q = 1, ..., m$.

Remark 1. The conclusion we draw from solving this system is that, when looking for a rough coefficient, a subset of numbers with the form

$$\alpha^S_{*\cdots\emptyset_{(q)}*\cdots*} \tag{10}$$

with $q = 1, ..., m$ are determined by its properties. This fact will be used in Section 5.

4.3. The concept of Rough Shapley function

If we fix a partition and a rough coefficient, we can obtain a value by taking the Shapley value of the rough version of each game.

Definition 4.3. Let $\mathcal{P}$ be a partition of N and a rough coefficient $\alpha \in RC^{\mathcal{P}}$. The α-Shapley value is defined as $\phi^{\mathcal{P},\alpha}(v) = \phi(v^\alpha)$ for all $v \in G^N$.

In order to obtain a coalitional function we must associate a value to each partition. But we will require some conditions, which are presented below.

Definition 4.4. Consider a family of rough coefficients $\alpha = \{\alpha(\mathcal{P})\}_{\mathcal{P}}$, one for each partition $\mathcal{P}$, satisfying

- Symmetry. If $\mathcal{P}$ is a partition and θ is a permutation of N, then $\alpha_{\theta(T)}^{\theta(S)}(\theta(\mathcal{P})) = \alpha_T^S(\mathcal{P})$.
- Reduction. If $\mathcal{P}$ is a partition, $A \in \mathcal{P}$ and $S \subseteq N \setminus A$, then $\alpha_T^S(\mathcal{P} \setminus \{A\}) = \alpha_T^S(\mathcal{P})$.

The rough Shapley function defined by α is the mapping ϕ^α that assigns to each partition $\mathcal{P}$ the value $\phi^{\mathcal{P},\alpha(\mathcal{P})}$ on G^N.

The condition of symmetry means that the coefficients are independent of the labeling of the players. The condition of reduction means that the coefficients are inherited from any partition $\mathcal{P}$ to partitions contained in $\mathcal{P}$.

For simplicity of notation, when a family of coefficients α and a partition $\mathcal{P}$ are fixed we will write α instead of $\alpha(\mathcal{P})$.

Notice that the classic Shapley function is a rough Shapley function. Indeed, if for each partition $\mathcal{P}$ we consider the crisp coefficient δ, it is clear that $\{\delta\}_\mathcal{P}$ satisfies symmetry and reduction. Besides, $\phi^{\mathcal{P},\delta} = \phi^N$ for any partition $\mathcal{P}$. The classic Shapley value is a rough solution that does not take into account the structure of groups.

5. Relation between both families of Shapley functions

Theorem 5.1. *Rough Shapley functions are coalitional Shapley functions.*

Proof. Consider a rough Shapley value ϕ^α where α is a family of rough coefficients. Our goal is to prove that ϕ^α is a coalitional Shapley function.
1) Efficiency. The Shapley value is efficient (S1), so for any partition $\mathcal{P}$ and $v \in G^N$,

$$\sum_{i\in N} \phi_i^{\mathcal{P},\alpha}(v) = \sum_{i\in N} \phi_i^N(v^\alpha) = v^\alpha(N) = v(N),$$

where the last equality follows from the fact that N is crisp and Proposition 4.2.
2) Linearity. The property follows from the linearity of the Shapley value (S2) and the fact that for any partition $\mathcal{P}$, games $v, w \in G^N$ and $a, b \in \mathbb{R}$

$$(av + bw)^\alpha = av^\alpha + bw^\alpha.$$

3) Compatible symmetry. Let $\mathcal{P}$ be a partition, θ a $\mathcal{P}$-compatible permutation and $v \in G^N$. The symmetry property of α implies that $\alpha_{\theta T}^{\theta S} = \alpha_T^S$ because $\theta\mathcal{P} = \mathcal{P}$. For each θS with $S \subseteq N$ we have that $\theta T \in [\mathring{\theta S}, \bar{\theta S}]$ if and only if $T \in [\mathring{S}, \bar{S}]$, and then

$$(\theta v)^\alpha(\theta S) = \sum_{\theta T\in[\mathring{\theta S},\bar{\theta S}]\}} \alpha_{\theta T}^{\theta S}\theta v(\theta T) = v^\alpha(S) = \theta v^\alpha(\theta S).$$

Since the Shapley value satisfies symmetry,

$$\phi_{\theta i}^{\mathcal{P},\alpha}(\theta v) = \phi_{\theta i}^N((\theta v)^\alpha) = \phi_{\theta i}^N(\theta v^\alpha) = \phi_i^N(v^\alpha) = \phi_i^{\mathcal{P},\alpha}(v).$$

4) Null group property. Let $\mathcal{P}$ be a partition and $v \in G^N$. Suppose that $A \in \mathcal{P}$ is a null group for v. By Proposition 4.2 all the players in A are null players in v^α. If $i \in A$, then $\phi_i^{\mathcal{P},\alpha}(v) = \phi_i^N(v^\alpha) = 0$ by (S3). Now take $i \in N \setminus A$. For each $S \subseteq N \setminus A$ we have,

by the reduction property of α, that

$$v^{\alpha(\mathcal{P})}(S) = \sum_{T\in[\mathring{S},\bar{S}]} \alpha_T^S(\mathcal{P})v(T) = \sum_{T\in[\mathring{S},\bar{S}]} \alpha_T^S(\mathcal{P}\setminus A)v(T) = v^{\alpha(\mathcal{P}\setminus A)}(S)$$

where $[\mathring{S},\bar{S}]$ is the same in $\mathcal{P}$ and $\mathcal{P}\setminus A$ because $S\subseteq N\setminus A$. Let $A=\{i_1,...,i_p\}$. We proved before that $i_1,...,i_p$ are null players in $v^{\alpha(\mathcal{P})}$. By (S5),

$$\begin{aligned}\phi_i^{\mathcal{P},\alpha}(v) &= \phi_i^N(v^{\alpha(\mathcal{P})}) = \phi_i^{N\setminus\{i_1\}}(v^{\alpha(\mathcal{P})}) = \cdots = \phi_i^{N\setminus A}(v^{\alpha(\mathcal{P})}) \\ &= \phi_i^{N\setminus A}(v^{\alpha(\mathcal{P}\setminus A)}) = \phi_i^{\mathcal{P}\setminus A,\alpha}(v).\end{aligned}$$

□

In the following example we will show the construction of a rough Shapley function, which will be a coalitional Shapley function different from the Owen value and from the classic Shapley function.

Example. We aim to define a particular family of rough coefficients α. To this end, we need a previous construction. Suppose that A is a finite set and S a nonempty set with $S\subsetneq A$. For all $T\subseteq A$ we define

$$h^A(S,T) = \frac{|S\cap T|}{|S|2^{|A|-1}},$$

and $h^A(A,A)=h^A(\emptyset,\emptyset)=1$. If we randomly (and with equal probability) choose an element in S and then randomly (and with equal probability) choose an subset of A that contains the element chosen, the number $h^A(S,T)$ is equal to the probability that the subset of A chosen is equal to T. The following equality holds:

$$\sum_{T\in[\emptyset,A]} h^A(S,T) = 1. \tag{11}$$

If $|S|=s$, $|A|=a$ and $p=|S\cap T|$ then

$$\begin{aligned}\sum_{T\in[\emptyset,A]} h^A(S,T) &= \sum_{T\in[\emptyset,A]} \frac{|S\cap T|}{|S|2^{|A|-1}} = \frac{1}{|S|2^{|A|-1}} \sum_{T\in[\emptyset,A]} |S\cap T| \\ &= \frac{1}{s2^{a-1}}\sum_{p=0}^{s}\binom{s}{p}2^{a-s}p = \frac{1}{s2^{s-1}}\sum_{p=1}^{s}\binom{s}{p}p \\ &= \frac{1}{s2^{s-1}}\sum_{p=1}^{s}\binom{s-1}{p-1}s = \frac{1}{2^{s-1}}\sum_{p=0}^{s-1}\binom{s-1}{p} = 1\end{aligned}$$

where we have used two well-known combinatorial identities[2]. Now consider the partition $\mathcal{P}=\{A_1,...,A_m\}$. For each $S\subseteq N$ and $T\in[\mathring{S},\bar{S}]$, we define

$$\alpha_T^S = \prod_{k\in H_{\mathcal{P}}(S)} h^{A_k}(S_k,T_k),$$

[2] $\sum_{k=0}^{r}\binom{r}{k}=2^r$, $k\binom{r}{k}=r\binom{r-1}{k-1}$

where $S_k = S \cap A_k$ and $H_{\mathcal{P}}(S) = \{k : S_k \neq \emptyset\}$. Let us prove that α is a rough coefficient.
Sharpness: since $H_{\mathcal{P}}(A_p) = \{p\}$ we have that $\alpha_{A_p}^{A_p} = h^{A_p}(A_p, A_p) = 1$.
Invariance: suppose that $S \subseteq N \setminus A_p$. Clearly, $H_{\mathcal{P}}(S \cup A_p) = H_{\mathcal{P}}(S) \cup \{p\}$, $(S \cup A_p)_p = A_p$ and $(S \cup A)_k = S_k$ with $k \neq p$. Moreover, if $T \in [\mathring{S}, \bar{S}]$ then $T \cup A_p$ satisfies the same properties. We have

$$\alpha_{T \cup A_p}^{S \cup A_p} = h^{A_p}(A_p, A_p) \prod_{k \in H_{\mathcal{P}}(S)} h^{A_k}(S_k, T_k) = \alpha_T^S.$$

Completeness: let $S \subseteq N \setminus A_p$, $R \in [\mathring{S}, \bar{S}]$ and $T \in (S, S \cup A_p)$. Let $T = S \cup K$ with $K \in (\emptyset, A_p)$. Since $H_{\mathcal{P}}(S \cup K) = H_{\mathcal{P}}(S) \cup \{p\}$ we have that, if $k \neq p$, then $(S \cup K)_k = S_k$ and $(R \cup L)_k = R_k$. Besides, $(S \cup K)_p = K$ and $(R \cup L)_p = L$. We have

$$\begin{aligned}
\sum_{L \in [\emptyset, A_p]} \alpha_{R \cup L}^{S \cup K} &= \sum_{L \in [\emptyset, A_p]} \prod_{k \in H_{\mathcal{P}}(S \cup K)} h^{A_k}((S \cup K)_k, (R \cup L)_k) \\
&= \sum_{L \in [\emptyset, A_p]} h^{A_p}(K, L) \prod_{k \in H_{\mathcal{P}}(S)} h^{A_k}(S_k, R_k) \\
&= \prod_{k \in H_{\mathcal{P}}(S)} h^{A_k}(S_k, R_k) \sum_{L \in [\emptyset, A_p]} h^{A_p}(K, L) \\
&= \prod_{k \in H_{\mathcal{P}}(S)} h^{A_k}(S_k, R_k) = \alpha_R^S = \alpha_{R \cup A_p}^{S \cup A_p},
\end{aligned}$$

where we have used (11) and invariance.

If we take the coefficient above $\alpha(\mathcal{P})$ for each partition $\mathcal{P}$ we obtain a family of rough coefficients, because, by construction, this family satisfies symmetry and reduction. Therefore, ϕ^α is a rough Shapley function, and, by Theorem 5.1, a coalitional Shapley function.

Let us calculate $\phi^\alpha(v)$ for a particular partition $\mathcal{P}$ and a particular game $v \in G^{\{1,2,3\}}$. Take $\mathcal{P} = \{\{1,2\},\{3\}\}$ and the unanimity game $u_{\{13\}} \in G^{\{1,2,3\}}$. We have that

$$h^{\{1,2\}}(\{1\}, \emptyset) = h^{\{1,2\}}(\{2\}, \emptyset) = h^{\{1,2\}}(\{1\}, \{2\}) = h^{\{1,2\}}(\{2\}, \{1\}) = 0,$$

$$h^{\{1,2\}}(\{1\}, \{1\}) = h^{\{1,2\}}(\{2\}, \{2\}) = h^{\{1,2\}}(\{1\}, \{1,2\}) = h^{\{12\}}(\{2\}, \{1,2\}) = 1/2.$$

Besides, by definition,

$$h^{\{1,2\}}(\emptyset, \emptyset) = h^{\{1,2\}}(\{1,2\}, \{1,2\}) = h^{\{3\}}(\emptyset, \emptyset) = h^{\{3\}}(\{3\}, \{3\}) = 1.$$

The rough version of a game $v \in G^N$ is

$$v^\alpha(\{1\}) = \frac{1}{2}v(\{1\}) + \frac{1}{2}v(\{1,2\}),\ v^\alpha(\{2\}) = \frac{1}{2}v(\{2\}) + \frac{1}{2}v(\{1,2\}),\ v^\alpha(\{3\}) = v(\{3\}),$$

$$v^\alpha(\{1,2\}) = v(\{1,2\}),\ v^\alpha(\{1,3\}) = \frac{1}{2}v(\{1,3\}) + \frac{1}{2}v(N),\ v^\alpha(\{2,3\}) = \frac{1}{2}v(\{2,3\}) + \frac{1}{2}v(N),$$

$$v^\alpha(N) = v(N).$$

In particular, $u^\alpha_{\{1,3\}}(\{1,3\}) = 1$, $u^\alpha_{\{1,3\}}(\{2,3\}) = 1/2$, $u^\alpha_{\{1,3\}}(N) = 1$ and $u^\alpha_{\{1,3\}}(S) = 0$ otherwise. The classical Shapley value and the Owen value coincide for this game, returning the payoff vector

$$\phi^N(u_{\{1,3\}}) = (1/2, 0, 1/2).$$

It is easy to check that

$$\phi^\alpha(u_{\{1,3\}}) = (4/12, 1/12, 7/12).$$

Notice that ϕ^α does not satisfy the null player property. Indeed, $\phi^\alpha_2(u_{\{1,3\}}) > 0$, although player 2 is null in $u_{\{1,3\}}$. This player receives a payoff for being part of a coalition, $\{1,2\}$, that is not irrelevant.

The following equality is a classic exercise of linear algebra that we will use in the proof of Theorem 5.2. Let $x_1, x_2..., x_n, y_1, y_2, ..., y_n \in \mathbb{R}$ and $n \in \mathbb{N}$. Then,

$$D_n = \begin{vmatrix} x_1 + y_1 & x_1 & \cdots & x_1 \\ x_2 & x_2 + y_2 & \cdots & x_2 \\ \vdots & \vdots & \cdots & \vdots \\ x_n & x_n & \cdots & x_n + y_n \end{vmatrix} = \prod_{k=1}^{n} y_k \left[1 + \sum_{k=1}^{n} \frac{x_k}{y_k} \right]. \qquad (12)$$

Moreover, $D_n = x_k \prod_{\{l=1, l\neq k\}}^{n} y_l + y_k D^k_{n-1}$, where D^k_{n-1} is the determinant obtained when we delete row k and column k in the expression of D_n

The last result of this paper is devoted to prove that the rough Shapley functions "almost" complete the family of coalitional Shapley functions. A partition $\mathcal{P}$ is said to be a partition *into large groups* if $|A| > 2$ for every $A \in \mathcal{P}$. Games with a priori large unions model situations which are interesting to analyze. For instance, the Owen value was applied to these games to study the power in the European Parliament [6] and in the International Monetary Fund [16].

Theorem 5.2. *If f is a coalitional Shapley function then it coincides with a rough Shapley function on partitions into large groups.*

Proof. Let f be a coalitional Shapley function. Let $\mathcal{P}$ be a partition into large groups, that is, $|A| > 2$ for all $A \in \mathcal{P}$. We will prove the result by induction on the cardinality $|\mathcal{P}|$ of the partition.

Base case, $|\mathcal{P}| = 1$. Consider f restricted to the games over the globalist partition $\mathcal{P} = \{N\}$. Notice that in this case compatible symmetry means symmetry because all the permutations over N are compatible with $\{N\}$. Hence, f is a value that satisfies efficiency, linearity and symmetry (an ELS-value[3] following [18]). Proposition 2 in [18] says that there is a coefficient b_s for each cardinality $s = 0, ..., n$ with $b_0 = 1$[4], $b_n = 1$

[3]Moreover, the null group property only says $f^{\{N\}}(0) = 0$. But combining efficiency with symmetry the same equality is achieved. Thus both families of values can be identified.

[4]Actually, in [18] took $b_0 = 0$ but this number is not significant.

and

$$f_i^{\{N\}}(v) = \sum_{\{S \subseteq N : i \in S\}} \gamma_s^n \left[b_s v(S) - b_{s-1} v(S \setminus \{i\}) \right].$$

We take, for each non-crisp coalition S, $\alpha_T^S = 0$ if $\emptyset \neq T \neq S$, $\alpha_S^S = b_{|S|}$ and $\alpha_\emptyset^S = 1 - b_{|S|}$. Observe that we have constructed a rough coefficient for partition $\{N\}$. Sharpness: the only crisp coalitions in this partition are N and $\emptyset$, and $b_n = b_0 = 1$. Invariance: it is true because $b_n = b_0$. Completeness: let $S = \emptyset$, $A = N$ and $T \in 2^N \setminus \{\emptyset, N\}$. We have

$$\sum_{L \in [\emptyset, N]} \alpha_L^T = \alpha_\emptyset^T + b_{|T|} = 1 = b_n.$$

Therefore, using this rough coefficient, we obtain $\phi^{\{N\},\alpha} = f^{\{N\}}$.

Induction step. Suppose that f coincides with a rough Shapley value for partitions with cardinality less than m, that is, there exists a family of rough coefficients α such that

$$\phi^{\mathcal{P},\alpha} = f^{\mathcal{P}} \tag{13}$$

for any partition $\mathcal{P}$ with $|\mathcal{P}| < m$.
Take a partition $\mathcal{P} = \{A_1, ..., A_m\}$. Let $N = \bigcup_{A \in \mathcal{P}} A = \{1, ..., n\}$. We look for a rough coefficient $\alpha(\mathcal{P}) = \alpha$ to define $f^{\mathcal{P}} = \phi^{\mathcal{P},\alpha}$.
Reduction condition implies that all the numbers α_T^S with $S \cap A_k = \emptyset$ (or $S \cap A_k = A_k$ by invariance) for $k = 1, ..., m$ are determined by (13). Following Section 3, coefficient α must satisfy the rough magic cube. By (10) all the numbers α_T^S with $T \cap A_k = \emptyset$ for some group $k = 1, ..., m$ are determined. Hence, only the numbers α_T^S with $T \cap A_k \neq \emptyset$ and $S \cap A_k \neq A_k$ for all $k = 1, ..., m$ are not determined.
The rough coefficient must satisfy the equations $\phi_i^{\mathcal{P},\alpha}(v) = f_i^{\mathcal{P}}(v)$ for each $i \in N$ and game v on N. If we use the linearity axiom of f we can reduce the system to the equations

$$\phi_i(u_H^\alpha) = \phi_i^{\mathcal{P},\alpha}(u_H) = f_i^{\mathcal{P}}(u_H) \tag{14}$$

for all $i \in N$ and $H \subseteq N$ nonempty. Moreover, we can assume $H \cap A_k \neq \emptyset$ for all $k = 1, ..., m$. Indeed, if $H \cap A_k = \emptyset$ for some $k = 1, ..., m$ then, by the null group property, $f_i^{\mathcal{P}}(u_H) = f_i^{\mathcal{P} \setminus A_k}(u_H)$ for any $i \in N \setminus A_k$ and $f_i^{\mathcal{P}}(u_H) = 0$ if $i \in A_k$. Therefore the payoffs in the games are determined. The same happens with any feasible $\phi^{\mathcal{P},\alpha}$.
Suppose $H = T_1 \cup T_2 \cup \cdots T_m$ with $T_k \subseteq A_k$ and $T_k \neq \emptyset$ for any $k = 1, ..., m$. We consider the sets T_k in H (and also in $\mathcal{P}$) such that:

- $T_k = A_k$ for $k = 1, ..., p$ ($p \in \{0, ..., m\}$),
- $T_k \neq A_k$ and $|T_k| > 1$ for $k = p+1, ..., p+q$ ($q \in \{0, ..., m-p\}$), and
- $|T_k| = 1$ for $k = p+q+1, ..., m$.

Now we use the axiom of compatible symmetry for $f^{\mathcal{P}}$ and $\phi^{\mathcal{P},\alpha}$ to reduce the number of equalities in (14). So, we obtain the following cases:

- For $k = 1, ..., p$: there is only one different payoff in A_k, $\phi_{i_k}(u_H^\alpha) = f_{i_k}^{\mathcal{P}}(u_H)$, because all the players are internal symmetric players in u_H.

- For $k = p+1, ..., m$: there are two different payoffs in A_k, $\phi_{i_k}(u_H^{\alpha}) = f_{i_k}^{\mathcal{P}}(u_H)$ and $\phi_{j_k}(u_H^{\alpha}) = f_{j_k}^{\mathcal{P}}(u_H)$ with $i_k \in T_k$ and $j_k \in A_k \setminus T_k$, because all the players in T_k are internal symmetric players in u_H and also all the players in $A_k \setminus T_k$.

So, we have five types of payoffs (one in the first kind of equations, two in the second one, and two in the third one). We reduce (14) to $p + 2(m-p)$ equations with the structure

$$\sum_{\{S \subseteq N : i_k \in S\}} \gamma_{|S|}^{n} \left[u_H^{\alpha}(S) - u_H^{\alpha}(S \setminus \{i_k\}) \right] = f_{i_k}^{\mathcal{P}}(u_H), \quad k = 1, ..., m \tag{15}$$

$$\sum_{\{S \subseteq N : j_k \in S\}} \gamma_{|S|}^{n} \left[u_H^{\alpha}(S) - u_H^{\alpha}(S \setminus \{j_k\}) \right] = f_{j_k}^{\mathcal{P}}(u_H), \quad k = p+1, ..., m \tag{16}$$

where

$$u_H^{\alpha}(S) = \sum_{R \in [\mathring{S} \cup H, \bar{S}]} \alpha_R^S.$$

We suppose them in the following order: for each $k = 1, ..., p$ the equation in (15) and for every $k = p+1, ..., m$ first the equation in (15) and second the equation in (16). We choose one player representing each group, that is, $d_k^H \in A_k$ for any $k = 1, ..., m$, such that $d_k^H \in T_k$ for all $k = 1, ..., p+q$ and $d_k^H \in A_k \setminus T_k$ for all $k = p+q+1, ..., m$. This election is feasible because the groups are large. Let

$$D^H = \{d_1^H, ..., d_m^H\}.$$

We select $p + 2(m-p)$ unknown numbers to solve (15,16):

- for $k = 1, ..., m$: $\alpha_H^{D^H \cup \{i_k^H\}}$, with $i_k^H \in T_k \setminus \{d_k^H\}$ for equation (15),
- for $k = p+1, ..., m$: $\alpha_H^{D^H \cup \{j_k^H\}}$, with $j_k^H \in A_k \setminus (T_k \cup \{d_k^H\})$ for equation (16).

We consider these unknowns ordered like their associated equations. Let $a_k = |A_k|$ and $t_k = |T_k|$ for $k = 1, ..., m$. We will take into account the symmetry condition of α in order to join equal numbers and get the structure of the matrix in Figure 2, where we focus on the coefficients of our unknown numbers in order to study their determinant. The explanation of how we get the different coefficients is in the Annex of the paper. We note that some numbers X_b^l with $l = 1, 2, 3, 4, 5$ appear in Figure 2, whose formulation can be seen in the Annex, which are independent of the unknowns and which are repeated in all the equations of each row of blocks. In a second step we take out X_b^q with $q = 1, ..., 5$ for the different rows, observe that $X_b^q \neq 0$ for all $k = 1, ..., 5$. We also take out the values

$$\frac{a_c - 1}{2}, \ \frac{t_c - 1}{2}, \ a_c - t_c, \ \frac{a_c - 2}{2}$$

from the columns of kinds 2,3,4 and 6 respectively. They are not zero because the groups are large. The reader must take into account the element of the main diagonal in each case. The result is Figure 3. We transform the main diagonal as

b	Main diagonal b	$\alpha_H^{D^H \cup \{i_c^H\}}$ $1,...,p$	$\alpha_H^{D^H \cup \{i_c^H\}}$ $p+1,...,p+q$	$\alpha_H^{D^H \cup \{j_c^H\}}$ $p+1,...,p+q$	$\alpha_H^{D^H \cup \{i_c^H\}}$ $p+q+1,...,m$	$\alpha_H^{D^H \cup \{j_c^H\}}$ $p+q+1,...,m$
$1,...,p$	$X_b^1 \frac{a_b-1}{2} \cdot$ $[2\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$X_b^1 \frac{a_c-1}{2} \cdot$ $[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$X_b^1 \frac{t_c-1}{2} \cdot$ $[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$X_b^1 (a_c - t_c) \cdot$ $[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$X_b^1 \cdot$ $[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$X_b^1 \frac{a_c-2}{2} \cdot$ $[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$
$p+1,...,p+q$ i_b	$X_b^2 \frac{t_b-1}{2} \cdot$ $[2\gamma^n_{m+1} - (t_b-2)\gamma^n_{m+2}]$	$X_b^2 \frac{a_c-1}{2} \cdot$ $[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$X_b^2 \frac{t_c-1}{2} \cdot$ $[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$X_b^2 (a_c - t_c) \cdot$ $[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$X_b^2 \cdot$ $[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$X_b^2 \frac{a_c-2}{2} \cdot$ $[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$
$p+1,...,p+q$ j_b	$X_b^3 \cdot$ $[\gamma^n_{m+1}$ $-(a_b - t_b - 1)\gamma^n_{m+2}]$	$X_b^3 \frac{a_c-1}{2} \cdot$ $[-\gamma^n_{m+2}]$	$X_b^3 \frac{t_c-1}{2} \cdot$ $[-\gamma^n_{m+2}]$	$X_b^3 (a_c - t_c) \cdot$ $[-\gamma^n_{m+2}]$	$X_b^3 \cdot$ $[-\gamma^n_{m+2}]$	$X_b^3 \frac{a_c-2}{2} \cdot$ $[-\gamma^n_{m+2}]$
$p+q+1,...,m$ i_b	$X_b^4 \cdot$ $[\gamma^n_{m+1}]$	$X_b^4 \frac{a_c-1}{2} \cdot$ $[-\gamma^n_{m+2}]$	$X_b^4 \frac{t_c-1}{2} \cdot$ $[-\gamma^n_{m+2}]$	$X_b^4 (a_c - t_c) \cdot$ $[-\gamma^n_{m+2}]$	$X_b^4 \cdot$ $[-\gamma^n_{m+2}]$	$X_b^4 \frac{a_c-2}{2} \cdot$ $[-\gamma^n_{m+2}]$
$p+q+1,...,m$ j_b	$X_b^5 \frac{a_b-2}{2} \cdot$ $[2\gamma^n_{m+1} - (a_b-3)\gamma^n_{m+2}]$	$X_b^5 \frac{a_c-1}{2} \cdot$ $[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$X_b^5 \frac{t_c-1}{2} \cdot$ $[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$X_b^5 (a_c - t_c) \cdot$ $[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$X_b^5 \cdot$ $[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$X_b^5 \frac{a_c-2}{2} \cdot$ $[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$

Figure 2. Kinds of rows and main diagonal. Step 1.

b	Main diagonal b	$\alpha_H^{D^H \cup \{i_c^H\}}$ $1,...,p$	$\alpha_H^{D^H \cup \{i_c^H\}}$ $p+1,...,p+q$	$\alpha_H^{D^H \cup \{j_c^H\}}$ $p+1,...,p+q$	$\alpha_H^{D^H \cup \{i_c^H\}}$ $p+q+1,...,m$	$\alpha_H^{D^H \cup \{j_c^H\}}$ $p+q+1,...,m$
$1,...,p$	$[2\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-1)\gamma^n_{m+2}]$
$p+1,...,p+q$ i_b	$[2\gamma^n_{m+1} - (t_b-2)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (t_b-1)\gamma^n_{m+2}]$
$p+1,...,p+q$ j_b	$\frac{1}{a_b - t_b} [\gamma^n_{m+1}$ $-(a_b - t_b - 1)\gamma^n_{m+2}]$	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$
$p+q+1,...,m$ i_b	γ^n_{m+1}	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$	$-\gamma^n_{m+2}$
$p+q+1,...,m$ j_b	$[2\gamma^n_{m+1} - (a_b-3)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$	$[\gamma^n_{m+1} - (a_b-2)\gamma^n_{m+2}]$

Figure 3. Kinds of rows and main diagonal. Step 2.

- For $b = 1, ..., p$,

$$2\gamma^n_{m+1} - \gamma^n_{m+2}(a_b - 2) = [\gamma^n_{m+1} - (a_b - 1)\gamma^n_{m+2}] + [\gamma^n_{m+1} + \gamma^n_{m+2}].$$

- For $b = p+1, ..., p+q$ and i_b,

$$2\gamma^n_{m+1} - (t_b - 2)\gamma^n_{m+2} = [\gamma^n_{m+1} - (t_b - 1)\gamma^n_{m+2}] + [\gamma^n_{m+1} + \gamma^n_{m+2}].$$

- For $b = p+1, ..., p+q$ and j_b,

$$\frac{1}{a_b - t_b}[\gamma^n_{m+1} - (a_b - t_b - 1)\gamma^n_{m+2}] = [-\gamma^n_{m+2}] + \frac{1}{a_b - t_b}[\gamma^n_{m+1} + \gamma^n_{m+2}].$$

- For $b = p+q+1, ..., m$ and i_b,

$$\gamma^n_{m+1} = [-\gamma^n_{m+2}] + [\gamma^n_{m+1} + \gamma^n_{m+2}].$$

- For $b = p+q+1, ..., m$ and j_b,

$$2\gamma^n_{m+1} - \gamma^n_{m+2}(a_b - 3) = [\gamma^n_{m+1} - (a_b - 2)\gamma^n_{m+2}] + [\gamma^n_{m+1} + \gamma^n_{m+2}].$$

Following equality (12), we obtain the determinant

$$\begin{aligned} D_{2m-p} &= \frac{\left(\gamma^n_{m+1} + \gamma^n_{m+2}\right)^{2m-p}}{\prod_{k=p+1}^{p+q}(a_k - t_k)} \left[1 + \sum_{k=1}^{p} \frac{\gamma^n_{m+1} - (a_k - 1)\gamma^n_{m+2}}{\gamma^n_{m+1} + \gamma^n_{m+2}} + \sum_{k=p+1}^{p+q} \frac{\gamma^n_{m+1} - (t_k - 1)\gamma^n_{m+2}}{\gamma^n_{m+1} + \gamma^n_{m+2}} \right. \\ &\quad \left. + \sum_{k=p+1}^{p+q} \frac{-\gamma^n_{m+2}(a_k - t_k)}{\gamma^n_{m+1} + \gamma^n_{m+2}} + \sum_{k=p+q+1}^{m} \frac{-\gamma^n_{m+2}}{\gamma^n_{m+1} + \gamma^n_{m+2}} + \sum_{k=p+q+1}^{m} \frac{\gamma^n_{m+1} - (a_k - 2)\gamma^n_{m+2}}{\gamma^n_{m+1} + \gamma^n_{m+2}} \right] \\ &= \frac{\left(\gamma^n_{m+1} + \gamma^n_{m+2}\right)^{2m-p-1}}{\prod_{k=p+1}^{p+q}(a_k - t_k)} \left[(m+1)\gamma^n_{m+1} + (m-n+1)\gamma^n_{m+2}\right]. \end{aligned}$$

It is easy to check that $D_{2m-p} = 0$, because

$$(m+1)\gamma^n_{m+1} + (m-n+1)\gamma^n_{m+2} = 0.$$

But there is a condition on the payoffs of $f^{\mathcal{P}}$, efficiency, that we have not used. We only needed to verify $2m-p-1$ equalities, that is, we can consider the same determinant with one less equation and one less unknown. If there exists k with $a_k \neq t_k$ then we delete row k and, by (12) (it can be multiplied by $a_k - t_k$),

$$D^k_{2m-p-1} = \gamma^n_{m+2}(a_k - t_k)\frac{(\gamma^n_{m+1} + \gamma^n_{m+2})^{2m-p-2}}{\prod_{b=p+1, b\neq k}^{p+q}(a_b - t_b)} \neq 0,$$

where D^k_{2m-p-1} is the determinant obtained when we delete row and column k in D_{2m-p}. Now we suppose $a_k = t_k$ for all $k = 1, ..., m$. In this case

$$D^k_{2m-p-1} = (\gamma^n_{m+1} - (a_k - 1)\gamma^n_{m+2})(\gamma^n_{m+1} + \gamma^n_{m+2})^{2m-p-2}.$$

If there is $k \in \{1, ..., m\}$ with $\gamma^n_{m+1} - (a_k - 1)\gamma^n_{m+2} \neq 0$, then we delete row k and we get $D^k_{2m-p-1} \neq 0$. If $\gamma^n_{m+1} - (a_k - 1)\gamma^n_{m+2} = 0$ for all $k = 1, ..., m$, then all the groups have the same size $a_k = \frac{n}{m+1}$. But if all the groups have the same size a then $am = n$ and this implies $a = \frac{n}{m}$. □

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Appendix A. Explication of Figure 2

We explain here the five different kinds of rows showed in Figure 2.

1) Consider an equation corresponding to $b \in \{1, ..., p\}$ in (15). Take $D^H \cup \{i\}$ with $i \notin A_b$. Observe that if there are K symmetric (in $\mathcal{P}$) coalitions S containing i_b to it in the equation, then there are also $(a_b - 1)K$ symmetric $S \setminus \{i_b\}$ options to it in the equation. So, the coefficient of number $\alpha_H^{D^H \cup \{i\}}$ is

$$K\left[\gamma_{m+1}^n - \gamma_{m+2}^n(a_b - 1)\right] \tag{A1}$$

Let $X_b^1 = \prod_{\substack{k=1\\k\neq b}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$. We look for the coefficient of the chosen numbers in this equation, analyzing the five kinds of numbers plus the number corresponding to the equation.

1.1 Number $\alpha_H^{D^H \cup \{i_b^H\}}$ (we cannot apply (A1) here). It appears for all the coalitions equivalent by symmetry to $S = D^H \cup \{i_b^H\}$, namely

$$(a_b - 1) \prod_{\substack{k=1\\k\neq b}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$$

times, and with coalitions equivalent by symmetry to $S \setminus \{i_b\} = D^H \cup \{i_b^H\}$, namely

$$\binom{a_b - 1}{2} \prod_{\substack{k=1\\k\neq b}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$$

times. So, the coefficient of this number is

$$X_b^1 \frac{a_b - 1}{2}\left[2\gamma_{m+1}^n - (a_b - 2)\gamma_{m+2}^n\right].$$

1.2 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, with $c \in \{1, ..., p\} \setminus \{b\}$. The quantity K of symmetric coalitions (compatible in $\mathcal{P}$) S containing i_b to $D^H \cup \{i_c^H\}$ is

$$K = \binom{a_c}{2} \prod_{\substack{k=1\\k\neq b,c}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

Since (A1) we get the coefficient of this number in the equation

$$X_b^1 \frac{a_c - 1}{2}\left[\gamma_{m+1}^n - (a_b - 1)\gamma_{m+2}^n\right].$$

1.3 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+1, ..., p+q\}$. We obtain in this case

$$K = \binom{t_c}{2} \prod_{\substack{k=1 \\ k \neq b}}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq c}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

Since (A1) we get the coefficient of this number

$$X_b^1 \frac{t_c - 1}{2} \left[\gamma_{m+1}^n - (a_b - 1)\gamma_{m+2}^n\right].$$

1.4 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+1, ..., p+q\}$. We get

$$K = t_c(a_c - t_c) \prod_{\substack{k=1 \\ k \neq b}}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq c}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1),$$

and then, from (A1), the coefficient of the number is

$$X_b^1 (a_c - t_c) \left[\gamma_{m+1}^n - (a_b - 1)\gamma_{m+2}^n\right].$$

1.5 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+q+1, ..., m\}$. It holds

$$K = \prod_{\substack{k=1 \\ k \neq b}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

Since (A1) the coefficient of the number is

$$X_b^1 \left[\gamma_{m+1}^n - (a_b - 1)\gamma_{m+2}^n\right].$$

1.6 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+q+1, ..., m\}$. We get

$$K = \binom{a_c - 1}{2} \prod_{\substack{k=1 \\ k \neq b}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq c}}^{m} (a_k - 1).$$

Using (A1), the coefficient of the number is

$$X_b^1 \frac{a_c - 2}{2} \left[\gamma_{m+1}^n - (a_b - 1)\gamma_{m+2}^n\right].$$

2) Consider an equation corresponding to $b \in \{p+1, ..., p+q\}$ in (15). Take $D^H \cup \{i\}$ with $i \notin T_b$. Observe that if there are K symmetric (in $\mathcal{P}$) coalitions S containing i_b to it in the equation, then there are also $(t_b - 1)K$ symmetric $S \setminus \{i_b\}$ options to it in the equation. So, the coefficient of number $\alpha_H^{D^H \cup \{i\}}$ is

$$K \left[\gamma_{m+1}^n - \gamma_{m+2}^n (t_b - 1)\right] \tag{A2}$$

Let $X_b^2 = \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1\\k\neq b}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$. We look for the coefficient of the chosen numbers in this equation.

2.1 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{1, ..., p\}$. The quantity K of symmetric coalitions (compatible in $\mathcal{P}$) S containing i_b to $D^H \cup \{i_c^H\}$ is

$$K = \binom{a_c}{2} \prod_{\substack{k=1\\k\neq c}}^{p} a_k \prod_{\substack{k=p+1\\k\neq b}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

Since (A2) we get the coefficient of this number in the equation

$$X_b^2 \frac{a_c - 1}{2} \left[\gamma_{m+1}^n - (t_b - 1)\gamma_{m+2}^n\right].$$

2.2 Number $\alpha_H^{D^H \cup \{i_b^H\}}$ (we cannot apply (A2) here). It appears for all the coalitions equivalent by symmetry to $S = D^H \cup \{i_b^H\}$, namely

$$(t_b - 1) \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1\\k\neq b}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$$

times, and with coalitions equivalent by symmetry to $S \setminus \{i_b\} = D^H \cup \{i_b^H\}$, namely

$$\binom{t_b - 1}{2} \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1\\k\neq b}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$$

times. So, the coefficient of this number is

$$X_b^2 \frac{t_b - 1}{2} \left[2\gamma_{m+1}^n - (t_b - 2)\gamma_{m+2}^n\right].$$

2.3 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+1, ..., p+q\} \setminus \{b\}$. We obtain in this case

$$K = \binom{t_c}{2} \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1\\k\neq b,c}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

Since (A2) we get the coefficient of this number

$$X_b^2 \frac{t_c - 1}{2} \left[\gamma_{m+1}^n - (t_b - 1)\gamma_{m+2}^n\right].$$

2.4 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+1, ..., p+q\}$. We get

$$K = t_c(a_c - t_c) \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq b,c}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1),$$

and then, from (A2), the coefficient of the number is

$$X_b^2 (a_c - t_c) \left[\gamma_{m+1}^n - (t_b - 1)\gamma_{m+2}^n \right].$$

2.5 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+q+1, ..., m\}$. It holds

$$K = \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq b}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

Since (A2) the coefficient of the number is

$$X_b^2 \left[\gamma_{m+1}^n - (t_b - 1)\gamma_{m+2}^n \right].$$

2.6 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+q+1, ..., m\}$. We get

$$K = \binom{a_c - 1}{2} \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq b}}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq c}}^{m} (a_k - 1).$$

Using (A2), the coefficient of the number is

$$X_b^2 \frac{a_c - 2}{2} \left[\gamma_{m+1}^n - (t_b - 1)\gamma_{m+2}^n \right].$$

3) Consider an equation corresponding to $b \in \{p+1, ..., p+q\}$ in (16). Observe that $j_b \notin D^H \cup \{i\}$ if $i \notin A_b \setminus T_b$. So, number $\alpha_H^{D^H \cup \{i\}}$ only appears as $S \setminus j_b$ in (16). Let $X_b^3 = \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$. We look for the coefficient of the chosen numbers in this equation.

3.1 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{1, ..., p\}$. The quantity of coalitions S containing i_b such that $S \setminus \{i_b\}$ was symmetric (compatible in $\mathcal{P}$) to $D^H \cup \{i_c^H\}$ is

$$\binom{a_c}{2} \prod_{\substack{k=1 \\ k \neq c}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

Hence we get the coefficient of this number in the equation

$$-X_b^3 \frac{a_c - 1}{2} \gamma_{m+2}^n.$$

3.2 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+1, ..., p+q\}$. We have the following quantity of options

$$\binom{t_c}{2} t_b \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq b,c}}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1).$$

So we get the coefficient of this number

$$-X_b^3 \frac{t_c - 1}{2} \gamma_{m+2}^n.$$

3.3 Number $\alpha_H^{D^H \cup \{j_b^H\}}$. It appears for all the coalitions equivalent by symmetry to $S = D^H \cup \{j_b^H\}$, namely

$$\prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$$

times, and with coalitions equivalent by symmetry to $S \setminus \{i_b\} = D^H \cup \{j_b^H\}$, namely

$$(a_b - t_b - 1) \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1)$$

times. So, the coefficient of this number is

$$X_b^3 \left[\gamma_{m+1}^n - (a_b - t_b - 1) \gamma_{m+2}^n \right].$$

3.4 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+1, ..., p+q\} \setminus \{b\}$. We get the number for $S \setminus \{j_b\}$ symmetric to $D^H \cup \{j_c^H\}$ in quantity

$$(a_c - t_c) \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{k=p+q+1}^{m} (a_k - 1),$$

and then the coefficient of the number is

$$-X_b^3 (a_c - t_c) \gamma_{m+2}^n.$$

3.5 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+q+1, ..., m\}$. The coefficient is

$$-X_b^3 \gamma_{m+2}^n.$$

3.6 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+q+1, ..., m\}$. We get symmetric versions of this coalition as $S \setminus \{j_b\}$

$$\binom{a_c - 1}{2} \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq c}}^{m} (a_k - 1).$$

Therefore the coefficient of the number is

$$-X_b^3 \frac{a_c - 2}{2} \gamma_{m+2}^n.$$

4) Consider an equation corresponding to $b \in \{p+q+1, ..., m\}$ in (15). This case is similar to 3), so let $X_b^4 = X_b^3$. We comment the coefficient of the chosen numbers in this equation.

4.1 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{1, ..., p\}$. The coefficient of this number in the equation is

$$-X_b^4 \frac{a_c - 1}{2} \gamma_{m+2}^n.$$

4.2 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+1, ..., p+q\}$. We have the coefficient

$$-X_b^4 \frac{t_c - 1}{2} \gamma_{m+2}^n.$$

4.3 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+1, ..., p+q\} \setminus \{b\}$. The coefficient of the number is

$$-X_b^4 (a_c - t_c) \gamma_{m+2}^n.$$

4.4 Number $\alpha_H^{D^H \cup \{i_b^H\}}$. This case is different to 3). It appears only for all the coalitions equivalent by symmetry to $S = D^H \cup \{i_b^H\}$, namely

$$(a_b - 1) \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b}}^{m} (a_k - 1)$$

times. So, the coefficient of this number is

$$X_b^4 \gamma_{m+1}^n.$$

4.5 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+q+1, ..., m\}$. The coefficient is

$$-X_b^4 \gamma_{m+2}^n.$$

4.6 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+q+1, ..., m\}$. The coefficient of the number is

$$-X_b^4 \frac{a_c - 2}{2} \gamma_{m+2}^n.$$

5) Consider an equation corresponding to $b \in \{p+q+1, ..., m\}$ in (16). Take $D^H \cup \{i\}$ with $i \notin A_b \setminus T_b$. Observe that if there are K symmetric (in $\mathcal{P}$) coalitions S containing j_b to it in the equation, then there are also $(a_b - 2)K$ symmetric $S \setminus \{j_b\}$ options to it in the equation. So, the coefficient of number $\alpha_H^{D^H \cup \{i\}}$ is

$$K \left[\gamma_{m+1}^n - \gamma_{m+2}^n (a_b - 2)\right] \tag{A3}$$

Let $X_b^5 = \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b}}^{m} (a_k - 1)$. We look for the coefficient of the chosen numbers in this equation.

5.1 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{1, ..., p\}$. The quantity K of symmetric coalitions (compatible in $\mathcal{P}$) S containing j_b to $D^H \cup \{i_c^H\}$ is

$$K = \binom{a_c}{2} \prod_{\substack{k=1 \\ k \neq c}}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b}}^{m} (a_k - 1).$$

Since (A3) we get the coefficient of this number in the equation

$$X_b^5 \frac{a_c - 1}{2} \left[\gamma_{m+1}^n - (a_b - 2)\gamma_{m+2}^n\right].$$

5.2 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+1, ..., p+q\}$. We obtain in this case

$$K = \binom{t_c}{2} \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq c}}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b}}^{m} (a_k - 1).$$

Since (A3) we get the coefficient of this number

$$X_b^5 \frac{t_c - 1}{2} \left[\gamma_{m+1}^n - (a_b - 2)\gamma_{m+2}^n\right].$$

5.3 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+1, ..., p+q\}$. We get

$$K = t_c(a_c - t_c) \prod_{k=1}^{p} a_k \prod_{\substack{k=p+1 \\ k \neq c}}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b}}^{m} (a_k - 1),$$

and then, from (A3), the coefficient of the number is

$$X_b^5 (a_c - t_c) \left[\gamma_{m+1}^n - (a_b - 2)\gamma_{m+2}^n\right].$$

5.4 Number $\alpha_H^{D^H \cup \{i_c^H\}}$, $c \in \{p+q+1, ..., m\}$. It holds

$$K = (a_c - 1) \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b,c}}^{m} (a_k - 1).$$

Since (A3) the coefficient of the number is

$$X_b^5 \left[\gamma_{m+1}^n - (a_b - 2)\gamma_{m+2}^n\right].$$

5.5 Number $\alpha_H^{D^H \cup \{j_b^H\}}$ (we cannot apply here (A3)). We get S symmetric to $D^H \cup \{j_b^H\}$ for

$$(a_b - 2) \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b}}^{m} (a_k - 1)$$

times and as $S \setminus \{i_b\}$ in

$$\binom{a_b - 2}{2} \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b}}^{m} (a_k - 1)$$

The coefficient of the number is

$$X_b^5 \frac{a_b - 2}{2} \left[2\gamma_{m+1}^n - (a_b - 3)\gamma_{m+2}^n\right].$$

5.6 Number $\alpha_H^{D^H \cup \{j_c^H\}}$, $c \in \{p+q+1, ..., m\} \setminus \{b\}$. We get

$$K = \binom{a_c - 1}{2} \prod_{k=1}^{p} a_k \prod_{k=p+1}^{p+q} t_k \prod_{\substack{k=p+q+1 \\ k \neq b,c}}^{m} (a_k - 1).$$

Using (A3), the coefficient of the number is

$$X_b^5 \frac{a_c - 2}{2} \left[\gamma_{m+1}^n - (a_b - 2)\gamma_{m+2}^n\right].$$