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Análisis Numérico de esquemas fraccionados en tiempo para Navier-Stokes $3D$ y Ecuaciones Primitivas

Memoria presentada por

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Índice

Table of contents

Introducción

1. Justificación y objetivo de la memoria

El presente trabajo estudia la resolución numérica, mediante diferentes esquemas de tipo fraccionario en tiempo y métodos de elementos finitos en espacio, del problema de Navier-Stokes incompresible (NS) y otros problemas relacionados dentro de la disciplina de la Mecánica de Fluidos, concretamente el problema de Ecuaciones Primitivas (EP).

Uno de los objetivos fundamentales de cualquier disciplina científica es la búsqueda del conocimiento de los fenómenos naturales que nos rodean, entendido como la comprensión y predicción de dichos fenómenos, con la máxima precisión posible. Es necesario, entre otras cosas, un profundo conocimiento de los mismos para hacer un control racional de los valiosos recursos naturales. En particular, el estudio, análisis y comprensión, del comportamiento de líquidos como el agua y de gases como el aire figura como un gran reto para la Ciencia.

Los científicos en el área de la Mecánica de Fluidos han ido construyendo y desarrollando diferentes modelos físicos, con sus formulaciones matemáticas correspondientes, que se adecuan a la realidad en el sentido de que pretenden simular su comportamiento. Realmente, si varios procedimientos físicos distintos pueden ser usados para simular el mismo fenómeno, es debido a que fundamentalmente, están descritos por las mismas ecuaciones. Además, si éstas están sometidas a las mismas ligaduras y condiciones iniciales, devuelven la misma solución independientemente del particular sentido de la variable, ya sea ésta la temperatura, el potencial eléctrico, el campo de velocidades de un fluido, la profundidad del agua, etc... Así, un modelo matemático responde a todos los modelos físicos equivalentes.

Muchos de estos modelos usan como herramienta matemática fundamental las Ecuaciones en Derivadas Parciales (EDP) y fueron inicialmente formulados a finales del siglo XIX y primeras décadas del siglo XX, por científicos como Stokes, Navier, Reynolds, Boussinesq y Rayleigh entre otros.

Evidentemente, para comprobar la fiabilidad de los distintos modelos se hace necesario compararlos con el comportamiento real. Para ello, por un lado se debe construir el modelo situándolo en las variables independientes espacio-tiempo y a continuación, identificar qué propiedades del sistema real son esenciales y cuales de ellas nos dan una descripción satisfactoria para llegar a una cuantificación de la realidad lo más exacta posible. Por otro lado, debemos disponer de un análisis cualitativo y cuantitativo de las soluciones de los problemas, en este caso de las soluciones de las EDP que describen los modelos. Es conocido que, en las situaciones más realistas, no es posible la obtención de expresiones explícitas de dichas soluciones, lo que conduce a la formulación de adecuados métodos numéricos para la obtención de soluciones aproximadas.

Estos métodos numéricos involucran tal cantidad de cálculos efectivos que muchos de ellos fueron inviables en la época en que fueron inicialmente propuestos.

La construcción de las primeras computadoras a mediados del siglo XX y el posterior desarrollo ha hecho que se produzca una verdadera explosión en la utilización y desarrollo de los métodos numéricos en general y, en particular, de los métodos diseñados para la aproximación de las soluciones de EDP. El vertiginoso desarrollo del material informático de cálculo, ha conllevado un desarrollo paralelo de los métodos numéricos.

Centrándonos en los esquemas numéricos de aproximación de EDP de evolución, estos incorporan alguna forma de discretización tanto de la variable temporal como de las variables espaciales. Un apropiado análisis numérico debe garantizar la convergencia teórica de los esquemas respecto al refinamiento de ambas discretizaciones o, debe permitir identificar las condiciones que han de satisfacer dichas discretizaciones para que se verifique la deseada convergencia (condiciones de estabilidad). Además, deben obtenerse estimaciones del error (normalmente respecto de soluciones regulares), que determinen la rapidez de convergencia.

La presente memoria recoge el estudio de diversos métodos numéricos aplicados a algunas de las ecuaciones más relevantes en la Dinámica de Fluidos, concretamente a las ecuaciones de evolución de Navier-Stokes (NS) y a las Ecuaciones Primitivas del Océano (EP).

Las aplicaciones de estas ecuaciones se desarrollan en muchos y tan variados campos como la Ingeniería Industrial, la Ingeniería Marina, la Oceanografía, la Areonaútica, la Metereología o la Astrofísica.

Sin duda, las ecuaciones de NS se han mostrado históricamente como un reto permanente y, en consecuencia, han actuado como motor de numerosas investigaciones, tanto desde el punto de vista teórico, para el desarrollo de numerosos métodos numéricos, como desde el punto de vista de las aplicaciones.

2. Las ecuaciones de Navier-Stokes incompresibles

Las ecuaciones de NS modelan el comportamiento de dos magnitudes físicas, el campo de velocidades $\mathbf{u}$ y la presión p, de un fluido viscoso e incompresible (con coeficiente de viscosidad cinemática constante y positivo) sometido a la acción mecánica de un campo de fuerzas exteriores $\mathbf{f}$, en un intervalo $[0, T]$ y en un dominio acotado Ω que vamos a considerar tridimensional.

Cuando se considera condición de no deslizamiento en la frontera, el problema está descrito por:

$$(NS) \qquad \begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{en } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{en } \Omega\times(0,T), \\ \mathbf{u} &= \mathbf{0} \quad \text{sobre } \partial\Omega\times(0,T), \\ \mathbf{u}_{|t=0} &= \mathbf{u}_0 \quad \text{en } \Omega. \end{cases}$$

Las ecuaciones de este problema son obtenidas (ver [25], [61], [32]) partiendo de las leyes de conservación de la Masa y de la Cantidad de Movimiento y suponiendo que el fluido es homogéneo (la densisad de masa sólo depende de t), incompresible (el volumen de un conjunto de partículas que se desplaza con el fluido permanece constante) y newtoniano viscoso. Para completar matemáticamente dichas ecuaciones se proporcionan condiciones iniciales (en el instante $t = 0$) y condiciones sobre la frontera $\partial\Omega$ del dominio Ω.

El término $(\mathbf{u}\cdot\nabla)\mathbf{u}$ recibe el nombre de término de transporte o de convección, mientras que el término $-\nu\,\Delta\mathbf{u}$ es el término de difusión o de viscosidad, siendo $\nu > 0$ la viscosidad cinemática del fluido. El término difusivo posee propiedades regularizantes para las soluciones, sin embargo, el término convectivo (no lineal) puede dar lugar a fuertes inestabilidades que contrarresten dichos efectos regularizadores.

Las principales dificultades desde el punto de vista teórico y numérico en la resolución del sistema (NS) son la condición de incompresibilidad $\nabla\cdot\mathbf{u} = 0$ y la presencia del término no lineal $(\mathbf{u}\cdot\nabla)\mathbf{u}$.

Desde el punto de vista teórico, los estudios de existencia, unicidad y regularidad de (NS) han servido para desarrollar nuevas técnicas matemáticas que hoy resultan cruciales para la teoría moderna de las ecuaciones en derivadas parciales ([55], [66], [77], [35], [60], [3], [86]). En la actualidad, como es conocido, permanece abierto el problema de la unicidad de solución débil de (NS) en dominios tridimensionales, debido a la falta de regularidad de dicha solución débil. Sólo algunos resultados parciales han sido obtenidos (ver [38], [64], [37]). Con respecto a soluciones fuertes (o clásicas), la existencia y unicidad ha sido probada sobre intervalos de tiempo locales dependiendo de los datos, en este sentido, desde el trabajo de Leray [62], se conoce que para datos suficientemente regulares existe una única solución regular local en tiempo. Desafortunadamente, la unicidad no se ha probado para la soluciones débiles cuya existencia se conoce de forma global en tiempo. Hay por tanto un salto entre el conjunto de soluciones cuya existencia se conoce y el conjunto más reducido de soluciones cuya unicidad está probada. De hecho, se ha demostrado que cualquier solución débil de (NS) pertenece a $L^{8/3}(0,T;\mathbf{L}^4(\Omega))$ y cualquier solución fuerte de (NS) pertenece a $L^8(0,T;\mathbf{L}^4(\Omega))$. La distancia entre estas clases de funciones es lo que parece que nos separa del resultado de existencia y unicidad global.

Por tanto, a nivel continuo, hay dos problemas fundamentales que permanecen abiertos: la unicidad de solución débil y la existencia global de solución fuerte. La importancia de estas cuestiones ha hecho que se hayan incluido dentro de “los siete problemas del milenio”, propuesto por The Clay Mathematics Institute [26] con un premio de 1,000,000 dólares.

Con respecto a los métodos numéricos para aproximar (NS), nos centramos en aquellos que en primer lugar realizan una discretización de la variable temporal y posteriormente realizan la discretización de las variables espaciales. Para la discretización del intervalo temporal $[0, T]$, podemos tomar un paso $k = T/M$ y una partición uniforme $\{t_m = m\,k\}$ con $m = 0, ..., M$. Los métodos construidos pretenden obtener aproximaciones $\mathbf{u}^m$ (y eventualmente p^m) de las soluciones exactas $\mathbf{u}$ (y p) en los instantes de tiempo t_m.

Los métodos clásicos secuenciales para la aproximación temporal son los métodos de un paso como los de Euler o Crank-Nicholson (que son respectivamente de primer y segundo orden respecto de k) o bien otros métodos más sofisticados como los de Gear o Runge-Kutta (de segundo y cuarto orden respecto de k). Estos métodos calculan cada aproximación en un instante de tiempo a partir de las aproximaciones en uno o varios instantes anteriores.

Otros métodos de aproximación temporal son los denominados métodos de paso fraccionado (fractional-step schemes), que consisten en descomponer cada paso de tiempo en varios pasos intermedios, resolviendo en cada uno de ellos diferentes problemas. Otros descomponen el operador diferencial de la ecuación diferencial en varios suboperadores, resolviendo varios problemas intermedios con cada suboperador, estos métodos son llamados de partición del operador (operator-splitting). Ejemplos clásicos de estos últimos son los métodos de proyección, que permiten desacoplar el cálculo del campo de velocidades y la presión en cada paso de tiempo, de esta forma el costo computacional es más bajo que el del problema original acoplado. Debido a esta eficacia computacional, estos métodos son actualmente muy populares.

Con cualquiera de estos dos métodos, paso fraccionado o partición del operador, se pretende pasar de un instante de tiempo t_m al siguiente t_{m+1} a través de la resolución consecutiva de varios subproblemas intermedios. De hecho, en la literatura muchas veces se identifican ambos métodos, porque habitualmente se combinan entre ellos (aunque en los métodos de paso fraccionado las aproximaciones intermedias están asociadas a pasos intermedios del intervalo $[t_m, t_{m+1}]$, mientras que en los otros las aproximaciones intermedias corresponden a aproximaciones auxiliares en $t = t_{m+1}$).

En esta memoria estudiaremos dos métodos de tipo splitting, conocidos como método de descomposición de la viscosidad y método de proyección con corrección de presión (o incremental en la presión), primero para NS y posteriormente para EP.

Respecto al problema de NS, en ambos métodos se separan las principales dificultades del problema: la condición de incompresibilidad y la no linealidad.

En la literatura, hay diversas formas de “separar” estas dos dificultades. Entre los esquemas más significativos están:

2.1. Los métodos de proyección

Las ideas originales de los métodos de proyección vienen del trabajo de Yanenko [94], donde se introduce el concepto de una partición del operador en las ecuaciones en pasos sucesivos.

Pero el origen de los métodos de proyección, es generalmente acreditado a los trabajos de Chorin ([23], [24], [21], [22]) y Temam ([87], [91], [88], [90]). En la actualidad, se engloba a estos métodos y sus variantes en los llamados métodos de segregación de la presión ([29] y [30]).

El método clásico de Temam es un esquema de dos pasos donde el segundo paso es de proyección en un espacio con divergencia nula. El esquema semidiscreto en tiempo consiste en:

Dada $(\mathbf{f}^m)_{m=1}^M$ una aproximación de $\mathbf{f}(t_m)$, definiremos $(\mathbf{u}^m, p^m)_{m=1}^M$ una aproximación de la solución $\{\mathbf{u}, p\}$ de (NS) en $t = t_m$, a través de un esquema a dos pasos como sigue:

Inicialización: $\mathbf{u}^0 = \widetilde{\mathbf{u}}^0 = \mathbf{u}(0)$

Paso de tiempo $m+1$:

Subetapa 1 : Dadas $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$, encontrar $\widetilde{\mathbf{u}}^{m+1} : \Omega \to I\!R^3$ solución de

$$(S_1)^{m+1} \qquad \left\{ \begin{array}{rcll} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + (\widetilde{\mathbf{u}}^m \cdot \nabla)\widetilde{\mathbf{u}}^{m+1} - \nu\Delta\widetilde{\mathbf{u}}^{m+1} & = & \mathbf{f}^{m+1} & \text{en } \Omega, \\ \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} & = & 0, & \end{array} \right.$$

(en el problema anterior hemos escrito la convección linealizada o semi-implícita).

Subetapa 2 : Dada $\widetilde{\mathbf{u}}^{m+1}$, encontrar $\mathbf{u}^{m+1}$ y p^{m+1} solución de

$$(S_2)^{m+1} \qquad \left\{ \begin{array}{rcll} \dfrac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}) + \nabla p^{m+1} & = & 0 & \text{en } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} & = & 0 & \text{en } \Omega, \\ \mathbf{u}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} & = & 0. & \end{array} \right.$$

La subetapa 2 es equivalente a decir que $\mathbf{u}^{m+1}$ es la proyección de $\widetilde{\mathbf{u}}^{m+1}$ en un espacio de tipo L^2 con divergencia nula. Esta es la razón por la que este método es usualmente conocido como método de proyección. La partición del operador en este caso consiste en separar los efectos de la incompresibilidad de los de la convección y difusión.

El principal interés del método de proyección es la posibilidad de desacoplar el cálculo de la presión y la velocidad. Así, tomando divergencia en $(S_2)^{m+1}$, obtenemos que la presión verifica el siguiente problema de Poisson

$$(\widetilde{S}_2)^{m+1} \qquad \left\{ \begin{array}{rcll} \Delta p^{m+1} & = & \dfrac{1}{k}\nabla \cdot \widetilde{\mathbf{u}}^{m+1} & \text{en } \Omega, \\ \nabla p^{m+1} \cdot \mathbf{n}|_{\partial\Omega} & = & 0. & \end{array} \right.$$

Hay que notar que la velocidad final $\mathbf{u}^{m+1}$ no satisface las condiciones de contorno completas del problema y la presión numérica satisface la condición de Neumann "artificial" $\nabla p^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = 0$, lo que introduce cierta inconsistencia en la aproximación frontera del esquema, que produce el fenómeno conocido como capa límite. Éste es uno de los puntos más conflictivos de los métodos clásicos de proyección y ha dado lugar a numerosas especulaciones sobre si la aproximación de la presión p^{m+1} es una buena aproximación a la solución exacta $p(t_{m+1})$ ([27], [76], [43]).

Otro aspecto importante del método de proyección es su inherente estabilidad. En el caso de un esquema completamente discreto, la versión discretizada por elementos finitos pasa por imponer la condición inf-sup discreta. Una alternativa al uso de elementos finitos estables satisfaciendo la condición inf-sup es utilizar las llamadas técnicas de estabilización (ver [27]). La idea esencial es modificar la forma variacional discreta para que la estabilidad de la presión pueda ser obtenida sin imponer la condición inf-sup. Este tipo de ideas puede también ser usado para evitar inestabilidades numéricas en flujos de convección dominante. En particular, se consiguen esquemas estabilizados con espacios discretos de Elementos Finitos de igual orden para la velocidad y presión (que no verifican la condición inf-sup). Sin embargo, la estabilidad va decayendo con el paso de tiempo. Este efecto está escondido en el análisis de la convergencia al imponer un paso de tiempo suficientemente pequeño. Un reciente estudio completo de la técnicas de estabilización que conducen a métodos estables totalmente discretizados, fue desarrollado por R. Codina en [28] y extendido por el mismo autor junto con S. Badia en [29].

Debido a que el gradiente de presión no aparece en el primer paso del método de proyección clásico, fue observado (probablemente en primer lugar) por Goda ([42]) que el añadir un término de gradiente de presión de la etapa anterior en el primer paso y corregir este término en el segundo paso, aumentaba la convergencia. Esta idea la hizo muy popular Van Kan, quien propuso un esquema de segundo orden con corrección de presión en [56]. En 1992, J. Shen [78] propuso un esquema de primer orden con corrección de presión (también llamado de presión incremental o esquema de Van Kan por lo referido antes), donde el término de presión aparece en los dos pasos. En este método, al igual que en el no incremental (o sin corrección de presión), aparecen condiciones de contorno artificiales para la presión.

El método con correción de la presión clásico propuesto por Shen en [78] puede escribirse como:

Inicialización: $\mathbf{u}^0 = \mathbf{u}(0)$

Paso de tiempo $m+1$:

Subetapa 1 : Dadas $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$ y p^m, encontrar $\widetilde{\mathbf{u}}^{m+1} : \Omega \to I\!R^3$ solución de

$$(S_1)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + (\widetilde{\mathbf{u}}^m \cdot \nabla)\widetilde{\mathbf{u}}^{m+1} - \nu\Delta\widetilde{\mathbf{u}}^{m+1} + \nabla p^m & = & \mathbf{f}^{m+1}, \\ \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} & = & 0, \end{array} \right.$$

Subetapa 2 : Dadas p^m y $\widetilde{\mathbf{u}}^{m+1}$, encontrar $\mathbf{u}^{m+1}$ y p^{m+1} solución de

$$(S_2)^{m+1} \qquad \left\{ \begin{array}{rcl} \frac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}) + \nabla(p^{m+1} - p^m) & = & 0, \\ \nabla \cdot \mathbf{u}^{m+1} & = & 0, \\ \mathbf{u}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} & = & 0. \end{array} \right.$$

Al igual que antes, la presión puede obtenerse a partir de un problema de Poisson

$$(S_2)_a^{m+1} \qquad \left\{ \begin{array}{l} \Delta(p^{m+1} - p^m) = \frac{1}{k}\nabla \cdot \widetilde{\mathbf{u}}^{m+1}, \\ \nabla(p^{m+1} - p^m) \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{array} \right.$$

Es importante observar que la introducción de la velocidad final $\mathbf{u}^m$ no es necesaria, puesto que la implementación puede ser realizada de la siguiente manera:

Dada $(p^{m-1}, \widetilde{\mathbf{u}}^m)$,

(a) Encontrar p^m como solución de $(S_2)_a^m$.

(b) Encontrar $\widetilde{\mathbf{u}}^{m+1}$ tal que

$$(S_2)_b^{m+1} \quad \frac{\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}^m}{k} + (\widetilde{\mathbf{u}}^m \cdot \nabla)\widetilde{\mathbf{u}}^{m+1} - \nu\Delta\,\widetilde{\mathbf{u}}^{m+1} + \nabla(2p^m - p^{m-1}) = \mathbf{f}^{m+1}, \quad \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0.$$

Entonces, el cálculo para la presión y la velocidad está desacoplado. Éste es el motivo por el que se conoce también al método de proyección como método de segregación de la presión. De hecho, $(S_2)_a^m$ es un problema de Poisson para la presión y $(S_2)_b^{m+1}$ es un problema de convección-difusión lineal para la velocidad.

Dicho método será objeto de estudio en la presente memoria, Capítulos 3 y 5.

Notemos que el método de proyección clásico puede ser también considerado como un método de segregación de la presión. La única diferencia es que en $(S_2)_b^{m+1}$ se cambia el término $\nabla(2p^m - p^{m-1})$ por ∇p^m.

Respecto al orden en tiempo, los métodos de corrección de presión que hemos presentado son esquemas de primer orden. Algunas alternativas han sido sugeridas para lograr métodos de segundo orden. La primera fue propuesta por Kim y Moin en [58], basado en un esquema donde el término convectivo es evaluado por un método de segundo orden de Adams-Bashforth y el término viscoso por un método de Crank-Nicolson.

Como comentamos antes, Van Kan en [56], propuso un método de segundo orden con corrección de presión. Este método es equivalente al método corregido por Shen, pero reemplazando Euler regresivo en la integración en tiempo por un esquema de segundo orden de Crank-Nicolson.

En 1989, Bell, Colella y Glaz introdujeron en [8] el primer método predictor-corrector. Ellos propusieron un esquema iterativo de segundo orden que converge al esquema "monolítico" de Crank-Nicolson. En cada iteración, se obtiene primero la velocidad intermedia, tratando el término convectivo de forma explícita y el término difusivo de manera implícita. Después, la velocidad final y la presión son calculadas de forma acoplada.

Existen además diversos trabajos en los que los esquemas de segundo orden con corrección de presión han sido estudiados analíticamente. E. y Liu estudian en [33] el esquema propuesto por Kim y Moin. Shen analizó el esquema de Van Kan en [83] y [84], obteniendo estimaciones de error óptimas hacia soluciones suficientemente regulares, suponiendo datos iniciales suficientemente convergentes para la velocidad y presión. En [74], estas estimaciones de error se consiguieron con hipótesis de regularidad más débiles sobre la solución exacta de (NS).

En cuanto a los métodos de tercer orden, estos han sido mucho menos estudiados. El primer autor que propuso un método de este tipo fue Gresho en [43]. Sin embargo, experimentos numéricos mostraron que este método era inestable en tiempo. En [85], Shen intentó explicar porqué los métodos con corrección de la presión de tercer orden no pueden estar uniformemente acotados en tiempo. De hecho, el autor concluyó que "todos los métodos con corrección de presión de orden más alto que dos son inestables". Sin embargo, de acuerdo al análisis de Badia y Codina en [7], la explicación de Shen en [85] no es apropiada porque se habían ignorado algunos términos. De hecho, Shen puntuó que el sistema continuo equivalente que él analizó lo obtuvo por intuición. En definitiva, la razón por la que los métodos de corrección de presión de orden más alto son inestables permanece como un problema abierto que todavía no ha sido totalmente explicado.

Existe otra clase de métodos de proyección, denominados esquemas con corrección de la velocidad, estudiados por Guermond y Shen en [46] y [48]. Estos fueron introducidos de diferentes formas (aunque equivalentes) en [70] y [57]. La principal idea es intercambiar el papel de la difusión y el gradiente de presión en los métodos de corrección de presión, esto es, el término viscoso es tratado de forma explícita (o ignorado) en el primer paso (que es el paso de proyección) y la velocidad es corregida en el segundo paso (de convección-difusión).

En 1996, Timmermans, Minev y Van de Vosse propusieron en [93] una versión modificada del método que conduce a mejores aproximaciones para la presión. Esta modificación consiste en corregir la presión con $\overline{p}^{m+1} = p^{m+1} - \nu \nabla \cdot \widetilde{\mathbf{u}}^{m+1}$. Esta corrección no causa esfuerzos adicionales numéricos y además, como fue observado por Guermond y Shen en [49], verifican una condición de contorno para la presión que es consistente para el problema de Stokes. Debido a que el operador rotacional juega un papel importante en esta mejora, ya que la idea del método viene de escribir $\Delta \mathbf{u} = \nabla \nabla \cdot \mathbf{u} - \nabla \times \nabla \times \mathbf{u}$, estos métodos son llamados en [49] métodos rotacionales con corrección de presión.

La mejora de las estimaciones de error para la presión tomando la forma rotacional ha sido estudiada en [47] y [74]. También en [74], Prohl realiza un estudio de algunas variantes de los métodos de proyección junto con su análisis numérico. Entre estas variantes están el método de Chorin-Uzawa y el método de Chorin-Penalty.

El método de Gauge, que fue introducido por E y Liu [33], y una variante de este esquema, el método de Gauge-Uzawa, que fue introducido y analizado por J.H. Pyo en [75], son otros esquemas considerados cuya intención es no incorporar inconsistencias o incompatibilidades entre las discretizaciones espacial y temporal.

Más recientemente, en [7], Badia y Codina han obtenido resultados de convergencia para el método de proyección original (sin corrección de presión) totalmente discreto en dos situaciones diferentes. El primer análisis se aplica a pares de elementos finitos que satisfacen una condición inf-sup discreta o de Ladyzhenskaya-Babuska-Brezzi ([60], [6], [16]). La segunda parte, analiza el método totalmente discreto de proyección usando el problema de Poisson para la presión, con igual interpolación para los espacios discretos de velocidad y presión (que no verifica la condición inf-sup). En ambos casos, estimaciones de error, sub-optimales para la presión, son obtenidas.

2.2. Los métodos de descomposición de la viscosidad

En los métodos de descomposición de la viscosidad no está totalmente desacoplada la incompresibilidad de la difusión. Claros ejemplos son los presentados por R. Naratajan [69] y los θ-esquemas de R. Glowinski [41].

Resultados de convergencia, estabilidad y estimaciones de error para estos últimos fueron obtenidos por E. Fernández-Cara y M. Marín en [34]. También, algunas versiones paralelizadas de estos esquemas fueron presentadas en [31], [2] y en la tesis de I. Albarreal [1].

Una variante de estos métodos, es el esquema presentado por J. Blasco y R. Codina ([12], [13]) y que fue originalmente motivado por un algoritmo predictor-multicorrector. Este esquema fue introducido para reforzar las condiciones de contorno para la velocidad de los métodos de proyección. En él se separan la no linealidad $(\mathbf{u}\cdot\nabla)\mathbf{u}$ y la condición de incompresibilidad $\nabla\cdot\mathbf{u} = 0$ en dos pasos diferentes, pero conservando términos de viscosidad y las condiciones de contorno en ambos pasos. El esquema semidiscreto en tiempo es el siguiente:

Inicialización: $\mathbf{u}^0 = \mathbf{u}(0)$

Paso de tiempo $m+1$:

Subetapa 1: Dada $\mathbf{u}^m$, encontrar $\mathbf{u}^{m+1/2}$ solución de

$$(S_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1/2} - \mathbf{u}^m) + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1/2} - \nu\Delta\mathbf{u}^{m+1/2} = \mathbf{f}^{m+1} \quad \text{en } \Omega, \\ \mathbf{u}^{m+1/2}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Subetapa 2: Dada $\mathbf{u}^{m+1/2}$, encontrar $\mathbf{u}^{m+1}$ y p^{m+1} solución de

$$(S_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) - \nu\Delta(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) + \nabla p^{m+1} = \mathbf{0} \quad \text{en } \Omega, \\ \nabla\cdot\mathbf{u}^{m+1} = 0 \quad \text{en } \Omega, \qquad \mathbf{u}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Este método también será objeto de estudio en la presente memoria.

2.3. Los métodos de pseudo-compresibilidad

La condición de incompresibilidad (que consiste básicamente en asumir que el volumen ocupado por un conjunto de partículas del fluido se conserva), es una de las principales dificultades de las ecuaciones de Navier-Stokes. En general, los métodos numéricos para flujos incompresibles pueden ser clasificados en tres categorías en función de cómo sea tratada la condición de incompresibilidad $\nabla \cdot \mathbf{u} = 0$: usando un subespacio de divergencia nula para la aproximación de la velocidad, usando un par de espacios discretos compatibles para la presión y velocidad y, por último, relajando la restricción de incompresibilidad de una forma adecuada. Esto último puede hacerse de varias maneras:

- Penalización ([82],[87]): $\nabla \cdot \mathbf{u}^\varepsilon + \varepsilon\, p^\varepsilon = 0$ en $\Omega \times (0,T)$.
- Compresibilidad artificial ([23], [89]): $\nabla \cdot \mathbf{u}^\varepsilon + \varepsilon\, p_t^\varepsilon = 0$ en $\Omega \times (0,T)$.
- Estabilización de la presión ([24], [90]): $\nabla \cdot \mathbf{u}^\varepsilon - \varepsilon\, \Delta p^\varepsilon = 0$ en $\Omega \times (0,T)$, $\partial_n p^\varepsilon|_{\partial\Omega} = 0$.
- Pseudo-compresibilidad ([81]): $\nabla \cdot \mathbf{u}^\varepsilon - \varepsilon\, \Delta p_t^\varepsilon = 0$ en $\Omega \times (0,T)$, $\partial_n p_t^\varepsilon|_{\partial\Omega} = 0$.

Las tres primeras versiones son bien conocidas y la última fue introducida más recientemente por Shen en [81]. El principal interés de estos métodos es el comportamiento cuando el parámetro $\varepsilon \to 0$. Un importante aspecto es que, en principio, todas ellas pueden emplearse con cualquier discretización espacial consistente, es decir, no se requiere la condición inf-sup, si bien el problema es que cuando $\varepsilon \to 0$ las constantes en las estimaciones pueden explotar.

En [82], J. Shen hace un revisión de todos estos aspectos. También en trabajos más recientes de R. Codina ([30]), se estudian métodos de estabilización de la presión, con la idea esencial de modificar la formulación variacional discreta para que la estabilidad de la presión se consiga con espacios de igual interpolación (que no verifican la condición inf-sup). Este tipo de métodos también puede ser usado para evitar inestabilidades numéricas provocadas por la convección.

Por otra parte, algunos de los métodos de proyección conocidos pueden ser reformulados como estos métodos eligiendo ε en función de k. Así, el método clásico de proyección no incremental (sin corrección de presión) puede ser observado como un método de estabilización de la presión (ver [76] y [79]) tomando $\varepsilon = k$ y el método de proyección incremental (con corrección de presión) como un método de pseudo-compresibilidad tomando $\varepsilon = k^2$. También, el método de Chorin-Uzawa está relacionado con un método de compresibilidad artificial y el método de Chorin-Penalty con un método de penalización (ver Prohl [74]).

3. Las Ecuaciones Primitivas del Océano

Al menos dos tercios de nuestro planeta están cubiertos por el agua de los océanos. Es por tanto el estudio de nuestros océanos, junto con el de la atmósfera que rodea la tierra, fundamental para predecir el tiempo y el clima, uno de los retos importantes que actualmente tiene el hombre.

El océano es considerado como un fluido ligeramente compresible, con fuerzas de Coriolis y centrípetas. El conjunto de ecuaciones que rigen lo que se conoce como "large scale ocean model" son: la ecuación de momentos, la ecuación de continuidad, la ecuación termodinámica o de la temperatura, la ecuación para la salinidad y la ecuación de estado (que relaciona la densidad con la temperatura y la salinidad).

Bajo ciertas simplificaciones, básicamente presión hidrostática (que relaciona la presión y la densidad del agua con la gravedad) y la hipótesis de "techo rígido", las ecuaciones de Navier-Stokes $3D$ derivan en las llamadas Ecuaciones Primitivas. Estas ecuaciones constituyen un modelo matemático general en el campo de los fluidos geofísicos [67], [73]. En particular, describen la circulación general del agua en lagos y océanos [68]. Por simplicidad, suponemos densidad constante, coordenadas cartesianas y que los efectos debidos a la temperatura y salinidad se pueden desacoplar del flujo dinámico.

El modelo diferencial regido por las ecuaciones primitivas se puede escribir como (ver [63, 67, 68]):

$$
(EP) \qquad \left\{
\begin{array}{rcll}
\partial_t \mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_H p & = & \mathbf{f} & \text{en } \Omega\times(0,T), \\
\partial_z p = -\rho\, g, \qquad \nabla\cdot\mathbf{U} & = & 0 & \text{en } \Omega\times(0,T), \\
\mathbf{u} = 0 \ \text{ sobre } \Gamma_l\times(0,T), \quad \mathbf{u} = u_3 n_3 & = & 0 & \text{sobre } \Gamma_b\times(0,T), \\
\nu\,\partial_z\mathbf{u} = \mathbf{g}_s, \quad u_3 & = & 0 & \text{sobre } \Gamma_s\times(0,T), \\
\mathbf{u}_{|t=0} & = & \mathbf{u}_0 & \text{en } \Omega,
\end{array}
\right.
$$

donde $\Omega = \{(\mathbf{x},z)\in I\!R^3 \ / \ \mathbf{x}=(x,y)\in S,\ -D(\mathbf{x})<z<0\}$ es el dominio $3D$ ocupado por el agua, con $S\subset I\!R^2$ el dominio superficial (un dominio acotado $2D$) y $D:\overline{S}\to I\!R_+$ (con $D>0$ en S) la función descrita por el fondo. Entonces, $\Gamma_s=\overline{S}\times\{0\}$ es la parte del contorno de Ω correspondiente a la superficie, $\Gamma_b=\{(\mathbf{x},-D(\mathbf{x})):\ \mathbf{x}\in S\}$ la parte correspondiente al fondo (con vector normal exterior $(\mathbf{n}_\mathbf{x},n_3)$) y $\Gamma_l=\{(\mathbf{x},z):\ \mathbf{x}\in\partial S, -D(\mathbf{x})<z<0\}$ la parte correspondiente a las paredes laterales.

Las incógnitas del problema son $\mathbf{U}=(\mathbf{u},u_3):\Omega\times(0,T)\to I\!R^3$ el campo de velocidades $3D$ (con $\mathbf{u}=(u_1,u_2)$ la correspondiente velocidad horizontal y u_3 la velocidad vertical) y $p:\Omega\times(0,T)\to I\!R$ la presión.

Además, $\mathbf{b}(\mathbf{u})=f\mathbf{u}^\perp$ representa los efectos de las Fuerzas de Coriolis, con $\mathbf{u}^\perp=(-u_2,u_1)^t$ y $f=2|w|\sin\theta$, donde w es la velocidad angular de la Tierra y $\theta=\theta(y)$ es la latitud, $\rho\in I\!R_+$ es la densidad del agua (que suponemos una constante positiva), $g\in I\!R_+$ es la aceleración de la gravedad (otra constante positiva), $\mathbf{f}:\Omega\times(0,T)\to I\!R^2$ es el campo de fuerzas horizontales externas (dependiendo por ejemplo de la salinidad y la temperatura) y $\mathbf{g}_s:\Gamma_s\times(0,T)\to I\!R^2$ representa los efectos de la fricción del viento sobre la superficie.

Finalmente, $\nabla = (\nabla_H, \partial_z)$ denota al operador gradiente tridimensional (con $\nabla_H = (\partial_x, \partial_y)$ su componente horizontal) y Δ el operador laplaciano tridimensional.

Por simplicidad, hemos considerado en (EP) difusión isótropa, la cuál se escribe como $-\nu\Delta\mathbf{u}$, donde $\nu > 0$ es un coeficiente de viscosidad. En general, debido a la diferencia entre las dimensiones horizontal y vertical del dominio, es frecuente considerar una difusión anisótropa, por ejemplo

$$-\nabla_H \cdot (\nu_h \nabla_H \mathbf{u}) - \nu_v\, \partial_z^2 \mathbf{u}$$

donde $\nu_h,\ \nu_v > 0$ son los coeficientes de viscosidad horizontal y vertical respectivamente, siendo $\nu_v \ll \nu_h$ ([73]). Los resultados obtenidos en la presente memoria pueden ser extendidos a este caso.

Si definimos $p_s(t;\mathbf{x}) = p(t;\mathbf{x},z) - \rho g z$, entonces $p_s : S \times (0,T) \to I\!R$ es una nueva variable (definida sólo sobre la superficie S), la cuál llamaremos presión superficial. El problema (EP) puede ser reformulado como el siguiente problema íntegro-diferencial ([63, 67, 68]):

$$(Q) \qquad \left\{ \begin{array}{rcll} \partial_t \mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_H p_s &=& \mathbf{f} & \text{en } \Omega \times (0,T), \\ \nabla_H \cdot \langle \mathbf{u} \rangle &=& 0 & \text{en } S \times (0,T), \\ \nu\partial_z \mathbf{u} = \mathbf{g}_s \quad \text{sobre } \Gamma_s \times (0,T), \qquad \mathbf{u} &=& 0 & \text{sobre } (\Gamma_b \cup \Gamma_l) \times (0,T), \\ \mathbf{u}_{|t=0} &=& \mathbf{u}_0 & \text{en } \Omega, \end{array} \right.$$

donde la velocidad vertical se obtiene de forma explícita como

$$u_3(t;\mathbf{x},z) = \int_z^0 \nabla_H \cdot \mathbf{u}(t;\mathbf{x},s)\, ds$$

y siendo el flujo total vertical de la velocidad horizontal denotado como

$$\langle \mathbf{u} \rangle (t;\mathbf{x}) = \int_{-D(\mathbf{x})}^0 \mathbf{u}(t;\mathbf{x},z)\, dz.$$

La existencia de una solucion débil $(\mathbf{u}, p_s)$ del problema (Q) es bien conocida, ver Lions-Temam-Wang [68] y Lewandowski [63], siempre en dominios con talud (i.e. $D \geq D_{min} > 0$ en $\overline{S}$). En estos trabajos, un resultado de compacidad se usa para obtener la velocidad $\mathbf{u}$ en un espacio con la restricción $\nabla_H \cdot \langle \mathbf{u} \rangle = 0$ y seguidamente, la presión superficial p_s es obtenida gracias a un específico lema de De Rham sobre la superficie S. En dominios sin talud, (i.e. cuando la función profundidad D puede degenerar a cero), la existencia de una solución débil $(\mathbf{u}, u_3, p_s)$ de (EP) se obtiene a través de un límite asintótico aplicado a las ecuaciones de Navier-Stokes con viscosidad anisótropa cuando el radio de profundidad sobre el diámetro horizontal del dominio tiende a cero; ver Besson-Laydi [11] para el caso estacionario y Azerad-Guillén [4, 5] para el caso evolutivo. La existencia de una solución débil del problema estacionario asociado a (Q) en dominios sin talud es demostrada por T. Chacón y F. Guillén [18] usando argumentos de aproximación interna: una formulación variacional mixta (velocidad-presión) se aproxima por un método de Elementos

Finitos conformes, que verifican la llamada "condición inf-sup hidrostática" (ver [18]). Por otro lado, F. Ortegón en [71], proporciona una generalización del lema de De Rham usado en [68] a dominios sin talud y en [72], estudia una versión regularizada del problema (Q) mediante perturbaciones monótonas, y obtiene otra demostración de la existencia de una solución de (Q).

En cuanto a los resultados de regularidad obtenidos, la existencia de solución fuerte para el problema lineal estacionario asociado a (Q) es tratado por M. Ziane en [95]. Este resultado es extendido en [52] al caso lineal evolutivo. Con respecto al problema no lineal, en [52], F. Guillén-González y M.A. Rodríguez-Bellido demuestran la existencia y unicidad de solución fuerte para dominios $2D$, global en tiempo para datos suficientemente pequeños o local en tiempo para profundidad suficientemente pequeña. La extensión y mejora de esta clase de resultados al caso de dominios $3D$ fue realizada por F. Guillén-González, N. Masmoudi y M.A. Rodríguez-Bellido en [50]. Finalmente, C. Cao y E.S. Titi en [17], suponiendo el fondo plano y con condición de contorno Neumann en el fondo, demostraron regularidad global en tiempo sin restricciones. Por otra parte, I. Kukavica y M. Ziane en [59], demuestran regularidad global sin restricciones con fondo Dirichlet.

Por otro lado, esquemas numéricos para el modelo de Ecuaciones Primitivas han sido presentados por T. Chacón y D. Rodríguez en [19, 20] usando un esquema estabilizado con elementos finitos, y por R. Bermejo en [9] y R. Bermejo y P. Galán en [10], usando un esquema semi-lagrangiano de proyección en tiempo junto con elementos finitos en espacio.

Con objeto de describir un esquema totalmente discreto, consideramos la siguiente reformulación del problema (EP):

$$(R)\qquad \left\{ \begin{array}{rcll} \partial_t \mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_H\, p_s & = & \mathbf{f} & \text{en } \Omega\times(0,T), \\ \partial_z^2 u_3 + \partial_z\,\nabla_H\cdot\mathbf{u} & = & 0 & \text{en } \Omega\times(0,T), \\ \nabla_H\cdot\langle\mathbf{u}\rangle & = & 0 & \text{en } S\times(0,T), \\ \nu\partial_z\mathbf{u} = \mathbf{g}_s \quad \text{sobre } \Gamma_s\times(0,T), \qquad \mathbf{u} & = & 0 & \text{sobre } (\Gamma_b\cup\Gamma_l)\times(0,T), \\ u_3 & = & 0 & \text{sobre } (\Gamma_s\cup\Gamma_b)\times(0,T). \\ \mathbf{u}_{|t=0} & = & \mathbf{u}_0 & \text{en } \Omega, \end{array} \right.$$

En los Capítulos 4 y 5 de esta memoria, pretendemos hacer un estudio numérico con el método de descomposición de la viscosidad y el método de proyección con corrección de la presión, que ya habremos usado para el caso de NS, ahora para las EP.

El principal inconveniente que nos encontraremos será la pérdida de regularidad de la velocidad vertical, lo que hace que debamos emplear nuevas técnicas para poder abordar con éxito la obtención de la convergencia, estabilidad y principalmente, las estimaciones de error.

4. Resumen de la memoria

En esta memoria nos centramos en el análisis numérico del método de descomposición de la viscosidad y del método de proyección incremental, aplicados a NS y EP posteriormente.

Por otro lado, en cuanto al método de proyección no incremental (sin corrección de presión), ya ha sido profundamente estudiado para el caso de Navier-Stokes por otros autores, en [7], [29] y [30], de dos maneras distintas: la primera usando una formulación mixta velocidad-presión y haciendo uso de la condición inf-sup para los espacios discretos aproximantes y la segunda, utilizando la formulación de la presión segregada sin imponer la condición inf-sup, si bien, en ambos casos no se consiguen estimaciones óptimas para el error de la presión, debido a que aunque el método no necesite una presión inicial para conseguir estabilidad y orden óptimo en velocidad, las condiciones de contorno artificiales para la presión generan una capa límite, que justifica la pérdida de aproximación para la presión del método.

La extensión del estudio realizado en la memoria para NS al caso de EP no es en absoluto directa, debido a las dificultades añadidas que nos encontramos en las EP, principalmente la pérdida de regularidad de la velocidad vertical y la integración global en vertical. De hecho, los resultados obtenidos para NS con el método de descomposición de la viscosidad no se pueden obtener para EP.

4.1. El problema de Navier-Stokes

Pretendemos obtener nuevas estimaciones de error para un esquema de paso fraccionado con descomposición de la viscosidad aplicado a NS, el cuál fue introducido y estudiado por J. Blasco y R. Codina en [12], [13], [14] y [15]. Este estudio corresponde a los Capítulos 1 y 2 de la presente memoria. Posteriormente en el Capítulo 3, estudiamos un método de segregación de la presión basado en un esquema de proyección incremental.

Fijada una partición regular de $[0,T]$ de diámetro $k = T/M$: $(t_m = m\,k)_{m=0}^{M}$, para un vector dado $u = (u^m)_{m=0}^{M}$ con $u^m \in X$ (un espacio de Banach), usaremos la siguiente notación para las normas discretas en tiempo:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^{M} \|u^m\|_X^2 \right)^{1/2} \quad \text{y} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\dots,M} \|u^m\|_X$$

Por simplicidad, denotaremos $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., y $\mathbf{H^1} = H^1(\Omega)^3$ etc. El producto escalar en $L^2(\Omega)$ será denotado por $(\cdot,\cdot)$. Una norma en un espacio X será denotada por $\|\cdot\|_X$.

Como ya hemos comentado, el método de descomposición de la viscosidad es un esquema a dos pasos que separa la no linealidad y la incompresibilidad del problema en pasos diferentes (pero conservando el término de viscosidad y las condiciones de contorno para la velocidad en ambos pasos). Básicamente, el esquema está descrito como sigue. Dada $\mathbf{u}^m$ una aproximación de $\mathbf{u}(t_m)$, primero se calcula una velocidad intermedia $\mathbf{u}^{m+1/2}$ (como una primera aproximación de $\mathbf{u}(t_{m+1})$) resolviendo un problema de convección-difusión, y después se calcula $(\mathbf{u}^{m+1}, p^{m+1})$ (como aproximación de $(\mathbf{u}(t_{m+1}), p(t_{m+1}))$) resolviendo un problema de tipo Stokes.

Esto nos permite reforzar las condiciones de contorno originales del problema en ambos pasos, lo que conduce a la convergencia de ambas velocidades hacia la misma función límite (una solución débil de (NS)) en $\mathbf{H}_0^1(\Omega)$, (ver [12], [13]). En efecto, en [12], [13], Blasco, Codina y Huerta prueban la convergencia de este esquema. Con este propósito, primero obtienen estimaciones a priori de estabilidad para ambas velocidades $\mathbf{u}^{m+1/2}$ y $\mathbf{u}^{m+1}$ en $l^\infty(\mathbf{L^2}) \cap l^2(\mathbf{H}^1)$ y posteriormente, aplicando resultados de compacidad para “controlar” el límite de los términos convectivos, un paso al límite conduce a la convergencia. Por otro lado, en [14] los autores obtuvieron estimaciones de error de orden $O(k)$ en $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ para la velocidad final $\mathbf{u}^{m+1}$ y orden $O(k^{1/2})$ en $l^2(L^2)$ para la presión.

Además, estas estimaciones en tiempo fueron usadas en [15] para obtener las siguientes estimaciones de error para un esquema totalmente discreto, cuyas soluciones denotamos $\mathbf{u}_h^{m+1/2}$ y $(\mathbf{u}_h^{m+1}, p_h^{m+1})$, basada en una aproximación de elementos finitos de orden $O(h)$ en $\mathbf{H}^1$ y orden $O(h^2)$ en $\mathbf{L}^2$ para las velocidades y $O(h)$ en L^2 para la presión:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(L^2)\cap l^2(H^1)} \leq C\,(k+h)$$

bajo la restricción $h^2 \leq C\,k$.

Por otro lado, en [13] se realizaron cálculos numéricos, que conducían a orden $O(k)$ tanto en velocidad como en presión.

Consecuentemente, existe un salto entre el análisis numérico (en donde se tenía $O(\sqrt{k})$ para la presión) y los cálculos numéricos (en los que se observa orden $O(k)$) respecto a la aproximación en tiempo para la presión.

Por otra parte, en [40], este esquema en tiempo de descomposición de la viscosidad es estudiado junto con varios métodos de elementos finitos de Galerkin discontinuos en espacio. Desde el punto de vista analítico, con $P_1 \times P_0$ como espacios discretos, fueron obtenidos orden $O(k+h)$ en $l^\infty(\mathbf{L}^2)$ para la velocidad y orden $O(\sqrt{k}+h)$ en $l^2(L^2)$ para la presión. Por otro lado, experimentos numéricos mostraron que las aproximaciones en espacio eran de orden $O(h)$ en una norma $\mathbf{H}^1$ discreta y $O(h^2)$ en $\mathbf{L}^2$ para la velocidad y orden $O(h)$ en L^2 para la presión, usando una aproximación $P_1 \times P_0$ (además, $O(h^2)$ en una norma $\mathbf{H}^1$ discreta y $O(h^3)$ en $\mathbf{L}^2$ para la velocidad y $O(h^2)$ en L^2 para la presión, usando espacios discretos $P_2 \times P_1$). Por tanto, de nuevo existe un salto entre los cálculos numéricos (que conducen a $O(h^2)$) y el análisis numérico (que prueba $O(h)$) respecto a la aproximación en espacio para la velocidad en norma $\mathbf{L}^2$.

En los Capítulos 1 y 2 pretendemos rellenar estos saltos demostrando analíticamente que se obtienen los resultados observados en los experimentos numéricos.

Esto lo conseguiremos en dos pasos, comparando primero la solución exacta y el esquema discreto en tiempo (Capítulo 1) y luego el esquema discreto en tiempo y el esquema completamente discreto (Capítulo 2). La prueba de las estimaciones óptimas para la presión, está basada en la obtención de estimaciones en normas fuertes para el error en velocidad.

1. Para el esquema semidiscreto en tiempo, pretendemos

   - Mejorar el orden de la estimación de error para la presión, de $O(\sqrt{k})$ a $O(k)$.
   - Mejorar la norma de las estimaciones de error en velocidad y presión, pasando de $l^\infty(\mathbf{L}^2)$ a $l^\infty(\mathbf{H}^1)$ en velocidad y de $l^2(L^2)$ a $l^\infty(L^2)$ en presión.

   Así, en el **Capítulo 1** de la presente memoria, hemos mejorado las estimaciones previas para el esquema semidiscreto en tiempo obteniendo las siguientes estimaciones óptimas de error:
$$\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{l^\infty(L^2)\cap l^2(H^1)} + \|p(t_m) - p^m\|_{l^2(L^2)} \leq C\, k.$$
   Además, suponiendo la siguiente hipótesis para la primera etapa, $\|\mathbf{u}(t_1) - \mathbf{u}^1\|_{L^2} \leq C\, k^2$, podemos mejorar la norma en las estimaciones de error en el siguiente sentido:
$$\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{l^\infty(H^1)} + \|p(t_m) - p^m\|_{l^\infty(L^2)} \leq C\, k.$$

2. En el **Capítulo 2**, usamos este esquema semidiscreto en tiempo como un problema auxiliar, con objeto de obtener las estimaciones de error para el esquema totalmente discreto con elementos finitos, que puede ser descrito como sigue. Dada $\mathbf{u}_h^m$ una aproximación de $\mathbf{u}(t_m)$, primero calculamos una velocidad intermedia $\mathbf{u}_h^{m+1/2}$ (como una primera aproximación de $\mathbf{u}(t_{m+1})$) por medio de un problema discreto de convección-difusión y, después, calculamos $(\mathbf{u}_h^{m+1}, p_h^{m+1})$ (aproximación de $(\mathbf{u}(t_{m+1}), p(t_{m+1}))$) resolviendo un problema de tipo Stokes discreto.

   Básicamente, los objetivos en este segundo paso son:

   - Extender el orden en las estimaciones de error, en velocidad y presión, del esquema semidiscreto en tiempo al esquema totalmente discreto, concretamente de $O(k)$ a $O(k+h)$.
   - Mejorar el orden en la estimación de error para la velocidad en norma $L^2(\mathbf{L}^2)$, de $O(k+h)$ (obtenida en [15]) a $O(k+h^2)$.

Juntando estos dos pasos anteriores y bajo la misma restricción para los parámetros de tiempo k y espacio h, $h^2 \leq C\, k$, impuesta en [15], obtendremos las siguientes estimaciones óptimas de error:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(L^2)\cap l^2(H^1)} + \|p(t_m) - p_h^m\|_{l^2(L^2)} \leq C\,(k+h),$$

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^2(L^2)} \le C\,(k + h^2)$$

Además, suponiendo la siguiente hipótesis para la primera etapa, $|(\mathbf{u}^1 - \mathbf{u}_h^1) - (\mathbf{u}^0 - \mathbf{u}_h^0)| \le C\,k\,h$, podemos mejorar la norma en las estimaciones de error:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(H^1)} + \|p(t_m) - p_h^m\|_{l^\infty(L^2)} \le C\,(k + h).$$

Notemos que, observando las estimaciones de error de orden $O(k + h)$ y $O(k + h^2)$ y la restricción $h^2 \le C\,k$, una elección apropiada del par (k, h) estaría entre $k = O(h)$ y $k = O(h^2)$ (siendo ambos casos válidos).

En el **Capítulo 3** nos centraremos en la aproximación del problema de Navier-Stokes pero ahora con un esquema de segregación de la presión inspirado en un método de proyección con presión incremental.

Con respecto a la convergencia del método de proyección de Chorin-Temam, en [88] se probó la convergencia del esquema semidiscreto en tiempo, mientras que en [24] la convergencia fue probada para el esquema totalmente discreto asociado a un problema con condiciones de contorno periódicas.

Posteriormente, fueron obtenidas estimaciones de error para el esquema semidiscreto en tiempo ([78], [80]) y para el esquema totalmente discreto ([45]). Básicamente, para el método de proyección sin corrección de presión se tienen ([78], [80]) estimaciones de error de orden $O(k^{1/2})$ en $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ y de orden $O(k)$ en $l^2(\mathbf{L}^2)$ para ambas velocidades y de orden $O(k^{1/2})$ en $l^2(L^2)$ para la presión. Para el método de proyección incremental estas estimaciones son mejoradas en [45] a orden $O(k)$ en $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ para la velocidad intermedia $\widetilde{\mathbf{u}}^{m+1}$ y a orden $O(k)$ en $l^2(L^2)$ para la presión. Hay que destacar que en [45] se considera una aproximación en espacio de elementos finitos en donde el paso de proyección está basado en una formulación mixta velocidad-presión. Además, en [45] se obtienen las estimaciones de error bajo la restricción de los parámetros $k^2 \le \alpha\,h$ (en el caso tridimensional). En este sentido, en el Capítulo 3 de la presente memoria obtenemos también estimaciones óptimas de error pero usando un esquema totalmente discreto desacoplado para los problemas de la velocidad y presión (llamado también de segregación de la presión) e imponiendo una restricción diferente, concretamente $h \le \alpha\,k$.

En [7] los autores realizaron un estudio del método de proyección no incremental totalmente discreto, observado como método de segregación de la presión, obteniendo resultados de convergencia y estimaciones de error de orden $O(k^{1/2} + h)$ bajo una restricción del tipo $\alpha\,h^2 \le k \le \beta\,h^2$, pero sin necesidad de imponer la condición inf-sup discreta para los espacios aproximantes. El Capítulo 3 sigue esta línea para el método de proyección incremental, en el sentido de que consideramos un esquema segregado para la presión y obtenemos estimaciones de error óptimas, de orden $O(k + h)$ para la presión, suponiendo que se verifica la condición inf-sup discreta e imponiendo la restricción $h \le \alpha\,k$.

Básicamente el Capítulo 3 consta de dos partes. En primer lugar, estudiamos el esquema semi-discreto en tiempo. Deduciremos la estabilidad y obtendremos estimaciones de error óptimas para la velocidad y la presión, concretamente, concluiremos la sección correspondiente con la obtención de estimaciones de error de orden $O(k)$ para la velocidad en $l^\infty(\mathbf{H}^1)$ y para la presión en $l^\infty(L^2)$, suponiendo una hipótesis adicional para el primer paso del esquema. En segundo lugar, nos centramos en el esquema totalmente discreto de segregación de la presión. Con respecto a las estimaciones de errores espaciales (comparando el esquema semi-discreto en tiempo con el esquema totalmente discreto), deduciremos estimaciones de error de orden $O(h)$ para la velocidad en $l^\infty(\mathbf{H}^1)$ y para la presión en $l^\infty(L^2)$, suponiendo de nuevo una hipótesis adicional sobre el primer paso del esquema y la restricción sobre los parámetros $h \le C\,k$. Combinando las dos secciones, obtenemos las estimaciones óptimas de los errores totales:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(H^1)} + \|p(t_m) - p_h^m\|_{l^\infty(L^2)} \le C\,(k+h).$$

4.1.1. Capítulo 1

Obtendremos en este capítulo nuevas estimaciones de error de orden $O(k^{1/2})$ para $\mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}$ y $\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}$ en $l^\infty(\mathbf{H}^1) \cap l^2(\mathbf{H}^2)$ y para $e_p^m = p(t_m) - p^m$ en $l^2(H^1)$. Estas estimaciones serán usadas para obtener orden $O(k^{1/2})$ en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ para las derivadas discretas en tiempo de $\mathbf{e}^{m+1/2}$ y $\mathbf{e}^{m+1}$, que conducirán al orden $O(k)$ para las derivadas discretas en tiempo de $\mathbf{e}^{m+1}$, bien en $l^2(\mathbf{L}^2)$ bien en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$, teniendo en este último caso que imponer una restricción para el primer paso del esquema (de hecho, estas dos estimaciones son obtenidas de forma independiente). Como consecuencia, la mejora para la estimación del error de presión e_p^m a orden $O(k)$ en $l^2(L^2)$ o en $l^\infty(L^2)$ se tienen respectivamente.

El esquema estaba descrito como sigue:

Inicialización: $\mathbf{u}^0 = \mathbf{u}_0$

Paso de tiempo $m+1$:

Subetapa 1: Dada $\mathbf{u}^m$, encontrar $\mathbf{u}^{m+1/2}$ solución de

$$(S_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1/2} - \mathbf{u}^m) + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1/2} - \nu\Delta\mathbf{u}^{m+1/2} = \mathbf{f}^{m+1} \quad \text{en } \Omega, \\ \mathbf{u}^{m+1/2}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Subetapa 2: Dada $\mathbf{u}^{m+1/2}$, encontrar $\mathbf{u}^{m+1}$ y p^{m+1} solución de

$$(S_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) - \nu\Delta(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) + \nabla p^{m+1} = \mathbf{0} \quad \text{en } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{en } \Omega, \qquad \mathbf{u}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Con respecto a la resolución efectiva del esquema, en cada paso de tiempo, será necesario para $(S_1)^{m+1}$ resolver tres ecuaciones de convección-difusión (puesto que el sistema es desacoplado por componentes) y para $(S_2)^{m+1}$ un problema de Stokes.

Sumando $(S_1)^{m+1}$ y $(S_2)^{m+1}$, llegamos a

$$(S_3)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m) + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1/2} - \nu\Delta\mathbf{u}^{m+1} + \nabla p^{m+1} = \mathbf{f}^{m+1} \quad \text{en } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{en } \Omega, \qquad \mathbf{u}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{array} \right.$$

Notemos que $(S_3)^{m+1}$ puede ser interpretado como unas relaciones de consistencia, puesto que la idea para probar la convergencia del esquema (ver [12]) es demostrar que $\mathbf{u}^{m+1/2}$ y $\mathbf{u}^{m+1}$ convergen al mismo límite. Por tanto, tomando límites en $(S_3)^{m+1}$, encontraremos una solución del problema continuo (NS) .

En el Capítulo 1 obtendremos estimaciones del error semi-discreto en tiempo (para la velocidad y la presión), con respecto a una solución suficientemente regular (única en particular) $(\mathbf{u}, p)$ de (NS) . Por simplicidad y sin pérdida de generalidad fijamos la constante de viscosidad $\nu = 1$.

Obtenemos los siguientes problemas diferenciales verificados por los errores:

$$(E_1)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1/2} - \mathbf{e}^m) - \Delta\mathbf{e}^{m+1/2} = -\nabla p(t_{m+1}) + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \\ \mathbf{e}^{m+1/2}|_{\partial\Omega} = \mathbf{0}, \end{array} \right.$$

$$(E_2)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}) - \Delta(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}) - \nabla p^{m+1} = \mathbf{0} \quad \text{en } \Omega \\ \nabla \cdot \mathbf{e}^{m+1} = 0 \quad \text{en } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{array} \right.$$

donde

$$\mathcal{E}^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t - t_m)\, \mathbf{u}_{tt}(t)\, dt - \left(\int_{t_m}^{t_{m+1}} \mathbf{u}_t \cdot \nabla \right) \mathbf{u}(t_{m+1}) := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1}$$

es el error de consistencia y

$$\mathbf{NL}^{m+1} = -\Big(\mathbf{e}^m \cdot \nabla\Big)\mathbf{u}(t_{m+1}) - (\mathbf{u}^m \cdot \nabla)\mathbf{e}^{m+1/2}$$

son los términos residuales que aparecen en las diferencias de los términos cuadráticos.

Finalmente, sumando $(E_1)^{m+1}$ y $(E_2)^{m+1}$, llegamos a:

$$(E_3)^{m+1} \qquad \left\{ \begin{array}{l} \delta_t\mathbf{e}^{m+1} - \Delta\mathbf{e}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{en } \Omega, \\ \nabla \cdot \mathbf{e}^{m+1} = 0 \quad \text{en } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{array} \right.$$

Para obtener las diferentes estimaciones de error, aparecerán las siguientes hipótesis para la (única) solución $(\mathbf{u}, p)$ de (NS) :

(**H1**) $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}), \quad p \in L^\infty(H^1), \quad \mathbf{u}_t \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^1), \quad \sqrt{t}\, \mathbf{u}_{tt} \in L^2(\mathbf{H}^{-1})$

(**H2**) $\sqrt{t}\mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$

(**H3**) $\mathbf{u}_{tt} \in L^2(\mathbf{V}')$

(**H4**) $p_t \in L^2(H^1), \quad \mathbf{u}_t \in L^\infty(\mathbf{H}^1) \cap L^2(\mathbf{H}^2), \quad \mathbf{u}_{tt} \in L^2(\mathbf{L}^2), \quad \sqrt{t}\mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1})$

(**H5**) $\mathbf{u}_{ttt} \in L^2((\mathbf{H}^2 \cap \mathbf{V})')$

(**H6**) $\mathbf{u}_{ttt} \in L^2(\mathbf{V}')$

(**H7**) $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1})$

Las hipótesis (**H1**)-(**H3**) pueden ser probadas suponiendo suficiente regularidad de los datos, pero desafortunadamente, para obtener las hipótesis (**H4**)-(**H7**), es necesario suponer que $\mathbf{u}_t(0) \in \mathbf{H}^1$, lo que implica una condición de compatibilidad no local para los datos $\mathbf{u}_0$ y $\mathbf{f}$ ([92]).

El hilo conductor del Capítulo 1 es el siguiente:

Partimos de las estimaciones de orden $O(k^{1/2})$ para ambas velocidades en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}_0^1)$ que ya fueron obtenidas en [12]:

Teorema 1 *([12]) Supponiendo la hipótesis* (**H1**)*, se tienen las siguientes estimaciones de error*

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leq C\, k^{1/2} \tag{1}$$

$$||\mathbf{e}^{m+1/2} - \mathbf{e}^{m+1}||_{l^2(\mathbf{L}^2)} + ||\mathbf{e}^{m+1/2} - \mathbf{e}^{m}||_{l^2(\mathbf{L}^2)} \leq C\, k. \tag{2}$$

La estimación (1) implica en particular las estimaciones uniformes para los errores

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \leq C$$

y, en consecuencia, como $\mathbf{u} \in L^\infty(\mathbf{H}^1)$, las estimaciones también se tienen para el esquema:

$$\|\mathbf{u}^{m+1/2}\|_{H^1} + \|\mathbf{u}^{m+1}\|_{H^1} \leq C, \quad \forall\, m.$$

Estas estimaciones para el esquema, nos permitirán acotar convenientemente los términos no lineales en las estimaciones en normas fuertes siguientes, donde usamos la notación

$$\delta_t \mathbf{e}^{m+1} = \frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{k}, \qquad \delta_t \mathbf{e}^{m+1/2} = \frac{\mathbf{e}^{m+1/2} - \mathbf{e}^{m-1/2}}{k},$$

para las derivadas discretas en tiempo de los errores.

Corolario 2 *Bajo las hipótesis del Teorema 1 y* (**H2**)*, se tiene*

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)} + \|e_p^{m+1}\|_{l^2(H^1)} \leq C\, k^{1/2}, \quad \|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{H}^1)} \leq C. \tag{3}$$

En particular, de (3) se tiene

$$\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^2)} + \|e_p^{m+1}\|_{l^\infty(H^1)} \leq C. \tag{4}$$

Corolario 3 *Suponiendo las hipótesis del Corolario 2, se obtiene*

$$\|\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)} \le C\,k, \quad \|\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} \le C\,k^{1/2}, \quad \|\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C. \tag{5}$$

En particular, usando (3), (4) y (5), obtenemos

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} \le C\,k^{1/2}, \quad \|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C,$$

y, finalmente, de $\mathbf{u} \in L^\infty(\mathbf{H}^2)$, podemos concluir

$$\|\mathbf{u}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C.$$

La estimación anterior, nos proporciona la regularidad necesaria para abordar la acotación de los términos no lineales, en los cuales la principal dificultad que aparece es la presencia de la velocidad intermedia $\mathbf{u}^{m+1/2}$ que, debido a la construcción del esquema, es la que a priori tendrá menos orden para las estimaciones de error.

En [14], las estimaciones para $(\mathbf{u}^{m+1}, p^{m+1})$ en $l^2(\mathbf{H}^2 \times H^1)$ y para $\mathbf{u}^{m+1/2}$ en $l^2(\mathbf{H}^2)$ fueron deducidas, pero bajo la restricción de k suficientemente pequeño. Ahora, estas estimaciones son mejoradas en dos sentidos: al pasar de $l^2(0,T)$ a $l^\infty(0,T)$ y al no imponer restricciones sobre k.

Con estos resultados, podemos mejorar la estimación de error para la velocidad final $\mathbf{u}^{m+1}$, obteniendo $O(k)$ para $\mathbf{e}^{m+1}$ en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1_0)$.

Teorema 4 *Suponiendo las hipótesis del Corolario 3 y* **(H3)***, se tienen las siguientes estimaciones:*

$$\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k, \tag{6}$$

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\,k^{1/2}.$$

Gracias a (5) y (6), llegamos a

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)} \le C\,k.$$

A continuación, obtenemos estimaciones de error para las derivadas discretas en tiempo de la siguiente manera. Para ello escribimos los problemas que verifican las derivadas discretas de los errores. Haciendo $\delta_t(E_1)^{m+1} := \dfrac{(E_1)^{m+1}-(E_1)^m}{k}$ y $\delta_t(E_2)^{m+1} := \dfrac{(E_2)^{m+1}-(E_2)^m}{k}$ obtenemos respectivamente

$$(D_1)^{m+1} \qquad \frac{1}{k}(\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^m) - \Delta \delta_t \mathbf{e}^{m+1/2} = -\nabla \delta_t p(t_{m+1}) + \delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1}$$

$$(D_2)^{m+1} \qquad \frac{1}{k}(\delta_t \mathbf{e}^{m+1} - \delta_t\, \mathbf{e}^{m+1/2}) - \Delta(\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2}) + \nabla(\delta_t e_p^{m+1} - \delta_t p(t_{m+1})) = \mathbf{0}.$$

Finalmente, sumando $(D_1)^{m+1}$ y $(D_2)^{m+1}$, llegamos a $(\forall\, m \geq 1)$:

$$(D_3)^{m+1} \qquad \frac{1}{k}(\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m) - \Delta \delta_t \mathbf{e}^{m+1} + \nabla \delta_t e_p^{m+1} = \delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1}$$

Primero obtenemos orden $O(k^{1/2})$ para $\delta_t \mathbf{e}^{m+1}$ y $\delta_t \mathbf{e}^{m+1/2}$ en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}_0^1)$.

Teorema 5 *Suponiendo las hipótesis del Teorema 4 y (***H4***), las siguientes estimaciones de error se tienen*

$$\|\delta_t \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}_0^1)} + \|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}_0^1)} \leq C\, k^{1/2},$$

$$\|\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} + \|\delta_t \mathbf{e}^{m+1/2} - \delta_t\, \mathbf{e}^{m}\|_{l^2(\mathbf{L}^2)} \leq C\, k.$$

Para mejorar a $O(k)$ las estimaciones anteriores, podemos razonar de dos formas distintas que nos llevan a dos resultados diferentes.

1. Usando un argumento de dualidad, obtenemos orden $O(k)$ para $\delta_t \mathbf{e}^{m+1}$ en $l^2(\mathbf{L}^2)$ y para e_p^{m+1} en $l^2(L^2)$, sin imponer hipótesis sobre la etapa inicial pero imponiendo el paso de tiempo k suficientemente pequeño.

   En este sentido, necesitamos usar el operador inverso A^{-1} del operador de Stokes A, es decir, $\mathbf{v} = A^{-1}\mathbf{u}$ es la solución (débil) del problema de Stokes (con una presión π)

   $$-\Delta \mathbf{v} + \nabla \pi = \mathbf{u} \ \text{ en } \Omega, \quad \nabla \cdot \mathbf{v} = 0 \ \text{ en } \Omega, \quad \mathbf{v} = \mathbf{0} \ \text{ sobre } \partial\Omega.$$

   Tenemos las siguientes igualdades

   $$(\nabla \mathbf{u}, \nabla(A^{-1}\mathbf{u})) = \|\mathbf{u}\|_{L^2}^2 \qquad \forall\, \mathbf{u} \in \mathbf{V}\ (\mathbf{u} \in \mathbf{H}_0^1,\, \nabla \cdot \mathbf{u} = 0)$$

   $$(\mathbf{u}, A^{-1}\mathbf{u}) = \|A^{-1}\mathbf{u}\|_{H^1}^2 \qquad \forall\, \mathbf{u} \in \mathbf{L}^2.$$

   Además, usando la regularidad H^2 del problema de Stokes, se tiene

   $$\|A^{-1}\, \mathbf{u}\|_{H^2} \leq C\, \|\mathbf{u}\|_{L^2} \quad \forall\, \mathbf{u} \in \mathbf{L}^2.$$

   Entonces, obtenemos el siguiente resultado

   Teorema 6 *Suponiendo las hipótesis del Teorema 5,* **(H5)** *y k suficientemente pequeño, se tienen las siguientes estimaciones:*

   $$\|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} \leq C\, k \ \ (\ y \ \ \|A^{-1}\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \leq C\, k).$$

   Una consecuencia directa del resultado anterior, mirando el problema $(E_3)^{m+1}$ como un problema de Stokes, es el siguiente corolario:

Corolario 7 *Suponiendo las hipótesis del Teorema 6, se tiene*

$$\|e_p^{m+1}\|_{l^2(L^2)} \le C\, k.$$

2. Por otra parte, suponiendo una hipótesis adicional para la etapa inicial del esquema, podemos obtener ahora sin la restricción de k suficientemente pequeño, el orden $O(k)$ para la derivada discreta del error final en velocidad $\delta_t \mathbf{e}^{m+1}$ pero en normas más fuertes, concretamente en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ y posteriormente $O(k)$ para $(\mathbf{e}^{m+1}, e_p^{m+1})$ en $l^\infty(\mathbf{H}^1\times L^2)$.

   El primer resultado obtenido es

   Teorema 8 *Suponiendo las hipótesis del Teorema 5,* **(H6)** *y la siguiente hipótesis de aproximación para la etapa inicial del esquema:*

   $$\|\delta_t \mathbf{e}^1\|_{L^2} \le C\, k,$$

   entonces

   $$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\, k.$$

   El resultado anterior nos lleva a las deseadas estimaciones de error para la presión en normas más fuertes que las obtenidas con el razonamiento dual:

   Corolario 9 *Suponiendo las hipótesis del Teorema 8 y* (**H7**)*, tenemos*

   $$\|e_p^{m+1}\|_{l^\infty(L^2)} \le C\, k \quad \text{y} \quad \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \le C\, k.$$

4.1.2. Capítulo 2

Ahora, usaremos el esquema semidiscreto en tiempo del Capítulo 1 como un problema auxiliar, para obtener el orden óptimo $O(k+h)$ para los errores en velocidad y presión en las normas $\mathbf{H}^1$ y L^2 respectivamente, y orden $O(k+h^2)$ para el error en velocidad en norma $\mathbf{L}^2$, para el esquema totalmente discreto por elementos finitos.

Concretamente, obtendremos estimaciones de error $O(h)$ para $\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}$ y $\mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}$ en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$, lo que implica estimaciones en $\mathbf{W}^{1,6}(\Omega)$ para las velocidades discretas bajo la restricción $h^2/k \le C$. A continuación, obtenemos orden $O(h)$ para las derivadas discretas en tiempo de $\mathbf{e}_d^{m+1}$ en $l^2(\mathbf{L}^2)$ y en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ obtenidas de forma independiente, donde una restricción sobre la etapa inicial del esquema deberá ser impuesta en el último caso. Además, se deducirán estimaciones de error de orden $O(k+h^2)$ para $\mathbf{e}_d^{m+1}$ en $l^2(\mathbf{L}^2)$. Por último, se obtienen estimaciones del error de presión de orden $O(h)$ en $l^2(L^2)$ y en $l^\infty(L^2)$ respectivamente, imponinedo de nuevo la restricción sobre la etapa inicial del esquema en el último caso.

Para todo ello, consideramos una discretización espacial de los problemas semidiscretos en tiempo $(S_1)^{m+1}$ y $(S_2)^{m+1}$ mediante el método de los elementos finitos. Introducimos tres espacios $\mathbf{X}_h$, $\mathbf{Y}_h \subset \mathbf{H}_0^1(\Omega)$ y $Q_h \subset L_0^2(\Omega)$ asociados a una familia de triangulaciones de elementos finitos del dominio Ω, siendo las funciones de $\mathbf{X}_h$, $\mathbf{Y}_h$ y Q_h localmente polinomios de grado al menos 1, 1 y 0, respectivamente.

Para el análisis del método, requeriremos las siguientes hipótesis:

- Los espacios $\mathbf{Y}_h$ y Q_h deben satisfacer la condición inf-sup estándar ([39]).
- Se verifican las siguientes propiedades de aproximación ([39]):

$$\|\mathbf{v} - I_h\mathbf{v}\|_{H^1} + \frac{1}{h}\|\mathbf{v} - I_h\mathbf{v}\|_{L^2} \le C\,h\,\|\mathbf{v}\|_{H^2} \qquad \forall\,\mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega),$$
$$\|q - J_h q\|_{L^2} \le C\,h\,\|q\|_{H^1} \quad \forall\,q \in H^1(\Omega) \cap L_0^2(\Omega),$$

  donde $(I_h, J_h) : \mathbf{H}_0^1(\Omega) \times L^2(\Omega) \to \mathbf{Y}_h \times Q_h$ es el operador de interpolación global definido como:

$$(I_h\mathbf{v}, J_h q) \in \mathbf{Y}_h \times Q_h : \begin{cases} (\nabla(I_h\mathbf{v} - \mathbf{v}), \nabla\,\mathbf{v}_h) - (J_h q - q, \nabla\cdot\mathbf{v}_h) = 0 \quad \forall\,\mathbf{v}_h \in \mathbf{Y}_h, \\ (\nabla\cdot(I_h\mathbf{v} - \mathbf{v}), q_h) = 0 \quad \forall\,q_h \in Q_h. \end{cases}$$

  Por otro lado,

$$\|\mathbf{v} - K_h\mathbf{v}\|_{H^1} + \frac{1}{h}\|\mathbf{v} - K_h\mathbf{v}\|_{L^2} \le C\,h\,\|\mathbf{v}\|_{\mathbf{H}^2} \qquad \forall\,\mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega),$$

  donde $K_h : \mathbf{H}_0^1(\Omega) \to \mathbf{X}_h$ es el operador de interpolación definido como:

$$K_h\mathbf{v} \in \mathbf{X}_h, \quad (\nabla(K_h\mathbf{v} - \mathbf{v}), \nabla\,\mathbf{v}_h) = 0 \quad \forall\,\mathbf{v}_h \in \mathbf{X}_h.$$

- Supondremos la siguiente restricción sobre el paso de tiempo k y el tamaño de la malla h:

  (H) Existe una constante $\alpha > 0$ (independiente de k y h) tal que $\dfrac{h^2}{k} \le \alpha$.

Como es usual, en el problema totalmente discreto, usaremos una forma trilineal anti-simétrica para el tratamiento del término convectivo $c(\cdot,\cdot,\cdot)$ que verifica

$$c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \qquad \forall\,\mathbf{u} \in \mathbf{H}_0^1, \;\; \forall\,\mathbf{v} \in \mathbf{H}^1, \tag{7}$$

El problema totalmente discreto está entonces descrito como sigue:

Inicialización: Sea $\mathbf{u}_h^0 \in \mathbf{Y}_h$ una aproximación de $\mathbf{u}^0$

Paso de tiempo $m+1$:

Subetapa 1: Dada $\mathbf{u}_h^m \in \mathbf{Y}_h$, calcular $\mathbf{u}_h^{m+1/2} \in \mathbf{X}_h$ tal que, para todo $\mathbf{v}_h \in \mathbf{X}_h$

$$(S_1)_h^{m+1} \qquad \frac{1}{k}(\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m, \mathbf{v}_h) + c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla\,\mathbf{u}_h^{m+1/2}, \nabla\,\mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h).$$

Subetapa 2: Dada $\mathbf{u}_h^{m+1/2} \in \mathbf{X}_h$, calcular $(\mathbf{u}_h^{m+1}, p_h^{m+1}) \in \mathbf{Y}_h \times Q_h$, tal que para todo $(\mathbf{v}_h, q_h) \in \mathbf{Y}_h \times Q_h$

$$(S_2)_h^{m+1} \quad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}), \nabla \mathbf{v}_h) - (p_h^{m+1}, \nabla \cdot \mathbf{v}_h) = 0, \\ (\nabla \cdot \mathbf{u}_h^{m+1}, q_h) = 0. \end{array} \right.$$

En la primera subetapa, debemos resolver un problema de convección-difusión desacoplado por componentes, mientras que la segunda subetapa es un problema de Stokes generalizado.

En [12], usando (7), se extienden los resultados de estabilidad y convergencia para el esquema semidiscreto en tiempo al caso totalmente discreto. Concretamente se obtiene

$$\|\mathbf{u}_h^m\|_{l^\infty(L^2)\cap l^2(H^1)} + \|\mathbf{u}_h^{m+1/2}\|_{l^\infty(L^2)\cap l^2(H^1)} \leq C.$$

Los errores espaciales

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \qquad \mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}$$

se descomponen en sus correspondientes partes discreta y de interpolación:

$$\mathbf{e}_d^{m+1} = \mathbf{e}_h^{m+1} + \mathbf{e}_i^{m+1}, \qquad \mathbf{e}_d^{m+1/2} = \mathbf{e}_h^{m+1/2} + \mathbf{e}_i^{m+1/2}, \qquad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1}$$

siendo $\mathbf{e}_i$ errores de interpolación y $\mathbf{e}_h$ errores espaciales discretos, concretamente

$$\mathbf{e}_h^{m+1} = I_h\mathbf{u}^{m+1} - \mathbf{u}_h^{m+1} \text{ y } \mathbf{e}_i^{m+1} = \mathbf{u}^{m+1} - I_h\mathbf{u}^{m+1},$$

$$\mathbf{e}_h^{m+1/2} = K_h\mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2} \text{ y } \mathbf{e}_i^{m+1/2} = \mathbf{u}^{m+1/2} - K_h\mathbf{u}^{m+1/2},$$

$$e_{p,h}^{m+1} = J_h p^{m+1} - p_h^{m+1} \text{ y } e_{p,i}^{m+1} = p^{m+1} - J_h p^{m+1}.$$

Entonces, comparando $(S_1)^{m+1}, (S_2)^{m+1}$ con $(S_1)_h^{m+1}, (S_2)_h^{m+1}$, tenemos los problemas variacionales que verifican los errores espaciales $\mathbf{e}_d^{m+1/2}$ y $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ respectivamente:

$$(E_1)_h^{m+1} \qquad \frac{1}{k}(\mathbf{e}_d^{m+1/2} - \mathbf{e}_d^m, \mathbf{v}_h) + (\nabla \mathbf{e}_d^{m+1/2}, \nabla \mathbf{v}_h) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{X}_h,$$

$$(E_2)_h^{m+1} \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}, \mathbf{v}_h) + (\nabla(\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}), \nabla \mathbf{v}_h) - (e_{p,d}^{m+1}, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{Y}_h \\ (\nabla \cdot \mathbf{e}_d^{m+1}, q_h) = 0, \quad \forall q_h \in Q_h \end{array} \right.$$

donde

$$\mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) = -c(\mathbf{e}_d^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, \mathbf{v}_h),$$

La problemática fundamental de este capítulo, será conseguir las estimaciones de error sin tener regularidad H^2 para las velocidades discretas, porque estamos con funciones que son sólo globalmente continuas.

El hilo conductor del Capítulo 2 es el siguiente.

Comenzamos obteniendo las estimaciones de error de orden $O(h)$ para ambas velocidades en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$.

Teorema 10 *Suponiendo las hipótesis del Corolario 7,* $(\mathbf{H})$ *y* $\|\mathbf{e}_d^0\|_{L^2}\le C\,h$*, se tiene*

$$\|\mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}+\|\mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}\le C\,h \tag{8}$$

$$\|\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^{m}\|^2_{l^2(\mathbf{L}^2)}+\|\mathbf{e}_d^{m+1}-\mathbf{e}_d^{m+1/2}\|^2_{l^2(\mathbf{L}^2)}\le C\,k\,h^2. \tag{9}$$

A partir de aquí, será necesario considerar los mismos espacios discretos para ambas velocidades $\mathbf{X}_h\equiv\mathbf{Y}_h$ (que denotamos $\mathbf{X}_h$).

Ahora, de (8), (9), (2) y la restricción $(\mathbf{H})$, se tiene la regularidad necesaria para conseguir la estabilidad en $\mathbf{W}^{1,3}\cap\mathbf{L}^\infty$ para la velocidad final discreta, lo cuál será primordial a la hora de acotar los términos convectivos de forma adecuada. Concretamente, obtenemos estimaciones no hilbertianas para el esquema, tanto en velocidad final como en velocidad intermedia:

Corolario 11 *Suponiendo las hipótesis del Teorema 10,*

$$\mathbf{u}_h^{m+1}\quad \textit{y}\quad \mathbf{u}_h^{m+1/2}\qquad \textit{están acotadas en}\qquad l^2(\mathbf{W}^{1,6}).$$

En consecuencia, usando las estimaciones de $\mathbf{u}^{m+1}$ y $\mathbf{u}^{m+1/2}$ en $l^\infty(\mathbf{H}^2)$,

$$\mathbf{e}_d^{m+1}\quad \text{y}\quad \mathbf{e}_d^{m+1/2}\qquad \text{están acotadas en}\qquad l^2(\mathbf{W}^{1,6}).$$

Seguidamente, se pasa a demostrar las estimaciones de error de orden $O(h)$ para $\delta_t\mathbf{e}_d^{m+1}$ y $\delta_t\mathbf{e}_d^{m+1/2}$ en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ y $(\mathbf{e}_d^{m+1},e_{p,d}^{m+1})$ en $l^\infty(\mathbf{H}^1\times L^2)$. Para ello, hacemos $\delta_t(E_1)_h^{m+1}$ y $\delta_t(E_2)_h^{m+1}$, obteniendo $(\forall m\ge 1)$:

$$(D_1)_h^{m+1}\qquad \frac{1}{k}(\delta_t\mathbf{e}_d^{m+1/2}-\delta_t\mathbf{e}_d^{m},\mathbf{v}_h)+(\nabla\delta_t\mathbf{e}_d^{m+1/2},\nabla\mathbf{v}_h)=\delta_t\,\mathbf{NL}_h^{m+1}(\mathbf{v}_h)\quad\forall\,\mathbf{v}_h\in\mathbf{X}_h$$

$$(D_2)_h^{m+1}\qquad \left\{\begin{array}{l}\dfrac{1}{k}(\delta_t\,\mathbf{e}_d^{m+1}-\delta_t\,\mathbf{e}_d^{m+1/2},\mathbf{w}_h)+(\nabla(\delta_t\mathbf{e}_d^{m+1}-\delta_t\mathbf{e}_d^{m+1/2}),\nabla\mathbf{w}_h)\\ -(\delta_t e_{p,d}^{m+1},\nabla\cdot\mathbf{w}_h)=0,\qquad \forall\mathbf{w}_h\in\mathbf{X}_h\\ (\nabla\cdot\delta_t\mathbf{e}_d^{m+1},q_h)=0,\quad \forall q_h\in Q_h\end{array}\right.$$

donde,

$$\delta_t\,\mathbf{NL}_h^{m+1}(\mathbf{v}_h)=c(\delta_t\mathbf{e}_d^{m},\,\mathbf{u}^{m+1/2},\mathbf{v}_h)+c(\delta_t\,\mathbf{u}_h^{m},\mathbf{e}_d^{m+1/2},\mathbf{v}_h)+c(\mathbf{e}_d^{m-1},\delta_t\,\mathbf{u}^{m+1/2},\mathbf{v}_h)+c(\mathbf{u}_h^{m-1},\delta_t\mathbf{e}_d^{m+1/2},\mathbf{v}_h)$$

Finalmente, sumando $(D_1)_h^{m+1}$ y $(D_2)_h^{m+1}$ se llega a que para todo $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times Q_h$:

$$(D_3)_h^{m+1}\quad \left\{\begin{array}{l}\dfrac{1}{k}(\delta_t\mathbf{e}_d^{m+1}-\delta_t\,\mathbf{e}_d^{m},\mathbf{v}_h)+(\nabla\delta_t\mathbf{e}_d^{m+1},\nabla\mathbf{v}_h)+(\delta_t e_{p,d}^{m+1},\nabla\cdot\mathbf{v}_h)=\delta_t\,\mathbf{NL}_h^{m+1}(\mathbf{v}_h),\\ (\nabla\cdot\delta_t\mathbf{e}_d^{m+1},q_h)=0.\end{array}\right.$$

Obtener estimaciones de error de orden $O(h)$ para las derivadas discretas en tiempo nos llevará a la obtención del orden óptimo del error en presión. Al igual que para el caso semidiscreto, estas estimaciones las obtenemos por dos caminos distintos, bien suponiendo una buena aproximación de la etapa inicial, o bien suponiendo el paso de tiempo k suficientemente pequeño.

Teorema 12 *Suponiendo las hipótesis del Teorema 10 y suponiendo la hipótesis para la etapa inicial del esquema*

$$\|\delta_t \mathbf{e}_d^1\|_{L^2} \leq C\,h,$$

entonces

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leq C\,h.$$

Usando este resultado, conseguimos una estimación de estabilidad para la velocidad final discreta de tipo l^∞ en tiempo:

Corolario 13 *Bajo las hipótesis del Teorema 12 y suponiendo* $\|\mathbf{u}_h^0\|_{W^{1,6}}^2 \leq C_0$ *(esto es,* $\mathbf{u}_0 \in \mathbf{W}^{1,6}$*), se tiene*

$$(\mathbf{u}_h^{m+1}) \quad \textit{está acotada en} \quad l^\infty(\mathbf{W}^{1,6}).$$

Con esto llegamos a la estimación óptima para el error de presión discreto:

Corolario 14 *Suponiendo las hipótesis del teorema 12 y Corolario 13, obtenemos*

$$\|e_{p,d}^{m+1}\|_{l^\infty(L^2)} \leq C\,h \quad y \quad \|\mathbf{e}_d^{m+1}\|_{l^\infty(H^1)} \leq C\,h.$$

Combinando el Corolario 9 y el Corolario 14, obtenemos las estimaciones para los errores totales

$$\|\mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1}\|_{l^\infty(H^1)} + \|p(t_{m+1}) - p_h^{m+1}\|_{l^\infty(L^2)} \leq C\,(k+h).$$

Al igual que en el caso semidiscreto en tiempo, usando un argumento de dualidad, podemos eliminar la hipótesis para la etapa inicial $|\delta_t \mathbf{e}_d^1| \leq C\,h$, pero suponiendo k suficientemente pequeño, obteniendo orden $O(h^2)$ en $l^2(\mathbf{L}^2)$ para $\mathbf{e}_d^{m+1}$ y $O(h)$ para $\delta_t \mathbf{e}_d^{m+1}$ en $l^2(\mathbf{L}^2)$ y para $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ en $l^2(\mathbf{H}^1 \times L^2)$.

Para esto, debemos considerar el operador inverso A_h^{-1} del operador discreto de Stokes A_h, definido como $\mathbf{w}_h = A_h^{-1}\mathbf{u}_h \in \mathbf{X}_h$ es la solución del siguiente problema de Stokes discreto (con una presión $\pi_h \in Q_h$):

$$\begin{cases} (\nabla\,\mathbf{w}_h, \nabla \mathbf{v}_h) - (\pi_h, \nabla\cdot\mathbf{v}_h) = (\mathbf{u}_h, \mathbf{v}_h) \quad \forall\,\mathbf{v}_h \in \mathbf{X}_h, \\ (\nabla\cdot\mathbf{w}_h, q_h) = 0 \quad \forall\,q_h \in Q_h. \end{cases}$$

En particular, se tiene

$$\|\mathbf{u}_h\|_{L^2}^2 = (\nabla\,A_h^{-1}\mathbf{u}_h, \nabla\,\mathbf{u}_h), \quad \text{si } (\nabla\cdot\mathbf{u}_h, q_h) = 0 \text{ para todo } q_h \in Q_h$$

y

$$\|\nabla A_h^{-1}\mathbf{u}_h\|_{L^2}^2 = (\mathbf{u}_h, A_h^{-1}\mathbf{u}_h).$$

Este operador nos permitirá (al igual que en el caso semidiscreto en tiempo), trabajar con funciones test más regulares, que facilitan la acotación de los términos convectivos.

Así, daremos los siguientes pasos. Primero, obtenemos la siguiente estimación inicial

Lema 15 (Estimación inicial) *Suponiendo las hipótesis del Teorema 10,* $\|\mathbf{u}_h^0\|_{W^{1,3}\cap L^\infty} \le C$ *y* $\|\mathbf{e}_h^0\| \le C\,h$*, entonces*

$$\|\delta_t \mathbf{e}_h^1\|_{L^2}^2 \le C\,\frac{h^2}{k}$$

Razonando entonces como en el Teorema 12 (sin aplicar $|\delta_t \mathbf{e}_d^1| \le C\,h$) y usando (**H**), llegamos a

$$\|\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}\|_{l^2(\mathbf{L}^2)}^2 + \|\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)}^2 \le C\,h^2.$$

y

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}^2 + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}^2 \le C\,\frac{h^2}{k} \le C$$

Seguidamente, razonando como en el Corolario 13, obtenemos la estimación

$$(\mathbf{u}_h^{m+1}) \quad \text{está acotada en} \quad l^\infty(\mathbf{W}^{1,6}),$$

que nos permite demostrar el orden $O(h^2)$ para $\mathbf{e}_d^{m+1}$ en $l^2(\mathbf{L}^2)$.

Teorema 16 *Suponiendo las hipótesis del Teorema 10 y el Lema 15 y* $\|\mathbf{e}_h^0\|_{H^{-1}} \le C\,h^2$*, se tiene*

$$\|\mathbf{e}_d^m\|_{l^2(\mathbf{L}^2)} \le C(k + h^2)$$

para k suficientemente pequeño.

Así, llegamos a la estimación para el error total

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^2(\mathbf{L}^2)} \le C(k + h^2)$$

A continuación, se llega al orden $O(h)$ para $\delta_t \mathbf{e}_d^{m+1}$ en $l^2(\mathbf{L}^2)$ y para $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ en $l^2(\mathbf{H}^1 \times L^2)$.

Teorema 17 *Bajo las hipótesis del Lema 15 y* $\|\mathbf{e}_h^0\|_{L^2} \le C\,k\,h^2$*, la siguiente estimación de error se tiene (para k suficientemente pequeño)*

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\,h.$$

Consecuencia de este resultado es la estimación

$$\|e_{p,d}^{m+1}\|_{l^2(L^2)} \le C\,h.$$

que nos conduce a

$$\|p(t_m) - p_h^m\|_{l^2(L^2)} \le C(k + h).$$

4.1.3. Capítulo 3

Aproximamos en este capítulo el problema de NS mediante un método de segregación de la presión, basado en un esquema de proyección con corrección de presión en tiempo y elementos finitos en espacio.

El objetivo del capítulo será la obtención de las estimaciones de error óptimas para el esquema totalmente dicreto. Para ello, estudiamos primero el caso semisdiscreto en tiempo, que usaremos posteriormente como un problema auxiliar para conseguir las estimaciones en el caso totalmente discreto.

Así, para el caso semidiscreto en tiempo, obtendremos resultados de dependencia continua y estabilidad del esquema y, seguidamente, estimaciones de error de orden $O(k)$ para la velocidad en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ y $O(1)$ para la presión en $l^\infty(H^1)$. Como consecuencia, la velocidad intermedia estará acotada en $l^\infty(\mathbf{H}^2)$. Después, obtendremos las estimaciones de error de orden $O(k)$ para las derivadas discretas en tiempo de las velocidades en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$, suponiendo una buena aproximación para el primer paso del esquema. Finalmente, se conseguirán las estimaciones de error de orden $O(k)$ para la velocidad en $l^\infty(\mathbf{H}^1)$ y para la presión en $l^\infty(L^2)$.

En segundo lugar, nos centraremos en el esquema totalmente discreto. Con respecto a las estimaciones de errores espaciales (comparando el esquema semi-discreto en tiempo con el esquema totalmente discreto), se obtendrán estimaciones de orden $O(h)$ para el error en velocidad en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ (junto con orden $O(h)$ para una velocidad discreta proyectada en $l^\infty(\mathbf{L}^2)$) que implicarán estimaciones en $\mathbf{W}^{1,6}(\Omega)$ de la velocidad, siempre que $h/k \le C$. Seguidamente, se obtendrán estimaciones de orden $O(h)$ para las derivadas discretas de la velocidad en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ (y para la velocidad proyectada en $l^\infty(\mathbf{L}^2)$). Finalmente, obtendremos orden $O(h)$ para la velocidad en $l^\infty(\mathbf{H}^1)$ y para la presión en $l^\infty(L^2)$.

Esquema semidiscreto en tiempo

Consideraremos por tanto el esquema semidiscreto en tiempo de un método de proyección incremental de presión de tipo Van-Kan [56]. Concretamente, se introduce un término explícito de presión en el problema de convección-difusión para la velocidad (subetapa 1), con una corrección implícita en el paso de proyección (subetapa 2).

Como la convección se considera sobre una velocidad que no es incompresible, usaremos una forma antisimétrica del término convectivo $c(\mathbf{u},\mathbf{v},\mathbf{w}) = \langle C(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle$.

El esquema semi-discreto estará descrito como sigue:

Inicialización: Sean $\widetilde{\mathbf{u}}^0 = \mathbf{u}(0)$ y p^0 dadas. Tomamos $\mathbf{u}^0 = \widetilde{\mathbf{u}}^0$.

Subetapa 1 : Dadas $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$ y p^m, encontrar $\widetilde{\mathbf{u}}^{m+1} : \Omega \to I\!R^3$ solución de

$$(S_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta\widetilde{\mathbf{u}}^{m+1} + \nabla p^m = \mathbf{f}^{m+1} \quad \text{en } \Omega, \\ \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0 \quad \text{sobre } \partial\Omega. \end{cases}$$

Notar que hemos escrito el término convectivo de forma semi-implícita en la velocidad intermedia, en contraposición al caso de descomposición de la viscosidad, cuya linealización se consideró con la velocidad final.

Subetapa 2 : Dadas p^m y $\widetilde{\mathbf{u}}^{m+1}$, encontrar $\mathbf{u}^{m+1} : \Omega \to I\!R^3$ y $p^{m+1} : \Omega \to I\!R$ solución de

$$(S_2)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}) + \nabla\left(p^{m+1} - p^m\right) = 0 \quad \text{en } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{en } \Omega, \quad \mathbf{u}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{array} \right.$$

La subetapa 2 es un paso de proyección. De hecho, $\mathbf{u}^{m+1} = P_{\mathbf{H}} \widetilde{\mathbf{u}}^{m+1}$ donde $P_{\mathbf{H}}$ es la L^2-proyección en $\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2 : \ \nabla \cdot \mathbf{u} = 0, \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$, porque

$$\left(\mathbf{u}^{m+1}, \nabla q\right) = 0 \quad \forall\, q \in H^1(\Omega).$$

Como es bien conocido, esta subetapa 2 es equivalente a los problemas desacoplados:

1. Encontrar $p^{m+1} : \Omega \to I\!R$ tal que

$$(S_2)_a^{m+1} \qquad \left\{ \begin{array}{l} k\,\Delta(p^{m+1} - p^m) = \nabla \cdot \widetilde{\mathbf{u}}^{m+1} \quad \text{en } \Omega \\ k\,\nabla(p^{m+1} - p^m) \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad \text{sobre } \partial\Omega. \end{array} \right.$$

2. Calcular $\mathbf{u}^{m+1}$ como:

$$(S_2)_b^{m+1} \qquad \mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m) \quad \text{en } \Omega.$$

Con respecto a la implementación del esquema, debemos introducir una presión artificial p^0 como aproximación de $p(0)$, que no es posible calcular en general. Consecuentemente, para implementar este esquema es necesario comenzar con etapas iniciales auxiliares con otro esquema. Este problema es inherente a todos los métodos incrementales de presión.

Por otro lado, sumando $(S_1)^{m+1}$ y $(S_2)^{m+1}$, obtenemos

$$(S_3)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta\widetilde{\mathbf{u}}^{m+1} + \nabla p^{m+1} = \mathbf{f}^{m+1} \quad \text{en } \Omega, \\ \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0, \quad \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{en } \Omega. \end{array} \right.$$

que puede interpretarse como unas relaciones de consistencia.

Respecto a este esquema, obtendremos resultados de dependencia continua y estabilidad incondicional.

Lema 18 *(Dependencia continua del esquema)*
a) *Dependencia continua con respecto a L^2:*
Suponiendo $\widetilde{\mathbf{u}}^{m+1}$ y $\mathbf{u}^m \in \mathbf{L}^2(\Omega)$, entonces existe una única $\mathbf{u}^{m+1} \in \mathbf{H}$. Además,

$$\|\mathbf{u}^{m+1}\|_{L^2} \le \|\widetilde{\mathbf{u}}^{m+1}\|_{L^2} \qquad y \qquad \|\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}\|_{L^2} \le \|\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m\|_{L^2}.$$

b) *Dependencia continua con respecto a H^1.*

Si $\mathbf{u}^m \in \mathbf{H}^1(\Omega) \cap \mathbf{H}$ y $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}^1_0(\Omega)$, entonces $\mathbf{u}^{m+1} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}$. Además, existe $C = C(\Omega) > 0$ tal que

$$\|\mathbf{u}^{m+1}\|_{H^1} \leq C\, \|\widetilde{\mathbf{u}}^{m+1}\|_{H^1}.$$

Lema 19 *(**Estabilidad del esquema**) Sea $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ ($\mathbf{H}^{-1}(\Omega)$ el espacio dual de $\mathbf{H}^1_0(\Omega)$) y $\mathbf{u}_0 \in \mathbf{H}$. Suponiendo la siguiente restricción sobre la presión inicial $\|k\nabla p^0\|_{L^2} \leq C_0$, entonces existe una constante $C = C(C_0, \mathbf{f}, \Omega) > 0$ tal que,*

$$\begin{aligned}
&\|\widetilde{\mathbf{u}}^{r+1}\|^2_{L^2} + \|\mathbf{u}^{r+1}\|^2_{L^2} + \|k\,\nabla p^{r+1}\|^2_{L^2} \leq C, \quad \forall\, r = 0, ..., M-1,\\
&k \sum_{m=0}^{M-1} \Big\{ \|\widetilde{\mathbf{u}}^{m+1}\|^2_{H^1} + \|\mathbf{u}^{m+1}\|^2_{H^1} \Big\} \leq C,\\
&\sum_{m=0}^{M-1} \Big\{ \|\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m\|^2_{L^2} + \|\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}\|^2_{L^2} \Big\} \leq C.
\end{aligned}$$

En la demostración de este resultado, se obtiene en primer lugar

$$\|\mathbf{u}^{m+1}\|_{l^\infty(L^2)} + \|\widetilde{\mathbf{u}}^{m+1}\|_{l^2(H^1)} + \|k\,\nabla p^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C$$

y teniendo en cuenta el Lema 18, obtenemos las estimaciones de estabilidad suplementarias

$$\|\widetilde{\mathbf{u}}^{m+1}\|_{l^\infty(L^2)} \leq C \quad \text{y} \quad \|\mathbf{u}^{m+1}\|_{l^2(H^1)} \leq C.$$

En cuanto a la convergencia de las aproximaciones en velocidades, ésta ya ha sido establecida (por ejemplo, ver [91]).

Pasamos a continuación a la obtención de las estimaciones de error (para la velocidad y presión) con respecto a una solución suficientemente regular $(\mathbf{u}, p)$ de (NS).

Introduciendo la siguientes notaciones para los errores en $t = t_{m+1}$:

$$\widetilde{\mathbf{e}}^{m+1} = \mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}^{m+1}, \quad \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad e_p^{m+1} = p(t_{m+1}) - p^{m+1},$$

y para las derivadas discretas de estos errores

$$\delta_t \mathbf{e}^{m+1} = \frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{k}, \quad \delta_t \widetilde{\mathbf{e}}^{m+1} = \frac{\widetilde{\mathbf{e}}^{m+1} - \widetilde{\mathbf{e}}^m}{k},$$

los problemas diferenciales descritos por los errores son los siguientes:

$$(E_1)^{m+1} \quad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1/2} - \mathbf{e}^m) - \Delta \widetilde{\mathbf{e}}^{m+1} + \nabla\,(e_p^m + k\,\delta_t\, p(t_{m+1})) = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{en } \Omega,\\ \widetilde{\mathbf{e}}^{m+1}|_{\partial\Omega} = \mathbf{0}, \end{array}\right.$$

$$(E_2)^{m+1} \quad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}) + \nabla\Big(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})\Big) = \mathbf{0} \quad \text{en } \Omega,\\ \nabla \cdot \mathbf{e}^{m+1} = 0 \quad \text{en } \Omega, \qquad \mathbf{e}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}. \end{array}\right.$$

donde

$$\mathcal{E}^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t-t_m)\,\mathbf{u}_{tt}(t)\,dt - \Big(\int_{t_m}^{t_{m+1}} \mathbf{u}_t\cdot\nabla\Big)\,\mathbf{u}(t_{m+1}) := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1}$$

es el error de consistencia y

$$\mathbf{NL}^{m+1} = -C\Big(\widetilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1})\Big) - C(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{e}}^{m+1}\Big) = -C\Big(\widetilde{\mathbf{e}}^m, \widetilde{\mathbf{u}}^{m+1}\Big) - C\Big(\mathbf{u}(t_m), \widetilde{\mathbf{e}}^{m+1}\Big).$$

son los términos residuales que aparecen en las diferencias de los términos cuadráticos.

De nuevo el problema $(E_2)^{m+1}$ puede ser descompuesto en los problemas siguientes:

$$(E_2)_a^{m+1} \qquad \left\{ \begin{array}{ll} k\,\Delta(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})) = \nabla\cdot\widetilde{\mathbf{e}}^{m+1} & \text{en } \Omega, \\ k\,\nabla(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1}))\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{0}, & \end{array}\right.$$

y

$$(E_2)_b^{m+1} \qquad \mathbf{e}^{m+1} = \widetilde{\mathbf{e}}^{m+1} - k\,\nabla(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})) \quad \text{en } \Omega.$$

Finalmente, sumando $(E_1)^{m+1}$ y $(E_2)^{m+1}$, llegamos a

$$(E_3)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1} - \mathbf{e}^m) - \Delta\widetilde{\mathbf{e}}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{en } \Omega, \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{en } \Omega, \qquad \mathbf{e}^{m+1}\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{0}. \end{array}\right.$$

Lema 20 *(Dependencia continua de los errores) Las siguientes desigualdades se tienen*

$$\|\mathbf{e}^{m+1}\|_{L^2} \le \|\widetilde{\mathbf{e}}^{m+1}\|_{L^2}, \qquad \|\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}\|_{L^2} \le \|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{L^2}.$$

Además, existe $C = C(\Omega) > 0$ tal que

$$\|\mathbf{e}^{m+1}\|_{H^1} \le C\,\|\widetilde{\mathbf{e}}^{m+1}\|_{H^1}.$$

Con objeto de obtener las diferentes estimaciones de error, suponemos las siguientes hipótesis de regularidad:

(H0) $\Omega \subset I\!R^3$ tal que el problema de Poisson en Ω tiene regularidad $\mathbf{H}^2(\Omega)$.

(H1) $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}), \quad p_t \in L^2(H^1), \quad \mathbf{u}_t \in L^2(\mathbf{L}^2), \quad \mathbf{u}_{tt} \in L^2(\mathbf{H}^{-1})$

(H2) $p_{tt} \in L^2(H^1), \quad \mathbf{u}_t \in L^\infty(\mathbf{L}^3) \cap L^3(\mathbf{H}^1), \quad \mathbf{u}_{tt} \in L^2(\mathbf{L}^2), \quad \mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1})$

(H3) $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1})$

Desafortunadamente, para obtener las hipótesis (**H1**)-(**H3**), es necesario suponer que $\mathbf{u}_t(0) \in \mathbf{H}^1$, que implica una condición global de compatibilidad para $\mathbf{u}_0$ y $\mathbf{f}(0)$.

El hilo conductor de la obtención de las estimaciones de error en el caso semidiscreto es el siguiente.

Obtenemos en primer lugar las estimaciones de error de orden $O(k)$ para las velocidades:

Teorema 21 *Suponiendo las hipótesis del Lema 19 y* **(H1)** *(en particular $\|\nabla e_p^0\|_{L^2} \le C$),*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k, \quad \|e_p^{m+1}\|_{l^\infty(H^1)} \le C$$

y

$$\|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{l^2(\mathbf{L}^2)} \le C\,k^{3/2}.$$

En la demostración, primero se obtiene $\|\mathbf{e}^{m+1}\|_{l^\infty(L^2)} + \|\widetilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \le C\,k$ y $\|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{l^2(L^2)} \le C\,k^{3/2}$. Las estimaciones $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)} \le C\,k$ y $\|\mathbf{e}^{m+1}\|_{l^2(H^1)} \le C$ se deducen posteriormente teniendo en cuenta el Lema 20.

Notemos que la estimación $\|\widetilde{\mathbf{e}}^m\|_{l^2(H^1)} \le C\,k$ implica que $\|\widetilde{\mathbf{e}}^m\|_{H^1} \le C$ para cada m.

Ahora, como consecuencia del resultado anterior y la regularidad H^2 del problema de Poisson $(E_1)^{m+1}$, podemos obtener el siguiente resultado:

Lema 22 *Bajo las hipótesis del Teorema 21, se tiene*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C, \qquad \forall m.$$

Podemos pasar entonces a la obtención de las estimaciones de orden $O(k)$ para la presión. Para ello, primero obtenemos las estimaciones de error para las derivadas discretas en tiempo de las velocidades. Para estas derivadas se tiene también el resultado de dependencia continua:

Lema 23 *(Dependencia continua de las derivadas discretas) Las siguientes estimaciones se tienen*

$$\|\delta_t\mathbf{e}^{m+1}\|_{L^2} \le \|\delta_t\widetilde{\mathbf{e}}^{m+1}\|_{L^2}, \qquad \|\delta_t\mathbf{e}^{m+1} - \delta_t\widetilde{\mathbf{e}}^{m+1}\|_{L^2} \le \|\delta_t\widetilde{\mathbf{e}}^{m+1} - \delta_t\mathbf{e}^m\|_{L^2}.$$

Además, existe $C = C(\Omega) > 0$ tal que

$$\|\delta_t\mathbf{e}^{m+1}\|_{H^1} \le C\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|_{H^1}.$$

Este lema será aplicado en la demostración del siguiente resultado

Teorema 24 *Suponiendo las hipótesis del Teorema 21,* (**H2**) *y la siguiente restricción sobre la etapa inicial*

$$\|\delta_t\mathbf{e}^1\|_{L^2} + \|k\,\nabla\delta_t e_p^1\|_{L^2} \le C\,k,$$

entonces se obtiene

$$\|\delta_t\mathbf{e}^{m+1}\|_{l^\infty(L^2)} + \|\delta_t\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C\,k$$

Finalmente, podemos obtener el orden óptimo para las estimaciones del error de la presión.

Teorema 25 *Suponiendo las hipótesis del Teorema 24 y* (**H3**)*, se obtienen las siguientes estimaciones de error*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} + \|e_p^{m+1}\|_{l^\infty(L^2)} \leq C\,k.$$

La demostración es una prueba a tres pasos. Primero, del Teorema 24 y la condición inf-sup aplicada a $(E_3)^{m+1}$, obtenemos $\|e_p^m\|_{l^2(L^2)} \leq C\,k$. Después, usando de nuevo el problema $(E_3)^{m+1}$, llegamos a $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} \leq C\,k$ y finalmente obtenemos $\|e_p^{m+1}\|_{l^\infty(L^2)} \leq C\,k$.

Esquema completamente discreto

Consideramos una aproximación de elementos finitos de los problemas semidiscretos $(S_1)^{m+1}$ y $(S_2)^{m+1}$. Consideramos dos espacios de elementos finitos $\mathbf{Y}_h \subset \mathbf{H}_0^1(\Omega)$ y $Q_h \subset H^1(\Omega) \cap L_0^2(\Omega)$ asociados a una familia regular de triangulaciones del dominio Ω. $\mathbf{Y}_h$ y Q_h son funciones globalmente continuas y localmente polinomios de grado al menos 1. Además, supondremos:

- La condición "*inf-sup*" estándar ([39]) para $(\mathbf{Y}_h, Q_h)$
- Las siguientes propiedades de los operadores de interpolación:
  - $I_h : \mathbf{L}^2 \to \mathbf{Y}_h$ tal que
  $$(\mathbf{u} - I_h\mathbf{u}, \nabla q_h) = 0, \quad \forall\, q_h \in Q_h,$$
  verificando
  $$\|\widetilde{\mathbf{u}} - I_h\widetilde{\mathbf{u}}\|_{H^{-1}} \leq C\,h\,\|\widetilde{\mathbf{u}} - I_h\widetilde{\mathbf{u}}\|_{L^2} \quad \forall\,\widetilde{\mathbf{u}} \in \mathbf{L}^2(\Omega),$$
  $$\|\widetilde{\mathbf{u}} - I_h\widetilde{\mathbf{u}}\|_{H^1} \leq C\,h\,\|\widetilde{\mathbf{u}}\|_{H^2} \quad \forall\,\widetilde{\mathbf{u}} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega),$$
  - $J_h : H^1 \to Q_h$ definido por
  $$(\nabla(J_h p - p), \nabla q_h) = 0 \quad \forall\, q_h \in Q_h,$$
  y verificando
  $$\|p - J_h p\|_{L^2} \leq C\,h\,\|p - J_h p\|_{H^1} \leq C\,h\,\|p\|_{H^1} \quad \forall\, p \in H^1(\Omega) \cap L_0^2(\Omega).$$

Ahora, para $\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m)$, definimos $K_{h,k}\mathbf{u}^{m+1} \in \mathbf{Y}_h + \nabla Q_h$ como:

$$K_{h,k}\mathbf{u}^{m+1} = I_h\widetilde{\mathbf{u}}^{m+1} - k\,\nabla J_h(p^{m+1} - p^m).$$

Entonces

$$\|\mathbf{u}^{m+1} - K_{h,k}\mathbf{u}^{m+1}\|_{L^2} \leq C\left(h\,\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} + k\,\|p^{m+1} - p^m\|_{H^1}\right) \leq C(k+h) \quad \forall m.$$

Finalmente, debemos suponer la siguiente restricción entre el paso de tiempo k y el tamaño de la malla h:

(H) Existe una constante $\alpha > 0$ (independiente de k y h) tal que $\dfrac{h}{k} \leq \alpha$.

El esquema totalmente discreto está descrito como sigue

Inicialización: Sea $(\widetilde{\mathbf{u}}_h^0, p_h^0) \in \mathbf{Y}_h \times Q_h$ una aproximación de $(\mathbf{u}^0, p^0)$. Tomar $\mathbf{u}_h^0 = \widetilde{\mathbf{u}}_h^0$.

Paso de tiempo $m+1$:

Subetapa 1: Dada $(\widetilde{\mathbf{u}}_h^m, p_h^m) \in \mathbf{Y}_h \times Q_h$ y $\mathbf{u}_h^m \in \mathbf{Y}_h + \nabla Q_h$, encontrar $\widetilde{\mathbf{u}}_h^{m+1} \in \mathbf{Y}_h$ tal que,

$(S_1)_h^{m+1}$

$$\frac{1}{k}(\widetilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m, \mathbf{v}_h) + c(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h) + (\nabla \widetilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h) + (\nabla p_h^m, \mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h).$$

Subetapa 2: Dada $(\widetilde{\mathbf{u}}_h^{m+1}, p_h^m) \in \mathbf{Y}_h \times Q_h$, encontrar $p_h^{m+1} \in Q_h$ tal que

$$(S_2)_{a,h}^{m+1} \qquad (k\,\nabla(p_h^{m+1} - p_h^m), \nabla q_h) = (\widetilde{\mathbf{u}}_h^{m+1}, \nabla q_h) \quad \forall q_h \in Q_h.$$

Ahora, definimos una velocidad final auxiliar $\mathbf{u}_h^{m+1} \in \mathbf{Y}_h + \nabla Q_h$ como

$$(S_2)_{b,h}^{m+1} \qquad \mathbf{u}_h^{m+1} = \widetilde{\mathbf{u}}_h^{m+1} - k\,\nabla(p_h^{m+1} - p_h^m).$$

Observemos que $\mathbf{u}_h^{m+1}$ verifica la propiedad de ortogonalidad en L^2:

$$(\mathbf{u}_h^{m+1}, \nabla q_h) = 0 \quad \forall\, q_h \in Q_h. \tag{10}$$

Finalmente, sumando las dos subetapas

$$(S_3)_h^{m+1} \quad \frac{1}{k}(\mathbf{u}_h^{m+1} - \mathbf{u}_h^m, \mathbf{v}_h) + c(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h) + (\nabla\widetilde{\mathbf{u}}_h^{m+1}, \nabla\mathbf{v}_h) + (\nabla p_h^{m+1}, \mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h)$$

La introducción de la velocidad final $\mathbf{u}_h^m$ no es necesaria, porque en la práctica la implementación del esquema puede realizarse como sigue: Dada $(p_h^{m-1}, \widetilde{\mathbf{u}}_h^m) \in Q_h \times \mathbf{Y}_h$,

(a) Encontrar $p_h^m \in Q_h$ tal que

$$(k\,\nabla(p_h^m - p_h^{m-1}), \nabla q_h) = (\widetilde{\mathbf{u}}_h^m, \nabla q_h) \quad \forall\, q_h \in Q_h.$$

(b) Encontrar $\widetilde{\mathbf{u}}_h^{m+1} \in \mathbf{Y}_h$ tal que $\forall\, \mathbf{v}_h \in \mathbf{Y}_h$:

$$\Big(\frac{\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}_h^m}{k}, \mathbf{v}_h\Big) + c(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h) + (\nabla \widetilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h) + (\nabla(2p_h^m - p_h^{m-1}), \mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h).$$

De nuevo, (a) es un problema de Poisson para la presión y (b) es un problema de convección-difusión para la velocidad (el cuál es también desacoplado en las componentes de $\widetilde{\mathbf{u}}_h^{m+1}$).

Notemos que tenemos que comenzar con una presión p_h^{-1} (para $m = 0$), que no tiene sentido. Por esto, o bien comenzamos con iteraciones previas con otro esquema o se comienza con un primer paso escrito como antes, i.e., con $\widetilde{\mathbf{u}}_h^0$, p_h^0 y $\mathbf{u}_h^0 = \widetilde{\mathbf{u}}_h^0$, suponiendo conocida una aproximación de la presión inicial p_h^0.

Es fácil extender los resultados obtenidos en la semi-discretización en tiempo ahora al caso totalmente discreto. Así, tenemos el resultado de dependencia continua de las velocidades. De $(S_2)_{b,h}^{m+1}$ y (10), obtenemos

$$\|\mathbf{u}_h^{m+1}\|_{L^2} \leq \|\widetilde{\mathbf{u}}_h^{m+1}\|_{L^2}.$$

Por otra parte, de $(S_2)_{a,h}^{m+1}$ obtenemos

$$\|\mathbf{u}_h^{m+1} - \widetilde{\mathbf{u}}_h^{m+1}\|_{L^2} \leq \|\widetilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m\|_{L^2}.$$

Además, usando la propiedad antisimétrica $c(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \widetilde{\mathbf{u}}_h^{m+1}) = 0$, podemos extender los resultados de estabilidad y convergencia para cada $r < N$:

$$\|\mathbf{u}_h^{r+1}\|_{l^\infty(L^2)} + \|\widetilde{\mathbf{u}}_h^{r+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|k\,\nabla p_h^{r+1}\|_{l^\infty(L^2)} \leq C$$
$$\sum_{m=0}^{r} \|\widetilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m\|_{L^2}^2 + \sum_{m=0}^{r} \|\mathbf{u}_h^{m+1} - \widetilde{\mathbf{u}}_h^{m+1}\|_{L^2}^2 \leq C$$

Presentamos a continuación el análisis de error para el esquema completamente discreto $(\widetilde{\mathbf{u}}_h^{m+1}, \mathbf{u}_h^{m+1}, p_h^{m+1})$ como aproximación al esquema semidiscreto en tiempo $(\widetilde{\mathbf{u}}^{m+1}, \mathbf{u}^{m+1}, p^{m+1})$. Consideramos entonces los siguientes errores:

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \qquad \widetilde{\mathbf{e}}_d^{m+1} = \widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}_h^{m+1}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}$$

Estos errores pueden ser descompuestos en sus partes discretas y de interpolación:

$$\mathbf{e}_d^{m+1} = \mathbf{e}_h^{m+1} + \mathbf{e}_i^{m+1}, \qquad \widetilde{\mathbf{e}}_d^{m+1} = \widetilde{\mathbf{e}}_h^{m+1} + \widetilde{\mathbf{e}}_i^{m+1}, \qquad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1}$$

siendo $\mathbf{e}_i$ errores de interpolación y $\mathbf{e}_h$ errores discretos espaciales, concretamente

$$\mathbf{e}_h^{m+1} = K_{h,k}\mathbf{u}^{m+1} - \mathbf{u}_h^{m+1} \text{ y } \mathbf{e}_i^{m+1} = \mathbf{u}^{m+1} - K_{h,k}\mathbf{u}^{m+1},$$

$$\widetilde{\mathbf{e}}_h^{m+1} = I_h\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}_h^{m+1} \text{ y } \widetilde{\mathbf{e}}_i^{m+1} = \widetilde{\mathbf{u}}^{m+1} - I_h\widetilde{\mathbf{u}}^{m+1},$$

$$e_{p,h}^{m+1} = J_h p^{m+1} - p_h^{m+1} \text{ y } e_{p,i}^{m+1} = p^{m+1} - J_h p^{m+1}.$$

De las igualdades $\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m)$ y $K_{h,k}\mathbf{u}^{m+1} = I_h\widetilde{\mathbf{u}}^{m+1} - k\,\nabla\,J_h(p^{m+1} - p^m)$, tenemos

$$\mathbf{e}_i^{m+1} = \widetilde{\mathbf{e}}_i^{m+1} - k\,\nabla(e_{p,i}^{m+1} - e_{p,i}^m). \tag{11}$$

En particular, usando esta igualdad reemplazando m por $m-1$,

$$\frac{1}{k}(\widetilde{\mathbf{e}}_i^{m+1} - \mathbf{e}_i^m) = e_i(\delta_t\widetilde{\mathbf{u}}^{m+1}) + \nabla(e_{p,i}^m - e_{p,i}^{m-1}). \tag{12}$$

Además, de la elección de los operadores de interpolación I_h y J_h,

$$\left(\mathbf{e}_i^{m+1}, \nabla q_h\right) = \left(\widetilde{\mathbf{e}}_i^{m+1}, \nabla q_h\right) - k\left(\nabla(e_{p,i}^{m+1} - e_{p,i}^m), \nabla q_h\right) = 0, \quad \forall\, q_h \in Q_h. \tag{13}$$

Por otro lado, de $(\mathbf{u}_h^{m+1}, \nabla q_h) = 0$, $\forall\, q_h \in Q_h$ y $(\mathbf{u}^{m+1}, \nabla q_h) = 0$, $\forall\, q_h \in H^1 \cap L^2_0$, entonces

$$(\mathbf{e}_d^{m+1}, \nabla q_h) = 0 \quad \forall q_h \in Q_h. \tag{14}$$

Finalmente, de (13) y (14), llegamos a

$$(\mathbf{e}_h^{m+1}, \nabla q_h) = 0 \quad \forall q_h \in Q_h.$$

Todo esto hará que, comparando $(S_1)^{m+1}, (S_2)^{m+1}$ y $(S_1)_h^{m+1}, (S_2)_{b,h}^{m+1}$, descomponiendo el error en sus partes discretas y de interpolación, y usando (11)-(12), los problemas que verifican los errores sean:

$$(E_1)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\, \widetilde{\mathbf{e}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla e_{p,h}^m, \mathbf{v}_h\Big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(e_i(\delta_t \widetilde{\mathbf{u}}^{m+1}), \mathbf{v}_h\Big) - \Big(\nabla \widetilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(\nabla(2\, e_{p,i}^m - e_{p,i}^{m-1}), \mathbf{v}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h \end{array} \right.$$

$$(E_2)_h^{m+1} \qquad \mathbf{e}_h^{m+1} = \widetilde{\mathbf{e}}_h^{m+1} - k\nabla(e_{p,h}^{m+1} - e_{p,h}^m).$$

Finalmente, sumando $(E_1)_h^{m+1}$ y $(E_2)_h^{m+1}$,

$$(E_3)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\, \widetilde{\mathbf{e}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla e_{p,h}^m, \mathbf{v}_h\Big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(e_i\Big(\delta_t \widetilde{\mathbf{u}}^{m+1}\Big), \mathbf{v}_h\Big) - \Big(\nabla \widetilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(\nabla(2\, e_{p,i}^m - e_{p,i}^{m-1}), \mathbf{v}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h. \end{array} \right.$$

Obtenemos a continuación las estimaciones de error de orden $O(h)$ para $\widetilde{\mathbf{e}}_h^{m+1}$ en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ y para $\mathbf{e}_h^{m+1}$ en $l^\infty(\mathbf{L}^2)$.

Teorema 26 *Suponiendo las hipótesis del Teorema 21,* $\|\mathbf{e}_h^0\|_{L^2} \leq C\, h$ *y* $\|k\nabla e_{p,h}^0\|_{L^2} \leq C\, h$, *entonces se tienen las siguientes estimaciones de error*

$$\|\widetilde{\mathbf{e}}_h^{m+1}\|^2_{l^\infty(L^2)\cap l^2(H^1)} + \|\mathbf{e}_h^{m+1}\|^2_{l^\infty(L^2)} + \|k\, \nabla e_{p,h}^{m+1}\|^2_{l^\infty(L^2)} \leq C\, h^2$$

$$\|\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m\|^2_{l^2(L^2)} \leq C\, k\, h^2.$$

Una consecuencia de este resultado y la restricción $(\mathbf{H})$, es la siguiente estimación en norma más regular que nos mejora la estimación de estabilidad para $\widetilde{\mathbf{u}}_h^{m+1}$:

Corolario 27 *Suponiendo las hipótesis del Teorema 26,* $(\mathbf{H})$ *y* $\|\mathbf{u}_h^0\|^2_{W^{1,6}} \leq C_0$ *(esto es,* $\mathbf{u}_0 \in \mathbf{W}^{1,6}$*), se tiene*

$$\widetilde{\mathbf{u}}_h^{m+1} \textit{ está acotada en } l^\infty(\mathbf{W}^{1,6}).$$

Seguidamente, obtenemos las estimaciones de error de orden $O(h)$ para $\delta_t \mathbf{e}_h^{m+1}$ en $l^\infty(\mathbf{L}^2)$, $\delta_t \widetilde{\mathbf{e}}_h^{m+1}$ en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ y $(\mathbf{e}_h^{m+1}, e_{p,d}^{m+1})$ en $l^\infty(\mathbf{H}^1 \times L^2)$.

Para ello, haciendo $\delta_t(E_1)_h^{m+1}$ y $\delta_t(E_2)_h^{m+1}$ y debido a la elección de los operadores de interpolación, se tiene $\forall m \geq 1$:

$$(D_1)_h^{m+1} \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\delta_t\widetilde{\mathbf{e}}_h^{m+1} - \delta_t\mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\delta_t\widetilde{\mathbf{e}}_h^{m+1}, \nabla\,\mathbf{v}_h\Big) + \Big(\nabla\delta_t e_{p,h}^m, \mathbf{v}_h\Big) = \delta_t\,\mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(\mathbf{e}_i(\delta_t\delta_t\widetilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\Big) - \Big(\nabla\delta_t\widetilde{\mathbf{e}}_i^{m+1}, \nabla\mathbf{v}_h\Big) - \Big(\nabla(2\,\delta_t e_{p,i}^m - \delta_t e_{p,i}^{m-1}), \mathbf{v}_h\Big) \quad \forall\,\mathbf{v}_h \in \mathbf{Y}_h \end{array} \right.$$

$$(D_2)_h^{m+1} \qquad \delta_t\mathbf{e}_h^{m+1} = \delta_t\widetilde{\mathbf{e}}_h^{m+1} - k\nabla(\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m).$$

De nuevo, se tiene la propiedad de ortogonalidad

$$\Big(\delta_t\mathbf{e}_h^{m+1}, \nabla q_h\Big) = 0, \quad \forall\, q_h \in Q_h.$$

El resultado correspondiente para las estimaciones de error de las derivadas discretas es el siguiente:

Teorema 28 *Suponiendo las hipótesis de los Teoremas 24 y 26, y suponiendo la siguiente hipótesis para la etapa inicial del esquema*

$$\|\delta_t\mathbf{e}_h^1\|_{L^2} + \|k\,\nabla\delta_t e_{p,h}^1\|_{L^2} \leq C\,h,$$

entonces

$$\|\delta_t\mathbf{e}_h^{m+1}\|_{l^\infty(L^2)} + \|\delta_t\widetilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|k\,\delta_t\nabla e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq C\,h.$$

Finalmente, con un razonamiento análogo al caso semidiscreto en tiempo obtenemos

Corolario 29 *Suponiendo las hipótesis del Teorema 28, las siguientes estimaciones de error se consiguen*

$$\|e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq C\,h \quad \text{y} \quad \|\mathbf{e}_h^{m+1}\|_{l^\infty(H^1)} \leq C\,h.$$

Por tanto, combinando el Teorema 25 y el Corolario 29, tenemos las siguientes estimaciones de los errores totales

$$\|\mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}_h^{m+1}\|_{l^\infty(H^1)} + \|p(t_{m+1}) - p_h^{m+1}\|_{l^\infty(L^2)} \leq C\,(k+h).$$

4.1.4. Conclusiones

Gracias a los resultados de los tres primeros capítulos de la memoria, podemos comparar el esquema de proyección con corrección de presión (incremental) correspondiente al Capítulo 3 y el esquema con descomposición de la viscosidad estudiado en los Capítulos 1 y 2.

En este sentido, en el método de la descomposición de la viscosidad, usando una formulación mixta en velocidad-presión, se ha conseguido estabilidad, convergencia y estimaciones óptimas de error (en norma $l^\infty(L^2)$ para la presión) partiendo de una inicialización buena de la velocidad del esquema y sin imponer una estimación inicial para la presión. Esto se debe a que en el esquema las condiciones de contorno se verifican de forma exacta. Además, con este método, ha sido posible emplear un razonamiento de dualidad que conduce a la estimación óptima de la presión en norma $l^2(L^2)$, sin imponer la hipótesis para la velocidad en el primer paso del esquema, pero suponiendo k suficientemente pequeño.

En cuanto al método de segregación de la presión (basado en un esquema de proyección con presión incremental), se ha conseguido estabilidad, convergencia y estimaciones óptimas de error pero partiendo de una buena estimación de la velocidad y presión inicial. Además, en este esquema no está claro como razonar con un argumento dual, para evitar la estimación inicial de la velocidad. Por otra parte, la implementación de este método tiene menos costo computacional que el primero, debido a que los problemas para la velocidad y presión están desacoplados.

Decir también que estos esquemas con descomposición de la viscosidad y proyección incremental, conservan los mismos resultados analíticos de los esquemas de tipo Euler [91], con la diferencia de que en los esquemas tipo Euler se obtienen estimaciones óptimas de error para la presión suponiendo soluciones menos regulares, si bien están también condicionadas por la compatibilidad global de la presión inicial. En contrapartida, el tratamiento numérico ha sido mejorado con los esquemas de descomposición de la viscosidad y segregación de la presión, debido a que las principales dificultades del problema (NS) se han separado.

4.2. Las Ecuaciones Primitivas

Respecto a las EP, pretendemos hacer un análisis numérico con los mismos tipos de esquemas que hemos usado en la presente memoria para abordar el problema de Navier-Stokes, a saber, el método de descomposición de la viscosidad (Capítulo 4) y el método de proyección con corrección de la presión (Capítulo 5).

En ambos casos, descomposición de la viscosidad y proyección, la diferencia y por tanto la problemática fundamental que presentan las EP con respecto al caso de las ecuaciones de NS, es la pérdida de regularidad de la velocidad vertical. Esto obligará a usar técnicas nuevas a las empleadas en el caso de Navier-Stokes, para controlar principalmente los términos convectivos. Así, el uso de estimaciones anisótropas, las cuales emplean espacios distintos en velocidad vertical y horizontal, la suposición de soluciones más regulares y restricciones de distinto tipo nos aparecerán ahora.

En efecto, la principal dificultad que nos encontramos para abordar el problema es la falta de regularidad de la velocidad vertical con respecto a la velocidad horizontal. Así, de

$$u_3(t;\mathbf{x},z) = \int_z^0 \nabla_H \cdot \mathbf{u}(t;\mathbf{x},s)\,ds$$

la velocidad vertical u_3 se obtiene en función de $\nabla_H \cdot \mathbf{u}$. En este proceso, no tenemos la regularidad L^2 para la derivadas horizontales de u_3, por lo que debemos definir el espacio de Hilbert (anisótropo):

$$H(\partial_z) = \{v \in L^2(\Omega) \ / \ \partial_z v \in L^2(\Omega)\}, \quad (\text{resp. } H^k(\partial_z) = \{v \in H^k(\Omega) \ / \ \partial_z v \in H^k(\Omega)\})$$

y $H_0(\partial_z) = \{v \in H(\partial_z) \ / \ v = 0 \text{ sobre } \Gamma_s \cup \Gamma_b\}$.

Debido a esta pérdida de regularidad de u_3 ($u_3 \in L^2$ pero $u_3 \notin H^1$), el término vertical convectivo $u_3 \partial_z \mathbf{u}$ no pertenece a $\mathbf{H}^{-1}(\Omega)$, por lo que debemos introducir funciones test más regulares en la formulación variacional de (R). En este sentido, es suficiente con tomar por ejemplo $\mathbf{v} \in \mathbf{H}^1(\Omega)$ tal que $\partial_z \mathbf{v} \in \mathbf{L}^3(\Omega)$, porque en este caso tenemos (ver [18]):

$$\Big\langle (\mathbf{U} \cdot \nabla)\mathbf{u}, \mathbf{v} \Big\rangle_\Omega = -\Big((\mathbf{U} \cdot \nabla)\mathbf{v}, \mathbf{u} \Big) < +\infty.$$

Usaremos la forma trilineal antisimétrica $c(\cdot,\cdot,\cdot)$ para el tratamiento del término convectivo. Entonces consideramos la siguiente formulación variacional de (R):

$\mathbf{U} = (\mathbf{u}, u_3) \in L^2(0,T; \mathbf{V} \times H_0(\partial_z))$ y $\mathbf{u} \in L^\infty(0,T;\mathbf{H})$ que satisface c.p.d. $t \in (0,T)$:

$$(R)_w \qquad \begin{cases} \Big(\mathbf{u}_t, \mathbf{v}\Big) + c\Big(\mathbf{U}, \mathbf{u}, \mathbf{v}\Big) + \nu\Big(\nabla \mathbf{u}, \nabla \mathbf{v}\Big) \\ \qquad = \Big\langle \mathbf{f}, \mathbf{v} \Big\rangle_\Omega + \Big\langle \mathbf{g}_s, \mathbf{v} \Big\rangle_{\Gamma_s}, \quad \forall\, \mathbf{v} \in \mathbf{V} \cap \mathbf{W}^{1,3}_{b,l} \cap \mathbf{L}^\infty, \\ \Big(\nabla_H \cdot \langle \mathbf{u} \rangle, q\Big)_S = 0, \quad \forall\, q \in L^2_0(S), \\ \Big(\partial_z u_3, \partial_z w\Big) = -\Big(\nabla_H \cdot \mathbf{u}, \partial_z w\Big), \quad \forall\, w \in H_0(\partial_z). \end{cases}$$

Introducimos algunos espacios anisótropos junto con estimaciones propias de dichos espacios. Dados $p, q \in [1, +\infty]$, una función u pertenece a $L^q_z L^p_{\mathbf{x}}(\Omega)$ si:

$$u(\cdot, z) \in L^p(S_z) \quad \text{y} \quad \|u(\cdot, z)\|_{L^p(S_z)} \in L^q(-D_{\max}, 0),$$

donde $S_z = \{\mathbf{x} \in S \ : \ (\mathbf{x}, z) \in \Omega\}$ y su norma viene dada por $\Big\| \|u(\cdot,z)\|_{L^p(S_z)} \Big\|_{L^q(-D_{\max},0)}$.

Sin pérdida de generalidad, denotamos $L^q_z L^p_{\mathbf{x}}$ en vez $L^q_z L^p_{\mathbf{x}}(\Omega)$, y L^p en vez de $L^p(\Omega)$. De forma similar definimos los espacios $H^1_z L^2_{\mathbf{x}} \equiv H^1(-D_{max}, 0; L^2(S_z))$, $\quad L^2_z H^1_{\mathbf{x}} \equiv L^2(-D_{max}, 0; H^1(S_z))$.. Notemos que $H^1_z L^2_{\mathbf{x}} = H(\partial_z)$.

Usaremos frecuentemente las siguientes desigualdades (ver [50]):

- Desigualdad horizontal de Gagliardo-Nirenberg:

$$\|u\|_{L^2_z L^4_{\mathbf{x}}} \le C \, \|u\|^{1/2}_{L^2} \|\nabla_H u\|^{1/2}_{L^2} \quad \forall\, u \in L^2_z H^1_{\mathbf{x}} \ \text{ tal que } u|_{\Gamma_b \cup \Gamma_l} = 0,$$

$$\|u\|_{L^2_z L^4_{\mathbf{x}}} \le C \, \|u\|^{1/2}_{L^2} \, \|u\|^{1/2}_{H^1} \quad \forall\, u \in H^1$$

- Desigualdad vertical de Poincaré:

$$\|v\|_{L^2} \le D^{1/2}_{max} \, \|\partial_z v\|_{L^2}, \quad \forall\, v \in H^1_z L^2_{\mathbf{x}} \ \text{ tal que } v|_{\Gamma_b} = 0 \ \text{ o } v|_{\Gamma_s} = 0. \tag{15}$$

- Desigualdad vertical de Gagliardo-Nirenberg:

$$\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \leq C\,(\|v\|_{L^2} + \|v\|^{1/2}_{L^2}\|\partial_z v\|^{1/2}_{L^2}), \quad \forall\, v \in H^1_z L^2_{\mathbf{x}}. \tag{16}$$

  Además, si $v|_{\Gamma_b} = 0$ o $v|_{\Gamma_s} = 0$, $\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \leq C\,\|v\|^{1/2}_{L^2}\|\partial_z v\|^{1/2}_{L^2}$.

  En particular, de (15) y (16), se tiene

$$\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \leq C\,\|\partial_z v\|_{L^2}, \quad \forall\, v \in H^1_z L^2_{\mathbf{x}} \text{ tal que } v|_{\Gamma_b} = 0 \text{ o } v|_{\Gamma_s} = 0. \tag{17}$$

En el **Capítulo 4** abordamos el método de descomposición de la viscosidad, pretendiendo obtener estimaciones de error óptimas para la presión; pero con las dificultades inherentes al problema de EP, no podremos hacer un razonamiento como el que hemos realizado para el caso de NS, ya que no podemos utilizar el problema semidiscreto en tiempo como un problema auxiliar para la obtención de dichas estimaciones de error. Concretamente, para la velocidad intermedia $\mathbf{u}_h^{m+1/2}$, que aparece en los términos convectivos, no podemos obtener una estimación con orden en $l^\infty(\mathbf{H}^1)$, lo que lleva a que no podamos conseguir una estimación para esta velocidad en $l^\infty(\mathbf{H}^2)$.

Por tanto, debemos comparar la solución exacta directamente con un esquema totalmente discreto. Se diseña entonces un esquema numérico para (R) basado en un esquema de descomposición de la viscosidad en tiempo y una aproximación espacial de elementos finitos, obteniendo estabilidad y convergencia, cuando $(k,h) \to 0$, hacia una solución débil de (R) y estimaciones de error respecto de una solución suficientemente regular de (R). En particular, el resultado de convergencia de este capítulo (ver Teorema 31 más adelante) puede ser interpretado como una nueva demostración de la existencia de solución débil de (R), en dominios sin talud.

Básicamente, en cada paso de tiempo m, tres subproblemas deben ser resueltos. Primero, dadas $(\mathbf{u}_h^m, p_{s,h}^m)$, calculamos la velocidad vertical $u_{3,h}^m$ en función de $\nabla_H \cdot \mathbf{u}_h^m$, después, obtenemos una velocidad intermedia $\mathbf{u}_h^{m+1/2}$ usando los términos convectivos pero sin considerar la restricción de tipo divergencia. Finalmente, obtenemos $\mathbf{u}_h^{m+1}$ y $p_{s,h}^{m+1}$ resolviendo un problema lineal tipo Stokes considerando la restricción $\nabla_H \cdot \langle \mathbf{u}_h^{m+1} \rangle = 0$ (los términos difusivos serán considerados en ambos pasos).

Después, suponiendo la existencia de una única solución suficientemente regular, obtendremos diferentes estimaciones de error tanto para las velocidades como para la presión. Concretamente, obtendremos en primer lugar, para $l = 1, 2$ (donde l es el orden de aproximación de los elementos finitos), estimaciones de error de orden $O(\sqrt{k} + h^l)$ para las velocidades $\mathbf{u}_h^{m+1/2}$ y $\mathbf{u}_h^{m+1}$, estimaciones de error mejoradas (de orden $O(k + h^l)$) para la velocidad final $\mathbf{u}_h^{m+1}$ y estimaciones de error de orden $O(\sqrt{k} + h^l)$ para la derivada discreta de la velocidad final en $l^2(\mathbf{L}^2)$, que conducen a estimaciones de error de orden $O(\sqrt{k} + h^l)$ para la presión $p_{s,h}^{m+1}$. Seguidamente, sólo para $l = 2$, obtendremos estimaciones de error de orden $O(\sqrt{k} + h^2)$ para las derivadas discretas de las velocidades $\mathbf{u}_h^{m+1/2}$ y $\mathbf{u}_h^{m+1}$ y estimaciones mejoradas (de orden $O(k+h^2)$) para la derivada discreta de la velocidad final $\mathbf{u}_h^{m+1}$, que conducen a estimaciones de error de orden $O(k+h^2)$ para la presión $p_{s,h}^{m+1}$. Finalmente, considerando una específica malla estructurada en vertical, obtendremos estimaciones de error de orden $O(k+h^{l+1})$ para la velocidad final $\mathbf{u}_h^{m+1}$ en $l^2(\mathbf{L}^2)$. Como consecuencia de este resultado, podremos ahora obtener estimaciones de error de orden $O(k+h)$ para $p_{s,h}$ en $l^2(L^2)$ en el caso $l=1$.

En el **Capítulo 5**, nos centraremos en el esquema semidiscreto en tiempo de proyección con corrección de la presión como aproximación al problema de EP, en la versión (Q). La diferencia fundamental con respecto al caso con descomposición de la viscosidad, es que ahora sí podemos obtener estimaciones de error para el esquema semidiscreto en tiempo, que pueden servir como etapa intermedia para la obtención de las estimaciones de los errores totales. En este sentido, se consigue una estimación de la velocidad $\widetilde{\mathbf{u}}^{m+1}$ en $l^\infty(\mathbf{H}^2)$, que nos permite abordar con éxito la obtención de las estimaciones de error óptimas.

El esquema semidiscreto en tiempo está descrito como sigue. Básicamente, en cada paso de tiempo m, deben ser resueltos tres subproblemas. Primero, se calcula una aproximación u_3^m para la velocidad vertical, después calcularemos una aproximación a la velocidad horizontal $\mathbf{u}$ en $t=t_{m+1}$ (que llamaremos $\widetilde{\mathbf{u}}^{m+1}$) y finalmente, calculamos una aproximación final $\mathbf{u}^{m+1}$ para la velocidad horizontal y una aproximación p_s^{m+1} para la presión p_s en $t=t_{m+1}$.

Para este esquema semidiscreto comenzaremos obteniendo resultados de estabilidad y convergencia hacia una solución débil del problema (Q).

Seguidamente, suponiendo la existencia de una solución suficientemente regular de (Q), obtenemos diferentes estimaciones de error. En este sentido, obtendremos primero estimaciones de error para las velocidades $\widetilde{\mathbf{u}}^{m+1}$ y $\mathbf{u}^{m+1}$ de orden $O(k)$, después estimaciones de orden $O(k)$ para las derivadas discretas de las velocidades y finalmente estimaciones óptimas del error de presión p_s^{m+1} de orden $O(k)$.

4.2.1. Capítulo 4

El propósito de este capítulo es diseñar un esquema numérico para (R) basado en un método de tipo splitting con descomposición de la viscosidad y Elementos Finitos en espacio. Para este esquema, demostraremos primero la estabilidad incondicional y la convergencia hacia una solución débil de (R) y, después, obtendremos estimaciones de error respecto a una solución suficientemente regular de (R), suponiendo la restricción $k \le C\,h^2$.

Desde el punto de vista del análisis numérico, en [18], se ha demostrado la convergencia de algunos esquemas de Elementos Finitos referidos al problema estacionario relativo a (Q), donde aparece la llamada *condición inf-sup hidrostática*.

Como ya hemos comentado, la extensión de los resultados obtenidos en el caso de Navier-Stokes, no son ahora en absoluto directos, ya que las dificultades que nos encontramos en las

Ecuaciones Primitivas hacen que debamos trabajar con otro tipo de espacios y formulaciones, no hilbertianas, anisótropas, y un cambio de técnicas para llegar a la obtención de las estimaciones de error óptimas, en situaciones diferentes que en el caso de NS. En este sentido, no es válido el razonamiento realizado para NS (que usa la semidiscretización en tiempo como un problema auxiliar para obtener las estimaciones de los errores totales), debido a que la falta de regularidad de la velocidad vertical obligaría en este caso a imponer hipótesis (inadmisibles) para el esquema, para controlar y acotar los términos convectivos. Por tanto, la idea ahora es discretizar el problema totalmente, y usar convenientes desigualdades inversas para acotar con suficiente orden los términos convectivos; esto nos conducirá a una restricción para los parámetros h (tamaño de la malla) y k (paso de tiempo) contraria a la que nos apareció en el Capítulo 2, en el sentido de que allí dicha restricción era $h^2 \leq \alpha\, k$ y ahora es $k \leq \alpha\, h^2$.

Vamos a considerar entonces una aproximación de elementos finitos del problema (R). Sean tres espacios de elementos finitos; $\mathbf{X}_h \subset \mathbf{H}^1_{b,l}(\Omega)$ para la velocidad horizontal, $Y_h \subset H_0(\partial_z)$ para la velocidad vertical y $Q_h \subset L^2_0(S)$ para la presión, asociados a una familia de triangulaciones del dominio Ω, que asumiremos regular y quasi-uniforme. No imponemos ninguna estructura especial para las triangulaciones, es decir, suponemos una malla no estructurada en general. Las funciones de $\mathbf{X}_h$ son globalmente continuas, las funciones de Y_h deben ser globalmente continuas sólo con respecto a la dirección vertical, mientras que las de Q_h podrían ser funciones discontinuas.

Para el análisis del método debemos requerir las siguientes hipótesis y propiedades:

(**H0**): Regularidad del dominio $\Omega \subset I\!R^3$ tal que el problema de Stokes hidrostático tiene regularidad $\mathbf{H}^2(\Omega) \times H^1(S)$ para la velocidad (horizontal) y la presión respectivamente.

(**H1**): Los espacios aproximantes $\mathbf{X}_h$ y Q_h satisfacen la condición inf-sup hidrostática ([18]).

(**H2**): Se verifican las siguientes desigualdades inversas para cada $\mathbf{u}_h \in \mathbf{X}_h$,

$$\|\mathbf{u}_h\|_{L^2_z L^4_{\mathbf{x}}} \leq C\, h^{-1/2} \|\mathbf{u}_h\|_{L^2}, \quad \|\mathbf{u}_h\|_{L^\infty_z L^4_{\mathbf{x}}} \leq C\, h^{-1/2} \|\mathbf{u}_h\|_{L^\infty_z L^2_{\mathbf{x}}}, \quad \|\mathbf{u}_h\|_{L^2_z L^\infty_{\mathbf{x}}} \leq C\, h^{-1} \|\mathbf{u}_h\|_{L^2}$$

$$\|\mathbf{u}_h\|_{L^3} \leq C\, h^{-1/2} \|\mathbf{u}_h\|_{L^2}, \quad \|\mathbf{u}_h\|_{H^1} \leq C\, h^{-1} \|\mathbf{u}_h\|_{L^2}, \quad \|\mathbf{u}_h\|_{W^{1,6}} \leq C\, h^{-1} \|\mathbf{u}_h\|_{H^1}$$

(**H3**): las propiedades de aproximación (para $l = 1$ o 2):

$$\|\mathbf{v} - I_h \mathbf{v}\|_{L^2} + h\, \|\mathbf{v} - I_h \mathbf{v}\|_{H^1} \leq C\, h^{l+1}\, \|\mathbf{v}\|_{\mathbf{H}^{l+1}} \qquad \forall\, \mathbf{v} \in \mathbf{H}^{l+1}(\Omega) \cap \mathbf{H}^1_{b,l}(\Omega),$$

$$\|\mathbf{v} - I_h \mathbf{v}\|_{L^2} \leq C\, h^{l}\, \|\mathbf{v}\|_{\mathbf{H}^{l}} \qquad \forall\, \mathbf{v} \in \mathbf{H}^{l} \cap \mathbf{H}^1_{b,l}(\Omega),$$

$$\|q - J_h q\|_{L^2} \leq C\, h^{l}\, \|q\|_{H^{l}} \qquad \forall\, q \in H^{l}(S) \cap L^2_0(S),$$

$$\|v_3 - K_h v_3\|_{H(\partial_z)} \leq C\, h^{l}\, \|v_3\|_{H^{l+1}} \qquad \forall\, v_3 \in H^{l+1}(\partial_z) \cap H_0(\partial_z), \tag{18}$$

donde $(I_h, J_h) : \mathbf{H}^1_{b,l} \times L^2 \to \mathbf{X}_h \times Q_h$ son los operadores globales definidos como:

$$(I_h\mathbf{v}, J_hq) \in \mathbf{X}_h \times Q_h : \begin{cases} \Big(\nabla(I_h\mathbf{v}-\mathbf{v}), \nabla\,\mathbf{v}_h\Big) - \Big(J_hq - q, \nabla_H \cdot \langle \mathbf{v}_h\rangle\Big)_S = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h, \\ \Big(\nabla_H \cdot \langle I_h\mathbf{v}-\mathbf{v}\rangle, q_h\Big)_S = 0 \quad \forall\, q_h \in Q_h, \end{cases}$$

y $K_h : H_0(\partial_z) \to Y_h$ es el operador definido como:

$$K_hv_3 \in Y_h \;:\; \Big(\partial_z(K_hv_3 - v_3), \partial_z y_h\Big) = 0 \quad \forall\, y_h \in Y_h.$$

Construimos entonces el esquema numérico, donde en cada paso de tiempo m, dadas $\{(\mathbf{f}^m, \mathbf{g}^m_s)\}_{m=1}^M$ aproximaciones de los datos $(\mathbf{f}, \mathbf{g}_s)$ en $t = t_m$, calculamos una sucesión $\{(\mathbf{u}^m_h, u^m_{3,h}, p^m_{s,h})\}_m$, que pretendemos sea una aproximación a una solución $(\mathbf{u}, u_3, p_s)$ de (R) en el instante $t = t_m$.

En el esquema, separamos las tres principales dificultades del problema (R):

- el cálculo de la velocidad vertical.
- los términos convectivos no lineales, $(\mathbf{U}\cdot\nabla)\mathbf{u}$ (en particular, la convección vertical $u_3\partial_z\mathbf{u}$ es menos regular que en el caso de Navier-Stokes),
- la restricción $\nabla_H \cdot \langle \mathbf{u}\rangle = 0$ en $S \times (0, T)$,

Así, dada $(\mathbf{u}^m_h, p^m_{s,h})$, primero calculamos la velocidad vertical $u^m_{3,h}$ en función de $\nabla_H \cdot \mathbf{u}^m_h$, después, obtenemos una velocidad intermedia $\mathbf{u}^{m+1/2}_h$ usando los términos convectivos pero no la restricción para la divergencia y, finalmente, obtenemos $\mathbf{u}^{m+1}_h$ y $p^{m+1}_{s,h}$ a través de un problema lineal tipo Stokes considerando la restricción $\nabla_H \cdot \langle \mathbf{u}^{m+1}_h\rangle = 0$ (donde los términos difusivos aparecen en ambos pasos, por ello es un método de descomposición de la viscosidad).

El esquema totalmente discreto es:

Inicialización: Sea $\mathbf{u}^0_h \in \mathbf{X}_h$ una aproximación de $\mathbf{u}(0)$.

Paso de tiempo $m + 1$:

Subetapa 0: Dada $\mathbf{u}^m_h \in \mathbf{X}_h$, calcular $u^m_{3,h} \in Y_h$ tal que, para todo $\overline{u}_{3,h} \in Y_h$

$$(S_0)^m_h \qquad \Big(\partial_z u^m_{3,h}, \partial_z\overline{u}_{3,h}\Big) = -\Big(\nabla_H \cdot \mathbf{u}^m_h, \partial_z\overline{u}_{3,h}\Big),$$

Subetapa 1: Dada $\mathbf{U}^m_h = (\mathbf{u}^m_h, u^m_{3,h}) \in \mathbf{X}_h \times Y_h$, encontrar $\mathbf{u}^{m+1/2}_h \in \mathbf{X}_h$ tal que,

$$(S_1)^{m+1}_h \qquad \begin{cases} \dfrac{1}{k}\Big(\mathbf{u}^{m+1/2}_h - \mathbf{u}^m_h, \mathbf{v}_h\Big) + c\Big(\mathbf{U}^m_h, \mathbf{u}^{m+1/2}_h, \mathbf{v}_h\Big) + \Big(\nabla\,\mathbf{u}^{m+1/2}_h, \nabla\,\mathbf{v}_h\Big) \\ = \Big\langle \mathbf{f}^{m+1}, \mathbf{v}_h\Big\rangle_\Omega + \Big\langle \mathbf{g}^m_s, \mathbf{v}_h\Big\rangle_{\Gamma_s} \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h \end{cases}$$

Subetapa 2: Dada $\mathbf{u}^{m+1/2}_h \in \mathbf{X}_h$, hallar $(\mathbf{u}^{m+1}_h, p^{m+1}_h) \in \mathbf{X}_h \times Q_h$, tal que,

$$(S_2)^{m+1}_h \quad \begin{cases} \dfrac{1}{k}\Big(\mathbf{u}^{m+1}_h - \mathbf{u}^{m+1/2}_h, \mathbf{v}_h\Big) + \Big(\nabla(\mathbf{u}^{m+1}_h - \mathbf{u}^{m+1/2}_h), \nabla\mathbf{v}_h\Big) - \Big(p^{m+1}_h, \nabla_H \cdot \langle \mathbf{v}_h\rangle\Big)_S = 0, \\ \Big(\nabla_H \cdot \langle \mathbf{u}^{m+1}_h\rangle, q_h\Big)_S = 0, \quad \forall\, (\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h \end{cases}$$

En la primera subetapa, debemos calcular un problema lineal elíptico. En la segunda, un problema desacoplado por componentes de convección-difusión y en la tercera, un problema generalizado de Stokes hidrostático, el cuál está bien definido imponiendo **(H1)**.

Sumando $(S_1)_h^{m+1}$ y $(S_2)_h^{m+1}$, obtenemos:

$$(S)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\mathbf{u}_h^{m+1} - \mathbf{u}_h^m, \mathbf{v}_h\Big) + c\Big(\mathbf{U}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h\Big) + \Big(\nabla\, \mathbf{u}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) \\ -\Big(p_h^{m+1}, \nabla_H \cdot \langle \mathbf{v}_h \rangle\Big)_S = \Big\langle \mathbf{f}^{m+1}, \mathbf{v}_h \Big\rangle_\Omega + \Big\langle \mathbf{g}_s^m, \mathbf{v}_h \Big\rangle_{\Gamma_s} \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h \end{array} \right.$$

El Capítulo 4 estará dividido en dos partes:

- La obtención de la estabilidad y convergencia incondicional del esquema hacia una solución débil del problema continuo (R).
- La obtención de estimaciones de error óptimas en velocidades y presión, respecto de soluciones suficientemente regulares, condicionadas a la restricción $k \le C\, h^2$.

Estabilidad y convergencia

Definimos los espacios $\mathbf{H} = \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \ / \ \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle = 0 \text{ en } S,\ \langle \mathbf{v} \rangle \cdot \mathbf{n}_{\partial S} = 0\}$ y $\mathbf{V} = \{\mathbf{v}_h \in \mathbf{H}_{b,l}^1(\Omega) \ / \ \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle = 0 \text{ en } S\}$.

Consideraremos la regularidad débil de los datos

$$(\mathbf{WR}) \qquad \mathbf{f} \in L^2(0,T;\mathbf{H}_{b,l}^{-1}(\Omega)), \quad \mathbf{g}_s \in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s)) \quad \text{y} \quad \mathbf{u}_0 \in \mathbf{H},$$

y elegimos

$$\mathbf{f}^{m+1} = \frac{1}{k}\int_{t_m}^{t_{m+1}} \mathbf{f}(t)\, dt, \quad \mathbf{g}_s^{m+1} = \frac{1}{k}\int_{t_m}^{t_{m+1}} \mathbf{g}_s(t)\, dt.$$

Lema 30 *(Estabilidad) Suponemos* $(\mathbf{WR})$ *y* $(\mathbf{H1})$. *Si* $(\mathbf{u}_h^0)$ *está acotado en* $\mathbf{L}^2$, *entonces*

$$\|\mathbf{u}_h^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|\mathbf{u}_h^{m+1/2}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C,$$

$$\|\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}\|_{l^2(L^2)} + \|\mathbf{u}_h^{m+1/2} - \mathbf{u}_h\|_{l^2(L^2)} \le C\, k,$$

$$\|u_{3h}^{m+1}\|_{l^2(H(\partial_z)\cap L_z^\infty L_{\mathbf{x}}^2)} \le C.$$

A continuación, definimos las siguientes sucesiones de funciones (definidas para todo $t \in [0,T]$)):

- ◇ $\mathbf{u}_{k,h}^{(i)} : [0,T] \to \mathbf{H}_{b,l}^1(\Omega)$, tal que $\mathbf{u}_{k,h}^{(i)}(t) = \mathbf{u}_h^{m+i/2}$ si $t \in (t_m, t_{m+1}]$, $i = 0,1,2$.
- ◇ $u_{3,k,h}^{(0)} : [0,T] \to L^2(\Omega)$, tal que $u_{3,k,h}^{(0)}(t) = u_{3,h}^m$ si $t \in (t_m, t_{m+1}]$.
- ◇ $\mathbf{u}_{k,h} : [0,T] \to \mathbf{V}$, continua, lineal por subintervalos y $\mathbf{u}_{k,h}(t_m) = \mathbf{u}_h^m$.

El resultado fundamental de esta primera parte es el relativo a la convergencia

Teorema 31 (Convergencia) *Suponiendo* (**WR**) *y* (**H0**)-(**H2**)*, entonces existe una subsucesión* (k',h') *de* (k,h)*, con* $(k',h') \downarrow 0$*, y una solución débil* $\mathbf{U} = (\mathbf{u}, u_3)$ *de* (R) *en* $(0,T)$*, tal que:* $(\mathbf{u}^{(i)}_{k',h'})$ *(para cada* $i = 0,1,2$*) y* $(\mathbf{u}_{k',h'})$ *convergen a* $\mathbf{u}$ *fuertemente en* $L^2(0,T;\mathbf{L}^2(\Omega))$*, débilmente-* en* $L^\infty(0,T;\mathbf{L}^2(\Omega))$ *y débilmente en* $L^2(0,T;\mathbf{H}^1_{b,l}(\Omega))$*, mientras que* $(u^{(0)}_{3,k',h'})$ *converge a* u_3 *débilmente en* $L^2(0,T;H_0(\partial_z))$*.*

La demostración de este resultado no es estándar respecto a la obtención de la compacidad necesaria en el paso al límite. Hace falta un argumento bastante técnico ya usado en [53] para el caso de un esquema de tipo Euler en modelos de cristales líquidos nemáticos.

Así, re-escribimos $(S)^{m+1}_h$ eliminando la presión como

$$\Big(\partial_t \mathbf{u}_{k,h}, \mathbf{v}_h\Big) + c\Big(\mathbf{U}^{(0)}_{k,h}, \mathbf{u}^{(1)}_{k,h}, \mathbf{v}_h\Big) + \Big(\nabla \mathbf{u}^{(2)}_{k,h}, \nabla \mathbf{v}_h\Big) = \Big(\mathbf{f}^{m+1}, \mathbf{v}_h\Big) + \Big(\mathbf{g}^{m+1}_s, \mathbf{v}_h\Big)_{\Gamma_s} \quad \forall \mathbf{v}_h \in \mathbf{X}_h \cap \mathbf{V}, \tag{19}$$

$$\Big(\partial_z u^{(0)}_{3,k,h}, \partial_z w_h\Big) = -\Big(\nabla_H \cdot \mathbf{u}^{(0)}_{k,h}, \partial_z w_h\Big) \quad \forall w_h \in Y_h \tag{20}$$

Para tomar límites en (19), basta obtener la compacidad de $(\mathbf{u}^{(2)}_{k,h})_{k,h}$ en $L^2(\mathbf{L}^2(\Omega))$. Para ello, definimos

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h \ / \ (\nabla_H \cdot \langle \mathbf{v}_h \rangle, q_h)_S = 0, \forall q_h \in Q_h\}$$

y $A_h^{-1} : \mathbf{V}_h \to \mathbf{V}_h$ el operador inverso del operador discreto de Stokes “hidrostático”, esto es, dada $\mathbf{u}_h \in \mathbf{V}_h$, $A_h^{-1}\mathbf{u}_h$ es la solución débil del problema:

$$A_h^{-1}\mathbf{u}_h \in \mathbf{V}_h \text{ tal que } \Big(\nabla A_h^{-1}\mathbf{u}_h, \nabla \mathbf{v}_h\Big) = \Big(\mathbf{u}_h, \mathbf{v}_h\Big) \quad \forall \mathbf{v}_h \in \mathbf{V}_h \tag{21}$$

Se tiene entonces que $|\nabla A_h^{-1}\mathbf{u}_h|$ y $\|\mathbf{u}_h\|_{V_h'}$ son normas equivalentes.

Obtenemos en primer lugar el siguiente resultado de tipo estimaciones fraccionarias en tiempo.

Lema 32 *Suponiendo las hipótesis del Teorema 31, se tiene*

$$\int_0^{T-\delta} \|\mathbf{u}^{(2)}_{k,h}(t+\delta) - \mathbf{u}^{(2)}_{k,h}(t)\|^2_{V_h'}\, dt \le C\,\delta, \quad \forall\, \delta : 0 < \delta < T,$$

donde $C > 0$ *depende sólo de los datos.*

Pero, esta derivada fraccionaria en tiempo ha sido acotada en la norma $\mathbf{V}_h'$, que varía con respecto al parámetro h. Sin embargo, los resultados de compacidad (ver por ejemplo J. Simon [86]) no se tienen en estas condiciones. Por tanto, siguiendo el razonamiento realizado en [53], debemos encontrar una norma independiente de h donde la derivada fraccionaria en tiempo pueda ser acotada. Para ello, consideramos la proyección ortogonal

$$R_h : \mathbf{V}_h \to \mathbf{V} \text{ definida como } \Big(\nabla(R_h \mathbf{v}_h - \mathbf{v}_h), \nabla \mathbf{w}\Big) = 0, \quad \forall \mathbf{w} \in \mathbf{V}.$$

El operador R_h tiene las siguientes propiedades ([53]):

$$\|R_h \mathbf{u}_h\|_{H^1} \le \|\mathbf{u}_h\|_{H^1} \quad \text{(dependencia continua en } H^1),$$

$$\|R_h \mathbf{u}_h - \mathbf{u}_h\|_{L^2} \le C\, h \|\nabla_H \cdot \langle \mathbf{u}_h \rangle\|_{L^2(S)} \quad \text{(estimación de error en } L^2)$$

y

$$\|R_h \mathbf{u}_h\|_{V'} \le \|\mathbf{u}_h\|_{V_h'} + C\, h.$$

Entonces llegamos a la estimación:

$$\int_0^{T-\delta} \|R_h\, \mathbf{u}_{h,k}^{(2)}(t+\delta) - R_h\, \mathbf{u}_{h,k}^{(2)}(t)\|_{V'}^2\, dt \le C\,(\delta + h).$$

Ahora sí podemos aplicar un resultado de compacidad (por perturbaciones) debido a P. Azérad y F. Guillén-González ([5]), obteniendo que $R_h \mathbf{u}_{k,h}^{(2)} \to \mathbf{u}$ en $L^2(0,T;\mathbf{L}^2)$-fuerte. De aquí tenemos $\mathbf{u}_{k,h}^{(2)} \to \mathbf{u}$ en $L^2(0,T;\mathbf{L}^2)$-fuerte y podemos concluir la demostración del Teorema 31.

Estimaciones de error

La principal diferencia con el caso de Navier-Stokes estudiado en los Capítulos 1 y 2, es la restricción que debemos imponer ahora entre el paso de tiempo k y el tamaño de la malla h, para obtener estimaciones óptimas de orden $O(k+h)$:

(H) Existe una constante $\alpha > 0$ (independiente de k y h) tal que $k \le \alpha\, h^2$.

Así, presentaremos un análisis de error para el esquema totalmente discreto $(\mathbf{u}_h^{m+1/2}, \mathbf{u}_h^{m+1}, p_h^{m+1})$ como una aproximación de $(\mathbf{u}(t_{m+1}), \mathbf{u}(t_{m+1}), p(t_{m+1}))$. Consideramos los errores totales:

$$\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1/2}, \qquad \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1}, \qquad e_p^{m+1} = p_s(t_{m+1}) - p_h^{m+1}$$

$$e_3^{m+1} = u_3(t_{m+1}) - u_{3,h}^{m+1}, \qquad \mathbf{E}^{m+1} = (\mathbf{e}^{m+1}, e_3^{m+1}),$$

que, de nuevo, pueden ser descompuestos en sus partes de interpolación y discreta:

$$\mathbf{e}^{m+1/2} = \mathbf{e}_i^{m+1/2} + \mathbf{e}_h^{m+1/2}, \qquad \mathbf{e}^{m+1} = \mathbf{e}_i^{m+1} + \mathbf{e}_h^{m+1}, \qquad e_p^{m+1} = e_{p,i}^{m+1} + e_{p,h}^{m+1}$$

$$e_3^{m+1} = e_{3,i}^{m+1} + e_{3,h}^{m+1}, \qquad \mathbf{E}^{m+1} = \mathbf{E}_h^{m+1} + \mathbf{E}_i^{m+1}$$

Entonces, los problemas variacionales para los errores espaciales $e_{3,h}^m$ y $\mathbf{e}_h^{m+1/2}$ son:

$$(E_0)_h^m \qquad \Big(\partial_z e_{3,h}^m, \partial_z v_{3,h}\Big) = -\Big(\nabla_H \cdot (\mathbf{e}_h^m + \mathbf{e}_i^m), \partial_z v_{3,h}\Big) \quad \forall v_{3,h} \in Y_h$$

(aquí se usa que $\Big(\partial_z\, e_{3,i}^{m+1}, \partial_z w_h\Big) = 0$, gracias a la elección del operador de interpolación K_h),

$$(E_1)_h^{m+1} \qquad \begin{cases} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\, \mathbf{e}_h^{m+1/2}, \nabla\, \mathbf{v}_h\Big) - \Big(p(t_{m+1}), \nabla_H \cdot \langle \mathbf{v}_h \rangle\Big)_S \\ = \mathbf{NL}^{m+1}(\mathbf{v}_h) + \Big(\mathcal{E}^{m+1}, \mathbf{v}_h\Big) \\ -\Big(\delta_t \mathbf{e}_i^{m+1}, \mathbf{v}_h\Big) - \Big(\nabla \mathbf{e}_i^{m+1}, \nabla \mathbf{v}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h \end{cases}$$

donde

$$\mathcal{E}^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t-t_m)\,\mathbf{u}_{tt}(t)\,dt - \Big(\int_{t_m}^{t_{m+1}} \mathbf{U}_t\cdot\nabla\Big)\,\mathbf{u}(t_{m+1})$$

es el error de consistencia, y

$$\mathbf{NL}^{m+1}(\mathbf{v}_h) = c\Big(\mathbf{E}_h^m, \mathbf{u}(t_{m+1}), \mathbf{v}_h\Big) - c\Big(\mathbf{U}_h^m, \mathbf{e}_h^{m+1/2}, \mathbf{v}_h\Big) + c\Big(\mathbf{E}_i^m, \mathbf{u}(t_{m+1}), \mathbf{v}_h\Big) - c\Big(\mathbf{U}_h^m, \mathbf{e}_i^{m+1}, \mathbf{v}_h\Big)$$

Por otro lado, de $(S_2)_h^{m+1}$ tenemos para cada $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$

$$(E_2)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}, \mathbf{v}_h\Big) + \Big(\nabla(\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}), \nabla\mathbf{v}_h\Big) = -\Big(p_h^{m+1}, \nabla_H\cdot\langle\mathbf{v}_h\rangle\Big)_S \\ \Big(\nabla_H\cdot\langle\mathbf{e}_h^{m+1}\rangle, q_h\Big)_S = 0. \end{array}\right.$$

Sumando$(E_1)_h^{m+1}$ y $(E_2)_h^{m+1}$, llegamos a:

$$(E)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\mathbf{e}_h^{m+1}, \nabla\mathbf{v}_h\Big) - \Big(e_{p,h}^{m+1}, \nabla_H\cdot\langle\mathbf{v}_h\rangle\Big)_S \\ \quad = -\dfrac{1}{k}\Big(\mathbf{e}_i^{m+1} - \mathbf{e}_i^m, \mathbf{v}_h\Big) + \mathbf{NL}^{m+1}(\mathbf{v}_h) + \Big(\mathcal{E}^{m+1}, \mathbf{v}_h\Big), \quad \forall\mathbf{v}_h\in\mathbf{X}_h \\ \Big(\nabla_H\cdot\langle\mathbf{e}_h^{m+1}\rangle, q_h\Big)_S = 0, \quad \forall q_h\in Q_h \end{array}\right.$$

Además, de $(E_0)_h^m$, se tiene:

$$\|\partial_z e_{3,h}^m\|_{L^2} \le C(\|\nabla_H\cdot\mathbf{e}_h^m\|_{L^2} + \|\nabla_H\cdot\mathbf{e}_i^m\|_{L^2}). \tag{22}$$

Por tanto, usando (17),

$$\|e_{3,h}^m\|^2_{L_z^\infty L_{\mathbf{x}}^2} \le C(\|\mathbf{e}_h^m\|_{H^1} + \|\mathbf{e}_i^m\|_{H^1}).$$

Por otro lado, la propiedad de aproximación (18) nos dice:

$$\|\partial_z e_{3,i}^m\|_{L^2} \le C\,h^l\,\|\partial_z u_3(t_m)\|_{H^l} \le C\,h^l\,\|\mathbf{u}(t_m)\|_{H^{l+1}} \le C\,h^l.$$

Usando de nuevo (17),

$$\|e_{3,i}^m\|_{L_z^\infty L_{\mathbf{x}}^2} \le C\,h^l.$$

Podemos ahora abordar la obtención de las estimaciones de error. Supondremos para ello que tenemos soluciones suficientemente regulares y comenzamos probando el orden $O(\sqrt{k}+h^l)$ para ambas velocidades en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$:

Teorema 33 *Suponiendo $|\mathbf{e}_h^0| \le C\,h^l$, para k suficientemente pequeño, se tiene*

$$\|\mathbf{e}_h^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,(\sqrt{k}+h^l)$$

$$\|\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m\|^2_{l^2(\mathbf{L}^2)} + \|\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}\|^2_{l^2(\mathbf{L}^2)} \le C\,k(k+h^{2l}).$$

A esta altura, no hace falta imponer la restricción $(\mathbf{H})$ sobre los parámetros (k,h) aunque debemos imponer k suficientemente pequeño.

Mejoramos seguidamente las estimaciones a orden $O(k+h^l)$ para $\mathbf{e}_h^{m+1}$ en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$.

Teorema 34 *Suponemos las hipótesis del Teorema 33 y* $(\mathbf{H})$*. Entonces,*

$$\|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leq C\,(k+h^l). \tag{23}$$

Notemos que, de (22) y (23), se tiene también

$$\|e_{3,h}^{m+1}\|_{l^2(H(\partial_z))} \leq C\,(k+h^l).$$

A continuación, obtenemos el orden $O(\sqrt{k}+h^l)$ para $\delta_t\mathbf{e}_h^{m+1}$ en $l^2(\mathbf{L}^2)$ y para $\mathbf{e}_h^{m+1}$ en $l^\infty(\mathbf{H}^1)$.

Teorema 35 *Suponiendo las hipótesis del Teorema 33,* $\|\mathbf{e}_h^0\|_{H^1} \leq C\,h^l$*, y la restricción sobre los parámetros* (k,h) *dada en* $(\mathbf{H})$*, se tienen las estimaciones de error*

$$\|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{H}^1)} + \|\delta_t\mathbf{e}_h^{m+1}\|_{L^2(\mathbf{L}^2)} \leq C\,(\sqrt{k}+h^l).$$

Este resultado nos lleva con un razonamiento estándar (aplicando la condición *Inf-Sup hidrostática* **(H1)**), al orden $O(\sqrt{k}+h^l)$ para las estimaciones de error de presión $e_{p,h}^{m+1}$ en $l^2(L^2)$:

Corolario 36 *Suponiendo que se tienen las hipótesis del Teorema 35, entonces*

$$\|e_{p,h}^{m+1}\|_{l^2(L^2)} \leq C\,(\sqrt{k}+h^l).$$

Los resultados anteriores son válidos para aproximaciones por elementos finitos lineales y cuadráticas ($l=1,2$). A continuación nos restringimos al caso $l=2$. Vamos a obtener primero orden $O(\sqrt{k}+h^2)$ para $\delta_t\mathbf{e}_h^{m+1}$ y $\delta_t\mathbf{e}_h^{m+1/2}$ en $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ y después $O(k+h^2)$ para $\delta_t\mathbf{e}_h^{m+1}$.

Para ello, consideremos los siguientes problemas resultantes de hacer $\delta_t(E_1)_h^{m+1}$ y $\delta_t(E_2)_h^{m+1}$ para cada $m\geq 1$, obteniendo $\forall\,\mathbf{v}_h\in\mathbf{X}_h$:

$(D_1)_h^{m+1}$

$$\left\{\begin{array}{l} \dfrac{1}{k}\Big(\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m},\mathbf{v}_h\Big)+\Big(\nabla\delta_t\mathbf{e}_h^{m+1/2},\nabla\,\mathbf{v}_h\Big)-\Big(\delta_t\,p_s(t_{m+1}),\nabla_H\cdot\langle\mathbf{v}_h\rangle\Big)_S \\ \quad =\Big(\delta_t\mathcal{E}^{m+1},\mathbf{v}_h\Big)+\delta_t\mathbf{NL}_h^{m+1}(\mathbf{v}_h)-\frac{1}{k}\Big((\delta_t\mathbf{e}_i^{m+1}-\delta_t\mathbf{e}_i^{m}),\mathbf{v}_h\Big)-\Big(\nabla\,\delta_t\mathbf{e}_i^{m+1},\nabla\,\mathbf{v}_h\Big) \end{array}\right.$$

donde

$$\begin{array}{rcl} \delta_t\,\mathbf{NL}_h^{m+1}(\mathbf{v}_h) &=& c\Big(\delta_t\mathbf{E}^m,\,\mathbf{u}(t_{m+1}),\mathbf{v}_h\Big)+c\Big(\delta_t\,\mathbf{U}_h^m,\mathbf{e}^{m+1/2},\mathbf{v}_h\Big) \\ &+& c\Big(\mathbf{E}^{m-1},\delta_t\mathbf{u}(t_{m+1}),\mathbf{v}_h\Big)+c\Big(\mathbf{U}_h^{m-1},\delta_t\mathbf{e}^{m+1/2},\mathbf{v}_h\Big) \end{array}$$

y, para todo $(\mathbf{w}_h,q_h)\in\mathbf{X}_h\times Q_h$,

$$(D_2)_h^{m+1}\qquad \left\{\begin{array}{l} \dfrac{1}{k}\Big(\delta_t\mathbf{e}_h^{m+1}-\delta_t\mathbf{e}_h^{m+1/2},\mathbf{w}_h\Big)+\Big(\nabla\,(\delta_t\mathbf{e}_h^{m+1}-\delta_t\mathbf{e}_h^{m+1/2}),\nabla\,\mathbf{w}_h\Big) \\ \qquad =-\Big(\delta_t p_{s,h}^{m+1},\nabla_H\cdot\langle\mathbf{w}_h\rangle\Big)_S \\ \Big(\nabla_H\cdot\langle\delta_t\mathbf{e}_h^{m+1}\rangle,q_h\Big)_S=0. \end{array}\right.$$

Finalmente, sumando $(D_1)_h^{m+1}$ y $(D_2)_h^{m+1}$ obtenemos, para todo $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$:

$$(D_3)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m}, \mathbf{v}_h\Big) + \Big(\nabla\, \delta_t \mathbf{e}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\delta_t e_{p,h}^{m+1}, \nabla_H \cdot \langle \mathbf{v}_h \rangle\Big)_S \\ \qquad = \Big(\delta_t \mathcal{E}^{m+1}, \mathbf{v}_h\Big) + \delta_t \mathbf{NL}_h^{m+1}(\mathbf{v}_h) - \dfrac{1}{k}\Big(\delta_t \mathbf{e}_i^{m+1} - \delta_t \mathbf{e}_i^{m}, \mathbf{v}_h\Big) \\ \Big(\nabla_H \cdot \langle \delta_t \mathbf{e}_h^{m+1} \rangle, q_h\Big)_S = 0. \end{array} \right.$$

Teorema 37 *Suponiendo las hipótesis del Teorema 34 con una aproximación por elementos finitos de orden $O(h^2)$ (es decir, $l = 2$) y suponiendo la siguiente hipótesis para la etapa inicial del esquema*

$$\|\delta_t \mathbf{e}_h^{1/2}\|_{L^2} \leq C\,(\sqrt{k} + h^2),$$

entonces

$$\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|\delta_t \mathbf{e}_h^{m+1/2}\|_{l^\infty(L^2)\cap l^2(H^1)} \leq C\,(\sqrt{k} + h^2),$$

$$\|\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}\|_{l^2(L^2)} + \|\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m}\|_{l^2(L^2)} \leq C\,\sqrt{k}\,(\sqrt{k} + h^2),$$

Por último, probamos el orden $O(k + h^2)$ para el error $\delta_t \mathbf{e}_h^{m+1}$ en $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$.

Teorema 38 *Suponiendo las hipótesis del Teorema 37 y la siguiente hipótesis para el primer paso del esquema*

$$\|\delta_t \mathbf{e}_h^{1}\|_{L^2} \leq C\,(k + h^2),$$

entonces

$$\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} \leq C\,(k + h^2).$$

Este resultado nos lleva a la obtención del orden óptimo para el error de presión, pero sólo en $l^2(L^2)$. No se consigue en norma $l^\infty(L^2)$, debido a que la estimación del error intermedio $\mathbf{e}_h^{m+1/2}$ no es óptima en $l^\infty(\mathbf{L}^2)$.

Corolario 39 *Suponiendo las hipótesis del Teorema 38, se tiene*

$$\|e_{p,h}^{m+1}\|_{l^2(L^2)} \leq C\,(k + h^2).$$

Finalmente, con un razonamiento dual, obtenemos estimaciones de error pero aproximando ahora el problema (Q) mediante una malla estructurada vertical de elementos finitos (por ejemplo mallas estructuradas formadas por prismas rectos). En este caso, cambiando el cálculo de la velocidad vertical a forma integral:

$$u_{3,h}^m(\mathbf{x}, z) = \int_z^0 \nabla_H \cdot \mathbf{u}_h^m(\mathbf{x}, s)\, ds, \tag{24}$$

el operador vertical de interpolación está definido como

$$K_h u_3(\mathbf{x}, z) = \int_z^0 \nabla_H \cdot I_h \mathbf{u}(\mathbf{x}, s)\, ds,$$

y de aquí, $e_{3,i}^m = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{e}_i^m$.

Entonces, para el esquema cambiando $(S_0)_h^m$ por (24), obtenemos estimaciones de error de orden $O(k + h^{l+1})$ para $\mathbf{e}_h^{m+1}$ en $l^2(\mathbf{L}^2)$.

Teorema 40 *Suponiendo las hipótesis del Teorema 34, y* $\|A_h^{-1}\mathbf{e}_h^0\|_{H^1} \le C\,h^3$ *(siendo A_h el operador hidrostático de Stokes definido en (21)), se tiene, para k suficientemente pequeño,*

$$\|\mathbf{e}_h^{m+1}\|_{l^2(L^2)} \le C(k + h^{l+1}).$$

Gracias a este resultado, podemos probar las estimaciones obtenidas en los Teoremas 37 y 38 también para el caso $l = 1$. Esto nos lleva a orden $O(k + h^l)$ en $l^2(L^2)$ para la presión:

Corolario 41 *Suponiendo las hipótesis de los Teoremas 40 y 38, se tiene*

$$\|e_{p,h}^{m+1}\|_{l^2(L^2)} \le C\,(k + h^l).$$

4.2.2. Capítulo 5

Presentamos en este capítulo un análisis numérico del esquema de proyección incremental semidiscreto en tiempo como aproximación de (Q), es decir, considerando la forma integral para la velocidad vertical.

El capítulo consta de dos partes. En la primera, describimos el esquema y se obtienen los resultados de estabilidad y convergencia. Posteriormente, demostramos estimaciones de error óptimas con respecto a una solución suficientemente regular del problema.

El esquema está descrito como sigue:

Inicialización: Sean $\widetilde{\mathbf{u}}^0 = \mathbf{u}(0)$ y p_s^0 dados. Tomar $\mathbf{u}^0 = \widetilde{\mathbf{u}}^0$.

Subetapa 0 : Dada $\widetilde{\mathbf{u}}^m$, calcular $\widetilde{u}_3^m$ como

$$(S_0)^m \qquad \widetilde{u}_3^m(\mathbf{x}, z) = \int_z^0 \nabla_H \cdot \widetilde{\mathbf{u}}^m(\mathbf{x}, s)\, ds.$$

Subetapa 1 : Dadas $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$, $\widetilde{u}_3^m$ y p_s^m, hallar $\widetilde{\mathbf{u}}^{m+1} : \Omega \to I\!R^2$ solución de

$$(S_1)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}) - \nu\Delta\widetilde{\mathbf{u}}^{m+1} + \nabla_H p_s^m &=& \mathbf{f}^{m+1}, \\ \nu\partial_z \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_s} = \mathbf{g}_s^{m+1}, \quad \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_b \cup \Gamma_l} &=& \mathbf{0}, \end{array} \right.$$

donde $\widetilde{\mathbf{U}}^m = (\widetilde{\mathbf{u}}^m, \widetilde{u}_3^m)$.

Subetapa 2 : Dadas p_s^m y $\widetilde{\mathbf{u}}^{m+1}$, hallar $\mathbf{u}^{m+1} : \Omega \to I\!R^2$ y $p_s^{m+1} : S \to I\!R^2$ solución de

$$(S_2)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}) + \nabla_H(p_s^{m+1} - p_s^m) &=& \mathbf{0} \quad \text{en } \Omega, \\ \nabla_H \cdot \langle \mathbf{u}^{m+1} \rangle = 0 \quad \text{en } S, \quad \langle \mathbf{u}^{m+1} \rangle \cdot \mathbf{n}|_{\partial S} &=& 0. \end{array} \right.$$

Sumando $(S_1)^{m+1}$ y $(S_2)^{m+1}$, obtenemos las relaciones de consistencia:

$$(S_3)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}) - \nu \Delta \widetilde{\mathbf{u}}^{m+1} + \nabla_H p_s^{m+1} & = & \mathbf{f}^{m+1} \quad \text{en } \Omega, \\ \nu \partial_z \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_s} = \mathbf{g}_s^{m+1}, \quad \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_b} = 0, \quad \nabla_H \cdot \langle \mathbf{u}^{m+1} \rangle & = & 0 \quad \text{en } S. \end{array} \right.$$

Notemos que en la implementación del esquema, la introducción de la velocidad final $\mathbf{u}^{m+1}$ no es necesaria y el cálculo de $\widetilde{\mathbf{u}}^{m+1}$ y p_s^{m+1} puede ser desacoplado. Esto es debido a que, en cada paso de tiempo, el esquema se implementa como sigue: Dados $(p_s^{m-1}, p_s^m, \widetilde{\mathbf{u}}^m)$,

(0) Encontrar $\widetilde{u}_3^m$ solución de $(S_0)^m$,

(1) Encontrar $\widetilde{\mathbf{u}}^{m+1}$ solución del problema de convección-difusión:

$$(\widetilde{S})^{m+1} \quad \left\{ \begin{array}{l} \dfrac{\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}^m}{k} + C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta \widetilde{\mathbf{u}}^{m+1} + \nabla_H (2\, p_s^m - p_s^{m-1}) = \mathbf{f}^{m+1} \quad \text{en } \Omega, \\ \nu \partial_z \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_s} = \mathbf{g}_s^{m+1}, \quad \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_b \cup \Gamma_l} = \mathbf{0}, \end{array} \right.$$

(2) Encontrar p_s^{m+1} solución del siguiente problema elíptico $2D$

$$(E)^{m+1} \qquad \left\{ \begin{array}{c} k \nabla_H \cdot (D\, \nabla_H (p_s^{m+1} - p_s^m)) = \nabla_H \cdot \langle \widetilde{\mathbf{u}}^{m+1} \rangle \quad \text{en } S \\ k\, D \nabla_H (p_s^{m+1} - p_s^m) \cdot \mathbf{n}_{\partial S} = 0 \quad \text{sobre } \partial S. \end{array} \right.$$

El problema $(\widetilde{S})^{m+1}$ se ha obtenido escribiendo $\mathbf{u}^m = \widetilde{\mathbf{u}}^m - k \nabla_H (p_s^m - p_s^{m-1})$ en $(S_1)^{m+1}$. Integrando verticalmente $(S_2)^{m+1}$ entre $z = -D(\mathbf{x})$ y $z = 0$, obtenemos

$$\langle \mathbf{u}^{m+1} \rangle - \langle \widetilde{\mathbf{u}}^{m+1} \rangle + k\ D\ \nabla_H (p_s^{m+1} - p_s^m) = 0 \qquad \text{en } S.$$

Tomando divergencia horizontal y multiplicando por $\mathbf{n}_{\partial S}$ (el vector normal exterior a ∂S), la incógnita $\mathbf{u}^{m+1}$ es eliminada y llegamos al problema elíptico con condición Neumann $(E)^{m+1}$ para la presión p_s^{m+1}.

Sólo para el análisis numérico, es interesante introducir la velocidad proyectada $\mathbf{u}^{m+1}$ en función de p_s^m, $\widetilde{\mathbf{u}}^{m+1}$ y p_s^{m+1}, como sigue:

$$\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k \nabla_H (p_s^{m+1} - p_s^m) \quad \text{en } \Omega.$$

Notemos que, al igual que en el caso de Navier-Stokes, para inicializar el esquema debemos conocer $\widetilde{\mathbf{u}}^0$, p^0 y p^{-1}. Entonces, como ya comentamos, debemos comenzar con unas etapas de tiempo previas usando otro esquema o comenzar con una aproximación de la presión inicial.

Estabilidad y Convergencia.

Obtenemos la estabilidad y convergencia hacia una solución débil del problema continuo (Q). Para ello, obtendremos algunas estimaciones a priori (estabilidad) y un posterior paso al límite (convergencia), donde deberemos aplicar resultados de compacidad para controlar el límite en los términos convectivos.

Comenzamos obteniendo estimaciones de estabilidad para $(\widetilde{\mathbf{u}}^{m+1})$ y $(\mathbf{u}^{m+1})$ en $L^\infty(0,T;\mathbf{L}^2(\Omega))\cap L^2(0,T;\mathbf{H}^1(\Omega))$, con la correspondiente estimación para $(\widetilde{u}_3^m)$ en $L^2(0,T;H(\partial_z))$.

Con respecto al paso de proyección $(S_2)^{m+1}$, se tiene el siguiente resultado:

Lema 42 *(Existencia, unicidad y dependencia continua de* $(S_2)^{m+1}$*).*
a) ***Dependencia continua en*** L^2**.** *El problema* $(S_2)^{m+1}$ *tiene una única solución* $(\mathbf{u}^{m+1},p_s^{m+1})\in \mathbf{H}\times(H^1(\Omega)\cap L^2_0(\Omega))$*. Además,*

$$\|\mathbf{u}^{m+1}\|_{L^2}\leq\|\widetilde{\mathbf{u}}^{m+1}\|_{L^2}\quad y\quad \|\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}\|_{L^2}\leq\|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m\|_{L^2}.$$

Por otra parte, usando la propiedad de ortogonalidad $\Big(\mathbf{u}^{m+1},\nabla_H q_s\Big)=0$ *para toda función* $q_s\in H^1(\Omega)$ *e independiente de* z*, tenemos*

$$\|\widetilde{\mathbf{u}}^{m+1}\|^2_{L^2}=\|\mathbf{u}^{m+1}\|^2_{L^2}+\|k\nabla_H(p_s^{m+1}-p_s^m)\|^2_{L^2}.$$

b) ***Dependencia continua en*** H^1**.** *Suponemos* $S\in C^3$*,* $D\in W^{1,\infty}(S)$ *y o bien* $D\geq D_{min}>0$ *en* S *o* $D>0$ *en* S *y* S *simplemente conexo. Si* $\widetilde{\mathbf{u}}^{m+1}\in\mathbf{H}^1_{b,l}(\Omega)$*, entonces* $(\mathbf{u}^{m+1},p_s^{m+1})\in\mathbf{H}^1(\Omega)\cap H^2(S)$*. Además, existe* $C=C(\Omega,D)>0$ *tal que*

$$\|\mathbf{u}^{m+1}\|_{H^1}\leq C\,\|\widetilde{\mathbf{u}}^{m+1}\|_{H^1}.$$

Este resultado será usado en la obtención de la estabilidad del esquema. Concretamente se obtiene

Lema 43 (Estabilidad) *Sea* $\mathbf{f}\in L^2(0,T;\mathbf{H}^{-1}_{b,l}(\Omega))$*,* $\mathbf{g}_s\in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$ *y* $\mathbf{u}_0\in\mathbf{V}$*. Si* $\|k\nabla_H p_s^0\|_{L^2}\leq C_0$*, entonces existe* $C=C(C_0,\nu,\mathbf{f},\mathbf{g}_s,\Omega)>0$ *tal que,*

$$\begin{gathered}\|\widetilde{\mathbf{u}}^{r+1}\|^2_{L^2}+\|\mathbf{u}^{r+1}\|^2_{L^2}\leq C,\qquad\forall\, r=0,...,M-1\\ k\sum_{m=0}^{M-1}\Big\{\|\widetilde{\mathbf{u}}^{m+1}\|^2_{H^1}+\|\mathbf{u}^{m+1}\|^2_{H^1}\Big\}\leq C,\\ \sum_{m=0}^{M-1}\Big\{\|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m\|^2_{L^2}+\|\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}\|^2_{L^2}\Big\}\leq C\end{gathered}$$

Con respecto a la velocidad vertical, usando la desigualdad de Poincaré vertical (15), existe una constante $C=C(C_0,\nu,\mathbf{f},\mathbf{g}_s,\Omega)>0$, tal que

$$k\sum_{m=0}^{M-1}\|\widetilde{u}_3^m\|^2_{H(\partial_z)}\leq C.$$

A continuación, obtendremos la convergencia del esquema. Para ello, definimos $\mathbf{u}_k^{(1)}$, $\widetilde{\mathbf{u}}_k^{(1)}$, $\mathbf{u}_k^{(0)}$, $\widetilde{\mathbf{u}}_k^{(0)}$, $\widetilde{u}_{3,k}^{(0)}$ las funciones definidas en $[0,T]$, constantes por subintervalos, iguales a $\mathbf{u}^{m+1}$, $\widetilde{\mathbf{u}}^{m+1}$, $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$ y $\widetilde{u}_3^m$ en $(t_m,t_{m+1}]$, respectivamente. Por otro lado, $\mathbf{u}_k$ es la función continua en $[0,T]$, lineal por subintervalos y tal que $\mathbf{u}_k(t_m)=\mathbf{u}^m$.

Teorema 44 (Convergencia) *Suponiendo las hipótesis del Lema 43, existe una subsucesión* (k') *de* (k)*, con* $k' \downarrow 0$*, y una solución débil* $\mathbf{U} = (\mathbf{u}, u_3)$ *de* (Q) *en* $(0,T)$*, tal que:* $\widetilde{\mathbf{u}}_{k'}^{(1)}$*,* $\mathbf{u}_{k'}^{(1)}$*,* $\widetilde{\mathbf{u}}_{k'}^{(0)}$*,* $\mathbf{u}_{k'}^{(0)}$ *y* $\mathbf{u}_{k'}$ *convergen a* $\mathbf{u}$ *débilmente-∗ en* $L^\infty(0,T;\mathbf{L}^2(\Omega))$*, débilmente en* $L^2(0,T;\mathbf{H}^1(\Omega))$ *y fuertemente en* $L^2(0,T;\mathbf{L}^2(\Omega))$*, mientras que* $\widetilde{u}_{3,k'}^{(0)}$ *converge a* u_3 *débilmente en* $L^2(0,T;H(\partial_z))$*.*

Estimaciones de error

Deducimos estimaciones óptimas de error, tanto para la velocidad como para la presión, con respecto a una solución suficientemente regular $\{\mathbf{u}, u_3, p_s\}$ del problema (Q). Los resultados que obtendremos pueden ser considerados como extensiones al caso de Ecuaciones Primitivas de los obtenidos en el Capítulo 3 para las Ecuaciones de Navier-Stokes. De hecho, seguimos el mismo argumento, acotando ahora convenientemente los términos convectivos. En este sentido, adecuadas estimaciones anisótropas deben ser consideradas para conservar el orden óptimo en dichas estimaciones.

Introducimos la siguientes notaciones para los errores en $t = t_{m+1}$:

$$\begin{aligned}
&\widetilde{\mathbf{e}}^{m+1} = \mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}^{m+1}, \qquad && \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}\\
&\widetilde{e}_3^{m+1} = u_3(t_{m+1}) - \widetilde{u}_3^{m+1}, \qquad && e_3^{m+1} = u_3(t_{m+1}) - u_3^{m+1}\\
&e_{p,s}^{m+1} = p_s(t_{m+1}) - p_s^{m+1}.
\end{aligned}$$

Entonces, los problemas diferenciales asociados a los errores son los siguientes:

$$(E_1)^{m+1} \left\{ \begin{array}{l} \dfrac{1}{k}(\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m) - \nu\Delta\widetilde{\mathbf{e}}^{m+1} + \nabla_H(e_{p,s}^m + k\,\delta_t p_s(t_{m+1})) = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{en } \Omega \\ \nu\,\partial_z\widetilde{\mathbf{e}}^{m+1}|_{\Gamma_s} = 0, \quad \widetilde{\mathbf{e}}^{m+1}|_{\Gamma_b\cup\Gamma_l} = 0, \end{array} \right.$$

$$(E_2)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}) + \nabla_H\Big(e_{p,s}^{m+1} - e_{p,s}^m - k\,\delta_t p_s(t_{m+1})\Big) &=& 0 \ \text{ en } \Omega \\ \nabla_H\cdot\langle\mathbf{e}^{m+1}\rangle = 0 \ \text{ en } S, \qquad \langle\mathbf{e}^{m+1}\rangle\cdot\mathbf{n}_{\partial S} &=& 0 \ \text{ sobre } \partial S. \end{array} \right.$$

donde

$$\mathcal{E}^{m+1} := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t-t_m)\,\partial_{tt}^2\mathbf{u}(t)\,dt - \left(\left(\int_{t_m}^{t_{m+1}}\partial_t\mathbf{U}\right)\cdot\nabla\right)\mathbf{u}(t_{m+1})$$

y

$$\mathbf{NL}^{m+1} = -C\Big((\widetilde{\mathbf{e}}^m, \widetilde{e}_3^m), \mathbf{u}(t_{m+1})\Big) - C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{e}}^{m+1})$$

El problema $(E_2)^{m+1}$ puede ser descompuesto en los dos problemas siguientes:

$$(E_2)_a^{m+1} \qquad \left\{ \begin{array}{ll} k\,\nabla_H\cdot(D\,\nabla_H(e_{p,s}^{m+1} - e_{p,s}^m - k\,\delta_t p(t_{m+1})) = \nabla\cdot\widetilde{\mathbf{e}}^{m+1} & \text{en } \Omega, \\ k\,\nabla_H(e_{p,s}^{m+1} - e_{p,s}^m - k\,\delta_t p_s(t_{m+1}))\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{0}, & \end{array} \right.$$

$(E_2)_b^{m+1}$ $$\mathbf{e}^{m+1} = \widetilde{\mathbf{e}}^{m+1} - k\,\nabla(e_{p,s}^{m+1} - e_{p,s}^{m} - k\,\delta_t p_s(t_{m+1})) \quad \text{en } \Omega.$$

Para obtener las estimaciones de error, debemos imponer las siguientes hipótesis de regularidad sobre el dominio y la solución $(\mathbf{u}, p_s)$ de (Q):

(H0) $\Omega \subset I\!R^3$ tal que se tiene regularidad $\mathbf{H}^2(\Omega)$ para el problema de Poisson $(E_1)^{m+1}$:

$$-\Delta \mathbf{u} = \mathbf{f} \quad \text{en } \Omega, \quad \partial_z \mathbf{u}|_{\Gamma_S} = \mathbf{g}, \quad \mathbf{u}|_{\Gamma_b \cup \Gamma_l} = 0$$

(H1) $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}),\ p_s \in L^\infty(H^1),\ \partial_t p_s \in L^2(H^1),\ \mathbf{u}_t \in L^2(\mathbf{H}^1),\ \mathbf{u}_{tt} \in L^2(\mathbf{H}_{b,l}^{-1})$

(H2) $\partial_{tt} p_s \in L^2(H^1), \quad \mathbf{u}_t \in L^\infty(\mathbf{W}^{1,3}), \quad \mathbf{u}_{tt} \in L^2(\mathbf{H}^1), \quad \mathbf{u}_{ttt} \in L^2(\mathbf{H}_{b,l}^{-1})$

(H3) $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}_{b,l}^{-1})$

Estas hipótesis sobre la solución son más fuertes que las impuestas para el mismo tipo de esquema en el caso de Navier-Stokes, debido a que tenemos que acotar de forma adecuada la parte vertical de los términos convectivos (que son ahora menos regulares), lo que obliga a exigir más regularidad a la solución $(\mathbf{u}, p_s)$.

Razonando como en el Lema 42, ahora para $(E_2)^{m+1}$, podemos obtener el siguiente resultado:

Lema 45 *(Dependencia continua de los errores).*
a) *Dependencia continua con respecto a L^2. Si $\widetilde{\mathbf{e}}^{m+1} \in L^2(\Omega)$, entonces $\mathbf{e}^{m+1} \in \mathbf{H}$. Además,*

$$\|\mathbf{e}^{m+1}\|_{L^2} \le \|\widetilde{\mathbf{e}}^{m+1}\|_{L^2} \quad y \quad \|\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}\|_{L^2} \le \|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^{m}\|_{L^2},$$

y se tiene

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{L^2}^2 = \|\mathbf{e}^{m+1}\|_{L^2}^2 + \|k\,\nabla_H(e_{p,s}^{m+1} - e_{p,s}^{m} - k\,\delta_t p_s(t_{m+1}))\|_{L^2}^2 = \|\mathbf{e}^{m+1}\|_{L^2}^2 + \|\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}\|_{L^2}^2.$$

b) *Dependencia continua con respecto a H^1. Suponiendo $S \in C^3$ y $D \in W^{1,\infty}(S)$. Si $\widetilde{\mathbf{e}}^{m+1} \in \mathbf{H}_{b,l}^1(\Omega)$, entonces $\mathbf{e}^{m+1} \in \mathbf{H}^1(\Omega)$. Además, existe $C = C(\Omega, D) > 0$ tal que*

$$\|\mathbf{e}^{m+1}\|_{H^1} \le C\,\|\widetilde{\mathbf{e}}^{m+1}\|_{H^1}.$$

Podemos ahora probar las estimaciones de error de orden $O(k)$ para las velocidades. Así, obtenemos

Teorema 46 *Suponiendo* **(H1)** *y $\|\nabla_H e_{p,s}^0\|_{L^2} \le C$. Entonces, para k suficientemente pequeño,*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k.$$

Además,

$$\|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^{m}\|_{l^2(\mathbf{L}^2)} \le C\,k^{3/2}.$$

Una consecuencia importante del resultado anterior es la obtención de las estimaciones de estabilidad de la velocidad en una norma más fuerte. Concretamente se obtiene

Lema 47 *Suponiendo las hipótesis del Teorema 46, se tiene para k suficientemente pequeño,*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C, \qquad \forall m.$$

De aquí concluimos la estimación de estabilidad para el esquema

$$\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} \le C \qquad \forall m,$$

que nos permitirá abordar la estimación de error de orden $O(k)$ para la presión. Para ello, primero obtenemos estimaciones de error para las derivadas discretas de las velocidades.

También ahora se tiene la dependencia continua para las derivadas discretas de los errores.

Lema 48 *(Dependencia continua de las derivadas discretas)*

$$\|\delta_t \mathbf{e}^{m+1}\|_{L^2} \le \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{L^2}, \qquad \|\delta_t \mathbf{e}^{m+1} - \delta_t \widetilde{\mathbf{e}}^{m+1}\|_{L^2} \le \|\delta_t \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^{m}\|_{L^2}.$$

Además, existe $C = C(\Omega) > 0$ tal que

$$\|\delta_t \mathbf{e}^{m+1}\|_{H^1} \le C \, \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{H^1}.$$

$$\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{L^2}^2 = \|\delta_t \mathbf{e}^{m+1}\|_{L^2}^2 + \|\delta_t \mathbf{e}^{m+1} - \delta_t \widetilde{\mathbf{e}}^{m+1}\|_{L^2}^2$$

Este resultado es utilizado en la obtención de las siguientes estimaciones

Teorema 49 *Suponiendo las hipótesis del Teorema 46,* (**H2**) *y la siguiente restricción sobre la aproximación inicial*

$$\|\delta_t \mathbf{e}^1\|_{L^2} + \|k \, \nabla_H \delta_t e_p^1\|_{L^2} \le C \, k,$$

entonces para k suficientemente pequeño,

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)} + \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2) \cap l^2(H^1)} \le C \, k.$$

Finalmente, podemos demostrar el orden óptimo para el error en velocidad $\widetilde{\mathbf{e}}^{m+1}$ en $l^\infty(\mathbf{H}^1)$ y para el error de presión en $l^\infty(L^2)$.

Teorema 50 *Suponiendo las hipótesis del Teorema 49 y* (**H3**)*, entonces*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} \, + \, \|e_{p,s}^{m+1}\|_{l^\infty(L^2)} \le C \, k.$$

4.2.3. Conclusiones

Las dificultades inherentes al problema de EP, hacen que debamos suponer hipótesis de regularidad más fuertes para la solución que en el caso de NS y debamos emplear técnicas anisótropas en la obtención de los resultados.

En cuanto a los dos métodos usados como aproximación al problema de EP, mientras que en el método de descomposición de la viscosidad debemos discretizar el problema totalmente y obtener los resultados bajo una restricción del tipo $k \leq \alpha\, h^2$ (debido al uso de desigualdades inversas), no pudiendo emplear la semidiscretización en tiempo como un problema auxiliar, en el caso de proyección esto sí es posible, pudiendo obtener estimaciones de error óptimas en el caso semidiscreto en tiempo. Además, en este último caso se han conseguido mejores estimaciones para la presión, ya que en el método con descomposición de la viscosidad eran estimaciones óptimas en $l^2(L^2)$, y ahora se tienen en $l^\infty(L^2)$. En contrapartida, en descomposición de la viscosidad es posible un argumento de dualidad para evitar las hipótesis sobre la etapa inicial, mientras que en proyección no funciona un razonamiento similar y además hay que imponer hipótesis sobre la presión inicial.

Por otra parte, en el caso de proyección, no está claro cómo discretizar en espacio la velocidad vertical. Podríamos cambiar la etapa $(S_0)^m$ por $(S_0)^m_h$ como hicimos en el caso con descomposición de la viscosidad para mallas no estructuradas en vertical, pero entonces no se corresponde con el esquema semidiscreto estudiado en el Capítulo 5. Dicho de otro modo, no está claro cómo el problema semidiscreto tendría una versión totalmente discreta en una malla no estructurada, porque tenemos que discretizar en espacio la velocidad vertical $\widetilde{u}_3^m$ (que no es cero en el fondo porque está en función de la velocidad $\widetilde{\mathbf{u}}^m$ que no verifica la restricción $\nabla_H \cdot \langle \widetilde{\mathbf{u}}^m \rangle = 0$).

Destacar también que en el caso de descomposición de la viscosidad, las estimaciones óptimas para las derivadas discretas, no funcionan con aproximación de orden $O(h)$, siendo necesario o bien considerar una aproximación cuadrática $O(h^2)$ o bien considerar una aproximación lineal $O(h)$ con una malla estructurada en vertical de elementos finitos y una modificación del esquema respecto al cálculo de la velocidad vertical.

5. Posibles extensiones y problemas abiertos

Ante los resultados obtenidos en esta memoria, cabe preguntarse por algunas cuestiones que quedan todavía pendientes.

- Respecto al problema de NS aproximado con el método de descomposición de la viscosidad (Capítulos 1 y 2), hemos obtenido, usando el caso semidiscreto en tiempo como un problema auxiliar para obtener las estimaciones de los errores totales, el orden óptimo para la velocidad y presión bajo una condición relativa a los parámetros $h^2 \leq \alpha\, k$.

Cabe preguntarse entonces qué tipo de condición aparecería si hubiéramos usado el argumento de discretizar totalmente el problema.

Observando los argumentos hechos en el caso del problema de Ecuaciones Primitivas (Capítulo 4) y quedándonos sólo con las cotas horizontales isótropas, parece que las estimaciones se pueden obtener bajo la restricción $k \leq \alpha\, h$.

- Respecto al problema de NS, para el método de proyección incremental segregando la presión, hemos obtenido en el Capítulo 3 (usando el problema semidiscreto en tiempo como problema auxiliar) las estimaciones bajo la restricción $h \leq \alpha\, k$. Por otra parte, Guermond y Quartapelle en [45] para el esquema con una formulación mixta velocidad-presión en el paso de proyección, obtuvieron las estimaciones de error bajo la condición $k^2 \leq \alpha\, h$. En esta caso cabe preguntarse, ¿para este método segregado es posible realizar una argumento discretizando totalmente el problema como lo han hecho Guermond y Quartapelle para una formulación mixta velocidad-presión?. Hay que tener en cuenta que en caso afirmativo y con una restriccion del tipo $k \leq \alpha\, h$, esto significaría que se tienen las estimaciones de error para el esquema sin restricciones sobre los parámetros de discretización.

- Respecto al problema de NS, a la vista de los resultados obtenidos para los dos esquemas estudiados, resulta que el análisis numérico funciona un poco mejor en el método con descomposición de la viscosidad, mientras que desde el punto de vista de la implementación tiene menor costo computacional el método de proyección, al desacoplar el cálculo de la velocidad y la presión. De hecho, el esquema de proyección tiene sentido usarlo incluso para el problema de Stokes, ya que separa la viscosidad de la incompresibilidad. Entonces, parece natural pensar en combinar ambos métodos, de manera que primero usemos el método de descomposición de la viscosidad y, después, aproximemos el problema de Stokes de la segunda subetapa mediante el método de proyección. Resulta interesante preguntarse por el análisis numérico de este esquema combinado.

- Para aproximar las EP con el método de descomposición de la viscosidad, se obtienen estimaciones óptimas de error bajo la restricción $k \leq \alpha\, h^2$. Ahora bien, no está claro cómo obtener estimaciones de error con hipótesis en cierto sentido contrarias, del tipo h pequeño en función de k.

- El estudio de las EP por el método de proyección incremental, ha sido realizado en el caso semidiscreto en tiempo. Entonces, ¿ podemos extender estas estimaciones al caso de un esquema totalmente discreto razonando como hemos hecho en NS ?

  Con los mismos argumentos que hemos realizado en NS parece que esto sí es posible, pero las estimaciones se consiguen para l^2 en tiempo, en vez de l^∞.

Algunos de los resultados obtenidos en la presente memoria, junto con las posibles extensiones y problemas abiertos anteriores, pueden observarse de forma esquemática en las siguientes tablas. En la Figura 1 para el problema de NS y en la Figura 2 para las EP.

Como **conclusiones finales**, observamos en la Figura 1, que para el caso de NS aproximado por el método de descomposición de la viscosidad, se pueden obtener estimaciones de error por dos caminos distintos (usando el problema semidiscreto en tiempo como problema auxiliar o discretizando totalmente de una vez) obteniendo las restricciones $h^2 \leq \alpha\, k$ y $k \leq \alpha\, h$ respectivamente. En consecuencia, se consiguen estimaciones de error sin restricciones para los parámetros, por tanto incondicionales.

Para el método de proyección incremental en presión, las restricción es $h \leq \alpha\, k$ y no está claro si se consiguen estimaciones de error con hipótesis del tipo $k \leq \alpha\, g(h)$. Por tanto, queda un rango de valores de los parámetros k y h para el que no han quedado demostradas las estimaciones óptimas de error, consiguiendo por tanto estimaciones de error condicionales.

En cuanto a las EP, observamos en la Figura 2, que con el método de descomposición de la viscosidad, la obtención de las estimaciones óptimas sólo ha sido posible discretizando totalmente el problema (esto era debido básicamente a que la velocidad final $\mathbf{u}^{m+1}$ es la mejor aproximación a la solución, y sin embargo, los términos convectivos dependen del gradiente de la velocidad intermedia $\mathbf{u}^{m+1/2}$). Para el esquema de proyección, sí es posible, puesto que la velocidad intermedia $\widetilde{\mathbf{u}}^{m+1}$ es ahora la mejor aproximación. De hecho, la velocidad proyectada puede ser tratada como una velocidad auxiliar para obtener los resultados de análisis numérico, que no es usada en la implementación efectiva del esquema.

Figura 1: **Navier-Stokes**: restricciones sobre parámetros y estimaciones de error

NS-**Descomposición viscosidad** (esquema auxiliar semidiscreto en tiempo)	$h^2 \leq \alpha\, k$	$O(k+h)$ $\mathbf{e}^{m+1}$ en $l^\infty(\mathbf{H}^1)$ e_p^{m+1} en $l^\infty(L^2)$
NS-**Descomposición viscosidad** (discretización total)	Parece: $k \leq \alpha\, h$ $(2D)$ $k \leq \alpha\, h^{3/2}$ $(3D)$	$O(k+h)$ $\mathbf{e}^{m+1}$ en $l^\infty(\mathbf{H}^1)$ e_p^{m+1} en $l^\infty(L^2)$
NS-**proyección no incremental** formulación mixta (problema auxiliar) con condición inf-sup. Badia-Codina[7]	$h^2 \leq \alpha\, k$	$O(k^{1/2}+h)$ $\widetilde{\mathbf{e}}^{m+1}$ en $l^\infty(\mathbf{H}^1)$ e_p^{m+1} en $l^\infty(L^2)$
NS-**proyección no incremental** problema segregado (problema auxiliar) sin condición inf-sup. Badia-Codina[7]	$\alpha\, h^2 \leq k \leq \beta\, h^2$	$O(k^{1/2}+h)$ $\widetilde{\mathbf{e}}^{m+1}$ en $l^\infty(\mathbf{H}^1)$ e_p^{m+1} en $l^\infty(L^2)$
NS-**proyección incremental** formulación mixta (discretización total) Guermond-Quartapelle[45]	$k^2 \leq \alpha\, h$	$O(k+h)$ $\widetilde{\mathbf{e}}^{m+1}$ en $l^\infty(\mathbf{H}^1)$ e_p^{m+1} en $l^\infty(L^2)$
NS-**proyección incremental** problema segregado (problema auxiliar semidiscreto en tiempo)	$h \leq \alpha\, k$	$O(k+h)$ $\widetilde{\mathbf{e}}^{m+1}$ en $l^\infty(\mathbf{H}^1)$ e_p^{m+1} en $l^\infty(L^2)$
NS-**proyección incremental** problema segregado (con discretización total)	¿$k \leq g(h)$?	¿?

Figura 2: **Ecuaciones Primitivas**: restricciones sobre parámetros y estimaciones de error

EP-**Descomposición viscosidad** (discretización total)	$k \leq \alpha\, h^2$	$O(k^{1/2} + h) = O(h)$ $(l = 1)$ o $O(k + h^2) = O(h^2)$ $(l = 2)$, $\mathbf{e}^{m+1}$ en $l^2(\mathbf{H}^1)$ e_p^{m+1} en $l^2(L^2)$
EP-**Descomposición viscosidad** (problema auxiliar semidiscreto en tiempo)	No parece posible: $h \leq f(k)$ Hipótesis esquema	¿?
EP-**Proyección** problema segregado (problema auxiliar semidiscreto en tiempo)	Parece: $h \leq \alpha\, k$ ¿$k \leq g(h)$?	Parece: $O(k + h)$ $\widetilde{\mathbf{e}}^{m+1}$ en $l^2(\mathbf{H}^1)$ e_p^{m+1} en $l^2(L^2)$

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New error estimates for a viscosity-splitting scheme in time for the $3D$ Navier-Stokes equations *

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Abstract

This work is devoted to the error analysis of a semi-discrete in time splitting scheme (using decomposition of the viscosity) for solving the incompressible time-dependent Navier-Stokes equations in 3D. This scheme has been previously studied by other authors but the originality of the present work is that it establishes for the first time an optimal error estimate for the pressure. This behavior had been observed numerically (with fully discrete schemes), but never hitherto proved. The proof is based on sharp estimates of second derivatives of the error.

Subject Classification 35Q30, 65N15, 76D05.
Keywords Navier-Stokes Equations, splitting in time schemes, error estimates, first order time scheme.

Introduction

We consider the Navier-Stokes system, associated to the dynamics of viscous and incompressible fluids filling a bounded domain $\Omega \subset I\!R^3$ in a time interval $(0,T)$:

$$(P) \qquad \left\{ \begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u} &=& \mathbf{0} & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}_{|t=0} &=& \mathbf{u}_0 & \text{in } \Omega. \end{array} \right.$$

where the unknowns are $\mathbf{u} : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R^3$ the velocity field and $p : (\mathbf{x},t) \in \Omega\times(0,T) \to I\!R$ the pressure, and data are $\nu > 0$ the viscosity coefficient (which simplicity is

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assumed constant) and $\mathbf{f} : (\mathbf{x}, t) \in \Omega \times (0,T) \to I\!R^3$ the external forces. We denote by ∇ the gradient operator and Δ the Laplace operator.

We consider a (regular) partition of $[0,T]$ of diameter $k = T/M$: $t_0 = 0, t_1 = k, \ldots, t_m = mk, \ldots, t_M = T$. If $u = (u^m)_{m=0}^M$ is a given vector with $u^m \in X$ (a Banach space), let us introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^{M} \|u^m\|_X^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max\nolimits_{m=0,\ldots,M} \|u^m\|_X$$

For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., and $\mathbf{H^1} = H^1(\Omega)^3$ etc.

The numerical analysis for the Navier-Stokes problem (P) has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties of this problem are the coupling between the pressure p and the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ and the nonlinearity of the convective terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

Fractional step methods are becoming widely used in this context, which split effects due to different operators appearing in the problem.

The origin of these methods is generally credited to the works of Chorin [5] and Temam [18]. They developed the well known projection method, which is a two step scheme where the second step is a free divergence projection step. The main drawback of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure verifies "artificial" boundary conditions. The convergence of this projection method, was proved in [19] for the time discrete scheme and in [6] for a fully discrete scheme associated to a problem with periodic boundary conditions.

More recently, error estimates for projection methods have been obtained (see [16], [17] for time discrete schemes and see [10] for a fully discrete scheme). Basically, for the so-called Chorin-Temam projection scheme, one has time error estimates of order $O(k^{1/2})$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ and of order $O(k)$ in $l^2(\mathbf{L}^2)$ for both velocities and order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure, improving to order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the intermediate velocity and order $O(k)$ in $l^2(L^2)$ for the pressure, for a modified projection scheme (called incremental pressure or Van-Kan scheme) where a pressure correction term is added in the projection step. Some variants of the projection scheme and its numerical analysis can be seen in [15].

Another class of fractional-step methods, called viscosity splitting methods (where viscosity is not fully decoupled from incompressibility), have also been studied. A fully discrete version of the so called θ-scheme (see [9]), was studied in [7] proving stability and convergence.

In this paper we provide new error estimates for a viscosity splitting fractional-step method, which was introduced and studied in [1], [2], [3] and [4]. It is a two-step scheme splitting the non-linearity and the incompressibility of the problem into different steps (but keeping viscosity term and boundary conditions in both steps). Basically, the corresponding time discrete scheme can be described as follows. Given $\mathbf{u}^m$ an approximation of $\mathbf{u}(t_m)$, first one computes an intermediate velocity $\mathbf{u}^{m+1/2}$ (as a first approximation of $\mathbf{u}(t_{m+1})$) by means of a convection-diffusion problem, and afterwards $(\mathbf{u}^{m+1}, p^{m+1})$ (as approximation of $(\mathbf{u}(t_{m+1}), p(t_{m+1}))$) is obtained solving a Stokes type problem. It allows to enforce, in both steps, the original boundary conditions of the problem, which leads to the convergence to a weak solution of (P) in the $\mathbf{H}_0^1(\Omega)$-norm (see [1], [2]). In fact, firstly a priori stability estimates for both velocities $\mathbf{u}^{m+1/2}$ and $\mathbf{u}^{m+1}$ in $l^\infty(\mathbf{L^2}) \cap l^2(\mathbf{H}^1)$ are obtained, and afterwards, a pass to the limit yields to the convergence, where compactness results must be applied to "control" the limit of the convective terms.

On the other hand, error estimates of order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the end-of-step velocity $\mathbf{u}^{m+1}$ and order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure p^{m+1} have also been obtained in [3]. Moreover, these estimates of the time scheme are used in [4] to obtain the following error estimates for a fully discrete scheme (which solutions are denoted as $\mathbf{u}_h^{m+1/2}$ and $(\mathbf{u}_h^{m+1}, p_h^{m+1})$), based on finite element approximations of order $O(h)$ in $\mathbf{H}^1$ for the velocity and $O(h)$ in L^2 for the pressure:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(L^2)\cap l^2(H^1)} \leq C\,(k+h)$$

under the constraint $h^2 \leq C\,k$.

In [8], this viscosity-splitting in time scheme is studied jointly with Galerkin discontinuous finite element in space (with $P_1 \times P_0$ discrete spaces), obtaining order $O(k+h)$ in $l^\infty(\mathbf{L}^2)$ for the velocity and order $O(\sqrt{k}+h)$ in $l^2(L^2)$ for the pressure.

On the other hand, numerical computations were done in [2], driving to order $O(k)$ in $L^2(\Omega)$ for velocity and pressure. Consequently, there is a gap between the numerical analysis (that gives $O(\sqrt{k})$) and the numerical computations (that gives $O(k)$) respect to the approximation in time for the pressure. In this paper we aim to fill this gap.

Basically, the objectives of this work are:

1. To improve the order of error estimate in pressure, from $O(\sqrt{k})$ to $O(k)$.
2. To improve the norm of error estimates in velocity and pressure, concretely from $l^\infty(\mathbf{L}^2)$ to $l^\infty(\mathbf{H}^1)$ in velocity and from $l^2(L^2)$ to $l^\infty(L^2)$ in pressure.

Due to these improvements, projection schemes with incremental pressure and the viscosity splitting scheme studied in this work, are fully comparable. In this sense, in [2] there are numerical computations comparing both schemes with respect to the time errors. Moreover, the viscosity splitting scheme studied in this work has the same analytical results as Euler-type schemes [20], improving their numerical treatment (since the main difficulties are split).

The extension of error estimates of this paper to the case of finite element approximation in space have also been obtained and will be the theme of a forthcoming paper [12]. The main results of this paper have been announced in [11].

This paper is organized as follows:

In Section 1, we describe the scheme, introducing the problems verified by the errors and the regularity hypotheses on the exact solution, which must be imposed throughout the paper.

In Section 2, some known results will be presented explaining the main ideas of the proofs (for the reader's convenience). Afterwards, we will obtain new $O(k^{1/2})$ error estimates for $\mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}$ and $\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}$ in $l^\infty(\mathbf{H}^1) \cap l^2(\mathbf{H}^2)$ and for $e_p^m = p(t_m) - p^m$ in $l^2(H^1)$. Previous error estimates will be used to obtain $O(k^{1/2})$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ for the discrete in time derivative of $\mathbf{e}^{m+1/2}$ and $\mathbf{e}^{m+1}$, which are applied to get $O(k)$ for the discrete in time derivative of $\mathbf{e}^{m+1}$, either in $l^2(\mathbf{L}^2)$ or in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$, where a constraint for the first step of the scheme must be imposed in the last case (in fact, these two estimates are obtained independently). As a consequence, the improvement of the pressure error estimates to order $O(k)$ either in $l^2(L^2)$ or in $l^\infty(L^2)$ hold, respectively.

In this paper, the following discrete Gronwall's lemma will be frequently used (for a proof, see [14, p. 369]):

Lemma 1 *Let k, B and a_m , b_m , c_m , γ_m be nonnegative numbers.*

a) (Discrete Gronwall inequality) *We assume*

$$a_{r+1} + k \sum_{m=0}^{r} b_m \le k \sum_{m=0}^{r} \gamma_m a_m + k \sum_{m=0}^{r} c_m + B \qquad \forall r \ge 0.$$

Then, one has

$$a_{r+1} + k \sum_{m=0}^{r} b_m \le \exp\left(k \sum_{m=0}^{r} \gamma_m\right) \left\{k \sum_{m=0}^{r} c_m + B\right\} \qquad \forall r \ge 0.$$

b) (Generalised discrete Gronwall inequality) *We assume*

$$a_r + k \sum_{m=0}^{r} b_m \le k \sum_{m=0}^{r} \gamma_m a_m + k \sum_{m=0}^{r} c_m + B \qquad \forall r \ge 0$$

such that $k\gamma_m < 1$ for all m. Then, setting $\sigma_m \equiv (1 - k\gamma_m)^{-1}$, one has

$$a_r + k \sum_{m=0}^{r} b_m \le \exp\left(k \sum_{m=0}^{r} \sigma_m \gamma_m\right) \left\{k \sum_{m=0}^{r} c_m + B\right\} \qquad \forall r \ge 0.$$

1 Time discrete scheme

1.1 Description of the scheme

Given a (uniform) partition of the time interval $[0, T]$ with diameter $k = T/M$, $\{t_m = m\,k\}_{m=0}^M$, and $(\mathbf{f}^m)_{m=1}^M$ an approximation of $\mathbf{f}(t_m)$, we will define $(\mathbf{u}^m, p^m)_{m=1}^M$ an approximation of the solution $\{\mathbf{u}, p\}$ of (P) in $t = t_m$, by means of a two-step scheme (introduced in [1, 3]) splitting the nonlinearity $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ into different steps (but keeping viscosity terms and boundary conditions in both steps):

Initialization: $\mathbf{u}^0 = \mathbf{u}_0$ (for simplicity, the exact initial condition is taken, although to change it by an adequate approximation is also possible).

Time step $m+1$:

Substep 1: Given $\mathbf{u}^m$, to find $\mathbf{u}^{m+1/2}$ solution of

$$(S_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1/2} - \mathbf{u}^m) + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1/2} - \nu\Delta\mathbf{u}^{m+1/2} = \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \mathbf{u}^{m+1/2}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Substep 2: Given $\mathbf{u}^{m+1/2}$, to find $\mathbf{u}^{m+1}$ and p^{m+1} solution of

$$(S_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) - \nu\Delta(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) + \nabla p^{m+1} = \mathbf{0} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{u}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

In respect of the effective resolution of this scheme, in each time step, it will be necessary to compute $(S_1)^{m+1}$ as three linear convection-diffusion equations (the system is uncoupled by components) and $(S_2)^{m+1}$ as a Stokes problem.

Adding $(S_1)^{m+1}$ and $(S_2)^{m+1}$, we arrive at

$$(S_3)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m) + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1/2} - \nu\Delta\mathbf{u}^{m+1} + \nabla\, p^{m+1} = \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{u}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Remark 2 $(S_3)^{m+1}$ *can be viewed as consistency relations, because the idea to prove the convergence of the scheme (see [1]) is to demonstrate that* $\mathbf{u}^{m+1/2}$ *and* $\mathbf{u}^{m+1}$ *converge to the same limit. Therefore, taking limits in* $(S_3)^{m+1}$*, a solution of the continuous problem* (P) *is found.*

1.2 Differential problems verified by the errors

We will obtain error estimates (for velocity and pressure) with respect to a sufficiently regular (unique in particular) solution $(\mathbf{u}, p)$ of (P). For simplicity and without loss of generality, we fix the viscosity constant $\nu = 1$.

We introduce the following notations for the errors at $t = t_{m+1}$:

$$\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}, \quad \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad e_p^{m+1} = p(t_{m+1}) - p^{m+1},$$

and for the discrete in time derivative of errors

$$\delta_t\mathbf{e}^{m+1} = \frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{k}, \qquad \delta_t\mathbf{e}^{m+1/2} = \frac{\mathbf{e}^{m+1/2} - \mathbf{e}^{m-1/2}}{k},$$

Subtracting $(S_1)^{m+1}$ with (P) in $t = t_{m+1}$, using the integral form of remainder and manipulating the convective terms, one has [1, 3] (for simplicity, we take $\mathbf{f}^{m+1} = \mathbf{f}(t_{m+1})$):

$$(E_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}^{m+1/2} - \mathbf{e}^m) - \Delta\mathbf{e}^{m+1/2} = -\nabla\, p(t_{m+1}) + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \\ \mathbf{e}^{m+1/2}|_{\partial\Omega} = \mathbf{0}, \end{cases}$$

where

$$\mathcal{E}^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t-t_m)\, \mathbf{u}_{tt}(t)\, dt - \Big(\int_{t_m}^{t_{m+1}} \mathbf{u}_t \cdot \nabla\Big)\, \mathbf{u}(t_{m+1}) := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1}$$

is the consistency error, and

$$\mathbf{NL}^{m+1} = -\Big(\mathbf{e}^m \cdot \nabla\Big)\mathbf{u}(t_{m+1}) - (\mathbf{u}^m \cdot \nabla)\mathbf{e}^{m+1/2}$$

are residual terms appearing in the differences of the quadratic terms. These terms can be also rewritten as

$$\mathbf{NL}^{m+1} = -\Big(\mathbf{e}^m \cdot \nabla\Big)\mathbf{u}^{m+1/2} - (\mathbf{u}(t_m) \cdot \nabla)\mathbf{e}^{m+1/2}$$

On the other hand, adding and subtracting the term $\mathbf{u}(t_{m+1})$ in $(S_2)^{m+1}$, we get

$$(E_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}) - \Delta(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}) - \nabla p^{m+1} = \mathbf{0} \quad \text{in } \Omega \\ \nabla \cdot \mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Finally, adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$, we arrive at:

$$(E_3)^{m+1} \qquad \begin{cases} \delta_t \mathbf{e}^{m+1} - \Delta \mathbf{e}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

1.3 Regularity hypotheses.

Let us introduce the following Hilbert spaces:

$$\begin{aligned} \mathbf{H} &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} \cdot \mathbf{n}_{\partial\Omega} = 0\}, \\ \mathbf{V} &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \ : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\}, \end{aligned}$$

where $\mathbf{n}_{\partial\Omega}$ the normal outwards vector to $\partial\Omega$. In the sequel, we will assume the following regularity hypothesis on Ω:

(H0) $\Omega \subset I\!R^3$ such that the Stokes problem in Ω has $\mathbf{H}^2 \times H^1$ regularity for velocity and pressure respectively.

In order to obtain the different error estimates, the following regularity hypotheses for the (unique) solution $(\mathbf{u}, p)$ of (P) will appear:

(H1) $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V})$, $\quad p \in L^\infty(H^1)$, $\quad \mathbf{u}_t \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^1)$, $\quad \sqrt{t}\, \mathbf{u}_{tt} \in L^2(\mathbf{H}^{-1})$

(H2) $\sqrt{t}\mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$

(H3) $\mathbf{u}_{tt} \in L^2(\mathbf{V}')$

(H4) $p_t \in L^2(H^1)$, $\quad \mathbf{u}_t \in L^\infty(\mathbf{H}^1) \cap L^2(\mathbf{H}^2)$, $\quad \mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$, $\quad \sqrt{t}\mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1})$

(H5) $\mathbf{u}_{ttt} \in L^2((\mathbf{H}^2 \cap \mathbf{V})')$

(H6) $\quad \mathbf{u}_{ttt} \in L^2(\mathbf{V}')$

(H7) $\quad \mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1})$

Here and in the sequel, we denote $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{V}'$ the dual space of $\mathbf{H}_0^1(\Omega)$ and $\mathbf{V}$ respectively. Note that $\mathbf{V}'$ and $(\mathbf{H}^2 \cap \mathbf{V})'$ are not spaces of distributions, but $\mathbf{u}_{tt}$ and $\mathbf{u}_{ttt}$ are distributions in $(0,T) \times \Omega$.

Hypothesis (**H1**) can be proved assuming enough regularity on the data. Concretely, one has (**H1**) assuming that ([20]):

$$\mathbf{u}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{V}, \quad \mathbf{f} \in L^\infty(\mathbf{H}), \quad \mathbf{f}_t \in L^2(\mathbf{V}').$$

On the other hand, (**H2**) and (**H3**) are obtained in [13] and [16] provided $\sqrt{t}\mathbf{f}_t \in L^2(\mathbf{L}^2)$ and $\mathbf{f}_t \in L^2(\mathbf{V}')$ respectively.

Unfortunately, in order to obtain hypotheses (**H4**)-(**H7**), it is necessary to assume that $\mathbf{u}_t(0) \in \mathbf{H}^1$, which implies a non local compatibility condition for the data $\mathbf{u}_0$ and $\mathbf{f}$. In particular, it is proved in [13] that this regularity statement can only be valid, if there exists $p_0 \in H^1$ (the initial pressure) solution of the overdetermined Elliptic problem

$$\begin{cases} \Delta p_0 = \nabla \cdot \Big(\mathbf{f}(0) - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 \Big) \quad \text{in } \Omega, \\ \nabla p_0|_{\partial\Omega} = \Big(\Delta \mathbf{u}_0 + \mathbf{f}(0) - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 \Big)|_{\partial\Omega}. \end{cases}$$

Notice that (**H4**) implies (**H5**), differentiating twice respect to the time the momentum system.

The norm and inner product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and $(\cdot,\cdot)$, whereas the semi-norm $|\nabla v|$, which is a norm in $H_0^1(\Omega)$, will be denoted by $\|v\|$. Any other norm in a space X will be denoted by $\|\cdot\|_X$

In the following, by C we will denote different constants, always independent of k.

2 Error estimates for the time scheme

2.1 $0(k^{1/2})$ for both velocities in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}_0^1)$

Theorem 3 *([1]) Assuming hypothesis* **(H1)**, *the following error estimates hold*

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\, k^{1/2} \tag{1}$$

$$||\mathbf{e}^{m+1/2} - \mathbf{e}^{m+1}||_{l^2(\mathbf{L}^2)} + ||\mathbf{e}^{m+1/2} - \mathbf{e}^m||_{l^2(\mathbf{L}^2)} \le C\, k. \tag{2}$$

Proof. The proof of Theorem 3 can be seen in [1]. The main idea of the proof is to consider

$$k \sum_{m=0}^{M-1} \Big\{ ((E_1)^{m+1}, \mathbf{e}^{m+1/2}) + ((E_2)^{m+1}, \mathbf{e}^{m+1}) \Big\},$$

using that $(\nabla p(t_{m+1}), \mathbf{e}^{m+1/2}) = (\nabla p(t_{m+1}), \mathbf{e}^{m+1/2} - \mathbf{e}^m)$ and $(\mathbf{u}^m \cdot \nabla \mathbf{e}^{m+1/2}, \mathbf{e}^{m+1/2}) = 0$. ■

Notice that, (1) implies in particular the uniform estimates for the errors

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \le C \tag{3}$$

and, since $\mathbf{u} \in L^\infty(\mathbf{H}^1)$, the following $l^\infty(\mathbf{H}^1_0)$ estimates for the scheme hold:

$$\|\mathbf{u}^{m+1/2}\| + \|\mathbf{u}^{m+1}\| \le C, \quad \forall\, m.$$

Corollary 4 *Under assumptions of Theorem 3 and* (**H2**)*, the following error estimates hold*

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)} + \|e_p^{m+1}\|_{l^2(H^1)} \le C\, k^{1/2}, \quad \|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{H}^1)} \le C. \tag{4}$$

Proof. The main idea is to consider $((E_3)^{m+1}, \delta_t \mathbf{e}^{m+1})$ and to use the $(\mathbf{H}^2 \times H^1)$ regularity of $(E_3)^{m+1}$. Indeed, thanks to the $(\mathbf{H}^2 \times H^1)$ regularity of the Stokes problem $(E_3)^{m+1}$:

$$\|\mathbf{e}^{m+1}\|^2_{\mathbf{H^2}} + \|e_p^{m+1}\|^2_{H^1} \le C\,(|\delta_t \mathbf{e}^{m+1}|^2 + |\mathcal{E}^{m+1}|^2 + |\mathbf{NL}^{m+1}|^2).$$

On the other hand, multiplying $(E_3)^{m+1}$ by $\delta_t \mathbf{e}^{m+1}$, we obtain

$$|\delta_t \mathbf{e}^{m+1}|^2 + \frac{1}{k}\,(\|\mathbf{e}^{m+1}\|^2 - \|\mathbf{e}^m\|^2 + \|\mathbf{e}^{m+1} - \mathbf{e}^m\|^2) \le C\,(|\mathcal{E}^{m+1}|^2 + |\mathbf{NL}^{m+1}|^2).$$

Then, combining adequately the two previous inequalities, one arrives at

$$\begin{aligned} &|\delta_t \mathbf{e}^{m+1}|^2 + \frac{C_1}{k}\,(\|\mathbf{e}^{m+1}\|^2 - \|\mathbf{e}^m\|^2 + \|\mathbf{e}^{m+1} - \mathbf{e}^m\|^2) + \|\mathbf{e}^{m+1}\|^2_{\mathbf{H^2}} + \|e_p^{m+1}\|^2_{H^1} \\ &\le C_2\,(|\mathcal{E}^{m+1}|^2 + |\mathbf{NL}^{m+1}|^2). \end{aligned} \tag{5}$$

We bound the terms on the right hand side as follows:

$$|\mathcal{E}^{m+1}|^2 \le C \Big\{ \int_{t_m}^{t_{m+1}} |\sqrt{t}\, \mathbf{u}_{tt}|^2 + k \left(\int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|^2_{\mathbf{L}^6} \right) \|\mathbf{u}(t_{m+1})\|^2_{\mathbf{W}^{1,3}} \Big\}$$

$$\begin{aligned} |\mathbf{NL}^{m+1}|^2 &\le C\Big(\|\mathbf{e}^m\| \|\mathbf{e}^m\|_{\mathbf{H}^2} + \|\mathbf{u}(t_m)\|^2_{\mathbf{L}^\infty} \Big) \|\mathbf{e}^{m+1/2}\|^2 + C\,\|\mathbf{u}(t_{m+1})\|^2_{\mathbf{W}^{1,3}}\,\|\mathbf{e}^m\|^2 \\ &\le \varepsilon\,\|\mathbf{e}^m\|^2_{\mathbf{H}^2} + C\,\|\mathbf{e}^m\|^2 + C\,\|\mathbf{e}^{m+1/2}\|^2 \end{aligned}$$

Here, we have used estimates given in (3) and the inequality $\|\mathbf{e}^m\|^2_{\mathbf{L}^\infty} \le C\,\|\mathbf{e}^m\| \|\mathbf{e}^m\|_{\mathbf{H}^2}$.

Now, multiplying by k and adding (5) from $m = 0$ to r, we get

$$\begin{aligned} C_1\,\|\mathbf{e}^{r+1}\|^2 &+ C_1 \sum_{m=0}^{r} \|\mathbf{e}^{m+1} - \mathbf{e}^m\|^2 + k \sum_{m=0}^{r} (|\delta_t \mathbf{e}^{m+1}|^2 + \|\mathbf{e}^{m+1}\|^2_{\mathbf{H^2}} + \|e_p^{m+1}\|^2) \\ &\le C\,k + C\,k^2 + \varepsilon\,k\,C_2 \sum_{m=0}^{r} \|\mathbf{e}^m\|^2_{\mathbf{H}^2} + C\,k \sum_{m=0}^{r} \Big(\|\mathbf{e}^{m+1/2}\|^2 + \|\mathbf{e}^m\|^2 \Big). \end{aligned}$$

Therefore, choosing ε small enough and applying (1), we obtain

$$C_1\,\|\mathbf{e}^{r+1}\|^2 + C_1 \sum_{m=0}^{r} \|\mathbf{e}^{m+1} - \mathbf{e}^m\|^2 + k \sum_{m=0}^{r} (|\delta_t \mathbf{e}^{m+1}|^2 + \|\mathbf{e}^{m+1}\|^2_{\mathbf{H^2}} + \|e_p^{m+1}\|^2) \le C\,k$$

hence (4) is deduced. ■

Notice that, (4) implies in particular,

$$\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^2)} + \|e_p^{m+1}\|_{l^\infty(H^1)} \le C \tag{6}$$

and, since $\mathbf{u} \in L^\infty(\mathbf{H}^2)$ and $p \in L^\infty(H^1)$, one also has

$$\|\mathbf{u}^{m+1}\|_{\mathbf{H}^2} + \|p^{m+1}\| \le C, \quad \forall\, m. \tag{7}$$

Corollary 5 *Assuming hypotheses of Corollary 4, the following error estimates hold*

$$\|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)} \le C\, k, \tag{8}$$

$$\|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} \le C\, k^{1/2}, \quad \|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C. \tag{9}$$

Proof. Multiplying $(E_2)^{m+1}$ by $k\,(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})$ and integrating in Ω, we obtain

$$\begin{aligned} |\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}|^2 + k\, \|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|^2 &= k(\nabla p^{m+1}, \mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}) \\ &\le \varepsilon |\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}|^2 + C\, k^2\, |\nabla p^{m+1}|^2 \end{aligned}$$

Therefore, using (7),

$$|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}|^2 + k\, \|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|^2 \le C\, k^2.$$

On the other hand, multiplying $(E_2)^{m+1}$ by $-k\,\Delta(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})$, we obtain

$$\|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|^2 + k\, |\Delta(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})|^2 \le \varepsilon\, k\, |\Delta(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})|^2 + C\, k\, |\nabla p^{m+1}|^2$$

hence, using again (7), we get

$$\|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|^2 + k\, |\Delta(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})|^2 \le C\, k.$$

■

In particular, using (4), (6) and (9), one has

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)} \le C\, k^{1/2}, \quad \|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C, \tag{10}$$

and, since $\mathbf{u} \in L^\infty(\mathbf{H}^2)$, one concludes

$$\|\mathbf{u}^{m+1/2}\|_{l^\infty(\mathbf{H}^2)} \le C. \tag{11}$$

Remark 6 *In [3], estimates for* $(\mathbf{u}^{m+1}, p^{m+1})$ *in* $l^2(\mathbf{H}^2 \times H^1)$ *and for* $\mathbf{u}^{m+1/2}$ *in* $l^2(\mathbf{H}^2)$ *are deduced, under the constraint* k *small enough. Now, these scheme estimates are improved from* $l^2(0,T)$ *to* $l^\infty(0,T)$ *and without constraints on* k.

Remark 7 *Assuming hypotheses of Corollary 4 and $p \in L^2(H^2_{loc})$, the following error estimates hold*

$$\|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|_{l^2(\mathbf{H}^1_{loc})} \leq C\,k, \quad \|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|_{l^2(\mathbf{H}^2_{loc})} \leq C\,k^{1/2},$$

i.e. for any compact $K \subset \Omega$ and ϕ_K a cut-off function (with $\phi_K = 1$ in K and support$(\phi_K) \subset\subset \Omega$), one has

$$\|\phi_K\,(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})\|_{L^2(\mathbf{H}^1)} \leq C(\phi_K)\,k, \quad \|\phi_K\,(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})\|_{L^2(\mathbf{H}^2)} \leq C(\phi_K)\,k^{1/2}. \quad (12)$$

Indeed, multiplying $(E_2)^{m+1}$ by $-k\,\phi_K\,\Delta(\phi_K\,(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}))$ and integrating by parts (all boundary terms vanish), one has

$$|\nabla(\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}))|^2 + k|\Delta(\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}))|^2$$

$$\leq \varepsilon\,k|\Delta(\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}))|^2 + C(\phi_K)\,k\left(|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}|^2 + |\nabla(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})|^2 + \|e_p^{m+1}\|^2_{H^1}\right)$$

$$+k\int_\Omega \nabla(\nabla(\phi_K\,p(t_{m+1})))\,\nabla(\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})) + k\int_\Omega \nabla(\nabla\phi_K\,p(t_{m+1}))\,\nabla(\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}))$$

$$\leq \varepsilon\,k|\Delta(\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}))|^2 + C(\phi_K)\,k\left(\|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|^2_{\mathbf{H}^1} + \|e_p^{m+1}\|^2_{H^1}\right)$$

$$+\varepsilon\,|\nabla(\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}))|^2 + C(\phi_K)\,k^2\Big(\|\phi_K p(t_{m+1})\|^2_{H^2} + \|p(t_{m+1})\|^2_{H^1}\Big)$$

If we choice ε small enough, multiplying by k and adding in m, one arrives at

$$\|\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})\|^2_{l^2(\mathbf{H}^1)} + k\|\phi_K(\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2})\|^2_{l^2(\mathbf{H}^2)}$$

$$\leq C(\phi_K)\,k\Big(\|\mathbf{e}^{m+1} - \mathbf{e}^{m+1/2}\|^2_{l^2(\mathbf{H}^1)} + \|e_p^{m+1}\|^2_{l^2(H^1)}\Big) + C(\phi_K)\,k^2\Big(\|\phi_K p\|^2_{l^2(H^2)} + \|p\|^2_{l^2(H^1)}\Big).$$

Therefore, we arrive at (12) using (4) and the regularity for the pressure $p \in L^2(H^1) \cap L^2(H^2_{loc})$.

Note that, the estimates of (12) are local in space because a boundary term (which is not possible to bound) appears in the exact pressure term. Since this boundary term vanishes provided $(\nabla p)|_{\partial\Omega} = \mathbf{0}$, in this particular case, these estimates hold until the boundary if we assume $p \in L^2(H^2)$.

2.2 $O(k)$ for $\mathbf{e}^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1_0)$

Theorem 8 *Under assumptions of Corollary 5 and* **(H3)**, *the following error estimate hold:*

$$\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leq C\,k, \quad (13)$$

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} \leq C\,k^{1/2}.$$

Proof. The proof of Theorem 8 can be seen in [3]. The main idea is to consider

$$k\sum_{m=0}^{M-1}((E_3)^{m+1},\mathbf{e}^{m+1})$$

using that the pressure term vanishes and bounding the nonlinear terms as follows

$$\begin{aligned} k(\mathbf{NL}^{m+1},\mathbf{e}^{m+1}) &= -k\,(\mathbf{e}^m\cdot\nabla\mathbf{u}^{m+1/2}+\mathbf{u}(t_m)\cdot\nabla\mathbf{e}^{m+1/2},\mathbf{e}^{m+1}) \\ &= k\,(\mathbf{e}^m\cdot\nabla\mathbf{e}^{m+1},\mathbf{u}^{m+1/2})+k\,(\mathbf{u}(t_m)\cdot\nabla\mathbf{e}^{m+1},\mathbf{e}^{m+1/2}-\mathbf{e}^{m+1}) \\ &\le \varepsilon\,k\,\|\mathbf{e}^{m+1}\|^2+C\,k\,\|\mathbf{u}^{m+1/2}\|^2_{\mathbf{L}^\infty}|\mathbf{e}^m|^2+C\,k\,\|\mathbf{u}(t_m)\|^2_{\mathbf{L}^\infty}\,|\mathbf{e}^{m+1/2}-\mathbf{e}^{m+1}|^2 \\ &\le \varepsilon\,k\,\|\mathbf{e}^{m+1}\|^2+C\,k\,(|\mathbf{e}^m|^2+|\mathbf{e}^{m+1/2}-\mathbf{e}^{m+1}|^2). \end{aligned}$$

For the last inequality, we use **(H1)** and scheme estimates (11). Therefore, the proof can be concluded thanks to discrete Gronwall's Lemma and (2).

■

From (8) and (13), one arrives at

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)}\le C\,k. \tag{14}$$

Remark 9 *Owing to (13) and Remark 7, if* $p\in L^2(H^2)$*, one arrives at*

$$\|\mathbf{e}^{m+1/2}\|_{l^2(\mathbf{H}^1_{loc})}\le C\,k.$$

Moreover, arguing as in the proof of Corollary 4 and Remark 7, assuming that $\mathbf{u}_{tt}\in L^2(\mathbf{L}^2_{loc})$*, it is easy to arrive at the following local in space error estimates*

$$\|\delta_t\mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2_{loc})}+\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1_{loc})\cap l^2(\mathbf{H}^2_{loc})}+\|e_p^{m+1}\|_{l^2(H^1_{loc})}\le C\,k.$$

2.3 $O(k^{1/2})$ for $\delta_t\mathbf{e}^{m+1}$ and $\delta_t\mathbf{e}^{m+1/2}$ in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1_0)$

Making $\delta_t(E_1)^{m+1}:=\dfrac{(E_1)^{m+1}-(E_1)^m}{k}$ and $\delta_t(E_2)^{m+1}:=\dfrac{(E_2)^{m+1}-(E_2)^m}{k}$ respectively, we obtain:

$$(D_1)^{m+1}\qquad \frac{1}{k}(\delta_t\mathbf{e}^{m+1/2}-\delta_t\mathbf{e}^m)-\Delta\delta_t\mathbf{e}^{m+1/2}=-\nabla\delta_t p(t_{m+1})+\delta_t\mathcal{E}^{m+1}+\delta_t\mathbf{NL}^{m+1}$$

$$(D_2)^{m+1}\qquad \frac{1}{k}(\delta_t\mathbf{e}^{m+1}-\delta_t\,\mathbf{e}^{m+1/2})-\Delta(\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2})+\nabla(\delta_t e_p^{m+1}-\delta_t p(t_{m+1}))=\mathbf{0}.$$

Finally, adding $(D_1)^{m+1}$ and $(D_2)^{m+1}$, we get ($\forall\,m\ge1$):

$$(D_3)^{m+1}\qquad \frac{1}{k}(\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^m)-\Delta\delta_t\mathbf{e}^{m+1}+\nabla\delta_t e_p^{m+1}=\delta_t\mathcal{E}^{m+1}+\delta_t\mathbf{NL}^{m+1}$$

Theorem 10 *We assume the hypotheses of Theorem 8 and (**H4**). Then the following error estimates hold*

$$\|\delta_t \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}_0^1)} + \|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}_0^1)} \le C\,k^{1/2},$$

$$\|\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)} + \|\delta_t \mathbf{e}^{m+1/2} - \delta_t\,\mathbf{e}^{m}\|_{l^2(\mathbf{L}^2)} \le C\,k.$$

Proof. We divided the proof into two parts:

1. *Initial error estimate :* $|\delta_t \mathbf{e}^1| \le C\,k^{1/2}$ *and* $\|\delta_t \mathbf{e}^1\| \le C$.

Multiplying $(E_3)^1$ by $k\,\mathbf{e}^1$ (recall that $\mathbf{e}^0 = \mathbf{0}$),

$$\begin{aligned}
|\mathbf{e}^1|^2 + k\,\|\mathbf{e}^1\|^2 &= k\,(\mathcal{E}^1, \mathbf{e}^1) + k\,(\mathbf{NL}^1, \mathbf{e}^1)\\
&\le \varepsilon|\mathbf{e}^1|^2 + C\,k^3\|\mathbf{u}_{tt}\|^2_{L^2(t_0,t_1;\mathbf{L}^2)} + C\,k^3\,\|\mathbf{u}(t_1)\|^2_{\mathbf{H}^2}\|\mathbf{u}_t\|^2_{L^2(t_0,t_1;\mathbf{L}^3)}\\
&+ \varepsilon\,k\,\|\mathbf{e}^1\|^2 + C\,k\,\|\mathbf{u}^0\|^2_{\mathbf{L}^\infty}|\mathbf{e}^{1/2}|^2\\
&\le \varepsilon|\mathbf{e}^1|^2 + C\,k^3 + \varepsilon\,k\,\|\mathbf{e}^1\|^2 + C\,k\,|\mathbf{e}^{1/2}|^2
\end{aligned}$$

Hence, using that $|\mathbf{e}^{1/2}|^2 \le C\,k^2$ (thanks to (14)), we conclude $|\delta_t \mathbf{e}^1|^2 + k\,\|\delta_t \mathbf{e}^1\|^2 \le C\,k$.

2. *Generic estimates for* $\delta_t \mathbf{e}^{m+1}$ *and* $\delta_t \mathbf{e}^{m+1/2}$ $(\forall m \ge 1)$.

Multiplying $(D_1)^{m+1}$ by $2\,k\,\delta_t \mathbf{e}^{m+1/2}$,

$$\begin{aligned}
&|\delta_t \mathbf{e}^{m+1/2}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^m|^2 + 2\,k\|\delta_t \mathbf{e}^{m+1/2}\|^2\\
&= 2\,k(-\nabla\delta_t p(t_{m+1}) + \delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1}, \delta_t \mathbf{e}^{m+1/2}) := I_0 + I_1 + I_2.
\end{aligned} \tag{15}$$

We bound the RHS as follows (using $\delta_t(a_r\,b_s) = a_r\,\delta_t b_s + \delta_t a_r\,b_{s-1}$):

$$I_0 = -2\,k(\nabla\delta_t p(t_{m+1}), \delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^m) \le \varepsilon\,|\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^m|^2 + C\,k\int_{t_m}^{t_{m+1}}\|p_t\|^2$$

$$\begin{aligned}
I_1 &= 2\,k\left(\delta_t\,\mathcal{E}_1^{m+1} + (\int_{t_m}^{t_{m+1}}\mathbf{u}_t)\cdot\nabla\delta_t \mathbf{u}(t_{m+1}) + \delta_t(\int_{t_m}^{t_{m+1}}\mathbf{u}_t)\cdot\nabla\mathbf{u}(t_m), \delta_t \mathbf{e}^{m+1/2}\right)\\
&\le \varepsilon\,k\|\delta_t \mathbf{e}^{m+1/2}\|^2 + C\,k\|\delta_t\,\mathcal{E}_1^{m+1}\|^2_{\mathbf{H}^{-1}}\\
&+ C\,\frac{1}{k}\left\|\int_{t_m}^{t_{m+1}}\mathbf{u}_t\right\|^4_{\mathbf{L}^4} + C\,k\,\|\mathbf{u}(t_m)\|^2_{\mathbf{L}^\infty}\left|\delta_t(\int_{t_m}^{t_{m+1}}\mathbf{u}_t)\right|^2\\
&\le \varepsilon\,k\|\delta_t \mathbf{e}^{m+1/2}\|^2 + C\,k\int_{t_{m-1}}^{t_{m+1}}\|\sqrt{t}\mathbf{u}_{ttt}\|^2_{\mathbf{H}^{-1}}\\
&+ C\,k\left(\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^2_{\mathbf{L}^4}\right)^2 + C\,k\int_{t_{m-1}}^{t_{m+1}}|\sqrt{t}\mathbf{u}_{tt}|^2
\end{aligned}$$

Here, we have applied the following estimates (obtained in [17]):

$$\|\delta_t\,\mathcal{E}_1^{m+1}\|^2_{\mathbf{H}^{-1}} \le C\int_{t_{m-1}}^{t_{m+1}}\|\sqrt{t}\mathbf{u}_{ttt}\|^2_{\mathbf{H}^{-1}} \quad \text{and} \quad \left|\delta_t(\int_{t_m}^{t_{m+1}}\mathbf{u}_t)\right|^2 \le C\int_{t_{m-1}}^{t_{m+1}}|\sqrt{t}\mathbf{u}_{tt}|^2$$

Moreover, we have written

$$k\Big(\int_{t_m}^{t_{m+1}}\mathbf{u}_t\Big)\cdot\nabla\,\delta_t\,\mathbf{u}(t_{m+1}) = \Big(\int_{t_m}^{t_{m+1}}\mathbf{u}_t\Big)\cdot\nabla\left(\int_{t_m}^{t_{m+1}}\mathbf{u}_t\right),$$

hence integrating by parts,

$$2\,k\left((\int_{t_m}^{t_{m+1}}\mathbf{u}_t)\cdot\nabla\,\delta_t\,\mathbf{u}(t_{m+1}),\delta_t\mathbf{e}^{m+1/2}\right)\le C\,\left\|\int_{t_m}^{t_{m+1}}\mathbf{u}_t\right\|^2_{\mathbf{L}^4}\|\delta_t\mathbf{e}^{m+1/2}\|$$

$$\le\varepsilon\,k\,\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,\frac{1}{k}\,\left\|\int_{t_m}^{t_{m+1}}\mathbf{u}_t\right\|^4_{\mathbf{L}^4}.$$

Finally, we have bounded as follows

$$\left\|\int_{t_m}^{t_{m+1}}\mathbf{u}_t\right\|^4_{\mathbf{L}^4}\le\left(\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|_{\mathbf{L}^4}\right)^4\le k^2\left(\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^2_{\mathbf{L}^4}\right)^2.$$

Now, we decompose I_2 (using $\delta_t(a_r\,b_s)=\delta_t a_r\,b_s+a_{r-1}\,\delta_t b_s$) as follows:

$$\begin{aligned}I_2\;&=\;-2\,k\,(\delta_t\mathbf{e}^m\cdot\nabla\mathbf{u}(t_{m+1}),\,\delta_t\mathbf{e}^{m+1/2})-2\,k(\delta_t\mathbf{u}^m\cdot\nabla\mathbf{e}^{m+1/2},\delta_t\mathbf{e}^{m+1/2})\\&-\;2\,k(\mathbf{e}^{m-1}\cdot\nabla\delta_t\mathbf{u}(t_{m+1}),\delta_t\mathbf{e}^{m+1/2})-2\,k\,(\mathbf{u}^{m-1}\cdot\nabla\delta_t\mathbf{e}^{m+1/2},\delta_t\mathbf{e}^{m+1/2}):=\sum_{i=1}^4 J_i\end{aligned}$$

Bounding each J_i term:

$$J_1\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,k\,\|\mathbf{u}(t_{m+1})\|^2_{L^\infty}\,|\delta_t\mathbf{e}^m|^2\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,k\,|\delta_t\mathbf{e}^m|^2$$

$$J_2=2\,k(\delta_t\mathbf{e}^m\,\cdot\nabla\mathbf{e}^{m+1/2},\delta_t\mathbf{e}^{m+1/2})-2\,k(\delta_t\mathbf{u}(t_m)\cdot\nabla\mathbf{e}^{m+1/2},\delta_t\mathbf{e}^{m+1/2}):=J_{21}+J_{22}$$

$$J_{21}\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,k\,\|\mathbf{e}^{m+1/2}\|^2_{\mathbf{H}^2}\,|\delta_t\mathbf{e}^m|^2\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,k\,|\delta_t\mathbf{e}^m|^2$$

$$J_{22}\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,\frac{1}{k}\,\|\mathbf{e}^{m+1/2}\|^2\left(\int_{t_{m-1}}^{t_m}\|\mathbf{u}_t\|_{\mathbf{L}^3}\right)^2\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,k\int_{t_{m-1}}^{t_m}\|\mathbf{u}_t\|^2_{\mathbf{L}^3}$$

$$J_3\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,\frac{1}{k}\,\|\mathbf{e}^{m-1}\|^2\,\left(\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|_{\mathbf{L}^3}\right)^2\le\varepsilon\,k\|\delta_t\mathbf{e}^{m+1/2}\|^2+C\,k\,\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^2_{\mathbf{L}^3}$$

(in order to obtain the last three estimates, we have applied error estimates for $\mathbf{e}^{m+1/2}$ in $l^\infty(\mathbf{H}^2)$ and $O(k^{1/2})$ for $\mathbf{e}^{m+1/2}$ and $\mathbf{e}^{m+1}$ in $l^\infty(\mathbf{H}^1)$ respectively),

$$J_4=0.$$

Applying previous bounds in (15) and choosing ε small enough,

$$\begin{aligned}&|\delta_t\mathbf{e}^{m+1/2}|^2-|\delta_t\mathbf{e}^m|^2+\frac{1}{2}|\delta_t\mathbf{e}^{m+1/2}-\delta_t\mathbf{e}^m|^2+k\,\|\delta_t\mathbf{e}^{m+1/2}\|^2\\&\le C\,k\int_{t_m}^{t_{m+1}}\|p_t\|^2+\,C\,k|\delta_t\mathbf{e}^m|^2+C\,k\int_{t_{m-1}}^{t_{m+1}}\left(\|\sqrt{t}\mathbf{u}_{ttt}\|^2_{\mathbf{H}^{-1}}+|\sqrt{t}\mathbf{u}_{tt}|^2\right)\\&+C\,k\,\Big(\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^2_{\mathbf{L}^4}\Big)^2+C\,k\int_{t_{m-1}}^{t_{m+1}}\|\mathbf{u}_t\|^2_{\mathbf{L}^3}\end{aligned}\tag{16}$$

On the other hand, multiplying $(D_2)^{m+1}$ by $2\,\delta_t\,\mathbf{e}^{m+1}$,

$$\begin{aligned}&|\delta_t\mathbf{e}^{m+1}|^2-|\delta_t\mathbf{e}^{m+1/2}|^2+|\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2}|^2\\&+k\Big\{\|\delta_t\mathbf{e}^{m+1}\|^2-\|\delta_t\mathbf{e}^{m+1/2}\|^2+\|\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2}\|^2\Big\}=0\end{aligned}\tag{17}$$

Making $\sum_{m=1}^{r}\{(16)+(17)\}$ (with any $r<M$), and taking into account that

$$k\sum_{m=1}^{r}\Big(\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|_{\mathbf{L}^4}^2\Big)^2 \le k\Big(\sum_{m=1}^{r}\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|_{\mathbf{L}^4}^2\Big)^2 \le k\,\|\mathbf{u}_t\|_{L^2(\mathbf{L}^4)}^4 \le C\,k$$

(using the Jensen's inequality $\sum_{m=1}^{r} a_m^2 \le (\sum_{m=1}^{r} a_m)^2$ for $a_m=\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|_{L^4}^2$), the Discrete Gronwall's Lemma can be applied, concluding

$$\begin{aligned}|\delta_t\mathbf{e}^{r+1}|^2 \quad &+ \quad \sum_{m=1}^{r}\{|\delta_t\mathbf{e}^{m+1}-\delta_t\,\mathbf{e}^{m+1/2}|^2+\frac{1}{2}|\delta_t\,\mathbf{e}^{m+1/2}-\delta_t\,\mathbf{e}^{m}|^2\}\\ &+ \quad k\sum_{m=1}^{r}\{\|\delta_t\mathbf{e}^{m+1}\|^2+\|\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2}\|^2\} \;\le\; |\delta_t\mathbf{e}^1|^2+C\,k \;\le\; C\,k,\end{aligned}$$

hence the estimates of this theorem can be deduced. ■

Remark 11 *In a similar way to the proof of Corollary 4, now combining* $\Big((D_3)^{m+1},\delta_t\delta_t\mathbf{e}^{m+1}\Big)$ *and the regularity* $\mathbf{H}^2\times H^1$ *of the Stokes problem* $(D_3)^{m+1}$*, we can obtain*

$$\|\delta_t\delta_t\mathbf{e}^{m+1}\|_{l^2(\mathbf{L}^2)}^2+\|\delta_t\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)}^2+\|\delta_t e_p^{m+1}\|_{l^2(H^1)}^2 \le C\,k + C\Big(\|\mathbf{u}_{tt}\|_{L^2(\mathbf{L}^2)}^2+\|\mathbf{u}_t\|_{L^2(\mathbf{H}^1)}^2\Big) \tag{18}$$

where $\delta_t\delta_t\mathbf{e}^{m+1}=\frac{1}{k}(\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^m)$*. Also, order* $O(k^{1/2})$ *is obtained if* $\sqrt{t}\,\mathbf{u}_{ttt}\in L^2(\mathbf{L}^2)$ *and* $\sqrt{t}\,\mathbf{u}_{tt}\in L^2(\mathbf{H}^1)$*.*

Taking into account (18) and arguing as in the proof of Corollary 5, that is making

$$\Big((D_2)^{m+1},-k\,\Delta(\delta_t\mathbf{e}^{m+1}-\delta_t\mathbf{e}^{m+1/2})\Big),$$

we can obtain

$$\|\delta_t\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)}\le C. \tag{19}$$

(and order $O(k^{1/2})$ *imposing higher regularity hypotheses on the exact solution* $\mathbf{u}$*.)*

Finally, from (18) and (19), and using that $\mathbf{u}_t\in L^\infty(\mathbf{H}^1)\cap L^2(\mathbf{H}^2)$*, we get*

$$\|\delta_t\mathbf{u}^{m+1}\|_{l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)}+\|\delta_t\mathbf{u}^{m+1/2}\|_{l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)}\le C.$$

2.4 $0(k)$ for $\delta_t\mathbf{e}^{m+1}$ in $l^2(\mathbf{L}^2)$ and e_p^{m+1} in $l^2(L^2)$

Let us denote by A^{-1} the inverse of the Stokes operator. Indeed, given $\mathbf{u}\in\mathbf{H}^{-1}$, we define $\mathbf{v}=A^{-1}\mathbf{u}\in\mathbf{V}$ as the weak solution of the following Stokes problem (with an associated pressure r)

$$-\Delta\mathbf{v}+\nabla r=\mathbf{u}\ \text{ in }\Omega,\quad \nabla\cdot\mathbf{v}=0\ \text{ in }\Omega,\quad \mathbf{v}=\mathbf{0}\ \text{ on }\partial\Omega.$$

The variational formulation of this problem in a "reduced" form (eliminating the pressure) is:

$$\text{To find }\mathbf{v}\in\mathbf{V}\ \text{ such that }\ (\nabla\mathbf{v},\nabla\mathbf{z})=(\mathbf{u},\mathbf{z})\quad\forall\,\mathbf{z}\in\mathbf{V}. \tag{20}$$

In particular, $\|\mathbf{u}\|_{\mathbf{V}'} = \|\mathbf{v}\| = \|A^{-1}\,\mathbf{u}\|$. In the sequel, we will use the following equalities:

$$(\nabla \mathbf{u}, \nabla(A^{-1}\mathbf{u})) = |\mathbf{u}|^2 \qquad \forall\, \mathbf{u} \in \mathbf{V}, \tag{21}$$

$$(\mathbf{u}, A^{-1}\mathbf{u}) = \|A^{-1}\mathbf{u}\|^2 (= \|\mathbf{u}\|^2_{\mathbf{V}'}) \qquad \forall\, \mathbf{u} \in \mathbf{L}^2. \tag{22}$$

which are obtained taking as test functions in (20) $\mathbf{z} = \mathbf{u}$ and $\mathbf{z} = A^{-1}\mathbf{u}$ respectively. Moreover, using H^2-regularity of the Stokes problem, one has

$$\|A^{-1}\,\mathbf{u}\|_{\mathbf{H}^2} \le C\,|\mathbf{u}| \qquad \forall\, \mathbf{u} \in \mathbf{L}^2.$$

Theorem 12 *Under assumptions of Theorems 10 and* **(H5)***, the following error estimate holds*

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{V}')\cap l^2(\mathbf{L}^2)} \le C\,k$$

provided k is small enough.

Proof. Again, we divide the proof into two steps:

1. *Initial estimate : $\|\delta_t \mathbf{e}^1\|_{\mathbf{V}'} \le C\,k$ and $|\delta_t \mathbf{e}^1| \le C\,k^{1/2}$.*

Multiplying $(E_3)^1$ by $k\,A^{-1}\mathbf{e}^1 \in \mathbf{V}$, and using (21)-(22),

$$\|\mathbf{e}^1\|^2_{\mathbf{V}'} + k\,|\mathbf{e}^1|^2 \le \, C\,k^2\,\|\mathcal{E}^1 + \mathbf{NL}^1\|^2_{\mathbf{V}'}$$

Since $|\mathbf{e}^{1/2}| \le C\,k$ then $\|\mathbf{NL}^1\|_{\mathbf{V}'} \le C\,k$. On the other hand, $\|\mathcal{E}^1\|_{\mathbf{V}'} \le C\,k$. Therefore,

$$\|\delta_t \mathbf{e}^1\|^2_{\mathbf{V}'} + k\,|\delta_t \mathbf{e}^1|^2 \le C\,k^2.$$

2. *Generic estimate of $\delta_t \mathbf{e}^{m+1}$ ($\forall\, m \ge 1$).*

Multiplying $(D_3)^{m+1}$ by $2\,k\,A^{-1}\,\delta_t \mathbf{e}^{m+1}$ (for each $m \ge 1$):

$$\begin{aligned}
&\|\delta_t \mathbf{e}^{m+1}\|^2_{\mathbf{V}'} - \|\delta_t \mathbf{e}^{m}\|^2_{\mathbf{V}'} + \|\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m}\|^2_{\mathbf{V}'} + 2\,k\,|\delta_t \mathbf{e}^{m+1}|^2 \\
&= 2\,k\,(\delta_t \mathcal{E}^{m+1}, A^{-1}\,\delta_t \mathbf{e}^{m+1}) + 2\,k\,(\delta_t \mathbf{NL}^{m+1}, A^{-1}\,\delta_t \mathbf{e}^{m+1}) := I_1 + I_2
\end{aligned}$$

Now, the pressure term vanishes and we bound the RHS as follows:

$$\begin{aligned}
I_1 &= 2\,k\left(\delta_t\,\mathcal{E}_1^{m+1} + (\int_{t_m}^{t_{m+1}} \mathbf{u}_t)\cdot\nabla\delta_t \mathbf{u}(t_{m+1}) + \delta_t(\int_{t_m}^{t_{m+1}} \mathbf{u}_t)\cdot\nabla\mathbf{u}(t_m), A^{-1}\,\delta_t \mathbf{e}^{m+1}\right) \\
&\le \varepsilon\,k|\delta_t \mathbf{e}^{m+1}|^2 + C\,k\|\delta_t\,\mathcal{E}_1^{m+1}\|^2_{(\mathbf{H}^2\cap\mathbf{V})'} \\
&+ C\,\frac{1}{k}\left\|\int_{t_m}^{t_{m+1}} \mathbf{u}_t\right\|^4_{\mathbf{L}^{12/5}} + C\,k\,\|\mathbf{u}(t_m)\|^2_{\mathbf{L}^\infty}\left\|\delta_t(\int_{t_m}^{t_{m+1}} \mathbf{u}_t)\right\|^2_{\mathbf{L}^{6/5}} \\
&\le \varepsilon\,k|\delta_t \mathbf{e}^{m+1}|^2 + C\,k^2\int_{t_{m-1}}^{t_{m+1}} \|\mathbf{u}_{ttt}\|^2_{(\mathbf{H}^2\cap\mathbf{V})'} \\
&+ C\,k^2\int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|^4_{\mathbf{L}^{12/5}} + C\,k^2\int_{t_{m-1}}^{t_{m+1}} \|\mathbf{u}_{tt}\|^2_{\mathbf{L}^{6/5}}
\end{aligned}$$

$$\begin{aligned} I_2 &= -2\,k\,(\delta_t \mathbf{e}^m \cdot \nabla \mathbf{u}(t_{m+1}),\, A^{-1}\,\delta_t \mathbf{e}^{m+1}) - 2\,k(\delta_t\,\mathbf{u}^m \cdot \nabla \mathbf{e}^{m+1/2},\, A^{-1}\,\delta_t \mathbf{e}^{m+1}) \\ &- 2\,k(\mathbf{e}^{m-1} \cdot \nabla \delta_t\,\mathbf{u}(t_{m+1})\,,\, A^{-1}\,\delta_t \mathbf{e}^{m+1}) - 2\,k\,(\mathbf{u}^{m-1} \cdot \nabla\,\delta_t \mathbf{e}^{m+1/2},\, A^{-1}\,\delta_t \mathbf{e}^{m+1}) := \sum_{i=1}^{4} J_i \end{aligned}$$

Bounding each J_i term:

$$J_1 \le \varepsilon\,k\,|\delta_t \mathbf{e}^m|^2 + C\,k\|\delta_t \mathbf{e}^{m+1}\|^2_{\mathbf{V}'}\|\mathbf{u}(t_{m+1})\|^2_{\mathbf{L}^\infty} \le \varepsilon\,k\,|\delta_t \mathbf{e}^m|^2 + C\,k\|\delta_t \mathbf{e}^{m+1}\|^2_{\mathbf{V}'}$$

$$J_2 = 2\,k\Big(\delta_t \mathbf{e}^m \cdot \nabla \mathbf{e}^{m+1/2}, A^{-1}\,\delta_t \mathbf{e}^{m+1}\Big) - 2\,k\Big(\delta_t\,\mathbf{u}(t_m) \cdot \nabla \mathbf{e}^{m+1/2}, A^{-1}\,\delta_t \mathbf{e}^{m+1}\Big) := J_{21} + J_{22}$$

$$J_{21} \le\ \varepsilon\,k\,|\delta_t \mathbf{e}^m|^2 + C\,k\,\|\delta_t \mathbf{e}^{m+1}\|^2_{\mathbf{V}'}\|\mathbf{e}^{m+1/2}\|^2_{\mathbf{L}^\infty} \le\ \varepsilon\,k\,|\delta_t \mathbf{e}^m|^2 + C\,k\,\|\delta_t \mathbf{e}^{m+1}\|^2_{\mathbf{V}'}$$

$$J_{22} \le \varepsilon\,k|\delta_t \mathbf{e}^{m+1}|^2 + C\,\frac{1}{k}\,|\mathbf{e}^{m+1/2}|^2\,\left(\int_{t_{m-1}}^{t_m} \|\mathbf{u}_t\|_{\mathbf{L}^3}\right)^2 \le \varepsilon\,k|\delta_t \mathbf{e}^{m+1}|^2 + C\,k^2\int_{t_{m-1}}^{t_m} \|\mathbf{u}_t\|^2_{\mathbf{L}^3}$$

(in J_{21} we use the estimate $\|\mathbf{e}^{m+1/2}\|_{\mathbf{L}^\infty} \le C$ given in (10) and in J_{22} the $O(k)$ of $\mathbf{e}^{m+1/2}$ in $l^\infty(\mathbf{L}^2)$ given in (14) is applied)

$$J_3 \le \varepsilon\,k|\delta_t \mathbf{e}^{m+1}|^2 + C\,\frac{1}{k}\,|\mathbf{e}^{m-1}|^2\,\left(\int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|_{\mathbf{L}^3}\right)^2 \le \varepsilon\,k|\delta_t \mathbf{e}^{m+1}|^2 + C\,k^2\int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|^2_{\mathbf{L}^3}$$

Now, the term J_4 is not zero, but we can bound it as follows

$$\begin{aligned} J_4 &\le\ C\,k\,\|\mathbf{u}^{m-1}\|_{\mathbf{L}^\infty}\|\,\|\delta_t \mathbf{e}^{m+1}\|_{\mathbf{V}'}\,|\delta_t \mathbf{e}^{m+1/2}| \\ &\le\ C\,k\,\|\mathbf{u}^{m-1}\|^2_{\mathbf{L}^\infty}\,\|\delta_t \mathbf{e}^{m+1}\|^2_{\mathbf{V}'} + \varepsilon\,k\,\Big\{|\delta_t \mathbf{e}^{m+1}|^2 + |\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^{m+1}|^2\Big\} \end{aligned}$$

Adding from $m = 1$ to r (with any $r < M$), taking into account estimates obtained in Theorem 8 and regularity hypotheses for the continuous solution, we get

$$\begin{aligned} &\|\delta_t \mathbf{e}^{r+1}\|^2_{\mathbf{V}'} + \sum_{m=1}^{r}\|\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m\|^2_{\mathbf{V}'} + k\sum_{m=1}^{r}|\delta_t \mathbf{e}^{m+1}|^2 \\ &\quad\le \|\delta_t \mathbf{e}^1\|^2_{\mathbf{V}'} + C\,k\sum_{m=1}^{r}\|\delta_t \mathbf{e}^{m+1}\|^2_{\mathbf{V}'} + C\,k\sum_{m=1}^{r}|\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2}|^2 + C\,k^2 \end{aligned}$$

Thanks to Theorem 10, one has $k\sum_{m=1}^{r}|\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2}|^2 \le C\,k^2$. Therefore, applying the generalized discrete Gronwall's Lemma, the desired estimates hold, for any k small enough. ■

Corollary 13 *Assuming hypotheses of Theorem 12, the following error estimate holds*

$$\|e_p^{m+1}\|_{l^2(L^2)} \le C\,k.$$

Proof. From $(E_3)^{m+1}$, $(\mathbf{e}^{m+1}, e_p^{m+1})$ verifies the following Stokes problem in Ω,

$$-\Delta \mathbf{e}^{m+1} + \nabla\,e_p^{m+1} = \mathbf{F}^{m+1}, \quad \nabla \cdot \mathbf{e}^{m+1} = 0, \quad \mathbf{e}^{m+1}_{|\partial\Omega} = \mathbf{0},$$

where

$$\mathbf{F}^{m+1} = -\delta_t \mathbf{e}^{m+1} + \mathcal{E}^{m+1} - (\mathbf{e}^m \cdot \nabla)\mathbf{u}(t_{m+1}) - (\mathbf{u}^m \cdot \nabla)\mathbf{e}^{m+1/2}.$$

In order to bound e_p^{m+1} in L^2, we consider the $\mathbf{H}^1 \times L^2$ regularity of the previous Stokes problem. Then, we have to bound $\mathbf{F}^{m+1}$ in the $\mathbf{H}^{-1}$-norm. Indeed, for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$:

$$\begin{array}{rcl} (\delta_t \mathbf{e}^{m+1}, \mathbf{v}) & \leq & \|\delta_t \mathbf{e}^{m+1}\|_{\mathbf{H}^{-1}} \|\mathbf{v}\| \leq C\, |\delta_t \mathbf{e}^{m+1}|\, \|\mathbf{v}\| \\ (\mathcal{E}^{m+1}, \mathbf{v}) & \leq & C\, k^{1/2} \left\{ \|\mathbf{u}_{tt}\|_{l^2(t_m, t_{m+1}; \mathbf{H}^{-1})} + \|\mathbf{u}_t\|_{l^2(t_m,t_{m+1};\mathbf{L}^2)} \|\mathbf{u}(t_{m+1})\|_{\mathbf{L}^\infty} \right\} \|\mathbf{v}\| \\ (\, \mathbf{e}^m \cdot \nabla \mathbf{u}(t_{m+1}), \mathbf{v}) & \leq & \|\mathbf{u}(t_{m+1})\|_{\mathbf{L}^\infty}\, |\mathbf{e}^m|\, \|\mathbf{v}\| \leq C\, |\mathbf{e}^m|\, \|\mathbf{v}\| \\ \Big(\mathbf{u}^m \cdot \nabla \mathbf{e}^{m+1/2}, \mathbf{v} \Big) & \leq & C\, \|\mathbf{u}^m\|_{\mathbf{L}^\infty}\, |\mathbf{e}^{m+1/2}|\, \|\mathbf{v}\| \leq C\, |\mathbf{e}^{m+1/2}|\, \|\mathbf{v}\| \end{array}$$

Accordingly, it is easy to deduce

$$\|\mathbf{F}^{m+1}\|_{l^2(\mathbf{H}^{-1})} \leq C\, k,$$

hence we can conclude the proof. ■

Remark 14 *When only $H^{3/2}(\Omega)$ regularity of Stokes problem is assumed instead of* (**H0**) *(for instance, this is the case of a general polygon or polyhedron domain without any additional conditions about its angles), we can repeat estimates of this Subsection, but changing $\mathbf{H}^{-1}$ by $(\mathbf{W}^{1,2+\varepsilon})'$, i.e. bounding the test functions in $\|\mathbf{v}\|_{\mathbf{W}^{1,2+\varepsilon}}$, obtaining that $\|e_p^{m+1}\|_{l^2(L^{2-\varepsilon})} \leq C\, k$. Consequently, the same error estimate as in Corollary 13, $\|e_p^{m+1}\|_{l^2(L^2)} \leq C\, k$, can be obtained assuming $H^{3/2+\varepsilon}$ regularity of Stokes problems, for ε small enough (avoiding additional conditions about the angles of the domain). For this, it is fundamental to use the embedding $H^{3/2+\varepsilon} \hookrightarrow \mathbf{W}^{1,3} \cap \mathbf{L}^\infty$, which is the appropriate space in order to bound the nonlinear terms.*

2.5 $0(k)$ for $\delta_t \mathbf{e}^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and for $(\mathbf{e}^{m+1}, e_p^{m+1})$ in $l^\infty(\mathbf{H}^1 \times L^2)$

In this section, we will prove again order $O(k)$ for discrete in time derivative of end of step velocity, but in higher norms than Theorem 12 (concretely, changing the $l^2(\mathbf{L}^2)$-norm by $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$). This improvement will require an additional hypothesis for the first step of the scheme.

Theorem 15 *We assume the hypotheses of Theorem 10 and* **(H6)**. *Assuming the following approximation hypothesis for the first step of the scheme*

$$|\delta_t \mathbf{e}^1| \leq C\, k,$$

the following error estimates hold

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} \leq C\, k.$$

Remark 16 *Comparing Theorems 12 and 15, in this last result k small enough is not necessary although the initial estimate $|\delta_t \mathbf{e}^1| \leq C\, k$ or equivalently $|\mathbf{e}^1| \leq C\, k^2$ must be imposed. It is not clear that this hypothesis be true in general, because $\mathbf{e}^1$ satisfies*

$$(I - k\Delta)\mathbf{e}^1 + k\nabla e_p^1 = k(\mathcal{E}^1 - \mathbf{u}^0 \cdot \nabla \mathbf{e}^{1/2}), \quad \nabla \cdot \mathbf{e}^1 = 0, \quad \mathbf{e}^1|_{\partial\Omega} = 0,$$

and the RHS is only of order $O(k)$ in the $L^2(\Omega)$-norm due to the term $\mathbf{u}^0 \cdot \nabla \mathbf{e}^{1/2}$. In fact, assuming that $\mathbf{u}_{tt} \in l^\infty(\mathbf{L}^2)$ and $\mathbf{u}_t \in l^\infty(\mathbf{L}^3)$ then $\mathcal{E}^1$ has optimal order $O(k)$ in the $L^2(\Omega)$-norm.

Consequently, if $\mathbf{u}(0) = 0$ then the constraint $|\mathbf{e}^1| \le C\,k^2$ imposed in Theorem 15 holds.

Remark 17 *In the particular case when the solution satisfies $\nabla p(0)|_{\partial\Omega} = \mathbf{0}$, there is other manner to avoid the hypothesis $|\delta_t \mathbf{e}^1| \le C\,k$, defining an "artificial" backward step giving sense to $\delta_t \mathbf{e}^0$ (see the proof of this assertion in Appendix). Moreover, this special solution must satisfy the following additional regularity hypotheses: $\mathbf{u}_{tt} \in L^\infty(\mathbf{L}^2)$ and $\sqrt{t}\mathbf{u}_{ttt} \in L^2(\mathbf{L}^2)$.*

Proof. [of Theorem 15]. Since initial estimate has been assumed, it is suffices to prove the generic estimate for $\delta_t \mathbf{e}^{m+1}$ (for each $m \ge 1$). Multiplying $(D_3)^{m+1}$ by $2\,k\,\delta_t \mathbf{e}^{m+1}$, pressure terms vanish, obtaining:

$$\begin{aligned} &|\delta_t \mathbf{e}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m|^2 + 2\,k \|\delta_t \mathbf{e}^{m+1}\|^2 \\ &\quad = 2\,k(\delta_t \mathcal{E}^{m+1}, \delta_t \mathbf{e}^{m+1}) + 2\,k(\delta_t \mathbf{NL}^{m+1}, \delta_t \mathbf{e}^{m+1}) := I_1 + I_2 \end{aligned} \tag{23}$$

We bound the RHS as follows:

$$\begin{aligned} I_1 &= 2\,k \left(\delta_t \mathcal{E}_1^{m+1} + (\int_{t_m}^{t_{m+1}} \mathbf{u}_t) \cdot \nabla \delta_t \mathbf{u}(t_{m+1}) + \delta_t (\int_{t_m}^{t_{m+1}} \mathbf{u}_t) \cdot \nabla \mathbf{u}(t_m), \delta_t \mathbf{e}^{m+1} \right) \\ &\le \varepsilon\,k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k \|\delta_t \mathcal{E}_1^{m+1}\|^2_{\mathbf{V}'} \\ &+ C \frac{1}{k} \left(\int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|_{\mathbf{L}^4} \right)^4 + C\,k\,\|\mathbf{u}(t_m)\|^2_{\mathbf{L}^\infty} \left| \delta_t (\int_{t_m}^{t_{m+1}} \mathbf{u}_t) \right|^2 \\ &\le \varepsilon\,k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k^2 \int_{t_{m-1}}^{t_{m+1}} \|\mathbf{u}_{ttt}\|^2_{\mathbf{V}'} \\ &+ C\,k^2 \int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|^4_{\mathbf{L}^4} + C\,k^2 \int_{t_{m-1}}^{t_{m+1}} |\mathbf{u}_{tt}|^2 \end{aligned}$$

$$\begin{aligned} I_2 &= -2\,k\,(\delta_t \mathbf{e}^m \cdot \nabla \mathbf{u}(t_{m+1}),\, \delta_t \mathbf{e}^{m+1}) - 2\,k(\delta_t \mathbf{u}^m \cdot \nabla \mathbf{e}^{m+1/2},\, \delta_t \mathbf{e}^{m+1}) \\ &- 2\,k(\mathbf{e}^{m-1} \cdot \nabla \delta_t\, \mathbf{u}(t_{m+1})\,,\, \delta_t \mathbf{e}^{m+1}) - 2\,k\,(\mathbf{u}^{m-1} \cdot \nabla\, \delta_t \mathbf{e}^{m+1/2},\, \delta_t \mathbf{e}^{m+1}) := \sum_{i=1}^4 J_i \end{aligned}$$

Bounding each J_i term:

$$J_1 \le \varepsilon\,k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k\,\|\mathbf{u}(t_{m+1})\|^2_{H^2}\,|\delta_t \mathbf{e}^m|^2$$

$$J_2 = 2\,k\Big(\delta_t \mathbf{e}^m \cdot \nabla \mathbf{e}^{m+1/2}, \delta_t \mathbf{e}^{m+1}\Big) - 2\,k\Big(\delta_t\,\mathbf{u}(t_m) \cdot \nabla \mathbf{e}^{m+1/2}, \delta_t \mathbf{e}^{m+1}\Big) := J_{21} + J_{22}$$

$$J_{21} \le \varepsilon\,k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k\,\|\mathbf{e}^{m+1/2}\|^2_{\mathbf{L}^\infty}\,|\delta_t \mathbf{e}^m|^2 \le \varepsilon\,k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k\,|\delta_t \mathbf{e}^m|^2$$

$$J_{22} \le \varepsilon\,k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k\,\|\delta_t\,\mathbf{u}(t_m)\|^2_{\mathbf{L}^\infty}\,|\mathbf{e}^{m+1/2}|^2 \le \varepsilon\,k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k^2 \int_{t_{m-1}}^{t_m} \|\mathbf{u}_t\|^2_{\mathbf{L}^\infty}$$

(here the $O(k)$-estimate of $\mathbf{e}^{m+1/2}$ in $l^\infty(\mathbf{L}^2)$ given in (14) has been used)

$$J_3 \le \varepsilon\, k \|\delta_t \mathbf{e}^{m+1}\|^2 + C\,\frac{1}{k}\,\|\mathbf{e}^{m-1}\|^2 \left(\int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|_{\mathbf{L}^3}\right)^2 \le \varepsilon\, k\|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k\,\|\mathbf{e}^{m-1}\|^2\,\|\mathbf{u}_t\|^2_{L^\infty(\mathbf{L}^3)}$$

(here it will be necessary $\mathbf{u}_t \in L^\infty(\mathbf{L}^3)$ and the $O(k)$-estimate of $\mathbf{e}^{m-1}$ in $l^2(\mathbf{H}^1)$).

Now, again the term J_4 is not zero, but we can bound it of the following way

$$\begin{aligned} J_4 &\le C\,k\,\|\mathbf{u}^{m-1}\|_{\mathbf{L}^\infty}\,\|\delta_t \mathbf{e}^{m+1}\|\,|\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^{m+1}| \\ &\le \varepsilon\, k\,\|\delta_t \mathbf{e}^{m+1}\|^2 + C\,k\,|\delta_t \mathbf{e}^{m+1/2} - \delta_t \mathbf{e}^{m+1}|^2 \end{aligned}$$

Adding from $m = 1$ to r (with any $r < M$), taking into account estimates obtained in Theorem 8 and regularity hypotheses for the solution $\mathbf{u}$, we get

$$\begin{aligned} |\delta_t \mathbf{e}^{r+1}|^2 &+ \sum_{m=1}^{r} |\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^m|^2 + k \sum_{m=1}^{r} \|\delta_t \mathbf{e}^{m+1}\|^2 \\ &\le C\,k \sum_{m=1}^{r} |\delta_t \mathbf{e}^{m+1} - \delta_t \mathbf{e}^{m+1/2}|^2 + C\,k \sum_{m=1}^{r} |\delta_t \mathbf{e}^m|^2 + C\,k^2. \end{aligned}$$

Thanks to Theorem 10 and applying the discrete Gronwall's Lemma, the proof can be concluded.

■

Corollary 18 *Assuming hypotheses of Theorem 15 and* (**H7**)*, the following error estimates hold*

$$\|e_p^{m+1}\|_{l^\infty(L^2)} \le C\,k \quad \textit{and} \quad \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \le C\,k.$$

Proof. In order to bound $(\mathbf{e}^{m+1}, e_p^{m+1})$ in $\mathbf{H}^1 \times L^2$, we consider again the weak regularity of the Stokes problem in Ω:

$$-\Delta \mathbf{e}^{m+1} + \nabla\, e_p^{m+1} = \mathbf{F}^{m+1}, \quad \nabla \cdot \mathbf{e}^{m+1} = 0, \quad \mathbf{e}^{m+1}{}_{|\partial\Omega} = \mathbf{0},$$

where

$$\mathbf{F}^{m+1} = -\delta_t \mathbf{e}^{m+1} + \mathcal{E}^{m+1} - (\mathbf{e}^m \cdot \nabla)\mathbf{u}(t_{m+1}) - (\mathbf{u}^m \cdot \nabla)\mathbf{e}^{m+1/2}.$$

Then, we have to bound $\mathbf{F}^{m+1}$ in the H^{-1}-norm. Indeed, for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$:

$$\begin{aligned} (\delta_t \mathbf{e}^{m+1}, \mathbf{v}) &\le \|\delta_t \mathbf{e}^{m+1}\|_{\mathbf{H}^{-1}}\|\mathbf{v}\| \le C\,|\delta_t \mathbf{e}^{m+1}|\,\|\mathbf{v}\| \\ (\mathcal{E}^{m+1}, \mathbf{v}) &\le C\,k\left\{\|\mathbf{u}_{tt}\|_{l^\infty(\mathbf{H}^{-1})} + \|\mathbf{u}_t\|_{l^\infty(\mathbf{L}^2)}\|\mathbf{u}\|_{l^\infty(\mathbf{H}^2)}\right\}\|\mathbf{v}\| \le C\,k\,\|\mathbf{v}\| \\ (\,\mathbf{e}^m \cdot \nabla \mathbf{u}(t_{m+1}), \mathbf{v}) &\le C\,\|\mathbf{u}(t_{m+1})\|_{\mathbf{L}^\infty}\,|\mathbf{e}^m|\,\|\mathbf{v}\| \le C\,|\mathbf{e}^m|\,\|\mathbf{v}\| \\ \left(\mathbf{u}^m \cdot \nabla \mathbf{e}^{m+1/2}, \mathbf{v}\right) &\le C\,\|\mathbf{u}^m\|_{\mathbf{L}^\infty}\,|\mathbf{e}^{m+1/2}|\,\|\mathbf{v}\| \le C\,|\mathbf{e}^{m+1/2}|\,\|\mathbf{v}\| \end{aligned}$$

Accordingly, applying $O(k)$ in $l^\infty(\mathbf{L}^2)$ of $\mathbf{e}^{m+1/2}$, $\mathbf{e}^m$ and $\delta_t \mathbf{e}^{m+1}$, one has

$$\|\mathbf{F}^{m+1}\|_{l^\infty(\mathbf{H}^{-1})} \le C\,k$$

and the proof is concluded. ■

Remark 19 *Arguing as in Corollary 18, but bounding* $\mathbf{F}^{m+1}$ *in* $l^\infty(\mathbf{L}^2)$ *instead of in* $l^\infty(\mathbf{H}^{-1})$, *we can obtain order* $O(k^{1/2})$ *for* $(\mathbf{e}^m, e_p^m)$ *in* $l^\infty(\mathbf{H}^2 \times H^1)$.

Appendix

Proof. (of Remark 17)

We consider the following backward step of the semidiscrete scheme:

Given $\mathbf{e}^0 = \mathbf{0}$ and $p^0 = p(0)$, to find $\mathbf{e}^{-1/2}$ such that

$$(E_2)^0 \qquad \left\{ \begin{array}{l} \dfrac{\mathbf{e}^0 - \mathbf{e}^{-1/2}}{k} - \Delta(\mathbf{e}^0 - \mathbf{e}^{-1/2}) - \nabla p^0 = \mathbf{0}, \\ \mathbf{e}^{-1/2}|_{\partial\Omega} = \mathbf{0}, \end{array} \right.$$

that is, $\mathbf{e}^{-1/2}$ is defines as the solution of the elliptic problem:

$$\mathbf{e}^{-1/2} - k\,\Delta \mathbf{e}^{-1/2} = -k\,\nabla\, p(0), \qquad \mathbf{e}^{-1/2}|_{\partial\Omega} = \mathbf{0}.$$

In particular, multiplying by $\mathbf{e}^{-1/2}$, one has

$$|\mathbf{e}^{-1/2}|^2 + k\,\|\mathbf{e}^{-1/2}\|^2 \le C\,k^2\,\|p(0)\|^2 \le C\,k^2.$$

On the other hand, multiplying by $-\Delta \mathbf{e}^{-1/2}$ and imposing $\nabla p(0) \in H_0^1(\Omega)$, one has

$$\|\mathbf{e}^{-1/2}\|^2 + k\,\|\mathbf{e}^{-1/2}\|^2_{\mathbf{H}^2} \le C\,k^2\,\|\nabla p(0)\|^2 \le C\,k^2. \tag{24}$$

On the other hand, given $\mathbf{e}^{-1/2}$, we define $\mathbf{e}^{-1}$ such that

$$(E_1)^0 \qquad \frac{\mathbf{e}^{-1/2} - \mathbf{e}^{-1}}{k} - \Delta\,\mathbf{e}^{-1/2} + \nabla\, p(0) = \mathcal{E}^0 + \mathbf{NL}^0,$$

(that is, $\mathbf{e}^{-1} = -k\,(\mathcal{E}^0 + \mathbf{NL}^0)$) where

$$\mathcal{E}^0 = \mathbf{u}_t(t_0) - \delta_t \mathbf{u}(t_0)$$

(here $\mathbf{u}(t_{-1})$ will be chosen later, see (30)) and, since $\mathbf{u}^0 = \mathbf{u}(0)$,

$$\mathbf{NL}^0 = \mathbf{u}(t_0)\cdot\nabla\mathbf{u}(t_0) - \mathbf{u}^0\cdot\nabla\,\mathbf{u}^{-1/2} = \mathbf{u}(t_0)\cdot\nabla\,\mathbf{e}^{-1/2}.$$

Notice that $\mathbf{u}^{-1}$ can be defined by $\mathbf{u}(t_{-1}) - \mathbf{e}^{-1}$. Moreover, in general $\nabla\cdot\mathbf{e}^{-1} \ne 0$ and $\mathbf{e}^{-1}|_{\partial\Omega} \ne \mathbf{0}$. From (24)

$$|\mathbf{NL}^0| \le \|\mathbf{u}(t_0)\|_{\mathbf{L}^\infty}\|\mathbf{e}^{-1/2}\| \le C\,k. \tag{25}$$

Finally, adding $(E_2)^0$ and $(E_1)^0$, we arrive at:

$$(E_3)^0 \qquad \frac{\mathbf{e}^0 - \mathbf{e}^{-1}}{k} - \Delta\,\mathbf{e}^0 = \mathcal{E}^0 + \mathbf{NL}^0$$

Then, since $\mathbf{e}^0 = 0$, one has

$$\delta_t \mathbf{e}^0 = \mathcal{E}^0 + \mathbf{NL}^0 = \mathbf{u}_t(t_0) - \delta_t \mathbf{u}(t_0) + \mathbf{u}(t_0) \cdot \nabla \mathbf{e}^{-1/2}. \tag{26}$$

On the other hand, making $\delta_t(E_1)^1 = \dfrac{(E_1)^1 - (E_1)^0}{k}$ and $\delta_t(E_2)^1 = \dfrac{(E_2)^1 - (E_2)^0}{k}$, we obtain respectively

$(D_1)^1$
$$\frac{\delta_t \mathbf{e}^{1/2} - \delta_t \mathbf{e}^0}{k} - \Delta\, \delta_t \mathbf{e}^{1/2} + \nabla\, \delta_t p(t_1) = \delta_t \mathcal{E}^1 + \delta_t \mathbf{NL}^1$$

$(D_2)^1$
$$\frac{\delta_t \mathbf{e}^1 - \delta_t \mathbf{e}^{1/2}}{k} - \Delta(\delta_t \mathbf{e}^1 - \delta_t \mathbf{e}^{1/2}) - \nabla\, \delta_t p^1 = \mathbf{0}.$$

Finally, adding $(D_1)^1$ and $(D_2)^1$, we get:

$(D_3)^1$
$$\frac{\delta_t \mathbf{e}^1 - \delta_t \mathbf{e}^0}{k} - \Delta\, \delta_t \mathbf{e}^1 + \nabla\, \delta_t e_p^1 = \delta_t \mathcal{E}^1 + \delta_t \mathbf{NL}^1$$

Since

$$\mathcal{E}^1 = \mathbf{u}_t(t_1) - \delta_t \mathbf{u}(t_1) - k\, \delta_t \mathbf{u}(t_1) \cdot \nabla\, \mathbf{u}(t_1)$$

then

$$\delta_t \mathcal{E}^1 = \frac{\mathcal{E}^1 - \mathcal{E}^0}{k} = \frac{\mathbf{u}_t(t_1) - \mathbf{u}_t\,(t_0)}{k} - \frac{\delta_t \mathbf{u}\,(t_1) - \delta_t \mathbf{u}\,(t_0)}{k} - \delta_t \mathbf{u}(t_1) \cdot \nabla\, \mathbf{u}(t_1) \tag{27}$$

On the other hand,

$$\delta_t \mathbf{NL}^1 = \frac{\mathbf{NL}^1 - \mathbf{NL}^0}{k} = -\mathbf{u}^0 \cdot \nabla \delta_t \mathbf{e}^{1/2}.$$

Now, in order to chose $\delta_t \mathbf{u}(t_0)$, we expand the functions $\mathbf{u}_t$ and $\mathbf{u}$ by convenient Taylor's developments around of $t_{1/2} = t_0 + \dfrac{k}{2}$:

$$\begin{aligned}
\mathbf{u}_t\,(t_1) &= \mathbf{u}_t\,(t_{1/2}) + \frac{k}{2}\, \mathbf{u}_{tt}\,(t_{1/2}) + \int_{t_{1/2}}^{t_1} (t_1 - t) \mathbf{u}_{ttt}\, dt \\
\mathbf{u}_t\,(t_0) &= \mathbf{u}_t\,(t_{1/2}) - \frac{k}{2}\, \mathbf{u}_{tt}\,(t_{1/2}) + \int_{t_{1/2}}^{t_0} (t_0 - t) \mathbf{u}_{ttt}\, dt
\end{aligned} \tag{28}$$

Then

$$\frac{\mathbf{u}_t\,(t_1) - \mathbf{u}_t\,(t_0)}{k} = \mathbf{u}_{tt}\,(t_{1/2}) + \frac{1}{k} \left(\int_{t_{1/2}}^{t_1} (t_1 - t) \mathbf{u}_{ttt}\, dt + \int_{t_0}^{t_{1/2}} (t_0 - t) \mathbf{u}_{ttt}\, dt \right) \tag{29}$$

In a similar way,

$$\mathbf{u}\,(t_1) = \mathbf{u}\,(t_{1/2}) + \frac{k}{2}\, \mathbf{u}_t\,(t_{1/2}) + \frac{1}{8}\, k^2\, \mathbf{u}_{tt}(t_{1/2}) + \frac{1}{2} \int_{t_{1/2}}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}\, dt$$

$$\mathbf{u}\,(t_0) = \mathbf{u}\,(t_{1/2}) - \frac{k}{2}\, \mathbf{u}_t\,(t_{1/2}) + \frac{1}{8}\, k^2\, \mathbf{u}_{tt}(t_{1/2}) + \frac{1}{2} \int_{t_{1/2}}^{t_0} (t_0 - t)^2 \mathbf{u}_{ttt}\, dt$$

Then

$$\delta_t \mathbf{u}(t_1) = \frac{\mathbf{u}\,(t_1) - \mathbf{u}\,(t_0)}{k} = \mathbf{u}_t\,(t_{1/2}) + \frac{1}{2\,k} \left(\int_{t_{1/2}}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}\, dt + \int_{t_0}^{t_{1/2}} (t_0 - t)^2 \mathbf{u}_{ttt}\, dt \right)$$

We choose $\mathbf{u}(t_{-1})$ such that

$$\delta_t \mathbf{u}(t_0) = \mathbf{u}_t(t_{1/2}) - k\, \mathbf{u}_{tt}(t_{1/2}) + k\, \delta_t \mathbf{u}(t_1) \cdot \nabla\, \mathbf{u}(t_1) \tag{30}$$

$\Big($i.e. $\mathbf{u}(t_{-1}) = \mathbf{u}(t_0) - k\, \mathbf{u}_t\,(t_{1/2}) + k^2\, \mathbf{u}_{tt}(t_{1/2}) - k^2\, \delta_t \mathbf{u}(t_1) \cdot \nabla\, \mathbf{u}(t_1)\ \Big)$. With this choice,

$$\frac{\delta_t \mathbf{u}\,(t_1) - \delta_t \mathbf{u}\,(t_0)}{k} = \mathbf{u}_{tt}(t_{1/2}) - \delta_t \mathbf{u}(t_1) \cdot \nabla\, \mathbf{u}(t_1) + \frac{1}{2\,k^2} \left(\int_{t_{1/2}}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}\, dt + \int_{t_0}^{t_{1/2}} (t_0 - t)^2 \mathbf{u}_{ttt}\, dt \right). \tag{31}$$

Therefore, from (27), (29) and (31)

$$\begin{aligned} \delta_t \mathcal{E}^1 &= \frac{1}{k} \left(\int_{t_{1/2}}^{t_1} (t_1 - t) \mathbf{u}_{ttt}\, dt - \int_{t_0}^{t_{1/2}} (t_0 - t) \mathbf{u}_{ttt}\, dt \right) \\ &- \frac{1}{2\,k^2} \left(\int_{t_{1/2}}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}\, dt - \int_{t_0}^{t_{1/2}} (t_0 - t)^2 \mathbf{u}_{ttt}\, dt \right). \end{aligned}$$

Therefore, for any norm $\|\cdot\|$, one has

$$\|\delta_t \mathcal{E}^1\| \le C \left(\int_{t_0}^{t_1} \|\sqrt{t} \mathbf{u}_{ttt}\|^2 \right)^{1/2} \tag{32}$$

On the other hand, from (28) and (30)

$$\mathcal{E}^0 = \mathbf{u}_t(t_0) - \delta_t \mathbf{u}(t_0) = \frac{k}{2} \mathbf{u}_{tt}(t_{1/2}) - k\, \delta_t \mathbf{u}(t_1) \cdot \nabla\, \mathbf{u}(t_1) + \int_{t_{1/2}}^{t_0} (t_0 - t) \mathbf{u}_{ttt}\, dt.$$

In particular,

$$|\mathcal{E}^0| \le \frac{k}{2} |\mathbf{u}_{tt}(t_{1/2})| + k\, \|\mathbf{u}_t(\chi)\|_{\mathbf{L}^3} \|\mathbf{u}(t_1)\|_{\mathbf{H}^2} + \frac{k}{\sqrt{2}} \left(\int_{t_{1/2}}^{t_0} |\sqrt{t} \mathbf{u}_{ttt}|^2 dt \right)^{1/2}.$$

Therefore, imposing that $\mathbf{u}_{tt} \in L^\infty(\mathbf{L}^2)$ and $\sqrt{t} \mathbf{u}_{ttt} \in L^2(\mathbf{L}^2)$, and recalling that $\mathbf{u}_t \in L^\infty(\mathbf{L}^3)$, one has

$$|\mathcal{E}^0| \le C\, k. \tag{33}$$

Finally, using (25), (26) and (33), one has $|\delta_t \mathbf{e}^0| \le C\, k$.

Now, multiplying $(D_1)^1$ by $\delta_t \mathbf{e}^{1/2}$,

$$\begin{aligned} & |\delta_t \mathbf{e}^{1/2}|^2 - |\delta_t \mathbf{e}^0|^2 + |\delta_t \mathbf{e}^{1/2} - \delta_t \mathbf{e}^0|^2 + k\, \|\delta_t \mathbf{e}^{1/2}\|^2 \\ & = k(\nabla\, \delta_t p(t_1), \delta_t \mathbf{e}^{1/2}) + k\, (\delta_t \mathcal{E}^1, \delta_t \mathbf{e}^{1/2}) - k\, (\mathbf{u}^0 \cdot \nabla \delta_t \mathbf{e}^{1/2}, \delta_t \mathbf{e}^{1/2}) \\ & \le \varepsilon\, k\, \|\delta_t \mathbf{e}^{1/2}\|^2 + C\, k \int_{t_0}^{t_1} \|p_t\|^2 + C\, k\, \|\delta_t \mathcal{E}^1\|^2_{\mathbf{H}^{-1}} \\ & \le \varepsilon\, k\, \|\delta_t \mathbf{e}^{1/2}\|^2 + C\, k \int_{t_0}^{t_1} \|p_t\|^2 + C\, k \Big(\int_{t_0}^{t_1} \|\sqrt{t}\, \mathbf{u}_{ttt}\|^2_{\mathbf{H}^{-1}}\, dt \Big) \end{aligned}$$

Here, we have used that $(\mathbf{u}^0 \cdot \nabla \delta_t \mathbf{e}^{1/2}, \delta_t \mathbf{e}^{1/2}) = 0$ and (32). Then, since $\delta_t \mathbf{e}^0$ is of order $O(k)$, one has $|\delta_t \mathbf{e}^{1/2}|^2 \le Ck$.

Finally, multiplying $(D_3)^1$ by $\delta_t \mathbf{e}^1$,

$$|\delta_t \mathbf{e}^1|^2 - |\delta_t \mathbf{e}^0|^2 + |\delta_t \mathbf{e}^1 - \delta_t \mathbf{e}^0|^2 + k\,\|\delta_t \mathbf{e}^1\|^2 = k\,(\delta_t \mathcal{E}^1, \delta_t \mathbf{e}^1) + k\,(\delta_t \mathbf{NL}^1, \delta_t \mathbf{e}^1) = J_1 + J_2$$

We bound as follows

$$J_1 \leq \varepsilon\,|\delta_t \mathbf{e}^1|^2 + C\,k^2 |\delta_t \mathcal{E}^1|^2 \leq \varepsilon\,|\delta_t \mathbf{e}^1|^2 + C\,k^2 \left(\int_{t_0}^{t_1} |\sqrt{t}\mathbf{u}_{ttt}|^2 \right)$$

$$J_2 = k(\mathbf{u}^0 \cdot \nabla\,\delta_t \mathbf{e}^1, \delta_t \mathbf{e}^{1/2}) \leq \varepsilon\,k\,\|\delta_t \mathbf{e}^1\|^2 + C\,k\,|\delta_t \mathbf{e}^{1/2}|^2$$

Therefore, since $|\delta_t \mathbf{e}^0|$ and $|\delta_t \mathbf{e}^{1/2}|^2$ are of order $O(k)$, we get $|\delta_t \mathbf{e}^1|^2 \leq C\,k^2$. ■

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Spatial error estimates for a finite element viscosity-splitting scheme for the Navier-Stokes equations*

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Abstract

In this paper, we obtain error estimates in space for a fully discrete first order time fractional-step scheme (using decomposition of the viscosity in time and finite elements in space), applied to the Navier-Stokes equations.

In [10], optimal first order error estimates (for velocity and pressure) for the corresponding time discrete scheme was obtained. Now, we use this time discrete scheme as an auxiliary problem to study the fully discrete finite element scheme, obtaining first order for the velocity and pressure in the norms of $H^1(\Omega)$ and $L^2(\Omega)$ respectively, and order two for the velocity in the $L^2(\Omega)$ norm.

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Introduction

We consider the Navier-Stokes system, modelling viscous and incompressible fluids filling a bounded domain $\Omega \subset I\!R^3$ in a time interval $(0,T)$:

$$(P) \qquad \left\{ \begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u} &=& 0 & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}_{|t=0} &=& \mathbf{u}_0 & \text{in } \Omega. \end{array} \right.$$

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where $\mathbf{u} : (\mathbf{x}, t) \in \Omega \times (0,T) \to I\!R^3$ the velocity field and $p : (\mathbf{x}, t) \in \Omega \times (0,T) \to I\!R^3$ the pressure are the unknowns, and data are $\nu > 0$ the viscosity coefficient (which is assumed constant for simplicity) and $\mathbf{f}(\mathbf{x}, t) \in \Omega \times (0,T) \to I\!R^3$ the external forces. We denote by ∇ the gradient operator and Δ the Laplace operator.

Considering a (regular) partition of $[0,T]$ of diameter $k = T/M$: $t_0 = 0, t_1 = k, \dots, t_m = mk, \dots, t_M = T$, for a given vector $u = (u^m)_0^M$ with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^{M} \|u^m\|_X^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\dots,M} \|u^m\|_X$$

For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., and $\mathbf{H^1} = H^1(\Omega)^3$ etc.

The numerical analysis for the Navier-Stokes problem (P) has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties in this problem are the coupling between the pressure and the incompressibility condition and the nonlinearity of the convective terms.

Fractional step methods are becoming widely used in this context, allowing us to separate the effects of different operators appearing in the problem. For instance, the projection schemes decompose the convection-diffusion operators to the incompresibility ([8], [13], [14], [12]).

Another fractional step method, called viscosity splitting methods (when viscosity is not fully decoupled from incompressibility), was introduced and studied in [1], [2], [3] and [4]. It is a two-step scheme splitting the nonlinearity and the incompressibility of the problem into different steps (but keeping viscosity term and boundary conditions in both steps). Given $\mathbf{u}_h^m$ an approximation of $\mathbf{u}(t_m)$, first one computes an intermediate velocity $\mathbf{u}_h^{m+1/2}$ (as a first approximation of $\mathbf{u}(t_{m+1})$) by means of a discrete convection-diffusion problem, and afterwards $(\mathbf{u}_h^{m+1}, p_h^{m+1})$ (as approximation of $(\mathbf{u}(t_{m+1}), p(t_{m+1}))$) is obtained solving a discrete Stokes type problem. In [1], [2], Blasco, Codina and Huerta prove the convergence of this semidiscrete in time scheme. On the other hand, also for the semidiscrete in time case, error estimates of order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the end-of-step velocity $\mathbf{u}^{m+1}$ and order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure p^{m+1} are obtained in [3]. Moreover, in [4] these estimates are used to obtain the following error estimates for a fully discrete scheme, based on finite element approximations in space of order $O(h)$ in $\mathbf{H}^1 \times L^2$ for the velocity and pressure:

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(L^2) \cap l^2(H^1)} \le C\,(k + h)$$

under the constraint $h^2 \le C\,k$.

It is important to remark that error estimates for the fully discrete pressure are not described in [4], although the estimate $\|p^m - p_h^m\|_{l^2(L^2)} \le C\,h/\sqrt{k}$ can be obtained with similar arguments.

On the other hand, in [2] numerical computations drive to order $O(k)$ in $L^2(\Omega)$ for velocity and pressure.

In [10], we have improved the previous error estimates for the semidiscrete in time scheme obtaining the following sharp error estimates for the pressure:

$$\|p(t_m) - p^m\|_{l^2(L^2)} \le C\, k.$$

Moreover, imposing $\|(\mathbf{u}(t_1) - \mathbf{u}^1) - (\mathbf{u}(0) - \mathbf{u}^0)\|_{L^2} \le C\, k^2$ one can improve the norm in the error estimates as follows,

$$\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{l^\infty(H^1)} + \|p(t_m) - p^m\|_{l^\infty(L^2)} \le C\, k.$$

Now, in this paper, we use this semidiscrete in time scheme as an auxiliary problem, in order to obtain error estimates for the fully discrete finite element scheme.

On the other hand, in [7], this viscosity-splitting in time scheme is studied jointly with several Galerkin discontinuous finite element methods in space. From the analytical point of view, with $P_1 \times P_0$ discrete spaces, order $O(k+h)$ in $l^\infty(\mathbf{L}^2)$ for the velocity and order $O(\sqrt{k}+h)$ in $l^2(L^2)$ for the pressure are obtained. On the other hand, by means of numerical computations, the following approximations in space are observed: order $O(h)$ in a discrete $\mathbf{H}^1$-norm and $O(h^2)$ in $\mathbf{L}^2$ for the velocity and order $O(h)$ in L^2 for the pressure using $P_1 \times P_0$ approximation (also, $O(h^2)$ in a discrete $\mathbf{H}^1$-norm and $O(h^3)$ in $\mathbf{L}^2$ for the velocity and order $O(h^2)$ in L^2 for the pressure using $P_2 \times P_1$ discrete spaces). Consequently, there is a gap between the numerical computatios (that gives $O(h^2)$) and the numerical analysis (that proves $O(h)$) respect to the approximation in space for the velocity in the $\mathbf{L}^2$-norm. In this paper, we pretend to fill this gap.

Basically, the objectives of this work are:

1. To extend the order in the error estimates, in velocity and pressure, from the semidiscrete in time scheme to a fully discrete scheme, concretely from $O(k)$ to $O(k+h)$.

2. To improve the order in the error estimate for velocity in norm $L^2(\mathbf{L}^2)$, from $O(k+h)$ (obtained in [4]) to $O(k+h^2)$.

More concretely, under the same constraint $h^2 \le C\, k$ imposed in [4], we will obtain the following sharp error estimates (for k small enough):

$$\|p(t_m) - p_h^m\|_{l^2(L^2)} \le C\,(k+h),$$

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^2(L^2)} \le C\,(k+h^2).$$

Moreover, imposing $|(\mathbf{u}^1 - \mathbf{u}_h^1) - (\mathbf{u}^0 - \mathbf{u}_h^0)| \le C\, k\, h$ (but not k small), one can improve the norm in the error estimates as follows,

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(H^1)} + \|p(t_m) - p_h^m\|_{l^\infty(L^2)} \le C\,(k+h).$$

Notice that, regarding the $O(k+h)$ and $O(k+h^2)$ error estimates and the constraint $h^2 \le C\,k$, an appropriate choice of the pair (k,h) is between $k = O(h)$ and $k = O(h^2)$ (being both cases valid).

Due to these improvements, projection scheme with pressure correction and viscosity splitting scheme, are fully comparable. Moreover, this scheme has the same analytical results than Euler's type schemes [15], improving their numerical treatment (since the main difficulties are split).

The main results of this paper have been announced in [9].

The paper is organized as follows:

In Section 1, we present the semi-discrete in time scheme, which, as we have said, it will be considered as an auxiliary scheme and the corresponding error estimates obtained in [10], explaining (for reader's convenience) the main ideas of the proofs. Basically, in [10], firstly $O(k^{1/2})$ error estimates for $\mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}$ and $\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}$ in $l^\infty(\mathbf{H}^1) \cap l^2(\mathbf{H}^2)$ and for $e_p^m = p(t_m) - p^m$ in $l^2(H^1)$ were obtained. These error estimates had been used to obtain $O(k^{1/2})$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ for the discrete in time derivative of $\mathbf{e}^{m+1/2}$ and of $\mathbf{e}^{m+1}$, which were applied to get $O(k)$ for the discrete in time derivative of $\mathbf{e}^{m+1}$, either in $l^2(\mathbf{L}^2)$ or in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$, imposing either k small enough or a constraint on the first step of the scheme respectively (in fact, these two estimates were obtained independently). Finally, $O(k)$ in $l^2(L^2)$ and in $l^\infty(L^2)$ for the pressure hold.

In Section 2, we study the fully discrete scheme. We present the finite elements spaces and their approximation properties. We describe the scheme and the problems verified by the discrete errors (comparing semidiscrete scheme and fully discrete scheme). Firstly, we will obtain $O(h)$ error estimates for $\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}$ and $\mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$, which imply estimates in $W^{1,6}(\Omega)$ for the discrete velocities whether $h^2/k \le C$. Afterwards, $O(h)$ for the discrete in time derivative of $\mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$ and in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ are obtained independently, where again k small enough is necessary in the first case or a constraint on the first step of the scheme in the last case. Moreover, $O(k+h^2)$ error estimates for $\mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$ is deduced. Finally, the pressure error estimates of order $O(h)$ in $l^2(L^2)$ and in $l^\infty(L^2)$ are obtained.

In this paper, the following discrete Gronwall's lemma will be frequently used (for a proof, see [11, p. 369]):

Lemma 1 *Let k, B and a_m , b_m , c_m , γ_m be nonnegative numbers.*

a) (Discrete Gronwall inequality) *We assume*

$$a_{r+1} + k\sum_{m=0}^{r} b_m \le k\sum_{m=0}^{r} \gamma_m a_m + k\sum_{m=0}^{r} c_m + B \qquad \forall r \ge 0.$$

Then, one has

$$a_{r+1} + k\sum_{m=0}^{r} b_m \le \exp\left(k\sum_{m=0}^{r} \gamma_m\right)\left\{k\sum_{m=0}^{r} c_m + B\right\} \qquad \forall r \ge 0.$$

b) (Generalized discrete Gronwall inequality) *We assume*

$$a_r + k\sum_{m=0}^{r} b_m \le k\sum_{m=0}^{r}\gamma_m a_m + k\sum_{m=0}^{r} c_m + B \qquad \forall r\ge 0$$

such that $k\gamma_m < 1$ *for all* m. *Then, setting* $\sigma_m \equiv (1-k\gamma_m)^{-1}$, *one has*

$$a_r + k\sum_{m=0}^{r} b_m \le \exp\left(k\sum_{m=0}^{r}\sigma_m\gamma_m\right)\left\{k\sum_{m=0}^{r} c_m + B\right\} \qquad \forall r\ge 0.$$

1 Semi-discrete in time scheme

1.1 Description of the scheme

Given a (uniform) partition of the time interval $[0,T]$ with diameter $k=T/M$, $\{t_m = m\,k\}_{m=0}^{M}$, and $(\mathbf{f}^m)_{m=1}^{M}$ an approximation of $\mathbf{f}(t_m)$ we have to define $(\mathbf{u}^m, p^m)_{m=1}^{M}$ an approximation of the solution $\{\mathbf{u}, p\}$ of (P) at the time $t=t_m$.

Initialization: $\mathbf{u}^0 = \mathbf{u}_0$

Time step $m+1$:

Substep 1: Given $\mathbf{u}^m$, to find $\mathbf{u}^{m+1/2}$ solution of

$$(S_1)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}^{m+1/2} - \mathbf{u}^m + (\mathbf{u}^m\cdot\nabla)\mathbf{u}^{m+1/2} - \nu\Delta\mathbf{u}^{m+1/2} = \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \mathbf{u}^{m+1/2}|_{\partial\Omega} = 0. \end{array}\right.$$

Substep 2: Give $\mathbf{u}^{m+1/2}$, to find $\mathbf{u}^{m+1}$ and p^{m+1} solution of

$$(S_2)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) - \nu\Delta(\mathbf{u}^{m+1} - \mathbf{u}^{m+1/2}) + \nabla p^{m+1} = 0 \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{u}^{m+1}|_{\partial\Omega} = 0. \end{array}\right.$$

1.2 Differential problems verified by the errors

For simplicity and without loss of generality, we fix the viscosity constant $\nu = 1$.

We consider the following notations for the errors in $t = t_{m+1}$:

$$\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}, \quad \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad e_p^{m+1} = p(t_{m+1}) - p^{m+1},$$

and for the discrete in time derivatives of errors

$$\delta_t\mathbf{e}^{m+1} = \frac{\mathbf{e}^{m+1}-\mathbf{e}^m}{k}, \qquad \delta_t\mathbf{e}^{m+1/2} = \frac{\mathbf{e}^{m+1/2}-\mathbf{e}^{m-1/2}}{k},$$

The errors verify the following problems:

$$(E_1)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1/2} - \mathbf{e}^m) - \Delta\mathbf{e}^{m+1/2} = -\nabla\, p(t_{m+1}) + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \\ \mathbf{e}^{m+1/2}|_{\partial\Omega} = 0, \end{array}\right.$$

where

$$\mathcal{E}^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t-t_m)\,\mathbf{u}_{tt}(t)\,dt - \Big(\int_{t_m}^{t_{m+1}} \mathbf{u}_t\cdot\nabla\Big)\,\mathbf{u}(t_{m+1})$$

is the consistency error, and

$$\mathbf{NL}^{m+1} = -\Big(\mathbf{e}^m\cdot\nabla\Big)\mathbf{u}(t_{m+1}) - (\mathbf{u}^m\cdot\nabla)\mathbf{e}^{m+1/2} = -\Big(\mathbf{e}^m\cdot\nabla\Big)\mathbf{u}^{m+1/2} - (\mathbf{u}(t_m)\cdot\nabla)\mathbf{e}^{m+1/2}$$

are residual terms appearing in the differences of the quadratic terms. On the other hand,

$$(E_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}) - \Delta(\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}) - \nabla p^{m+1} = 0 \quad \text{in } \Omega \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = 0. \end{cases}$$

Adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$, we arrive at:

$$(E_3)^{m+1} \qquad \begin{cases} \delta_t\mathbf{e}^{m+1} - \Delta\mathbf{e}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = 0. \end{cases}$$

1.3 Known results.

Let us to introduce the following Hilbert spaces:

$$\begin{aligned} \mathbf{H} &= \{\mathbf{v}\in\mathbf{L}^2(\Omega) \;:\; \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega,\; \mathbf{v}\cdot\mathbf{n}_{\partial\Omega} = 0\}, \\ \mathbf{V} &= \{\mathbf{v}\in\mathbf{H}_0^1(\Omega) \;:\; \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\}, \end{aligned}$$

being $\mathbf{n}_{\partial\Omega}$ the normal outwards vector of $\partial\Omega$.

We denote $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{V}'$ the dual space of $\mathbf{H}_0^1(\Omega)$ and $\mathbf{V}$ respectively. The norm and scalar product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and $(\cdot,\cdot)$, whereas the norm in $H_0^1(\Omega)$ of the gradient in $L^2(\Omega)$ will be denoted by $\|\cdot\|$. Any other norm in a space X will be denoted by $\|\cdot\|_X$.

On the other hand, by C we will denote different constants, always independent of k (and h).

In the sequel, we will assume the following regularity hypothesis on Ω:

(H0) $\quad \Omega\subset I\!R^3$ such that the Stokes problem in Ω has $\mathbf{H}^2\times H^1$ regularity for velocity and pressure respectively,

and the following regularity for the exact solution $(\mathbf{u}, p)$

$$\mathbf{u}\in L^\infty(\mathbf{H}^2\cap\mathbf{V}),\; p\in L^\infty(H^1),\; \mathbf{u}_t\in L^\infty(\mathbf{L}^2)\cap L^2(\mathbf{H}^1),\; \mathbf{u}_{tt}\in L^2(\mathbf{V}').$$

In these hypotheses, one has the following error estimates ([3]):

$$\|\mathbf{e}^{m+1/2}\|_{l^2(H^1)} \le C\sqrt{k}, \quad \|\mathbf{e}^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C\,k \quad \text{and} \quad \|e_p^{m+1}\|_{l^2(L^2)} \le C\sqrt{k} \tag{1}$$

$$\|\mathbf{e}^{m+1/2}-\mathbf{e}^m\|^2_{l^2(\mathbf{L}^2)} + \|\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}\|^2_{l^2(\mathbf{L}^2)} \le C\,k^2 \tag{2}$$

whether k is small enough.

On the other hand, in [10], the following improvement for the pressure error is obtained:

Theorem 2 *Assuming additional regularity hypotheses:* $p_t \in L^2(H^1)$, $\mathbf{u}_t \in L^\infty(\mathbf{H}^1) \cap L^2(\mathbf{H}^2)$, $\mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$, $\mathbf{u}_{ttt} \in L^2(\mathbf{V}')$ *and* $\sqrt{t}\mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1})$, *the following error estimate holds*

$$\|e_p^{m+1}\|_{l^2(L^2)} \leq C\, k$$

for k *small enough.*

Proof: The main ideas in the proof are based in the following three steps:

1. H^2 error estimates. Using the $H^2 \times H^1$-regularity of Stokes problem verified by $(\mathbf{e}^{m+1}, e_p^{m+1})$ and the H^2-regularity of the Poisson-Dirichlet problem verified by $\mathbf{e}^{m+1/2}$, one has

$$\mathbf{e}^{m+1/2},\ \mathbf{e}^{m+1} \text{ bounded in } l^\infty(\mathbf{H}^2) \tag{3}$$

   On the other hand, we have

$$\|\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)} \leq C\, k \tag{4}$$

2. Making $(\delta_t(E_1)^{m+1}, \delta_t \mathbf{e}^{m+1/2}) + (\delta_t(E_2)^{m+1}, \delta_t \mathbf{e}^{m+1})$, one gets

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leq C\sqrt{k}$$

3. Duality argument. Making $(\delta_t(E_3)^{m+1}, A^{-1}\, \delta_t \mathbf{e}^{m+1})$, being A^{-1} the inverse of the Stokes operator, one has, for k small enough,

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{V}')\cap l^2(\mathbf{L}^2)} \leq C\, k. \tag{5}$$

Theorem 3 *[10] Assuming* $|\delta_t \mathbf{e}^1| \leq C\, k$ *and* $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1})$, *the following error estimates hold*

$$\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \leq C\, k \quad \text{and} \quad \|e_p^{m+1}\|_{l^\infty(L^2)} \leq C\, k.$$

The proof of this Theorem is based in the error estimate

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leq C\, k$$

obtained making $(\delta_t(S_3)^{m+1}, \delta_t \mathbf{e}^{m+1})$.

Remark 4 *When only* $H^{3/2}$ *regularity of Stokes problem is assumed instead of* (**H0**) *(for instance, this is the case of a general polygon or polyhedron domain without any additional conditions about its angles), we can obtain ([10])* $\|e_p^{m+1}\|_{l^2(L^{2-\varepsilon})} \leq C\, k$, *for any* $\varepsilon > 0$ *(and the error estimate* $\|e_p^{m+1}\|_{l^2(L^2)} \leq C\, k$, *is obtained assuming* $H^{3/2+\varepsilon}$ *regularity of Stokes problems, for* ε *small enough, whence there is not additional conditions about the angles of the domain).*

2 Fully discrete scheme

2.1 Finite element approximation and fully discrete scheme

We consider a finite element approximation of the semidiscrete problems $(S_1)^{m+1}$ and $(S_2)^{m+1}$. We restrict ourselves to the case where Ω is a $2D$ polygon or a $3D$ polyhedron satisfying (**H0**). We consider three families of finite element spaces $\mathbf{X}_h$, $\mathbf{Y}_h \subset \mathbf{H}_0^1(\Omega)$ and $Q_h \subset L_0^2(\Omega)$ associated to a family of triangulations of the domain Ω of mesh size h, which it will be assumed regular and quasi-uniform (in the sense of Ciarlet [5]). The finite element functions in $\mathbf{X}_h$, $\mathbf{Y}_h$ and Q_h are locally polynomials of degree at least 1, 1 and 0, respectively. Moreover, the approximating spaces $\mathbf{Y}_h$ and Q_h are thus required to satisfy the standard "*inf − sup*" condition ([6]):

There exists $\beta > 0$ independent of h such that, for all $h > 0$,

$$\inf_{q_h \in Q_h \setminus \{0\}} \left(\sup_{\mathbf{v}_h \in \mathbf{Y}_h \setminus \{0\}} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\| \, |q_h|} \right) \geq \beta.$$

In such a way, the following approximating properties hold ([6]):

$$\|\mathbf{v} - I_h \mathbf{v}\| + \frac{1}{h} |\mathbf{v} - I_h \mathbf{v}| \leq C\, h \, \|\mathbf{v}\|_{\mathbf{H}^2} \qquad \forall\, \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega),$$

$$|q - J_h q| \leq C\, h\, \|q\| \qquad \forall\, q \in H^1(\Omega) \cap L_0^2(\Omega),$$

where $(I_h, J_h) : \mathbf{H}^2 \times H^1 \to \mathbf{Y}_h \times Q_h$ is the vectorial operator defined as:

$$(I_h \mathbf{v}, J_h q) \in \mathbf{Y}_h \times Q_h : \left\{ \begin{array}{rcl} (\nabla (I_h \mathbf{v} - \mathbf{v}), \nabla \mathbf{v}_h) - (J_h q - q, \nabla \cdot \mathbf{v}_h) & = & 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h, \\ (\nabla \cdot (I_h \mathbf{v} - \mathbf{v}), q_h) & = & 0 \quad \forall\, q_h \in Q_h. \end{array} \right. \tag{6}$$

On the other hand,

$$\|\mathbf{v} - K_h \mathbf{v}\| + \frac{1}{h} |\mathbf{v} - K_h \mathbf{v}| \leq C\, h\, \|\mathbf{v}\|_{\mathbf{H}^2} \qquad \forall\, \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega),$$

where $K_h : \mathbf{H}^2 \to \mathbf{X}_h$ is the (scalar) operator defined as:

$$K_h \mathbf{v} \in \mathbf{X}_h, \quad (\nabla (K_h \mathbf{v} - \mathbf{v}), \nabla \mathbf{v}_h) \;=\; 0 \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h.$$

Finally, the following constraint between the time step size k and the mesh size h will also be assumed:

(**H**) There exists a constant $\alpha > 0$ (independent of k and h) such that $\dfrac{h^2}{k} \leq \alpha$.

This assumption does not impose an upper bound on the time step size, hence the scheme remains unconditionally stable.

As usual, in the fully discrete problem, we will use the following skew-symmetric part of the trilinear form for the treatment of the convective term:

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \int_\Omega \Big\{ (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} - (\mathbf{u} \cdot \nabla)\mathbf{w} \cdot \mathbf{v} \Big\}, \qquad \forall\, \mathbf{u} \in \mathbf{H}_0^1,\ \mathbf{v} \in \mathbf{H}^1,\ \mathbf{w} \in \mathbf{H}^1$$

or equivalently

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega \Big\{ (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w} \Big\}$$

or equivalently

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = -\int_\Omega \Big\{ (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w} \Big\}$$

Previous equalities hold even in the discrete case, hence we can use, in the sequel, any of these three possibilities. Obviously, $c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}$ whether $\nabla\cdot\mathbf{u} = 0$.

The trilinear form $c(\cdot,\cdot,\cdot)$ verifies

$$c(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \qquad \forall\, \mathbf{u}\in\mathbf{H}^1_0, \quad \forall\, \mathbf{v}\in\mathbf{H}^1, \tag{7}$$

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) \le C \left\{ \begin{array}{l} \|\mathbf{u}\|\, \|\mathbf{v}\|_{W^{1,3}\cap L^\infty}\, |\mathbf{w}| \\ \|\mathbf{u}\|_{L^3}\, \|\mathbf{v}\|\, \|\mathbf{w}\| \end{array} \right.$$

where the role of $\mathbf{u},\mathbf{v},\mathbf{w}$ can be interchanged, using the appropriate expression of $c(\cdot,\cdot,\cdot)$. The fully discrete scheme remains as follows:

Initialization: Let $\mathbf{u}_h^0 \in \mathbf{Y}_h$ be an approximation of $\mathbf{u}_0$

Step of time $m+1$:

Subtep 1: Given $\mathbf{u}_h^m \in \mathbf{Y}_h$, to compute $\mathbf{u}_h^{m+1/2} \in \mathbf{X}_h$ such that, for all $\mathbf{v}_h \in \mathbf{X}_h$

$$(S_1)_h^{m+1} \quad \frac{1}{k}(\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m, \mathbf{v}_h) + c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla\,\mathbf{u}_h^{m+1/2}, \nabla\,\mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h)$$

Substep 2: Given $\mathbf{u}_h^{m+1/2} \in \mathbf{X}_h$, to compute $(\mathbf{u}_h^{m+1}, p_h^{m+1}) \in \mathbf{Y}_h \times Q_h$, such that for all $(\mathbf{v}_h, q_h) \in \mathbf{Y}_h \times Q_h$

$$(S_2)_h^{m+1} \quad \left\{ \begin{array}{l} \frac{1}{k}(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}), \nabla\mathbf{v}_h) - (p_h^{m+1}, \nabla\cdot\mathbf{v}_h) = 0, \\ (\nabla\cdot\mathbf{u}_h^{m+1}, q_h) = 0. \end{array} \right.$$

In the first substep, a decoupled linear convection-diffusion scheme must be computed, whereas the second substep can be seen as a (generalized) Stokes problem.

Notice that, using (7), one can extend results of stability and convergence for the semidiscrete in time schemes to this fully discrete scheme (see [1]). In fact, making $\Big((S_1)_h^{m+1}, \mathbf{u}_h^{m+1/2}\Big) + \Big((S_2)_h^{m+1}, \mathbf{u}_h^{m+1}\Big)$, one can obtain

$$\|\mathbf{u}_h^m\|_{l^\infty(L^2)\cap l^2(H^1)} + \|\mathbf{u}_h^{m+1/2}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C. \tag{8}$$

2.2 Problems related to the space discrete errors

We will present an error analysis for the fully discrete scheme $(\mathbf{u}_h^{m+1/2}, \mathbf{u}_h^{m+1}, p_h^{m+1})$ as an approximation of the semidiscrete scheme $(\mathbf{u}^{m+1/2}, \mathbf{u}^{m+1}, p^{m+1})$. Consequently, we consider the following space errors:

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \qquad \mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}$$

These errors can be decomposed as follows (splitting the discrete and the interpolation parts):

$$\mathbf{e}_d^{m+1} = \mathbf{e}_h^{m+1} + \mathbf{e}_i^{m+1}, \qquad \mathbf{e}_d^{m+1/2} = \mathbf{e}_h^{m+1/2} + \mathbf{e}_i^{m+1/2}, \qquad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1}$$

being $\mathbf{e}_i$ interpolation errors and $\mathbf{e}_h$ space discrete errors, concretely

$$\mathbf{e}_h^{m+1} = I_h\mathbf{u}^{m+1} - \mathbf{u}_h^{m+1} \text{ and } \mathbf{e}_i^{m+1} = \mathbf{u}^{m+1} - I_h\mathbf{u}^{m+1},$$

$$\mathbf{e}_h^{m+1/2} = K_h\mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2} \text{ and } \mathbf{e}_i^{m+1/2} = \mathbf{u}^{m+1/2} - K_h\mathbf{u}^{m+1/2},$$

$$e_{p,h}^{m+1} = J_h p^{m+1} - p_h^{m+1} \text{ and } e_{p,i}^{m+1} = p^{m+1} - J_h p^{m+1}.$$

Comparing $(S_1)^{m+1}, (S_2)^{m+1}$ and $(S_1)_h^{m+1}, (S_2)_h^{m+1}$, we have the following variational problems verified by the space errors $\mathbf{e}_d^{m+1/2}$ and $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ respectively:

$$(E_1)_h^{m+1} \qquad \frac{1}{k}(\mathbf{e}_d^{m+1/2} - \mathbf{e}_d^m, \mathbf{v}_h) + (\nabla \mathbf{e}_d^{m+1/2}, \nabla \mathbf{v}_h) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{X}_h,$$

where

$$\mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) = -c(\mathbf{e}_d^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, \mathbf{v}_h),$$

and

$$(E_2)_h^{m+1} \qquad \begin{cases} \frac{1}{k}(\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}, \mathbf{v}_h) + (\nabla(\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}), \nabla \mathbf{v}_h) \\ -(e_{p,d}^{m+1}, \nabla\cdot\mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{Y}_h \\ (\nabla\cdot\mathbf{e}_d^{m+1}, q_h) = 0, \quad \forall q_h \in Q_h \end{cases}$$

2.3 $O(h)$-error estimates for both velocities in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$

Theorem 5 *We assume hypotheses of Theorem 2,* $(\mathbf{H})$ *and* $|\mathbf{e}_d^0| \le C\,h$*. Then, the following error estimates hold*

$$\|\mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,h \tag{9}$$

$$\|\mathbf{e}_d^{m+1/2} - \mathbf{e}_d^m\|_{l^2(\mathbf{L}^2)} + \|\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}\|_{l^2(\mathbf{L}^2)} \le C\,\sqrt{k}\,h. \tag{10}$$

Remark 6 *From (1), (4) and Theorem 5, we can bound the total error as follows*

$$\|\mathbf{u}(t_{m+1})-\mathbf{u}_h^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)}+\|\mathbf{u}(t_{m+1})-\mathbf{u}_h^{m+1/2}\|_{l^\infty(L^2)}\le C(k+h)$$

$$\|\mathbf{u}(t_{m+1})-\mathbf{u}_h^{m+1/2}\|_{l^2(H^1)}\le C(\sqrt{k}+h).$$

Proof. [of Theorem 5]. Theorem 5 is announced in [4]. Here, for the convenience's reader, we give an outline of the proof.

The main idea is to make

$$2\,k\sum_{m=0}^{M-1}\Big\{((E_1)_h^{m+1},\mathbf{e}_h^{m+1/2})+((E_2)_h^{m+1},\mathbf{e}_h^{m+1})\Big\}.$$

In fact, making $2\,k\,((E_1)_h^{m+1},\mathbf{e}_h^{m+1/2})$ and using $\mathbf{e}_h^{m+1/2}=\mathbf{e}_d^{m+1/2}-\mathbf{e}_i^{m+1/2}$, we arrive at

$$\begin{aligned}
&|\mathbf{e}_d^{m+1/2}|^2-|\mathbf{e}_d^m|^2+|\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m|^2+2\,k\,\|\mathbf{e}_d^{m+1/2}\|^2\\
=\;&2\,k\,c(\mathbf{e}_d^m,\mathbf{u}^{m+1/2},\mathbf{e}_d^{m+1/2})+2\,k\,c(\mathbf{e}_d^m,\mathbf{u}^{m+1/2},\mathbf{e}_i^{m+1/2})-2\,k\,c(\mathbf{u}_h^m,\mathbf{e}_d^{m+1/2},\mathbf{e}_d^{m+1/2})\\
-\;&2\,k\,c(\mathbf{u}_h^m,\mathbf{e}_d^{m+1/2},\mathbf{e}_i^{m+1/2})+2(\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m,\mathbf{e}_i^{m+1/2})+2\,k\,(\nabla\mathbf{e}_d^{m+1/2},\nabla\mathbf{e}_i^{m+1/2})
\end{aligned}$$

We bound only the main terms related to $\mathbf{e}_i^{m+1/2}$ of the RHS (using that $\|\mathbf{u}^{m+1/2}\|_{H^2}\le C$):

$$\begin{aligned}
2(\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m,\mathbf{e}_i^{m+1/2})&\le\varepsilon\,|\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m|^2+C|\mathbf{e}_i^{m+1/2}|^2\le\varepsilon\,|\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m|^2+C\,h^4\,\|\mathbf{u}^{m+1/2}\|_{H^2}^2\\
&\le\varepsilon\,|\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m|^2+C\,h^4
\end{aligned}$$

$$\begin{aligned}
2\,k\,c(\mathbf{u}_h^m,\mathbf{e}_d^{m+1/2},\mathbf{e}_i^{m+1/2})&\le C\,k\,\|\mathbf{u}_h^m\|^2\,\|\mathbf{e}_i^{m+1/2}\|_{L^3}^2+\varepsilon\,k\|\mathbf{e}_d^{m+1/2}\|^2\\
&\le C\,k\,\|\mathbf{u}_h^m\|^2\,|\mathbf{e}_i^{m+1/2}|\,\|\mathbf{e}_i^{m+1/2}\|+\varepsilon\,k\|\mathbf{e}_d^{m+1/2}\|^2\\
&\le C\,k\,h^3\,\|\mathbf{u}_h^m\|^2\;\|\mathbf{u}^{m+1/2}\|_{H^2}^2+\varepsilon\,k\|\mathbf{e}_d^{m+1/2}\|^2\\
&\le C\,k\,h^3\,\|\mathbf{u}_h^m\|^2+\varepsilon\,k\|\mathbf{e}_d^{m+1/2}\|^2
\end{aligned}$$

On the other hand, making $2\,k\,((E_2)_h^{m+1},\mathbf{e}_h^{m+1})$, and using $\mathbf{e}_h^{m+1}=\mathbf{e}_d^{m+1}-\mathbf{e}_i^{m+1}$, we arrive at

$$\begin{aligned}
&|\mathbf{e}_d^{m+1}|^2-|\mathbf{e}_d^{m+1/2}|^2+|\mathbf{e}_d^{m+1}-\mathbf{e}_d^{m+1/2}|^2+2\,k\Big\{\|\mathbf{e}_d^{m+1}\|^2-\|\mathbf{e}_d^{m+1/2}\|^2+\|\mathbf{e}_d^{m+1}-\mathbf{e}_d^{m+1/2}\|^2\Big\}\\
=\;&2(\mathbf{e}_d^{m+1}-\mathbf{e}_d^{m+1/2},\mathbf{e}_i^{m+1})+2\,k\,(\nabla(\mathbf{e}_d^{m+1}-\mathbf{e}_d^{m+1/2}),\nabla\,\mathbf{e}_i^{m+1})+2\,k\,(e_{p,d}^{m+1},\nabla\cdot(\mathbf{e}_d^{m+1}-\mathbf{e}_i^{m+1}))
\end{aligned}$$

The RHS can be bounded in a similar way, take into account that $\Big(e_{p,i}^{m+1},\nabla\cdot(\mathbf{e}_d^{m+1}-\mathbf{e}_i^{m+1})\Big)=0.$

Adding from $m=0$ to r (with any $r<M$), we can get (applying discrete Gronwall's Lemma)

$$\|\mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}+\|\mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}^2\le C\,h^2(1+h^2/k)+C\,k\,h^3\sum_m\|\mathbf{u}_h^m\|^2$$

Finally, the first term of RHS is bounded by $C\,h^2$ using (**H**) and the last term is bounded by $C\,h^3$ using (8). ■

Remark 7 *In [4], the same discrete space for both velocities is considered, although the result of Theorem 5 is valid without constraints between* $\mathbf{X}_h$ *and* $\mathbf{Y}_h$.

Remark 8 *If* $\mathbf{X}_h \equiv \mathbf{Y}_h$ *and* $I_h = K_h$ *is the* L^2*-orthogonal projector, the constraint* $(\mathbf{H})$ *can be avoid in Theorem 5. Indeed, the treatment for the discrete in time derivative is changed taking into account that*

$$2\,k\left(\frac{\mathbf{e}_i^{m+1/2}-\mathbf{e}_i^m}{k},\mathbf{e}_h^{m+1/2}\right)=0 \quad \text{and} \quad 2\,k\left(\frac{\mathbf{e}_i^{m+1}-\mathbf{e}_i^{m+1/2}}{k},\mathbf{e}_h^{m+1}\right)=0.$$

■

In the sequel, to consider the constraint $(\mathbf{H})$ and the same discrete space $\mathbf{X}_h \equiv \mathbf{Y}_h$ (which we only will denote as $\mathbf{X}_h$) will be necessary.

From (9) and constraint $(\mathbf{H})$, one has in particular

$$\|\mathbf{u}_h^{m+1/2}\|^2+\|\mathbf{u}_h^{m+1}\|^2 \le C\Big(\|\mathbf{e}_d^{m+1/2}\|^2+\|\mathbf{e}_d^{m+1}\|^2+\|\mathbf{u}^{m+1/2}\|^2+\|\mathbf{u}^{m+1}\|^2\Big) \le C\left(\frac{h^2}{k}+1\right) \le C. \tag{11}$$

On the other hand, from (10) and using again $(\mathbf{H})$, one has

$$\|\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m\|_{l^2(\mathbf{L}^2)}^2+\|\mathbf{e}_d^{m+1}-\mathbf{e}_d^{m+1/2}\|_{l^2(\mathbf{L}^2)}^2 \le C\,k^2$$

that can be re-written as:

$$\frac{\mathbf{e}_d^{m+1/2}-\mathbf{e}_d^m}{k} \quad \text{and} \quad \frac{\mathbf{e}_d^{m+1}-\mathbf{e}_d^{m+1/2}}{k} \quad \text{are bounded in } l^2(\mathbf{L}^2).$$

Finally, using (2) and that $\dfrac{\mathbf{u}(t_{m+1})-\mathbf{u}(t_m)}{k}$ is bounded in $l^2(\mathbf{L}^2)$, we get $\dfrac{\mathbf{u}^{m+1}-\mathbf{u}^{m+1/2}}{k}$ and $\dfrac{\mathbf{u}^{m+1/2}-\mathbf{u}^m}{k}$ are bounded in $l^2(\mathbf{L}^2)$. Consequently,

$$\frac{\mathbf{u}_h^{m+1/2}-\mathbf{u}_h^m}{k} \quad \text{and} \quad \frac{\mathbf{u}_h^{m+1}-\mathbf{u}_h^{m+1/2}}{k} \quad \text{are bounded in } l^2(\mathbf{L}^2). \tag{12}$$

Lemma 9 *Assume the inverse inequality* $\|\mathbf{v}_h\|_{W^{1,6}} \le C\,h^{-1}\|\mathbf{v}_h\|$ *for each* $\mathbf{v}_h \in \mathbf{X}_h$*. Let* $\mathbf{g} \in \mathbf{L}^2$*. If* $(\mathbf{u}_h, q_h) \in \mathbf{X}_h \times Q_h$ *is the solution of the problem*

$$\begin{cases} (\nabla\,\mathbf{u}_h, \nabla\,\mathbf{v}_h) - (p_h, \nabla\cdot\,\mathbf{v}_h) = (\mathbf{g}, \mathbf{v}_h) & \forall\,\mathbf{v}_h \in \mathbf{X}_h \\ (\nabla\cdot\mathbf{u}_h, q_h) = 0 \qquad \forall\,q_h \in Q_h \end{cases}$$

or respectively $\mathbf{u}_h \in \mathbf{X}_h$ *is the solution of the problem*

$$(\nabla\,\mathbf{u}_h, \nabla\,\mathbf{v}_h) = (\mathbf{g}, \mathbf{v}_h) \quad \forall\,\mathbf{v}_h \in \mathbf{X}_h.$$

then (in both cases)

$$\|\mathbf{u}_h\|_{W^{1,6}} \le C\,|\mathbf{g}|.$$

Proof: See Appendix.

Corollary 10 *Assuming hypotheses of Theorem 5, one has*

$$\mathbf{u}_h^{m+1} \quad \textit{and} \quad \mathbf{u}_h^{m+1/2} \qquad \textit{are bounded in} \quad l^2(\mathbf{W}^{1,6}).$$

Consequently, using the estimates of $\mathbf{u}^{m+1}$ *and* $\mathbf{u}^{m+1/2}$ *in* $l^\infty(\mathbf{H}^2)$ *(obtained from (3)),*

$$\mathbf{e}_d^{m+1} \quad \textit{and} \quad \mathbf{e}_d^{m+1/2} \qquad \textit{are bounded in} \quad l^2(\mathbf{W}^{1,6}).$$

Proof: Adding $(S_1)_h^{m+1}$ and $(S_2)_h^{m+1}$, one has for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$:
$(S_3)_h^{m+1}$

$$\begin{cases} (\nabla \mathbf{u}_h^{m+1}, \nabla \mathbf{v}_h) - (p_h^{m+1}, \nabla \cdot \mathbf{v}_h) = (\delta_t \mathbf{u}_h^{m+1} - \mathbf{u}_h^m \cdot \nabla \mathbf{u}_h^{m+1/2} + \frac{1}{2}(\nabla \cdot \mathbf{u}_h^m)\, \mathbf{u}_h^{m+1/2} + \mathbf{f}^{m+1}, \mathbf{v}_h) \\ (\nabla \cdot \mathbf{u}_h^{m+1}, q_h) = 0. \end{cases}$$

Therefore, using Lemma 9 (here $\mathbf{X}_h \equiv \mathbf{Y}_h$ is necessary) and (11),

$$\begin{aligned} \|\mathbf{u}_h^{m+1}\|_{W^{1,6}}^2 &\leq C(|\delta_t \mathbf{u}_h^{m+1}|^2 + |\mathbf{f}^{m+1}|^2 + \|\mathbf{u}_h^m\|_{W^{1,3}\cap L^\infty}^2 \|\mathbf{u}_h^{m+1/2}\|^2) \\ &\leq C(|\delta_t \mathbf{u}_h^{m+1}|^2 + |\mathbf{f}^{m+1}|^2 + \|\mathbf{u}_h^m\|^{1-\delta} \|\mathbf{u}_h^m\|_{W^{1,6}}^{1+\delta}) \\ &\leq \varepsilon \|\mathbf{u}_h^m\|_{W^{1,6}}^2 + C\,(|\delta_t \mathbf{u}_h^{m+1}|^2 + |\mathbf{f}^{m+1}|^2 + 1). \end{aligned}$$

Since (12) implies in particular that $\delta_t \mathbf{u}_h^{m+1}$ is bounded in $l^2(\mathbf{L}^2)$, then

$$\mathbf{u}_h^{m+1} \text{ is bounded in } l^2(\mathbf{W}^{1,6}). \tag{13}$$

On the other hand, re-writting $(S_1)_h^{m+1}$ as :

$$(S_1)_h^{m+1} \qquad (\nabla \mathbf{u}_h^{m+1/2}, \nabla \mathbf{v}_h) \;=\; -\frac{1}{k}(\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m, \mathbf{v}_h) - c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\mathbf{f}^{m+1}, \mathbf{v}_h)$$

then, by using again Lemma 9,

$$\|\mathbf{u}_h^{m+1/2}\|_{W^{1,6}}^2 \leq C\left(\left|\frac{1}{k}(\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m)\right|^2 + |\mathbf{f}^{m+1}|^2 + \|\mathbf{u}_h^m\|_{W^{1,3}\cap L^\infty}^2 \|\mathbf{u}_h^{m+1/2}\|^2\right).$$

Finally, by using (11), (12) and (13), the estimate of $\mathbf{u}_h^{m+1/2}$ in $l^2(\mathbf{W}^{1,6})$ holds. ■

Remark 11 *When only* $\mathbf{H}^{3/2+\varepsilon}$ *regularity of* Ω *is assumed (see Remark 4), estimates of Corollary 10 hold, but changing* $\mathbf{W}^{1,6}$ *by* $\mathbf{W}^{1,3} \cap \mathbf{L}^\infty$*. In fact, estimates of* $\mathbf{u}_h^{m+1}$, $\mathbf{u}_h^{m+1/2}$ *in* $l^2(\mathbf{W}^{1,3} \cap \mathbf{L}^\infty)$ *will be sufficient in the sequel.*

2.4 $O(h)$-error estimates for $\delta_t \mathbf{e}_d^{m+1}$ and $\delta_t \mathbf{e}_d^{m+1/2}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ in $l^\infty(\mathbf{H}^1 \times L^2)$

Making $\delta_t(E_1)_h^{m+1}$ and $\delta_t(E_2)_h^{m+1}$, one obtains ($\forall m \geq 1$):

$$(D_1)_h^{m+1} \qquad \frac{1}{k}(\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^m, \mathbf{v}_h) + (\nabla \delta_t \mathbf{e}_d^{m+1/2}, \nabla \mathbf{v}_h) = \delta_t \, \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{X}_h$$

where,

$$\begin{aligned} \delta_t \mathbf{NL}_h^{m+1}(\mathbf{v}_h) &= c(\delta_t \mathbf{e}_d^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) + c(\delta_t \mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, \mathbf{v}_h) \\ &+ c(\mathbf{e}_d^{m-1}, \delta_t \mathbf{u}^{m+1/2}, \mathbf{v}_h) + c(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_d^{m+1/2}, \mathbf{v}_h) \end{aligned}$$

and, for all $(\mathbf{w}_h, q_h) \in \mathbf{X}_h \times Q_h$,

$$(D_2)_h^{m+1} \qquad \left\{ \begin{array}{l} \frac{1}{k}(\delta_t \, \mathbf{e}_d^{m+1} - \delta_t \, \mathbf{e}_d^{m+1/2}, \mathbf{w}_h) + (\nabla (\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}), \nabla \mathbf{w}_h) \\ -(\delta_t e_{p,d}^{m+1}, \nabla \cdot \mathbf{w}_h) = 0, \\ (\nabla \cdot \delta_t \mathbf{e}_d^{m+1}, q_h) = 0. \end{array} \right.$$

Finally, adding $(D_1)_h^{m+1}$ and $(D_2)_h^{m+1}$ we obtain, for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$:

$$(D_3)_h^{m+1} \quad \left\{ \begin{array}{l} \frac{1}{k}(\delta_t \mathbf{e}_d^{m+1} - \delta_t \, \mathbf{e}_d^m, \mathbf{v}_h) + (\nabla \delta_t \mathbf{e}_d^{m+1}, \nabla \mathbf{v}_h) + (\delta_t e_{p,d}^{m+1}, \nabla \cdot \mathbf{v}_h) = \delta_t \, \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \\ (\nabla \cdot \delta_t \mathbf{e}_d^{m+1}, q_h) = 0. \end{array} \right.$$

Theorem 12 *Under the hypotheses of Theorems 2 and 5, assuming the hypothesis for the first step of the scheme*

$$|\delta_t \mathbf{e}_d^1| \leq C\, h,$$

then

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} \leq C\, h.$$

Proof: Since the initial estimate $|\delta_t \mathbf{e}_d^1| \leq C\, h$ has been assumed, it is suffices to prove the generic estimate for $\delta_t \mathbf{e}_d^{m+1}$ and $\delta_t \mathbf{e}_d^{m+1/2}$, for each $m \geq 1$.

Taking $2\, k\, \delta_t \mathbf{e}_h^{m+1/2} \in \mathbf{X}_h$ as test function in $(D_1)_h^{m+1}$, which is decomposed as $2\, k(\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_i^{m+1/2})$, one has

$$\begin{array}{l} |\delta_t \mathbf{e}_d^{m+1/2}|^2 - |\delta_t \mathbf{e}_d^m|^2 + |\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^m|^2 + 2\; k \|\delta_t \mathbf{e}_d^{m+1/2}\|^2 \\ = 2\, k \left(\frac{1}{k}(\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^m), \delta_t \mathbf{e}_i^{m+1/2} \right) - 2\, k \left(\nabla\, \delta_t \mathbf{e}_d^{m+1/2}, \nabla \delta_t \mathbf{e}_i^{m+1/2} \right) \\ +2\, \delta_t \mathbf{NL}_h^{m+1} \left(\delta_t \mathbf{e}_h^{m+1/2} \right) := I_1 + I_2 + I_3. \end{array} \tag{14}$$

We bound the RHS of (14) as follows:

$$I_1 \leq \varepsilon\, |\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^m|^2 + C\, |\delta_t \mathbf{e}_i^{m+1/2}|^2 \leq \varepsilon\, |\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^m|^2 + C\, h^4\, \|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}^2$$

$$I_2 \leq \varepsilon\, k\, \|\delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\, k\, \|\delta_t \mathbf{e}_i^{m+1/2}\|^2 \leq \varepsilon\, k\, \|\delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\, k\, h^2\, \|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}^2$$

With respect to the nonlinear terms, for $m \geq 1$,

$$\begin{aligned} I_3 &= 2\,k\,c(\delta_t \mathbf{e}_d^m,\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) + 2\,k\,c(\delta_t\,\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) \\ &+ 2\,k\,c(\mathbf{e}_d^{m-1}, \delta_t\,\mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) + 2\,k\,c\,(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_h^{m+1/2}) := \sum_{i=1}^{4} J_i \end{aligned}$$

Bounding each J_i term:

$$J_1 = 2\,k\,c(\delta_t \mathbf{e}_d^m,\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_d^{m+1/2}) + 2\,k\,c(\delta_t \mathbf{e}_d^m,\, \mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_i^{m+1/2}) = J_{11} + J_{12}$$

$$J_{11} \leq \varepsilon\,k \|\delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\,k\,\|\mathbf{u}^{m+1/2}\|_{H^2}^2\,|\delta_t \mathbf{e}_d^m|^2$$

$$J_{12} \leq \varepsilon\,k \|\delta_t \mathbf{e}_d^m\|^2 + C\,k\,\|\mathbf{u}^{m+1/2}\|^2\,\|\delta_t \mathbf{e}_i^{m+1/2}\|^2 \leq \varepsilon\,k \|\delta_t \mathbf{e}_d^m\|^2 + C\,k\,h^2\,\|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}^2$$

$$J_2 = -2\,k\,c(\delta_t\,\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_d^{m+1/2}) + 2\,k\,c(\delta_t\,\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_i^{m+1/2}) = J_{21} + J_{22}$$

$$\begin{aligned} J_{21} &= 2\,k\,c(\delta_t\,\mathbf{e}_d^m, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_d^{m+1/2}) - 2\,k\,c(\delta_t \mathbf{u}^m, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_d^{m+1/2}) \\ &\leq 2\,k\,|\delta_t \mathbf{e}_d^m|\,\|\mathbf{e}_d^{m+1/2}\|_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty}\,\|\delta_t \mathbf{e}_d^{m+1/2}\| + 2\,k\,C\,\|\delta_t \mathbf{u}^m\|_{L^3}\,\|\mathbf{e}_d^{m+1/2}\|\,\|\delta_t \mathbf{e}_d^{m+1/2}\| \\ &\leq \varepsilon\,k\,\|\delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\,k\,\|\mathbf{e}_d^{m+1/2}\|_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty}^2\,|\delta_t \mathbf{e}_d^m|^2 + C\,k\,\|\mathbf{e}_d^{m+1/2}\|^2 \end{aligned}$$

Here, we have used that $\delta_t \mathbf{u}^m$ is bounded in $l^\infty(\mathbf{L}^3)$ (that is obtained from the previous estimates of (5)).

Reasoning as in J_{21}, we bound J_{22} as follows

$$\begin{aligned} J_{22} &= 2\,k\,c(\delta_t\,\mathbf{e}_d^m, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_i^{m+1/2}) + 2\,k\,c(\delta_t\,\mathbf{u}^m, \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_i^{m+1/2}) \\ &\leq C\,k\,\|\mathbf{e}_d^{m+1/2}\|_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty}^2\,|\delta_t \mathbf{e}_d^m|^2 + C\,k\,\|\mathbf{e}_d^{m+1/2}\|^2 + C\,k\,\|\delta_t \mathbf{e}_i^{m+1/2}\|^2 \\ &\leq C\,k\,\|\mathbf{e}_d^{m+1/2}\|_{\mathbf{W}^{1,3}\cap\mathbf{L}^\infty}^2\,|\delta_t \mathbf{e}_d^m|^2 + C\,k\,\|\mathbf{e}_d^{m+1/2}\|^2 + C\,k\,h^2 \|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}^2 \end{aligned}$$

$$J_3 = 2\,k\,c(\mathbf{e}_d^{m-1}, \delta_t\,\mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_d^{m+1/2}) + 2\,k\,c(\mathbf{e}_d^{m-1}, \delta_t\,\mathbf{u}^{m+1/2},\, \delta_t \mathbf{e}_i^{m+1/2}) = J_{31} + J_{32}$$

$$J_{31} \leq C\,k\,\|\mathbf{e}_d^{m-1}\|\,\|\delta_t \mathbf{u}^{m+1/2}\|_{L^3}\,\|\delta_t \mathbf{e}_d^{m+1/2}\| \leq \varepsilon\,k\,\|\delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\,k\,\|\mathbf{e}_d^{m-1}\|^2$$

$$J_{32} \leq C\,k\,\|\delta_t \mathbf{e}_i^{m+1/2}\|^2 + C\,k\,\|\mathbf{e}_d^{m-1}\|^2 \leq C\,k\,h^2\,\|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}^2 + C\,k\,\|\mathbf{e}_d^{m-1}\|^2$$

$$J_4 = 2\,k\,c(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_d^{m+1/2}) + 2\,k\,c(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_d^{m+1/2},\, \delta_t \mathbf{e}_i^{m+1/2}) = J_{41} + J_{42}$$

$$J_{41} = 0$$

$$\begin{aligned} J_{42} &\leq C\,k\,\|\mathbf{u}_h^{m-1}\|\,\|\delta_t \mathbf{e}_i^{m+1/2}\|_{L^3}\,\|\delta_t \mathbf{e}_d^{m+1/2}\| \\ &\leq \varepsilon\,k\,\|\delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\,k\,h^3 \|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}^2 \end{aligned}$$

Here, we have used the error interpolation $\|\delta_t \mathbf{e}_i^{m+1/2}\|_{L^3} \le C\, h^{3/2} \|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}$.

Now, taking $2\,k\,\delta_t \mathbf{e}_h^{m+1} \in \mathbf{X}_h$ as test function in $(D_2)_h^{m+1}$, which is decomposed as $2\,k\,(\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_i^{m+1})$, one has:

$$
\begin{aligned}
&|\delta_t \mathbf{e}_d^{m+1}|^2 - |\delta_t \mathbf{e}_d^{m+1/2}|^2 + |\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}|^2 \\
&+ k\Big\{ \|\delta_t \mathbf{e}_d^{m+1}\|^2 - \|\delta_t \mathbf{e}_d^{m+1/2}\|^2 + \|\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}\|^2 \Big\} \\
&= 2(\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}, \delta_t \mathbf{e}_i^{m+1}) + 2\ k\,(\nabla(\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}^{m+1/2}), \nabla\, \delta_t \mathbf{e}_i^{m+1}) \\
&-2\,k\,(\delta_t e_{p,d}^{m+1}, \nabla \cdot \delta_t \mathbf{e}_i^{m+1}) + 2\,k\,(\delta_t e_{p,d}^{m+1}, \nabla \cdot \delta_t \mathbf{e}_d^{m+1}) := L_1 + L_2 + L_3 + L_4
\end{aligned}
\tag{15}
$$

We bound the RHS of (15) as follows:

$$
L_1 \le \varepsilon\, |\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}|^2 + C\, |\delta_t \mathbf{e}_i^{m+1}|^2 \le \varepsilon\, |\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}|^2 + C\, h^4\, \|\delta_t \mathbf{u}^{m+1}\|_{H^2}^2
$$

$$
L_2 \le \varepsilon\, k\, \|\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\, k\, \|\delta_t \mathbf{e}_i^{m+1}\|^2 \le \varepsilon\, k\, \|\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}\|^2 + C\, k\, h^2 \|\delta_t \mathbf{u}^{m+1}\|_{H^2}^2
$$

$$
\begin{aligned}
L_3 &= -2\,k(\delta_t\, e_{p,h}^{m+1}, \nabla \cdot \delta_t \mathbf{e}_i^{m+1}) - 2\,k\,(\delta_t\, e_{p,i}^{m+1}, \nabla \cdot \delta_t \mathbf{e}_i^{m+1}) \\
&\le C\, k\, |\delta_t e_{p,i}^{m+1}|^2 + \varepsilon\, k\, \|\delta_t \mathbf{e}_i^{m+1}\|^2 \le C\, k\, h^2 \|\delta_t p^{m+1}\|^2 + \varepsilon\, k\, h^2\, \|\delta_t \mathbf{u}^{m+1}\|_{H^2}^2
\end{aligned}
$$

(here we have used that $\Big(\delta_t e_{p,h}^{m+1}, \nabla \cdot (\delta_t \mathbf{e}_i^{m+1})\Big) = 0$)

$$
\begin{aligned}
L_4 &= 2\,k\,(\delta_t e_{p,i}^{m+1}, \nabla \cdot \delta_t \mathbf{e}_d^{m+1}) + 2\,k\,(\delta_t e_{p,h}^{m+1}, \nabla \cdot \delta_t \mathbf{e}_d^{m+1}) \\
&\le C\, k\, |\delta_t e_{p,i}^{m+1}|^2 + \varepsilon\, k\, \|\delta_t \mathbf{e}_d^{m+1}\|^2 \le C\, k\, h^2 \|\delta_t p^{m+1}\|^2 + \varepsilon\, k\, \|\delta_t \mathbf{e}_d^{m+1}\|^2
\end{aligned}
$$

(here we have used that $\Big(\delta_t e_{p,h}^{m+1}, \nabla \cdot \delta_t \mathbf{e}_d^{m+1}\Big) = 0$)

Making $\sum_{m=1}^{r}\{(14)+(15)\}$ (with any $r < M$), taking in account the estimates of the Theorem 5 and the estimate $k \sum_m (\|\delta_t \mathbf{u}^{m+1/2}\|_{H^2}^2 + \|\delta_t \mathbf{u}^{m+1}\|_{H^2}^2) \le C$ (obtained in [10]), we get

$$
\begin{aligned}
|\delta_t \mathbf{e}_d^{r+1}|^2 \quad &+ \quad \sum_{m=1}^{r} \{ |\delta_t \mathbf{e}_d^{m+1} - \delta_t\, \mathbf{e}_d^{m+1/2}|^2 + |\delta_t\, \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^{m}|^2 \} \\
&+ \quad k \sum_{m=1}^{r} \{ \|\delta_t \mathbf{e}_d^{m+1}\|^2 + \|\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}\|^2 \} \\
&\le \quad |\delta_t \mathbf{e}_d^{1}|^2 + C\, k \sum_{m=1}^{r} |\delta_t \mathbf{e}_d^{m}|^2 + C\, k \sum_{m=1}^{r} \|\mathbf{e}_d^{m+1/2}\|_{\mathbf{W}^{1,3}\cap \mathbf{L}^\infty}^2\, |\delta_t \mathbf{e}_d^{m}|^2 + C\left(1 + \frac{h^2}{k}\right) h^2.
\end{aligned}
$$

Therefore, taking in account that $k \sum_m \|\mathbf{e}_d^{m+1/2}\|_{\mathbf{W}^{1,3}\cap \mathbf{L}^\infty}^2 \le C$, the condition (**H**) and applying the Discrete Gronwall's Lemma, we obtain

$$
\begin{aligned}
&\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)}^2 + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(L^2)\cap l^2(H^1)}^2 \\
&+ \frac{1}{k}\Big(\|\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}\|_{l^2(L^2)}^2 + \|\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^{m+1}\|_{l^2(L^2)}^2 \Big) \le |\delta_t \mathbf{e}_d^1|^2 + C\, h^2.
\end{aligned}
\tag{16}
$$

Finally, using the hypothesis $|\delta_t \mathbf{e}_d^1| \le C\, h$, the proof is finished. ■

Corollary 13 *Under hypotheses of Theorem 12 and assuming* $\|\mathbf{u}_h^0\|_{W^{1,6}}^2 \le C_0$ *(that is* $\mathbf{u}_0 \in \mathbf{W}^{1,6}$*), one has*

$$(\mathbf{u}_h^{m+1}) \quad \text{is bounded in} \quad l^\infty(\mathbf{W}^{1,6}).$$

Proof: The proof is similar to Corollary 10, using now that, from Theorem 12, $\delta_t \mathbf{u}_h^{m+1}$ is bounded in $L^\infty(\mathbf{L}^2)$. Indeed, we have

$$\|\mathbf{u}_h^{m+1}\|_{W^{1,6}}^2 \le \varepsilon \|\mathbf{u}_h^m\|_{W^{1,6}}^2 + C\left(|\delta_t \mathbf{u}_h^{m+1}|^2 + |\mathbf{f}^{m+1}|^2 + 1\right) \le \varepsilon \|\mathbf{u}_h^m\|_{W^{1,6}}^2 + C_1,$$

hence, if we choose ε verifying $\varepsilon\, C_0 \le C_1$ and $\varepsilon\, 2\, C_1 \le C_1$, one has

$$\|\mathbf{u}_h^m\|_{W^{1,6}} \le 2\, C_1 \quad \forall\, m.$$

■

Lemma 14 *Let* $(\mathbf{u}, p) \in \mathbf{H}_0^1 \times L_0^2$ *and* $\mathbf{g} \in \mathbf{H}^{-1}$*. If* $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times Q_h$ *is the solution of*

$$\begin{cases} (\nabla(\mathbf{u} - \mathbf{u}_h), \nabla \mathbf{v}_h) - (p - p_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{g}, \mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{X}_h \\ (\nabla \cdot (\mathbf{u} - \mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h \end{cases}$$

then

$$\|\mathbf{u} - \mathbf{u}_h\| + |p - p_h| \le C\, h\, (\|\mathbf{u}\|_{H^2} + \|p\|) + C\, \|\mathbf{g}\|_{H^{-1}}.$$

The proof of this Lemma can be seen in [6] in the case $\mathbf{g} = \mathbf{0}$. For the more general case $\mathbf{g} \neq \mathbf{0}$, the weak estimates are also valid, but the duality argument to obtain the estimate in the L^2 norm is not clear.

Corollary 15 *Assuming hypotheses of Theorem 12, the following error estimates hold*

$$\|e_{p,d}^{m+1}\|_{l^\infty(L^2)} \le C\, h \quad \text{and} \quad \|\mathbf{e}_d^{m+1}\|_{l^\infty(H^1)} \le C\, h.$$

Proof: Adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$, one has for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$,

$$(E_3)_h^{m+1} \qquad \begin{cases} (\nabla\, \mathbf{e}_d^{m+1}, \nabla\, \mathbf{v}_h) - (e_{p,d}^{m+1}, \nabla \cdot \mathbf{v}_h) = -(\delta_t \mathbf{e}_d^{m+1}, \mathbf{v}_h) + \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ (\nabla \cdot \mathbf{e}_d^{m+1}, q_h) = 0. \end{cases}$$

Therefore, using the Lemma 14, it is necessary to bound $\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(H^{-1})}$ and $\|\mathbf{NL}_h^{m+1}\|_{l^\infty(H^{-1})}$. Thanks to Theorem 12, $\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(L^2)} \le C\, h$. Therefore, it is suffices to bound $\|\mathbf{NL}_h^{m+1}\|_{l^\infty(H^{-1})} \le C\, h$. In fact, for each $\mathbf{v} \in \mathbf{H}_0^1$,

$$\begin{aligned} \mathbf{NL}_h^{m+1}(\mathbf{v}) &\le C\left(|\mathbf{e}_d^m|\, \|\mathbf{u}^{m+1/2}\|_{\mathbf{H}^2} + \|\mathbf{u}_h^m\|_{\mathbf{W}^{1,3} \cap \mathbf{L}^\infty}\, |\mathbf{e}_d^{m+1/2}|\right) \|\mathbf{v}\| \\ &\le C\left(|\mathbf{e}_d^m| + |\mathbf{e}_d^{m+1/2}|\right) \|\mathbf{v}\| \le C\, h\, \|\mathbf{v}\| \end{aligned}$$

hence the desired estimates hold. ■

Remark 16 *Combining Theorem 3 and Corollary 15, the following error estimate for the total error holds*

$$\|\mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1}\|_{l^\infty(H^1)} + \|p(t_{m+1}) - p_h^{m+1}\|_{l^\infty(L^2)} \le C\, (k + h).$$

2.5 A Duality Argument

The task of this subsection is to eliminate the hypothesis about the initial step $|\delta_t \mathbf{e}_d^1| \le C\,h$ of the previous subsection, and to obtain $O(h^2)$ in $l^2(L^2)$ for $\mathbf{e}_d^{m+1}$ and $O(h)$ for $\delta_t \mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$ and for $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ in $l^2(\mathbf{H}^1 \times L^2)$.

Proposition 17 (Initial estimate) *Under hypotheses of Theorem 5, if we assume* $\|\mathbf{u}_h^0\|_{W^{1,3}\cap L^\infty} \le C$ *and* $\|\mathbf{e}_h^0\|^2 \le C\,h^2$, *then*

$$|\delta_t \mathbf{e}_h^1|^2 \le C\,\frac{h^2}{k} \tag{17}$$

In particular, the same estimate holds for $\delta_t \mathbf{e}_d^1$.

Proof: Re-writting $(E_3)_h^1$ in function of the errors $\mathbf{e}_h$ and $\mathbf{e}_i$, and taking as function test $\mathbf{v}_h = \delta_t \mathbf{e}_h^1$,

$$\begin{aligned} |\delta_t \mathbf{e}_h^1|^2 \;&+\; \frac{1}{2\,k}\Big(\|\mathbf{e}_h^1\|^2 - \|\mathbf{e}_h^0\|^2\Big) + \frac{1}{2}\,k\,\|\delta_t \mathbf{e}_h^1\|^2 \\ &\le\; \varepsilon\,|\delta_t \mathbf{e}_h^1|^2 + C|\delta_t \mathbf{e}_i^1|^2 + \varepsilon\,k\,\|\delta_t \mathbf{e}_h^1\|^2 + \frac{C}{k}\,\|\mathbf{e}_i^1\|^2 \\ &+\; \frac{C}{k}\|\mathbf{u}_h^0\|^2_{W^{1,3}\cap L^\infty}\,|\mathbf{e}_d^{1/2}|^2 + \frac{C}{k}\|\mathbf{u}^{1/2}\|^2_{W^{1,3}\cap L^\infty}\,|\mathbf{e}_d^0|^2. \end{aligned}$$

From $|\mathbf{e}_d^{1/2}| + |\mathbf{e}_d^0| \le C\,h$, $|\delta_t \mathbf{e}_i^1|^2 \le C\,h^2\|\delta_t \mathbf{u}^1\|^2$, $\|\mathbf{e}_i^1\|^2 \le C\,h^2\|\mathbf{u}^1\|^2_{H^2}$ and $\|\mathbf{u}_h^0\|_{W^{1,3}\cap L^\infty} + \|\mathbf{u}^{1/2}\|_{W^{1,3}\cap L^\infty} + \|\delta_t \mathbf{u}^1\| + \|\mathbf{u}^1\|_{H^2} \le C$, we get the desired estimate. ■

Arguing as in Theorem 12 (without applying $|\delta_t \mathbf{e}_d^1| \le C\,h$) we can arrive at (16). Then, from (17), one has

$$\|\delta_t \mathbf{e}_d^{m+1} - \delta_t \mathbf{e}_d^{m+1/2}\|^2_{l^2(\mathbf{L}^2)} + \|\delta_t \mathbf{e}_d^{m+1/2} - \delta_t \mathbf{e}_d^{m+1}\|^2_{l^2(\mathbf{L}^2)} \le C\,h^2. \tag{18}$$

and

$$\|\delta_t \mathbf{e}_d^{m+1}\|^2_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_d^{m+1/2}\|^2_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,\frac{h^2}{k} \le C$$

where we have used (**H**) in the last bound.

Then, arguing as in Corollary 13, but changing schemes $\mathbf{u}_h^{m+1}$ and $\mathbf{u}_h^{m+1/2}$ by errors $\mathbf{e}_h^{m+1}$ and $\mathbf{e}_h^{m+1/2}$, one can deduce

$$\mathbf{e}_h^{m+1} \quad \text{is bounded in} \quad l^\infty(\mathbf{W}^{1,6}). \tag{19}$$

Indeed, splitting the fully discrete error and the interpolation error in $(E_3)_h^{m+1}$, one has for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$,

$$\left\{ \begin{aligned} &(\nabla\,\mathbf{e}_h^{m+1}, \nabla\,\mathbf{v}_h) - (e_{p,h}^{m+1}, \nabla\cdot\mathbf{v}_h) = -(\delta_t \mathbf{e}_d^{m+1}, \mathbf{v}_h) \\ &\qquad + c(\mathbf{e}_d^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, \mathbf{v}_h) - (\nabla\,\mathbf{e}_i^{m+1}, \nabla\,\mathbf{v}_h) + (e_{p,i}^{m+1}, \nabla\cdot\mathbf{v}_h) \\ &(\nabla\cdot\mathbf{e}_d^{m+1}, q_h) = 0 \end{aligned} \right.$$

Using the interpolation operators defined in (6), $(\nabla \mathbf{e}_i^{m+1}, \nabla \mathbf{v}_h) - (e_{p,i}^{m+1}, \nabla \cdot \mathbf{v}_h) = 0$. Then, from Lemma 9, we get

$$\begin{aligned}\|\mathbf{e}_h^{m+1}\|_{W^{1,6}}^2 &\leq C(|\delta_t \mathbf{e}_d^{m+1}|^2 + \|\mathbf{e}_d^m\|^2 \, \|\mathbf{u}^{m+1/2}\|_{W^{1,3}\cap L^\infty}^2 + \|\mathbf{u}_h^m\|_{W^{1,3}\cap L^\infty}^2 \, \|\mathbf{e}_h^{m+1/2}\|^2) \\ &\leq C(1 + \|\mathbf{u}_h^m\|^{1-\delta} \, \|\mathbf{u}_h^m\|_{W^{1,6}}^{1+\delta}) \leq \varepsilon \, \|\mathbf{u}_h^m\|_{W^{1,6}}^2 + C,\end{aligned}$$

hence estimate (19) holds.

In particular, using continuous dependence of interpolation operator I_h, from (19) we get

$$(\mathbf{u}_h^{m+1}) \quad \text{is bounded in} \quad l^\infty(\mathbf{W}^{1,6}). \tag{20}$$

Now, let us denote by A_h^{-1} the inverse of the discrete Stokes operator. Indeed, given $\mathbf{u}_h \in \mathbf{X}_h$, we define $\mathbf{w}_h = A_h^{-1}\mathbf{u}_h \in \mathbf{X}_h$ as the weak solution of the following discrete Stokes problem (with a pressure $r_h \in Q_h$)

$$\begin{cases} (\nabla \mathbf{w}_h, \nabla \mathbf{v}_h) - (r_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{u}_h, \mathbf{v}_h) \quad \forall \, \mathbf{v}_h \in \mathbf{X}_h, \\ (\nabla \cdot \mathbf{w}_h, q_h) = 0 \quad \forall \, q_h \in Q_h. \end{cases}$$

In particular, taking $\mathbf{v}_h = \mathbf{u}_h \in \mathbf{X}_h$, since $(\nabla \cdot \mathbf{u}_h, q_h) = 0$ for all $q_h \in Q_h$, one has

$$|\mathbf{u}_h|^2 = (\nabla \mathbf{w}_h, \nabla \mathbf{u}_h) = (\nabla A_h^{-1}\mathbf{u}_h, \nabla \mathbf{u}_h),$$

and taking $\mathbf{v}_h = \mathbf{w}_h \in \mathbf{X}_h$,

$$|\nabla \mathbf{w}_h|^2 = (\mathbf{u}_h, \mathbf{w}_h), \quad \text{i.e.} \quad |\nabla A_h^{-1}\mathbf{u}_h|^2 = (\mathbf{u}_h, A_h^{-1}\mathbf{u}_h).$$

2.5.1 $O(h^2)$ for $\mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$

Theorem 18 *Assuming hypotheses of Theorem 5 and Proposition 17 and $\|\mathbf{e}_h^0\|_{H^{-1}} \leq C\, h^2$, for each k small enough, one has*

$$\|\mathbf{e}_d^m\|_{l^2(\mathbf{L}^2)} \leq C(k + h^2).$$

Proof: Re-writting $(E_3)_h^{m+1}$ in function of the errors $\mathbf{e}_h$ and $\mathbf{e}_i$ and using

$$(\nabla \mathbf{e}_i^{m+1}, \nabla \mathbf{v}_h) - (e_{p,i}^{m+1}, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \, \mathbf{v}_h \in \mathbf{Y}_h,$$

we have

$$(\delta_t \mathbf{e}_h^{m+1}, \mathbf{v}_h) + (\nabla \mathbf{e}_h^{m+1}, \nabla \mathbf{v}_h) - (e_{p,h}^{m+1}, \nabla \cdot \mathbf{v}_h) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) - (\delta_t \mathbf{e}_i^{m+1}, \mathbf{v}_h).$$

Taking $\mathbf{v}_h = A_h^{-1}\mathbf{e}_h^{m+1} \in \mathbf{X}_h$, we obtain

$$\begin{aligned}&\frac{1}{2}\Big(\|A_h^{-1}\,\mathbf{e}_h^{m+1}\|^2 - \|A_h^{-1}\,\mathbf{e}_h^m\|^2 + \|A_h^{-1}\,\mathbf{e}_h^{m+1} - A_h^{-1}\,\mathbf{e}_h^m\|^2\Big) + k|\mathbf{e}_h^{m+1}|^2 \\ &\leq k(\mathbf{NL}_h^{m+1}, A_h^{-1}\,\mathbf{e}_h^{m+1}) - k(\delta_t \mathbf{e}_i^{m+1}, A_h^{-1}\,\mathbf{e}_h^{m+1}) := I_1 + I_2.\end{aligned}$$

Bounding the terms of the RHS as follows

$$
\begin{aligned}
I_1 &= k\, c(\mathbf{e}_d^m, \mathbf{u}^{m+1/2}, A_h^{-1}\, \mathbf{e}_h^{m+1}) + k\, c(\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, A_h^{-1}\, \mathbf{e}_h^{m+1})\\
&\le C\, k(\|\mathbf{u}^{m+1/2}\|^2_{W^{1,3}\cap L^\infty} + \|\mathbf{u}_h^m\|^2_{W^{1,3}\cap L^\infty})\, \|A_h^{-1}\, \mathbf{e}_h^{m+1}\|^2 + \varepsilon k |\mathbf{e}_d^m|^2 + \varepsilon\, k\, |\mathbf{e}_d^{m+1/2}|^2\\
&\le C\, k \|A_h^{-1}\, \mathbf{e}_h^{m+1}\|^2 + \varepsilon k |\mathbf{e}_h^m|^2 + \varepsilon\, k\, |\mathbf{e}_i^m|^2\\
&\quad + \varepsilon\, k\, \{|\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}|^2 + |\mathbf{e}_h^{m+1}|^2 + |\mathbf{e}_i^{m+1}|^2\}\\
&\le C\, k \|A_h^{-1}\, \mathbf{e}_h^{m+1}\|^2 + \varepsilon k |\mathbf{e}_h^m|^2 + \varepsilon\, k\, h^4 \|\mathbf{u}^m\|^2_{H^2}\\
&\quad + \varepsilon\, k\, \{|\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}|^2 + |\mathbf{e}_h^{m+1}|^2 + h^4 \|\mathbf{u}^{m+1}\|^2_{H^2}\}
\end{aligned}
$$

(in the above inequality, we have used that $\|\mathbf{u}^{m+1/2}\|_{W^{1,3}\cap L^\infty} \le C$ and $\|\mathbf{u}_h^m\|_{W^{1,3}\cap L^\infty} \le C$ thanks to (20))

$$
\begin{aligned}
I_2 &= -k(\delta_t \mathbf{e}_i^{m+1}, A_h^{-1}\, \mathbf{e}_h^{m+1}) \le C\, k \|A_h^{-1} \mathbf{e}_h^{m+1}\|^2 + \varepsilon\, k |\mathbf{e}_i(\delta_t \mathbf{u}^{m+1})|^2\\
&\le C\, k \|A_h^{-1} \mathbf{e}_h^{m+1}\|^2 + \varepsilon\, k\, h^4 \|\delta_t \mathbf{u}^{m+1}\|^2_{H^2}.
\end{aligned}
$$

Now, adding from $m=0$ to r (with any $r < M$), we get

$$
\begin{aligned}
\|A_h^{-1}\, \mathbf{e}_h^{r+1}\|^2 \;&+\; k \sum_{m=0}^{r} |\mathbf{e}_h^{m+1}|^2\\
&\le \|A_h^{-1} \mathbf{e}_h^0\|^2 + C\, k \sum_{m=0}^{r} \|A_h^{-1} \mathbf{e}_h^{m+1}\|^2 + \varepsilon\, h^4\, k \sum_{m=0}^{r} \|\delta_t \mathbf{u}^{m+1}\|^2_{H^2}\\
&+\; C\, k\, h^4 \sum_{m=0}^{r} \{\|\mathbf{u}^m\|^2_{H^2} + \|\mathbf{u}^{m+1}\|^2_{H^2}\} + \varepsilon\, k \sum_{m=0}^{r} |\mathbf{e}_h^m|^2 + \varepsilon\, k \sum_{m=0}^{r} |\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}|^2
\end{aligned}
$$

Then, using firstly (10) and estimates of $\mathbf{u}^m$, $\mathbf{u}^{m+1}$ and $\delta_t \mathbf{u}^{m+1}$ in $l^2(\mathbf{H}^2)$ and afterwards ($\mathbf{H}$), one has

$$
\begin{aligned}
\|A_h^{-1}\, \mathbf{e}_h^{r+1}\|^2 + k \sum_{m=0}^{r} |\mathbf{e}_h^{m+1}|^2 &\le C\, \|\mathbf{e}_h^0\|^2_{-1} + C \left(k \sum_{m=0}^{r} \|A_h^{-1}\, \mathbf{e}_h^{m+1}\|^2 + k\, h^2 + h^4\right) + \varepsilon\, k |\mathbf{e}_h^0|^2\\
&\le C \left(k \sum_{m=0}^{r} \|A_h^{-1}\, \mathbf{e}_h^{m+1}\|^2 + k^2 + h^4\right)
\end{aligned}
$$

Finally, applying the generalized discrete Gronwall's Lemma, we obtain (whether k is small enough)

$$
\|\mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\, (k + h^2).
$$

In particular,

$$
\|\mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\, (k + h^2).
$$

■

Remark 19 *From (1) and Theorem 18, one has*

$$
\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^2(\mathbf{L}^2)} \le C(k + h^2)
$$

2.5.2 $O(h)$ for $\delta_t \mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$ and for $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ in $l^2(\mathbf{H}^1 \times L^2)$

Theorem 20 *Under assumptions of Proposition 17 and $|\mathbf{e}_h^0| \le C\,k\,h^2$, the following error estimate holds (for each k small enough)*

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\,h.$$

Proof: Using interpolation errors, it is suffices to prove the result for $\delta_t \mathbf{e}_h^{m+1}$. Indeed, using $|\delta_t \mathbf{e}_i^{m+1}| \le C\,h\,\|\delta_t \mathbf{u}^{m+1}\|$ and that $\delta_t \mathbf{u}^{m+1}$ is bounded in $l^2(\mathbf{H}^1)$, one has

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\,h + \|\delta_t \mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)}.$$

Therefore, we will prove that $\|\delta_t \mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\,h$. We divide this proof into two steps:

1. *Initial estimate : $\|A_h^{-1}\,\delta_t \mathbf{e}_h^1\| \le C\,h$ and $|\delta_t \mathbf{e}_h^1| \le C$.*

Re-writting $(E_3)_h^1$ in function of the errors $\mathbf{e}_h$ and $\mathbf{e}_i$, we have

$$(\delta_t \mathbf{e}_h^1, \mathbf{v}_h) + (\nabla\, \mathbf{e}_h^1, \nabla \mathbf{v}_h) - (e_{p,h}^1, \nabla\cdot \mathbf{v}_h) = \mathbf{NL}_h^1(\mathbf{v}_h) - (\delta_t \mathbf{e}_i^1, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{X}_h$$

Taking $\mathbf{v}_h = A_h^{-1}\,\delta_t \mathbf{e}_h^1$, we obtain

$$\begin{aligned}&\|A_h^{-1}\,\delta_t \mathbf{e}_h^1\|^2 + \frac{1}{2\,k}\Big(|\mathbf{e}_h^1|^2 - |\mathbf{e}_h^0|^2 + |\mathbf{e}_h^1 - \mathbf{e}_h^0|^2\Big)\\ &\le (\mathbf{NL}_h^1, A_h^{-1}\,\delta_t \mathbf{e}_h^1) - (\delta_t \mathbf{e}_i^1, A_h^{-1}\,\delta_t \mathbf{e}_h^1) := I_1 + I_2.\end{aligned}$$

Bounding the terms of the RHS as follows

$$I_1 \le \varepsilon \|A_h^{-1}\,\delta_t \mathbf{e}_h^1\|^2 + C\|\mathbf{u}_h^0\|^2_{W^{1,3}\cap L^\infty} |\mathbf{e}_d^{1/2}|^2 + C\|\mathbf{u}^{1/2}\|^2_{W^{1,3}\cap L^\infty} |\mathbf{e}_d^0|^2 \le \varepsilon \|A_h^{-1}\,\delta_t \mathbf{e}_h^1\|^2 + C\,h^2$$

(in the above inequality, we have used that $|\mathbf{e}_d^0| \le C\,h$ by hypothesis and $|\mathbf{e}_d^{1/2}| \le C\,h$ as a particular consequence of (9)),

$$I_2 \le \varepsilon \|A_h^{-1}\,\delta_t \mathbf{e}_h^1\|^2 + C\,h^2\,\|\delta_t \mathbf{u}^1\|^2.$$

Hence, using the hypothesis $|\mathbf{e}_h^0| \le C\,k\,h^2$, we arrive at

$$\|A_h^{-1}\,\delta_t \mathbf{e}_h^1\|^2 + \frac{1}{k}|\mathbf{e}_h^1|^2 \le \frac{1}{k}\,|\mathbf{e}_h^0|^2 + C\,h^2 \le C\,h^2.$$

2. *Generic estimate of $\delta_t \mathbf{e}_h^{m+1}$ ($\forall\, m \ge 1$).*

Writting $(D_3)_h^{m+1}$ in function of the errors $\mathbf{e}_h$ and $\mathbf{e}_i$,

$$\begin{aligned}&\Big(\frac{1}{k}(\delta_t \mathbf{e}_h^{m+1} - \delta_t\, \mathbf{e}_h^m), \mathbf{v}_h\Big) + \Big(\nabla \delta_t \mathbf{e}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + (\delta_t e_{p,h}^{m+1}, \nabla\cdot \mathbf{v}_h)\\ =\;& \delta_t\, \mathbf{NL}_h^{m+1}(\mathbf{v}_h) - \Big(\frac{1}{k}(\delta_t \mathbf{e}_i^{m+1} - \delta_t\, \mathbf{e}_i^m), \mathbf{v}_h\Big) + (\nabla \delta_t \mathbf{e}_i^{m+1}, \nabla\, \mathbf{v}_h) - (\delta_t e_{p,i}^{m+1}, \nabla\cdot \mathbf{v}_h)\\ =\;& \delta_t\, \mathbf{NL}_h^{m+1}(\mathbf{v}_h) - \Big(\frac{1}{k}(\delta_t \mathbf{e}_i^{m+1} - \delta_t\, \mathbf{e}_i^m), \mathbf{v}_h\Big)\end{aligned}$$

Taking $\mathbf{v}_h = 2\,k\,A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1}$ (for each $m \geq 1$):

$$\begin{aligned} & \|A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1}\|^2 - \|A_h^{-1}\,\delta_t \mathbf{e}_h^{m}\|^2 + \|A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1} - A_h^{-1}\,\delta_t \mathbf{e}_h^{m}\|^2 + 2\,k\,|\delta_t \mathbf{e}_h^{m+1}|^2 \\ = \ & 2\,k\,(\delta_t \mathbf{NL}^{m+1}, A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1}) + C\,k\Big(-\frac{1}{k}(\delta_t \mathbf{e}_i^{m+1} - \delta_t\,\mathbf{e}_i^{m}), A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1}\Big) \end{aligned}$$

We are going only to bound the more complicated term of the RHS:

$$\begin{aligned} 2\,k\,c(\mathbf{u}_h^{m-1}, \delta_t \mathbf{e}_h^{m+1/2}, A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1}) &\leq C\,k\,\|\mathbf{u}_h^{m-1}\|^2_{W^{1,3}\cap L^\infty}\,\|A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1}\|^2 + \varepsilon\,k\,|\delta_t \mathbf{e}_h^{m+1/2}|^2 \\ &\leq C\,k\,\|A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1}\|^2 + \varepsilon\,k\,\{|\delta_t \mathbf{e}_h^{m+1}|^2 + |\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m+1}|^2\} \end{aligned}$$

where (20) is used in order to bound $\|\mathbf{u}_h^{m-1}\|^2_{W^{1,3}\cap L^\infty}$.

Adding from $m = 1$ to r, last term in the above inequality can be bounded by $C\,h^2$ thanks to (18).

Reasoning with similar arguments to Theorem 2 ([10]), applying the generalized discrete Gronwall's Lemma, we arrive at

$$\|\delta_t \mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)} \leq C\,h.$$

for k small enough. ■

Corollary 21 *Assuming hypotheses of Theorem 20, one has*

$$\|e_{p,d}^{m+1}\|_{l^2(L^2)} \leq C\,h.$$

The proof is based in $(E_3)_h^{m+1}$, in a similar manner as in the proof of Corollary 15.

Remark 22 *Combining Theorem 2 and Corollary 21, we arrive at*

$$\|p(t_m) - p_h^m\|_{l^2(L^2)} \leq C(k + h).$$

Appendix

Proof [of Lemma 9]:

We consider $\big(\mathbf{u}(h), p(h)\big)$ the solution of Stokes Problem (or respectively Poisson Problem) with second member $\mathbf{g}$. This solution verifies ([6])

$$\|\mathbf{u}(h)\|_{H^2} + \|p(h)\|_{H^1} \leq C\,|\mathbf{g}| \quad \text{(or respectively } \|\mathbf{u}(h)\|_{H^2} \leq C\,|\mathbf{g}|)$$

and

$$\|\mathbf{u}(h) - \mathbf{u}_h\| \leq C\,h\,\{\|\mathbf{u}(h)\|_{H^2} + \|p(h)\|_{H^1}\} \quad \text{(or respectively } \|\mathbf{u}(h) - \mathbf{u}_h\| \leq C\ h\,\|\mathbf{u}(h)\|_{H^2})$$

In particular,

$$\|\mathbf{u}(h) - \mathbf{u}_h\| \leq C\,h\,|\mathbf{g}| \tag{21}$$

On the other hand, we decompose $\mathbf{u}_h = \mathbf{u}_h - I_h\mathbf{u}(h) + I_h\mathbf{u}(h) - \mathbf{u}(h) + \mathbf{u}(h)$. Then,

$$\|\mathbf{u}_h\|_{W^{1,6}} \le \|\mathbf{u}_h - I_h\mathbf{u}(h)\|_{W^{1,6}} + \|I_h\mathbf{u}(h) - \mathbf{u}(h)\|_{W^{1,6}} + \|\mathbf{u}(h)\|_{W^{1,6}}$$

We bound the RHS as follows

$$\|\mathbf{u}(h)\|_{W^{1,6}} \le C\|\mathbf{u}(h)\|_{H^2} \le C\,|\mathbf{g}|$$

$$\|I_h\mathbf{u}(h) - \mathbf{u}(h)\|_{W^{1,6}} \le C\,\|\mathbf{u}(h)\|_{W^{1,6}} \le C\|\mathbf{u}(h)\|_{H^2} \le C\,|\mathbf{g}|$$

(here, the stability property $\|I_h\mathbf{u}\|_{W^{1,6}} \le \|\mathbf{u}\|_{W^{1,6}}$ has been applied)

$$\|\mathbf{u}_h - I_h\mathbf{u}(h)\|_{W^{1,6}} \le \frac{C}{h}\|\mathbf{u}_h - I_h\mathbf{u}(h)\| \le \frac{C}{h}\{\|\mathbf{u}_h - \mathbf{u}(h)\| + \|I_h\mathbf{u}(h) - \mathbf{u}(h)\|\}$$

(here we have used the following inverse inequality $\|\mathbf{v}_h\|_{W^{1,6}} \le C\,h^{-1}\|\mathbf{v}_h\|$ for each $\mathbf{v}_h \in \mathbf{X}_h$).

Finally, applying (21) and error interpolation inequality $\|I_h\mathbf{u}(h) - \mathbf{u}(h)\| \le C\,h\,\|\mathbf{u}(h)\|_{H^2}$, we arrive at

$$\|I_h\mathbf{u}(h) - \mathbf{u}_h\|_{W^{1,6}} \le C\,|\mathbf{g}|.$$

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Optimal error estimates of a pressure segregation scheme for the $3D$ Navier-Stokes equations via an incremental pressure projection method *

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Abstract

This work is devoted to obtain optimal error analysis of a fully discrete scheme for the incompressible time-dependent Navier-Stokes equations in three-dimensional domains, which decouples the computations of velocity and pressure, solving a linear convection-diffusion system for the velocity and a Poisson problem for the pressure.

The main idea used here, is to introduce an artificial projection step in order to rewrite the scheme as the pressure incremental projection method.

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Keywords: Navier-Stokes Equations, pressure segregation, projection schemes, error estimates, first order time scheme, finite elements.

Introduction

We consider the Navier-Stokes system, associated to the dynamics of viscous and incompressible fluids filling a bounded domain $\Omega \subset I\!R^3$ in a time interval $(0,T)$:

$$(P) \qquad \left\{ \begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u} &=& \mathbf{0} & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}_{|t=0} &=& \mathbf{u}_0 & \text{in } \Omega. \end{array} \right.$$

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where the unknowns are $\mathbf{u} : (\mathbf{x}, t) \in \Omega \times (0,T) \to I\!R^3$ the velocity field and $p : (\mathbf{x}, t) \in \Omega \times (0,T) \to I\!R$ the pressure, and data are $\nu > 0$ the viscosity coefficient (which simplicity is assumed constant) and $\mathbf{f} : (\mathbf{x}, t) \in \Omega \times (0,T) \to I\!R^3$ the external forces. We denote by ∇ the gradient operator and Δ the Laplace operator.

We consider a (regular) partition of $[0,T]$ of diameter $k = T/M$: $t_0 = 0, t_1 = k, \ldots, t_m = mk, \ldots, t_M = T$. If $u = (u^m)_{m=0}^M$ is a given vector with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^{M} \|u^m\|_X^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\ldots,M} \|u^m\|_X$$

For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., and $\mathbf{H^1} = H^1(\Omega)^3$ etc.

The numerical analysis for the Navier-Stokes Problem (P) has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties of this problem are the coupling between the pressure p and the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ and the nonlinearity of the convective terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

Fractional step methods are becoming widely used in this context, which split effects due to different operators appearing in the problem.

The origin of these methods is generally credited to the works of Chorin [4] and Temam [21]. They developed the well known *Chorin-Temam projection method*, which is a two step scheme, computing an intermediate velocity via a convection-diffusion problem and the second step is a free divergence projection step to obtain a pair velocity-pressure. Afterwards, a modified projection scheme (called *incremental pressure or Van-Kan scheme*) where an explicit pressure term is added in the first step and a pressure correction term in the projection step was developed. The main drawback of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure verifies “artificial” boundary conditions.

There are other variants of projection methods: rotational pressure-correction schemes ([24], [11]), in which the rotational operator plays a key role, velocity-correction schemes ([9], [10]). Other variants can be seen in [16] and [17].

The convergence of the *Chorin-Temam projection method*, was proved in [22] for the time discrete scheme and in [5] for a fully discrete scheme associated to a problem with periodic boundary conditions.

More recently, error estimates for projection methods have been obtained (see [19], [20] for time discrete schemes and see [8] for a fully discrete scheme). Basically, for the Chorin-Temam projection scheme, one has time error estimates of order $O(k^{1/2})$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ and of order $O(k)$ in $l^2(\mathbf{L}^2)$ for both velocities and of order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure. For the incremental pressure scheme, the error estimates are improved to order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$

for the intermediate velocity and order $O(k)$ in $l^2(L^2)$ for the pressure (this last estimate in the linear case) [19], [20]. In fact, these optimal estimates for the time discrete scheme, are also obtained in [8] for a fully discrete scheme by stable finite elements, solving the projection step by means of a mixed velocity-pressure formulation, under the constraint $k^2 \le \alpha h$ in three-dimensional domains or $k^2 \le \alpha(1 + \log(h^{-1}))$ in two-dimensional cases. The argument is based on the use of some inverse inequalities, hence a regular and quasi-uniform family of triangulations of of the domain Ω must be considered. By the contrary, in the present paper, we will obtain optimal error estimates under the completely different constraint $h \le \alpha\, k$ and without to consider a quasi-uniform triangulation, for a FEM decoupled scheme, which can be rewritten as the incremental pressure projection method but solving the projection step by means of a Poisson problem for the pressure.

This particular property that these projection methods (without and with pressure correction), can be rewritten as pressure segregation methods (decoupled the computations for velocity and pressure), was observed in [18], [19], in order to justify heuristically the better approximation of the first step velocity than the free-divergence one. Very recently, for the segregation scheme associated to the non-incremental projection method, in [1] the convergence and sub-optimal error estimates for the pressure (of order $O(k^{1/2}+h)$) are obtained without to impose the inf-sup condition for the finite element spaces, under the more exigent constraint $\alpha\, h^2 \le k \le \beta\, h^2$.

The work of the present paper follows this line, in the sense that we use the pressure segregation formulation, but related to the incremental projection method, obtaining optimal error estimates (of order $O(k+h)$) also for the pressure, assuming the inf-sup condition for the discrete spaces and under the constraint $h \le \alpha k$.

This paper is organized as follows:

In Section 1, we study the time discrete scheme. Firstly, the scheme is described and its stability is deduced, introducing the problems verified by the errors and the regularity hypotheses imposed on the exact solution. Afterwards, we will obtain $O(k)$ error estimates for the velocity in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and $O(1)$ for the pressure in $l^\infty(H^1)$. As a consequence, the intermediate velocity is bounded in $l^\infty(\mathbf{H}^2)$. After that, we will get $O(k)$ error estimates for the discrete in time derivatives of velocity in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$. Finally, $O(k)$ error estimates for the velocity in $l^\infty(\mathbf{H}^1)$ and for the pressure in $l^\infty(L^2)$ hold.

Section 2 is devoted to the fully discrete scheme, which decouples the computations of the pressure and for the velocity. We present the finite elements spaces and their approximation properties, the scheme and the problems verified by the errors (comparing the time discrete scheme given in Section 1 and this fully discrete scheme). With respect to the spatial error estimates, we will obtain firstly $O(h)$ for the velocity in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ (jointly with $O(h)$ for an auxiliary discrete projected velocity in $l^\infty(\mathbf{L}^2)$), which imply $W^{1,6}(\Omega)$-estimates for the velocity whether $h/k \le C$. Afterwards, $O(h)$ for the discrete derivative of velocity in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$

(and for the projected velocity in $l^\infty(\mathbf{L}^2)$) are obtained. Finally, $O(h)$-error estimates for the velocity in $l^\infty(\mathbf{H}^1)$ and for the pressure in $l^\infty(L^2)$ hold.

1 Time discrete scheme

1.1 Description of the scheme

Given a (uniform) partition of the time interval $[0,T]$ with diameter $k = T/M$, $\{t_m = m\,k\}_{m=0}^M$, and $(\mathbf{f}^m)_{m=1}^M$ an approximation of $\mathbf{f}(t_m)$, we will define $(\mathbf{u}^m, p^m)_{m=1}^M$ an approximation of the solution $\{\mathbf{u}, p\}$ of (P) at the time $t = t_m$, by means of a two-step scheme splitting the nonlinearity $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and the incompressibility condition $\nabla\cdot\mathbf{u} = 0$ into different steps (but keeping pressure terms in both steps):

We consider a pressure correction projection scheme of Van-Kan type [15]. Concretely, an explicit pressure term is introduced in the convection-diffusion problem for the velocity (step 1), with an implicit correction in the free-divergence projection step (step 2).

In the sequel, as usual, we will use the following skew-symmetric form of the convective term:

$$C(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\,\mathbf{v} \quad \forall\,\mathbf{u}\in\mathbf{H}_0^1,\ \mathbf{v}\in\mathbf{H}^1,$$

and

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega C(\mathbf{u},\mathbf{v})\cdot\mathbf{w} = \int_\Omega \Big\{(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}\Big\}, \qquad \forall\,\mathbf{u}\in\mathbf{H}_0^1,\ \mathbf{v}\in\mathbf{H}^1,\ \mathbf{w}\in\mathbf{H}^1$$

or equivalently

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\int_\Omega \Big\{(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\Big\} = -\int_\Omega \Big\{(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}\Big\}$$

Previous equalities hold even in the fully discrete case, hence we can use, in the sequel, any of these three possibilities. Obviously, $c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}$ whether $\nabla\cdot\mathbf{u} = 0$.

The trilinear form $c(\cdot,\cdot,\cdot)$ verifies

$$c(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \qquad \forall\,\mathbf{u}\in\mathbf{H}_0^1,\ \ \forall\,\mathbf{v}\in\mathbf{H}^1, \tag{1}$$

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) \le C \begin{cases} \|\mathbf{u}\|\,\|\mathbf{v}\|_{W^{1,3}\cap L^\infty}\,|\mathbf{w}| \\ \|\mathbf{u}\|_{L^3}\,\|\mathbf{v}\|\,\|\mathbf{w}\| \end{cases}$$

where the role of $\mathbf{u},\mathbf{v},\mathbf{w}$ can be interchanged, using the appropriate expression of $c(\cdot,\cdot,\cdot)$.

For simplicity and without loss of generality, we fix the viscosity constant $\nu = 1$.

The semi-discrete scheme is defined as follows:

Initialization: Let $\widetilde{\mathbf{u}}^0 = \mathbf{u}(0)$. Let p^0 be given and to take $\mathbf{u}^0 = \widetilde{\mathbf{u}}^0$.

Sub-step 1 : Given $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$ and p^m, to find $\widetilde{\mathbf{u}}^{m+1} : \Omega \to I\!R^3$ solution of

$$(S_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta \widetilde{\mathbf{u}}^{m+1} + \nabla p^m = \mathbf{f}^{m+1}, & \text{in } \Omega \\ \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0, \quad \text{on } \partial\Omega \end{cases}$$

Notice that we have written the convection term in a linear or semi-implicit form. For simplicity, we take $\mathbf{f}^{m+1} = \mathbf{f}(t_{m+1})$.

Sub-step 2 : Given p^m and $\widetilde{\mathbf{u}}^{m+1}$, to find $\mathbf{u}^{m+1} : \Omega \to I\!R^3$ and $p^{m+1} : \Omega \to I\!R$ solution of

$$(S_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}) + \nabla\left(p^{m+1} - p^m\right) = 0 \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{in } \Omega, \quad \mathbf{u}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{cases}$$

This step is a projection step. In fact, $\mathbf{u}^{m+1} = P_{\mathbf{H}} \widetilde{\mathbf{u}}^{m+1}$ where $P_{\mathbf{H}}$ is the L^2-projection onto $\mathbf{H}$, because

$$\left(\mathbf{u}^{m+1}, \nabla q\right) = 0 \quad \forall\, q \in H^1(\Omega). \tag{2}$$

It is well known that Sub-step 2 is equivalent to the two following (decoupled) problems:

1. To find $p^{m+1} : \Omega \to I\!R$ such that

$$(S_2)_a^{m+1} \qquad \begin{cases} k\,\Delta(p^{m+1} - p^m) = \nabla \cdot \widetilde{\mathbf{u}}^{m+1} \quad \text{in } \Omega \\ k\,\nabla(p^{m+1} - p^m) \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

2. To find $\mathbf{u}^{m+1} : \Omega \to I\!R^3$ as

$$(S_2)_b^{m+1} \qquad \mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m) \quad \text{in } \Omega.$$

With respect to the practical implementation of this scheme, an initial pressure p^0 must be introduced as approximation of $p(0)$, which is not possible to compute (see Remark 5 below). Consequently, to implement this scheme, is necessary to begin with several auxiliary initial steps with another scheme. This problem is inherent to each pressure incremental scheme.

On the other hand, adding $(S_1)^{m+1}$ and $(S_2)^{m+1}$, we get

$$(S_3)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta \widetilde{\mathbf{u}}^{m+1} + \nabla p^{m+1} = \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0, \quad \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{in } \Omega. \end{cases}$$

$(S_3)^{m+1}$ can be viewed as consistence relations, because if we could demonstrate that $\widetilde{\mathbf{u}}^{m+1}$ and $\mathbf{u}^{m+1}$ converge to the same limit function $\mathbf{u}$ and the convergence is sufficiently strong, taking limits in $(S_3)^{m+1}$, we will find that $\mathbf{u}$ is a solution of the continuous problem (P).

1.2 Unconditional stability of the time discrete scheme

Let us to introduce the following Hilbert spaces:

$$\begin{aligned} \mathbf{H} &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} \cdot \mathbf{n}_{\partial\Omega} = 0\}, \\ \mathbf{V} &= \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) \ : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\}, \end{aligned}$$

being $\mathbf{n}_{\partial\Omega}$ the normal outwards vector to $\partial\Omega$.

In this Section, by C we will denote different constants, always independent of k.

The norm and inner product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and $\Big(\cdot,\cdot\Big)$, whereas the norm in $H^1_0(\Omega)$ of the gradient in $L^2(\Omega)$ will be denoted by $\|\cdot\|$. Any other norm in a space X will be denoted by $\|\cdot\|_X$

Lemma 1 *(Continuous dependence of the scheme)*
a) *Continuous dependence with respect to L^2.*

Assuming $\widetilde{\mathbf{u}}^{m+1}$ y $\mathbf{u}^m \in \mathbf{L}^2(\Omega)$, then there exists an unique $\mathbf{u}^{m+1} \in \mathbf{H}$. Moreover,

$$|\mathbf{u}^{m+1}| \le |\widetilde{\mathbf{u}}^{m+1}| \qquad \textit{and} \qquad |\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}| \le |\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|.$$

b) *Continuous dependence with respect to H^1.*

If $\mathbf{u}^m \in \mathbf{H}^1(\Omega) \cap \mathbf{H}$ and $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}^1_0(\Omega)$, then $\mathbf{u}^{m+1} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}$. Moreover, there exists $C = C(\Omega) > 0$ such as

$$\|\mathbf{u}^{m+1}\| \le C\, \|\widetilde{\mathbf{u}}^{m+1}\|.$$

Proof.

a) From $\mathbf{u}^{m+1} = P_{\mathbf{H}}\widetilde{\mathbf{u}}^{m+1}$, one has $|\mathbf{u}^{m+1}| \le |\widetilde{\mathbf{u}}^{m+1}|$.

On the other hand, by multiplying $(S_2)^{m+1}$ by $\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}$, we obtain

$$\begin{aligned} |\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}|^2 &= \Big(k\,\nabla(p^{m+1}-p^m), \mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}\Big) = \Big(k\,\nabla(p^{m+1}-p^m), \mathbf{u}^{m} - \widetilde{\mathbf{u}}^{m+1}\Big) \\ &\le |k\,\nabla(p^{m+1}-p^m)|\,|\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m| \le |\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}|\,|\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|, \end{aligned}$$

hence we get $|\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}| \le |\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|$. Notice that this last estimate can be obtained directly as consequence from the L^2-projection property of best approximation, because $|\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}| = \min\limits_{\mathbf{u}\in\mathbf{H}} |\mathbf{u} - \widetilde{\mathbf{u}}^{m+1}|$

b) By applying the regularity of the problem $(S_2)^{m+1}_a$, there exists $p^{m+1} - p^m \in H^2 \cap L^2_0$ and it satisfies

$$k\,\|\nabla(p^{m+1}-p^m)\|_{H^1} \le C\|\widetilde{\mathbf{u}}^{m+1}\|,$$

where $C = C(\Omega, D) > 0$. Then, $\mathbf{u}^{m+1} \in \mathbf{H}^1(\Omega)$ and

$$\|\mathbf{u}^{m+1}\| \le C\Big\{\|\widetilde{\mathbf{u}}^{m+1}\| + k\,\|\nabla(p^{m+1}-p^m)\|_{H^1}\Big\}$$

obtaining the desired estimates.

Notice that **b**) can be understood as the $\mathbf{H}^1$-stability of the $\mathbf{L}^2$-projection.

■

Lemma 2 *(***Stability of the scheme***) Let* $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ *(*$\mathbf{H}^{-1}(\Omega)$ *being the dual space of* $\mathbf{H}^1_0(\Omega)$*) and* $\mathbf{u}_0 \in \mathbf{H}$*. Assuming the following constraint on the initial discrete pressure* $k\,|\nabla p^0| \leq C_0$*, then there exists a constant* $C = C(C_0, \mathbf{f}, \Omega) > 0$ *such that,*

$$
\begin{aligned}
&|\widetilde{\mathbf{u}}^{r+1}|^2 + |\mathbf{u}^{r+1}|^2 + |k\,\nabla p^{r+1}|^2 \leq C, \quad \forall\, r = 0, ..., M-1,\\
&k \sum_{m=0}^{M-1} \left\{ \|\widetilde{\mathbf{u}}^{m+1}\|^2 + \|\mathbf{u}^{m+1}\|^2 \right\} \leq C,\\
&\sum_{m=0}^{M-1} \left\{ |\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|^2 + |\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}|^2 \right\} \leq C.
\end{aligned}
$$

Proof. (An outline of the proof) By making $2k\left((S_1)^{m+1}, \widetilde{\mathbf{u}}^{m+1}\right) + k\left((S_2)^{m+1}, \widetilde{\mathbf{u}}^{m+1} + \mathbf{u}^{m+1} + k\,(\nabla p^{m+1} + \nabla p^m)\right)$, one can obtain (following the same lines that in the proof of Theorem 6),

$$
\|\mathbf{u}^{m+1}\|_{l^\infty(L^2)} + \|\widetilde{\mathbf{u}}^{m+1}\|_{l^2(H^1)} + \|k\,\nabla p^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C.
$$

After that, taking into account Lemma 1, we obtain the supplementary stability estimates

$$
\|\widetilde{\mathbf{u}}^{m+1}\|_{l^\infty(L^2)} \leq C \quad \text{and} \quad \|\mathbf{u}^{m+1}\|_{l^2(H^1)} \leq C.
$$

■

Starting of the previous stability estimates, the convergence of the velocity approximations have already been established (for instance, see [23]). Concretely, defining $\mathbf{u}_k$ as the piecewise constant functions taking values $\mathbf{u}^{m+1}$ on $(t_m, t_{m+1}]$, the following result hold:

Proposition 3 (Convergence) *Under conditions of Lemma 2, there exists a subsequence* (k') *of* (k)*, with* $k' \downarrow 0$*, and a weak solution* $\mathbf{u}$ *of* (P) *in* $(0,T)$*, such that:* $\mathbf{u}_{k'} \to \mathbf{u}$ *weakly-∗ in* $L^\infty(0,T;\mathbf{L}^2(\Omega))$*, weakly in* $L^2(0,T;\mathbf{H}^1(\Omega))$ *and strongly in* $L^2(0,T;\mathbf{H})$*.*

1.3 Differential problems verified by the errors

We will obtain error estimates (for velocity and pressure) with respect to a sufficiently regular (unique in particular) solution $(\mathbf{u}, p)$ of (P). For this, we introduce the following notations for the errors in $t = t_{m+1}$:

$$
\widetilde{\mathbf{e}}^{m+1} = \mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}^{m+1}, \quad \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad e_p^{m+1} = p(t_{m+1}) - p^{m+1},
$$

and for the discrete in time derivative of errors

$$
\delta_t \mathbf{e}^{m+1} = \frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{k}, \quad \delta_t \widetilde{\mathbf{e}}^{m+1} = \frac{\widetilde{\mathbf{e}}^{m+1} - \widetilde{\mathbf{e}}^m}{k}.
$$

Subtracting $(S_1)^{m+1}$ with (P) in $t = t_{m+1}$, using the integral rest and manipulating the convective terms, one has [2, 3]:

$$(E_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m) - \Delta\widetilde{\mathbf{e}}^{m+1} + \nabla\left(e_p^m + k\,\delta_t\, p(t_{m+1})\right) = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} & \text{in } \Omega, \\ \widetilde{\mathbf{e}}^{m+1}|_{\partial\Omega} = \mathbf{0}, \end{cases}$$

where

$$\mathcal{E}^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t - t_m)\,\mathbf{u}_{tt}(t)\,dt - \left(\int_{t_m}^{t_{m+1}} \mathbf{u}_t \cdot \nabla\right)\mathbf{u}(t_{m+1}) := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1}$$

is the consistency error, and

$$\mathbf{NL}^{m+1} = -C\Big(\widetilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1})\Big) - C\Big(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{e}}^{m+1}\Big)$$

are the residual terms appearing in the differences of the quadratic terms. These terms can be rewritten as

$$\mathbf{NL}^{m+1} = -C\Big(\widetilde{\mathbf{e}}^m, \widetilde{\mathbf{u}}^{m+1}\Big) - C\Big(\mathbf{u}(t_m), \widetilde{\mathbf{e}}^{m+1}\Big)$$

On the other hand, adding and subtracting the term $\mathbf{u}(t_{m+1})$ in $(S_2)^{m+1}$, we get

$$(E_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}) + \nabla\Big(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})\Big) = \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Again, the problem $(E_2)^{m+1}$ can be decomposed into two problems as follows:

$$(E_2)_a^{m+1} \qquad \begin{cases} k\,\Delta\Big(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})\Big) = \nabla\cdot\widetilde{\mathbf{e}}^{m+1} \quad \text{in } \Omega, \\ k\,\nabla\Big(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})\Big)\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{0}, \end{cases}$$

and

$$(E_2)_b^{m+1} \qquad \mathbf{e}^{m+1} = \widetilde{\mathbf{e}}^{m+1} - k\,\nabla(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})) \quad \text{in } \Omega.$$

Finally, adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$, we arrive at:

$$(E_3)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{e}^{m+1} - \mathbf{e}^m) - \Delta\widetilde{\mathbf{e}}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}\cdot\mathbf{n}|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

Lemma 4 *(Continuous dependence of the errors) The following inequalities hold*

$$|\mathbf{e}^{m+1}| \le |\widetilde{\mathbf{e}}^{m+1}|, \qquad |\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}| \le |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|.$$

Moreover, there exists $C = C(\Omega) > 0$ such that

$$\|\mathbf{e}^{m+1}\| \le C\,\|\widetilde{\mathbf{e}}^{m+1}\|.$$

Proof. The proof is similar to Lemma 1, by using that $\mathbf{e}^{m+1} = P_{\mathbf{H}}\widetilde{\mathbf{e}}^{m+1}$. ∎

1.4 Regularity hypotheses.

In the sequel, we will assume the following regularity hypothesis on Ω:

($\mathbf{H0}$) $\Omega \subset I\!R^3$ such that the Poisson problem in Ω has $\mathbf{H}^2(\Omega)$ regularity.

In order to obtain the different error estimates, the following regularity hypotheses for the (unique) solution $(\mathbf{u}, p)$ of (P) will appear:

($\mathbf{H1}$) $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}), \quad p_t \in L^2(H^1), \quad \mathbf{u}_t \in L^2(\mathbf{L}^2), \quad \mathbf{u}_{tt} \in L^2(\mathbf{H}^{-1})$

($\mathbf{H2}$) $p_{tt} \in L^2(H^1), \quad \mathbf{u}_t \in L^\infty(\mathbf{L}^3) \cap L^3(\mathbf{H}^1), \quad \mathbf{u}_{tt} \in L^2(\mathbf{L}^2), \quad \mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1})$

($\mathbf{H3}$) $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1})$

Remark 5 *Unfortunately, to obtain hypotheses* ($\mathbf{H1}$)-($\mathbf{H3}$) *is necessary to assume that* $\mathbf{u}_t(0) \in \mathbf{H}^1$*, which implies a non local compatibility condition for the data* $\mathbf{u}_0$ *and* $\mathbf{f}$*. In particular, it is proved in [14] that this regularity statement can only be valid, if there exists* $p_0 \in H^1$ *(the initial pressure) solution of the overdetermined Elliptic problem*

$$\begin{cases} \Delta p_0 = \nabla \cdot \Big(\mathbf{f}(0) - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0\Big) \quad \text{in } \Omega, \\ \nabla p_0|_{\partial\Omega} = \Big(\Delta \mathbf{u}_0 + \mathbf{f}(0) - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0\Big)|_{\partial\Omega}. \end{cases}$$

1.5 $O(k)$-error estimates for the velocities

Theorem 6 *Under conditions of Lemma 2,* **(H1)** *and the constraint* $|\nabla e_p^0| \le C$*, the following error estimates hold:*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k \quad \text{and} \quad \|e_p^{m+1}\|_{l^\infty(H^1)} \le C. \tag{3}$$

Moreover,

$$\|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{l^2(\mathbf{L}^2)} \le C\,k^{3/2}. \tag{4}$$

Proof.

By multiplying $(E_1)^{m+1}$ by $2\,k\,\widetilde{\mathbf{e}}^{m+1}$ and integrating in Ω, one has:

$$\begin{aligned} &|\widetilde{\mathbf{e}}^{m+1}|^2 - |\mathbf{e}^m|^2 + |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + 2\,k\|\widetilde{\mathbf{e}}^{m+1}\|^2 + 2\,k\left(\nabla(e_p^m + k\,\delta_t\,p(t_{m+1})), \widetilde{\mathbf{e}}^{m+1}\right) \\ &\le 2\,k\,\langle \mathcal{E}^{m+1}, \widetilde{\mathbf{e}}^{m+1}\rangle + 2\,k(\mathbf{NL}^{m+1}, \widetilde{\mathbf{e}}^{m+1}) \end{aligned} \tag{5}$$

The consistency error $\mathcal{E}^{m+1} = \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1}$ can be bounded as follows:

$$2\,k\,\langle \mathcal{E}_1^{m+1}, \widetilde{\mathbf{e}}^{m+1}\rangle \;\le\; \frac{k}{3}\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k^2 \int_{t_m}^{t_{m+1}} \|\mathbf{u}_{tt}\|^2_{\mathbf{H}^{-1}}\,dt,$$

$$2\,k\left(\mathcal{E}_2^{m+1}, \widetilde{\mathbf{e}}^{m+1}\right) \le 2\,k\left(\int_{t_m}^{t_{m+1}} |\mathbf{u}_t|\right) \|\nabla \mathbf{u}(t_{m+1})\|_{L^3} \|\widetilde{\mathbf{e}}^{m+1}\|_{L^6} \le \frac{k}{3}\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k^2 \int_{t_m}^{t_{m+1}} |\mathbf{u}_t|^2.$$

With respect to the convective terms,

$$2\,k\,\mathbf{NL}^{m+1}(\widetilde{\mathbf{e}}^{m+1}) = 2\,k\,c\Big(\widetilde{\mathbf{e}}^{m}, \mathbf{u}(t_{m+1}), \widetilde{\mathbf{e}}^{m+1}\Big) \le \frac{k}{3}\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\mathbf{u}(t_{m+1})\|^2_{L^\infty\cap W^{1,3}}|\widetilde{\mathbf{e}}^{m}|^2$$

(here the antisymmetry property $c(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{e}}^{m+1}, \widetilde{\mathbf{e}}^{m+1}) = 0$ has been used).

Taking into account these estimates in (5), we arrive at

$$\begin{aligned}&|\widetilde{\mathbf{e}}^{m+1}|^2 - |\mathbf{e}^m|^2 + |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + k\|\widetilde{\mathbf{e}}^{m+1}\|^2 + 2\,k\left(\nabla(e_p^m + k\,\delta_t\,p(t_{m+1})), \widetilde{\mathbf{e}}^{m+1}\right)\\ &\le C\,k\,|\widetilde{\mathbf{e}}^m|^2 + C\,k^2\int_{t_m}^{t_{m+1}}(\|\mathbf{u}_{tt}\|^2_{\mathbf{H}^{-1}} + |\mathbf{u}_t|^2)dt.\end{aligned} \tag{6}$$

On the other hand, multiplying $(E_2)^{m+1}$ by $k(\mathbf{e}^{m+1}+\widetilde{\mathbf{e}}^{m+1}) + k^2\left(\nabla e_p^{m+1} + \nabla e_p^m\right)$ and using that $\left(\mathbf{e}^{m+1}, \nabla e_p^{m+1}\right) = 0 = \left(\mathbf{e}^{m+1}, \nabla e_p^m\right) = \left(\mathbf{e}^{m+1}, \nabla\delta_t p(t_{m+1})\right)$ (owing to (2)), we obtain

$$\begin{aligned}|\mathbf{e}^{m+1}|^2 - |\widetilde{\mathbf{e}}^{m+1}|^2 + \left(|k\,\nabla e_p^{m+1}|^2 - |k\,\nabla e_p^m|^2\right) - 2\,k\left(\widetilde{\mathbf{e}}^{m+1}, \nabla e_p^m\right)\\ = k^2\left(\widetilde{\mathbf{e}}^{m+1}, \nabla\delta_t p(t_{m+1})\right) + k^3\Big(\nabla\delta_t p(t_{m+1}), \nabla e_p^{m+1} + \nabla e_p^m\Big)\end{aligned} \tag{7}$$

Now, we bound the RHS of the above equality as follows:

$$k^2\left(\widetilde{\mathbf{e}}^{m+1}, \nabla\delta_t p(t_{m+1})\right) = k^2\left(\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m, \nabla\delta_t p(t_{m+1})\right) \le \varepsilon\,|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + C\,k^4\,|\nabla\delta_t p(t_{m+1})|^2$$

(here, we have used that $(\mathbf{e}^m, \nabla\delta_t p(t_{m+1})) = 0$)

$$k^3\Big(\nabla\delta_t p(t_{m+1}), \nabla e_p^{m+1} + \nabla e_p^m\Big) = k^3\left(\nabla\delta_t p(t_{m+1}), \nabla e_p^{m+1} - \nabla e_p^m\right) + k^3\left(\nabla\delta_t p(t_{m+1}), 2\,\nabla e_p^m\right) = I_1 + I_2$$

By using $(E_2)_b^{m+1}$, the term I_1 can be rewritten as

$$\begin{aligned}I_1 &\le k^2\left(\nabla\delta_t p(t_{m+1}), \widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^{m+1}\right) + k^4\,|\nabla\delta_t p(t_{m+1})|^2 \le \varepsilon\,|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^{m+1}|^2 + C\,k^3\int_{t_m}^{t_{m+1}}|\nabla p_t|^2\\ &\le \varepsilon\,|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + C\,k^3\int_{t_m}^{t_{m+1}}|\nabla p_t|^2\end{aligned}$$

We bound the I_2 term:

$$I_2 \le C\,k\,|k\,\nabla e_p^m|^2 + C\,k^3\,|\nabla\delta_t p(t_{m+1}))|^2 \le C\,k\,|k\,\nabla e_p^m|^2 + C\,k^2\int_{t_m}^{t_{m+1}}|\nabla p_t|^2$$

Adding up $\sum_{m=0}^{r}\{(6)_m + (7)_m\}$ the term $2\,k\Big(\widetilde{\mathbf{e}}^{m+1}, \nabla e_p^m\Big)$ vanish, hence taking into account the above estimates, we obtain

$$\begin{aligned}&|\mathbf{e}^{r+1}|^2 + |k\,\nabla e_p^{r+1}|^2 + \tfrac{1}{2}\sum_{m=0}^{r}\left|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\right|^2 + k\sum_{m=0}^{r}\|\widetilde{\mathbf{e}}^{m+1}\|^2 \le |\mathbf{e}^0|^2 + |k\,\nabla e_p^0|^2\\ &+C\,k\sum_{m=0}^{r}|\widetilde{\mathbf{e}}^m|^2 + k\sum_{m=0}^{r}|k\,\nabla e_p^m|^2 + C\,k^2\,\{\|\mathbf{u}_{tt}\|^2_{L^2(H^{-1})} + \|\mathbf{u}_t\|^2_{L^2(L^2)} + \|\nabla p_t\|^2_{L^2(L^2)}\}\end{aligned} \tag{8}$$

Now, bounding the third term of the RHS of (8) as follows (using again $(E_2)_b^{m+1}$ and the L^2-orthogonality property (2))

$$C\,k\sum_{m=0}^{r}|\widetilde{\mathbf{e}}^m|^2 \le C\,k\sum_{m=0}^{r}|\mathbf{e}^m|^2 + C\,k\sum_{m=0}^{r}\{|k\,\nabla e_p^m|^2 + |k\,\nabla e_p^{m-1}|^2\} + C\,k^3\sum_{m=0}^{r}|\nabla\delta_t p(t_{m+1})|^2,$$

then, applying the discrete Gronwall inequality, for each $m \geq 1$, we arrive at

$$\|\mathbf{e}^{m+1}\|_{l^\infty(L^2)} + \|\widetilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \leq C\,k, \quad \|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{l^2(L^2)} \leq C\,k^{3/2} \quad \text{and} \quad \|e_p^{m+1}\|_{l^\infty(H^1)} \leq C.$$

The estimate $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)} \leq C\,k$ is obtained from $|\widetilde{\mathbf{e}}^{m+1}| \leq |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m| + |\mathbf{e}^m|$ and the previous estimates. Finally, taking into account the Lemma 4, we obtain

$$\|\mathbf{e}^{m+1}\|_{l^2(H^1)} \leq C\,k \qquad \text{and} \qquad \sum |\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}|^2 \leq C\,k^2.$$

■

Notice that the estimate $\|\widetilde{\mathbf{e}}^m\|_{l^2(H^1)} \leq C\,k$ implies in particular $\|\widetilde{\mathbf{e}}^m\|_{H^1} \leq C$ for each m.

Lemma 7 *Under hypotheses of Theorem 6, one has*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \leq C, \qquad \forall m.$$

Proof. From the H^2-regularity of the Poisson problem $(E_1)^{m+1}$, one has

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2}^2 \leq C\left(\left|\frac{\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m}{k}\right|^2 + |\nabla e_p^m|^2 + k^2|\nabla\,\delta_t p(t_{m+1})|^2 + |\mathcal{E}^{m+1}|^2 + |\mathbf{NL}^{m+1}|^2\right).$$

To bound the first and the second term of the RHS, we use that $|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m| \leq C\,k$ from (4) and $\|e_p^{m+1}\|_{l^\infty(H^1)} \leq C$ from (3). It is easy to bound the third and the forth terms of the RHS. Finally, we bound the more complicate term of the nonlinear term $\mathbf{NL}^{m+1} = C(\widetilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1})) + C(\widetilde{\mathbf{e}}^m, \widetilde{\mathbf{e}}^{m+1}) + C(\mathbf{u}(t_m), \widetilde{\mathbf{e}}^{m+1})$ as follows:

$$|C(\widetilde{\mathbf{e}}^m, \widetilde{\mathbf{e}}^{m+1})|^2 \leq C\,\|\widetilde{\mathbf{e}}^m\|\,\|\widetilde{\mathbf{e}}^{m+1}\|_{L^\infty\cap W^{1,3}} \leq C\,\|\widetilde{\mathbf{e}}^m\|\,\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \leq \varepsilon\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2}^2 + C,$$

where in the last estimate, we have used that $\|\widetilde{\mathbf{e}}^m\| \leq C$. ■

Remark 8 *As a consequence of the L^∞ in time estimates $\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \leq C$ and $\|e_p^{m+1}\| \leq C$, $\forall m$, one has*

$$\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} \leq C \quad \textit{and} \quad \|p^{m+1}\| \leq C \quad \forall m.$$

1.6 $O(k)$-error estimates for the pressure

First, we are going to obtain error estimates for discrete time derivative of velocity. Afterwards, we will obtain the optimal $O(k)$ estimates for the pressure.

Lemma 9 *(Continuous dependence of discrete derivatives)*

$$|\delta_t \mathbf{e}^{m+1}| \leq |\delta_t \widetilde{\mathbf{e}}^{m+1}|, \qquad |\delta_t \mathbf{e}^{m+1} - \delta_t \widetilde{\mathbf{e}}^{m+1}| \leq |\delta_t \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m|.$$

Moreover, there exists $C = C(\Omega) > 0$ such as

$$\|\delta_t \mathbf{e}^{m+1}\| \leq C\,\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|.$$

Proof. The proof is similar to Lemma 1 and Lemma 4, by using that $\delta_t \mathbf{e}^{m+1} = P_{\mathbf{H}}(\delta_t \widetilde{\mathbf{e}}^{m+1})$. ■

Theorem 10 *Assuming hypotheses of the Theorem 6,* (**H2**) *and the following constraint on the initial approximation*

$$|\delta_t \mathbf{e}^1| + |k\,\nabla \delta_t e_p^1| \le C\,k,$$

then one obtains

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C\,k$$

Remark 11 *Reasoning with similar arguments as in [12], we can get* $|\delta_t \mathbf{e}^1| \le C\,k^{1/2}$.

Proof. By making $\delta_t(E_1)^{m+1}$ and $\delta_t(E_2)^{m+1}$, one obtains $\forall m$,

$$(D_1)^{m+1} \qquad \left\{ \; \frac{\delta_t \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m}{k} - \Delta \delta_t \widetilde{\mathbf{e}}^{m+1} + \nabla(\delta_t e_p^m + k\,\delta_t\delta_t p(t_{m+1})) = \delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1} \right.$$

where $\delta_t\delta_t p(t_{m+1}) = \dfrac{1}{k}(\delta_t p(t_{m+1}) - \delta_t p(t_m))$, and

$$(D_2)^{m+1} \qquad \left\{ \; \frac{\delta_t \mathbf{e}^{m+1} - \delta_t \widetilde{\mathbf{e}}^{m+1}}{k} + \nabla(\delta_t e_p^{m+1} - \delta_t e_p^m - k\,\delta_t\delta_t p(t_{m+1})) = 0. \right.$$

The proof follows similar lines of [8] and [19]. By multiplying $(D_1)^{m+1}$ by $2\,k\,\delta_t\,\widetilde{\mathbf{e}}^{m+1}$, we get:

$$|\delta_t \widetilde{\mathbf{e}}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m|^2 + 2\,k\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2$$

$$+2\,k\Big(\nabla(\delta_t e_p^m + k\,\delta_t\delta_t p(t_{m+1})), \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) = 2\,k\,\Big(\delta_t \mathcal{E}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\,\Big(\delta_t \mathbf{NL}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big)$$

We bound the RHS as follows:

$$2\,k\Big(\delta_t\,\mathcal{E}_1^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1}\Big) \le \varepsilon\,k\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k^2 \int_{t_{m-1}}^{t_{m+1}} \|\mathbf{u}_{ttt}\|_{H^{-1}}^2$$

$$\begin{aligned} 2\,k\Big(\delta_t\,\mathcal{E}_2^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1}\Big) &= 2\,k\Big(\delta_t \mathbf{u}(t_{m+1}) \cdot \nabla(\mathbf{u}(t_{m+1}) - \mathbf{u}(t_m)), \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) \\ &+ 2\,k\Big((\delta_t \mathbf{u}(t_{m+1}) - \delta_t \mathbf{u}(t_m)) \cdot \nabla \mathbf{u}(t_m), \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) := I_1 + I_2 \end{aligned}$$

$$I_1 \le \varepsilon k\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\|\delta_t \mathbf{u}(t_{m+1})\|_{L^3}^2 \| \int_{t_m}^{t_{m+1}} \partial_t \mathbf{u}\|_{H^1}^2 \le \varepsilon k\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k^2\|\mathbf{u}_t\|_{L^\infty(L^2)} \int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|_{H^1}^3$$

$$I_2 \le \varepsilon\,k\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\nabla \mathbf{u}(t_m)\|_{L^3}^2 \left|\frac{1}{k}\Big(\int_{t_m}^{t_{m+1}} \mathbf{u}_t - \int_{t_{m-1}}^{t_m} \mathbf{u}_t\Big)\right|^2 \le \varepsilon\,k\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k^2 \int_{t_{m-1}}^{t_{m+1}} |\mathbf{u}_{tt}|^2$$

(in the above inequality we have used estimates obtained in [20])

Now, we bound the non-linear terms:

$$2\,k\,\Big(\delta_t \mathbf{NL}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) = 2\,k\,c\Big(\delta_t \widetilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1}), \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\,c\Big(\delta_t \widetilde{\mathbf{u}}^m, \widetilde{\mathbf{e}}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big)$$

$$+2\,k\,c\Big(\widetilde{\mathbf{e}}^{m-1}, \delta_t \mathbf{u}(t_{m+1}), \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\,c\Big(\widetilde{\mathbf{u}}^{m-1}, \delta_t \widetilde{\mathbf{e}}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) := \sum_{i=1}^4 L_i$$

$$L_1 \le k\,|\delta_t\,\widetilde{\mathbf{e}}^m|\,\|\mathbf{u}(t_{m+1})\|_{L^\infty\cap W^{1,3}}\,\|\delta_t\,\widetilde{\mathbf{e}}^{m+1}\| \le \varepsilon\,k\,\|\delta_t\,\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,|\delta_t\,\widetilde{\mathbf{e}}^m|^2$$

$$L_2 = 2\,k\,c\Big(\delta_t\,\widetilde{\mathbf{e}}^m, \widetilde{\mathbf{e}}^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\,c\Big(\delta_t\mathbf{u}(t_m), \widetilde{\mathbf{e}}^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1}\Big) := L_{21} + L_{22}$$

$$L_{21} \le 2\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2}\,|\delta_t\widetilde{\mathbf{e}}^m|\,\|\delta_t\,\widetilde{\mathbf{e}}^{m+1}\|^2 \le \varepsilon\,k\,\|\delta_t\,\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,|\delta_t\,\widetilde{\mathbf{e}}^m|^2$$

where we have used that $\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C$,

$$L_{22} \le k\,\|\delta_t\mathbf{u}(t_m)\|_{L^3}\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|\,\|\widetilde{\mathbf{e}}^{m+1}\| \le \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2$$

(in the above estimate we have used the regularity $\mathbf{u}_t \in L^\infty(\mathbf{L}^3)$), and from similar way,

$$L_3 \le \varepsilon\,k\,\|\delta_t\,\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\widetilde{\mathbf{e}}^{m-1}\|^2.$$

Finally,

$$L_4 = 0.$$

Taking into account the above estimates and taking ε small enough, we arrive at

$$\begin{aligned}&|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 - |\delta_t\mathbf{e}^m|^2 + |\delta_t\widetilde{\mathbf{e}}^{m+1} - \delta_t\mathbf{e}^m|^2 + k\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + 2\,k\Big((\nabla(\delta_t e_p^m + k\,\delta_t\delta_t p(t_{m+1})), \delta_t\widetilde{\mathbf{e}}^{m+1}\Big)\\ &\le C\,k|\delta_t\widetilde{\mathbf{e}}^m|^2 + C\,k^2\int_{t_{m-1}}^{t_{m+1}}\Big(\|\mathbf{u}_{ttt}\|^2_{H^{-1}} + |\mathbf{u}_{tt}|^2\Big) + C\,k^2\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^3 + C\,k\,(\|\widetilde{\mathbf{e}}^{m-1}\|^2 + \|\widetilde{\mathbf{e}}^{m+1}\|^2)\end{aligned} \tag{9}$$

On the other hand, multiplying $(D_2)^{m+1}$ by $k\,(\delta_t\mathbf{e}^{m+1} + \delta_t\widetilde{\mathbf{e}}^{m+1}) + k^2\,(\nabla\delta_t e_p^{m+1} + \nabla\delta_t e_p^m)$ we obtain

$$\begin{aligned}&|\delta_t\mathbf{e}^{m+1}|^2 - |\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 + |k\,\nabla\delta_t e_p^{m+1}|^2 - |k\,\nabla\delta_t e_p^m|^2 - 2\,k\,\Big(\delta_t\widetilde{\mathbf{e}}^{m+1}, \nabla\delta_t e_p^m\Big)\\ &\quad = k^2\,\Big(\delta_t\widetilde{\mathbf{e}}^{m+1}, \nabla\delta_t\delta_t p(t_{m+1})\Big) + k^3\,\Big(\nabla\delta_t e_p^{m+1} + \nabla\delta_t e_p^m, \nabla\delta_t\delta_t p(t_{m+1})\Big).\end{aligned} \tag{10}$$

Reasoning as in Theorem 6 making $\displaystyle\sum_{m=1}^{r}\{(9) + (10)\}$ and using error estimates of Theorem 6, we arrive at

$$|\delta_t\mathbf{e}^{r+1}|^2 + |k\,\nabla\delta_t e_p^{r+1}|^2 + \frac{1}{2}\sum_{m=1}^{r}|\delta_t\widetilde{\mathbf{e}}^{m+1} - \delta_t\mathbf{e}^m|^2 + \frac{k}{2}\sum_{m=1}^{r}\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2$$

$$\le |\delta_t\mathbf{e}^1|^2 + |k\,\nabla\delta_t e_p^1|^2 + C\,k\sum_{m=1}^{r}|\delta_t\widetilde{\mathbf{e}}^m|^2 + C\,k\sum_{m=1}^{r}|k\,\nabla\delta_t e_p^m|^2 + C\,k^2$$

Now, we bound the third term of the RHS as follows (using again $(D_2)^m$ and the L^2-orthogonality)

$$C\,k\sum_{m=0}^{r}|\delta_t\widetilde{\mathbf{e}}^m|^2 \le C\,k\sum_{m=0}^{r}|\delta_t\mathbf{e}^m|^2 + C\,k\sum_{m=0}^{r}(|k\,\nabla\delta_t e_p^m|^2 + |k\,\nabla\delta_t e_p^{m-1}|^2) + C\,k^3\sum_{m=0}^{r}|\nabla\delta_t\delta_t p(t_{m+1})|^2.$$

Then, applying the discrete Gronwall Lemma, we arrive at

$$\|\delta_t\mathbf{e}^{m+1}\|_{l^\infty(L^2)} \le C\,k, \quad \sum_{m=1}^{r}|\delta_t\widetilde{\mathbf{e}}^{m+1} - \delta_t\mathbf{e}^m|^2 \le C\,k^2, \quad \|\delta_t\widetilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \le C\,k$$

$$\text{and} \quad \|\delta_t e_p^{m+1}\|_{l^\infty(H^1)} \le C.$$

After that, taking into account Lemma 9,

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^2(H^1)} \le C\,k \quad \text{and} \quad \sum |\delta_t \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^{m+1}|^2 \le C\,k^2.$$

Finally $\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)} \le C\,k$ is deduced. ∎

Theorem 12 *Under hypothesis of Theorem 10 and* (**H3**)*, the following error estimates hold*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} + \|e_p^{m+1}\|_{l^\infty(L^2)} \le C\,k.$$

Proof. From Theorem 10 and the continuous inf-sup condition applied to $(E_3)^{m+1}$, we can deduce the estimate $\|e_p^m\|_{l^2(L^2)} \le C\,k$. Indeed, rewritten $(E_3)^{m+1}$ as

$$-\nabla e_p^{m+1} = \delta_t \mathbf{e}^{m+1} - \Delta \widetilde{\mathbf{e}}^{m+1} - \mathcal{E}^{m+1} - \mathbf{NL}^{m+1}, \qquad e_p^{m+1} \in L_0^2(\Omega),$$

then, applying the continuous inf-sup condition, one has

$$\|e_p^{m+1}\|_{L^2} \le \|\delta_t \mathbf{e}^{m+1}\|_{H^{-1}} + \|\widetilde{\mathbf{e}}^{m+1}\| + C\Big\{k\|\widetilde{\mathbf{e}}^{m+1}\| + \frac{k^{3/2}}{\sqrt{3}}\Big(\int_{t_m}^{t_{m+1}} \|\mathbf{u}_{tt}\|_{H^{-1}}^2\,dt\Big)\Big\}. \tag{11}$$

Taking into account that $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \le C\,k$ and $\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)} \le C\ k$ we arrive at

$$\|e_p^{m+1}\|_{l^2(L^2)} \le C\,k.$$

On the other hand, of $(E_3)^{m+1}$ we have

$$-\Delta \widetilde{\mathbf{e}}^{m+1} = -\delta_t \mathbf{e}^{m+1} - \nabla e_p^{m+1} + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1}, \qquad \widetilde{\mathbf{e}}^{m+1}|_{\partial\Omega} = 0.$$

Multiplying the last inequality by $2\,k\,\delta_t \widetilde{\mathbf{e}}^{m+1}$, we obtain

$$|\nabla \widetilde{\mathbf{e}}^{m+1}|^2 - |\nabla \widetilde{\mathbf{e}}^m|^2 + |\nabla \widetilde{\mathbf{e}}^{m+1} - \nabla \widetilde{\mathbf{e}}^m|^2$$

$$= -2\,k\Big(\nabla e_p^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) - 2\,k\Big(\delta_t \mathbf{e}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\Big(\mathcal{E}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\Big(\mathbf{NL}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\Big)$$

$$\le k\,|e_p^{m+1}|^2 + k\,|\nabla \delta_t \widetilde{\mathbf{e}}^{m+1}|^2 + C\,k\,|\delta_t \mathbf{e}^{m+1}|^2 + C\,k\,|\delta_t \widetilde{\mathbf{e}}^{m+1}|^2 + 2\,k\,|\mathcal{E}^{m+1}|^2 + 2\,k\,|\mathbf{NL}^{m+1}|^2$$

Now, we bound the two last terms on the right side hand as follows

$$|\mathcal{E}^{m+1}|^2 \le C\,k^2$$

$$|\mathbf{NL}^{m+1}|^2 \le C(\|\widetilde{\mathbf{e}}^m\|^2 + \|\widetilde{\mathbf{e}}^{m+1}\|^2)$$

where in the second inequality, we have used $\|\widetilde{\mathbf{u}}^m\|_{L^\infty \cap W^{1,3}} \le C$ and $\|\mathbf{u}(t_{m+1})\|_{L^\infty \cap W^{1,3}} \le C$.

Adding from $m = 1$ to r we obtain

$$|\nabla \widetilde{\mathbf{e}}^{r+1}|^2 \le C\,k \sum_{m=1}^{r} \Big(|e_p^{m+1}|^2 + |\nabla \delta_t \widetilde{\mathbf{e}}^{m+1}|^2 + |\delta_t \mathbf{e}^{m+1}|^2 + |\delta_t \widetilde{\mathbf{e}}^{m+1}|^2 + \|\widetilde{\mathbf{e}}^m\|^2 + \|\widetilde{\mathbf{e}}^{m+1}\|^2\Big) + C\,k^2$$

hence, we arrive at $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} \le C\,k$ applying the estimates of precedent results.

On the other hand, by using the inequality (11) and taking into account that

$$\|\delta_t \mathbf{e}^{m+1}\|_{H^{-1}} \le C\,|\delta_t \mathbf{e}^{m+1}| \le C\,k, \quad \|\widetilde{\mathbf{e}}^{m+1}\| \le C\,k \quad \text{and} \quad \mathbf{u}_{tt} \in L^\infty(H^{-1})$$

we arrive at

$$\|e_p^{m+1}\|_{l^\infty(L^2)} \le C\,k.$$

$\blacksquare$

2 Fully discrete scheme

In this section, by C we will denote different constants, always independent of k and h.

2.1 Finite element approximation and fully discrete scheme

We consider a finite element approximation of the time discrete scheme given above. We restrict ourselves to the case where Ω is a $2D$ polygon or a $3D$ polyhedron satisfying the regularity hypothesis of (**H0**). We consider two finite element spaces $\mathbf{Y}_h \subset \mathbf{H}^1_0(\Omega)$ and $Q_h \subset H^1(\Omega)\cap L^2_0(\Omega)$ associated to a regular family of triangulations $\mathcal{T}_h$ of the domain Ω of mesh size h (in the sense of Ciarlet [6]). The finite element $\mathbf{Y}_h$ and Q_h are globally continuous functions and locally polynomials of degree at least 1. Moreover, we will assume:

- The "*inf-sup*" condition ([7]) for $(\mathbf{Y}_h, Q_h)$:

  There exists $\beta > 0$ independent of h such that,

$$\inf_{q_h\in Q_h\setminus\{0\}} \left(\sup_{\mathbf{v}_h\in \mathbf{Y}_h\setminus\{0\}} \frac{(q_h, \nabla\cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|\,|q_h|} \right) \ge \beta.$$

- There exists certain interpolation operators with the following properties:

  – $I_h : \mathbf{L}^2 \to \mathbf{Y}_h$ such as

$$(\mathbf{u} - I_h\mathbf{u}, \nabla q_h) = 0, \quad \forall\, q_h \in Q_h \tag{12}$$

  and satisfying the following approximating properties:

$$\|\tilde{\mathbf{u}} - I_h\tilde{\mathbf{u}}\|_{H^{-1}} \le C\,h\,\|\tilde{\mathbf{u}} - I_h\tilde{\mathbf{u}}\|_{L^2} \quad \forall\,\tilde{\mathbf{u}} \in \mathbf{L}^2(\Omega),$$

$$\|\tilde{\mathbf{u}} - I_h\tilde{\mathbf{u}}\|_{H^1} \le C\,h\,\|\tilde{\mathbf{u}}\|_{H^2} \quad \forall\,\tilde{\mathbf{u}} \in \mathbf{H}^2(\Omega)\cap \mathbf{H}^1_0(\Omega),$$

  – $J_h : H^1 \to Q_h$ defined by

$$\Big(\nabla(J_h p - p), \nabla q_h\Big) = 0 \quad \forall\, q_h \in Q_h,$$

  satisfying

$$|p - J_h p| \le C\,h\,\|p - J_h p\|_{H^1} \le C\,h\,\|p\|_{H^1} \quad \forall\, p \in H^1(\Omega)\cap L^2_0(\Omega),$$

Remark 13 *For instance, if we consider the* $I\!P_1$*-bubble* $\times$ $I\!P_1$ *approximation to construct the space* $\mathbf{Y}_h \times Q_h$*, then a possible manner to choice* I_h *is as follows: Let* $\widetilde{I}_h$ *be a regularization interpolation operator (of Clément or Scot-Zhang type) onto the globally continuous and locally* $I\!P_1$ *finite element space, that is* $I_h\mathbf{u}|_T \in I\!P_1$ *for each* $T \in \mathcal{T}_h$*, then* $\widetilde{I}_h$ *satisfies*

$$|\widetilde{I}_h\mathbf{u}| \le C\,|\mathbf{u}|, \quad |\widetilde{I}_h\mathbf{u} - \mathbf{u}| \le C\,h\,\|\mathbf{u}\|, \quad \|\widetilde{I}_h\mathbf{u} - \mathbf{u}\| \le C\,h\,\|\mathbf{u}\|_{H^2}.$$

We define $I_h\mathbf{u} = \widetilde{I}_h\mathbf{u} + \sum_T b_T\,\alpha_T(\mathbf{u})$ *with* b_T *a bubble function and* $\alpha_T \in I\!R$ *such as* $\int_T(\mathbf{u} - I_h\mathbf{u}) = 0$ *for each* $T \in \mathcal{T}_h$*, that is*

$$\alpha_T(\mathbf{u}) = \frac{\int_T(\mathbf{u} - \widetilde{I}_h\mathbf{u})}{\int_T b_T} \quad \forall\,T \in \mathcal{T}_h.$$

Then, in particular (12) holds. Moreover, it is known ([7]), by means of a duality argument, that

$$\|\mathbf{u} - I_h\mathbf{u}\|_{H^{-1}} \le C\,h\,|\mathbf{u} - I_h\mathbf{u}|.$$

Now, if I_h *is* L^2*-stable, we obtain the desired estimates. For this, since* $\widetilde{I}_h$ *is* L^2*-stable, it suffices that the bubble part* $\sum_T b_T\,\alpha_T(\mathbf{u})$ *be* L^2*-stable. Indeed,*

$$\Big|\sum_T b_T\,\alpha_T(\mathbf{u})\Big|^2 = \sum_T \Big|\alpha_T(\mathbf{u})\Big|^2 \Big(\int_T b_T^2\Big) = \sum_T \Big(\int_T \mathbf{u} - \widetilde{I}_h\mathbf{u}\Big)^2 \frac{\int_T b_T^2}{\Big(\int_T b_T\Big)^2}$$

$$\le C\sum_T |T|\Big(\int_T |\mathbf{u} - \widetilde{I}_h\mathbf{u}|^2\Big)\frac{|T|}{|T|^2} \le C\,|\mathbf{u} - \widetilde{I}_h\mathbf{u}|^2 \le C|\mathbf{u}|^2$$

where in the last estimate, the L^2*-stability of* $\widetilde{I}_h$ *has been used.*

Now, following the equality $\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m)$, we define $K_{h,k}\mathbf{u}^{m+1} \in \mathbf{Y}_h + \nabla Q_h$ by:

$$K_{h,k}\mathbf{u}^{m+1} = I_h\widetilde{\mathbf{u}}^{m+1} - k\,\nabla J_h(p^{m+1} - p^m). \tag{13}$$

Then, we can obtain

$$|\mathbf{u}^{m+1} - K_{h,k}\,\mathbf{u}^{m+1}| \le C\,\Big(h\,\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} + k\,\|p^{m+1} - p^m\|\Big) \le C(k+h) \quad \forall m.$$

Indeed, due to the definition (13) of $K_{h,k}\mathbf{u}^{m+1}$,

$$\mathbf{u}^{m+1} - K_{h,k}\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - I_h\widetilde{\mathbf{u}}^{m+1} - k\,\nabla\Big((p^{m+1} - J_hp^{m+1}) - (p^m - J_hp^m)\Big)$$

$$\le C\,(h\,\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} + k\,\|p^{m+1} - p^m\|) \le C(k+h),$$

where the approximation properties for I_h and J_h have been used.

Finally, the following constraint between the time step size k and the mesh size h will also be assumed:

$(\mathbf{H})$ There exists a constant $\alpha > 0$ (independent of k and h) such that $\dfrac{h}{k} \le \alpha$.

The fully discrete scheme remains as follows:

Initialization: Let $(\widetilde{\mathbf{u}}_h^0, p_h^0) \in \mathbf{Y}_h \times Q_h$ be an approximation of $(\mathbf{u}^0, p^0)$. Put $\mathbf{u}_h^0 = \widetilde{\mathbf{u}}_h^0$.

Step of time $m+1$:

Sub-step 1: Given $(\widetilde{\mathbf{u}}_h^m, p_h^m) \in \mathbf{Y}_h \times Q_h$ and $\mathbf{u}_h^m \in \mathbf{Y}_h + \nabla Q_h$, to find $\widetilde{\mathbf{u}}_h^{m+1} \in \mathbf{Y}_h$ such that,

$(S_1)_h^{m+1}$

$$\frac{1}{k}\Big(\widetilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m, \mathbf{v}_h\Big) + c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h\Big) + \Big(\nabla \widetilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h\Big) + \Big(\nabla p_h^m, \mathbf{v}_h\Big) = \Big(\mathbf{f}^{m+1}, \mathbf{v}_h\Big).$$

Sub-step 2: Given $(\widetilde{\mathbf{u}}_h^{m+1}, p_h^m) \in \mathbf{Y}_h \times Q_h$, to find $p_h^{m+1} \in Q_h$ such that

$$(S_2)_{a,h}^{m+1} \qquad \Big(k\,\nabla(p_h^{m+1} - p_h^m), \nabla q_h\Big) = \Big(\widetilde{\mathbf{u}}_h^{m+1}, \nabla q_h\Big) \quad \forall q_h \in Q_h.$$

Now, we define $\mathbf{u}_h^{m+1} \in \mathbf{Y}_h + \nabla Q_h$ by

$$(S_2)_{b,h}^{m+1} \qquad \mathbf{u}_h^{m+1} = \widetilde{\mathbf{u}}_h^{m+1} - k\,\nabla(p_h^{m+1} - p_h^m).$$

Observe that, $\mathbf{u}_h^{m+1}$ verifies the L^2-orthogonality property:

$$\Big(\mathbf{u}_h^{m+1}, \nabla q_h\Big) = 0 \quad \forall\, q_h \in Q_h. \tag{14}$$

Notice that, adding both sub-steps, we obtain:

$$(S_3)_h^{m+1} \quad \frac{1}{k}\Big(\mathbf{u}_h^{m+1} - \mathbf{u}_h^m, \mathbf{v}_h\Big) + c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h\Big) + \Big(\nabla\widetilde{\mathbf{u}}_h^{m+1}, \nabla\mathbf{v}_h\Big) + \Big(\nabla p_h^{m+1}, \mathbf{v}_h\Big) = \Big(\mathbf{f}^{m+1}, \mathbf{v}_h\Big)$$

We introduce the end-of-step velocity $\mathbf{u}_h^m$ only to make the numerical analysis. This is not necessary for the practical implementation of this scheme, which can be realized as follows:

Given $(p_h^{m-1}, \widetilde{\mathbf{u}}_h^m) \in Q_h \times \mathbf{Y}_h$,

(a) To find $p_h^m \in Q_h$ such that

$$\Big(k\,\nabla(p_h^m - p_h^{m-1}), \nabla q_h\Big) = \Big(\widetilde{\mathbf{u}}_h^m, \nabla q_h\Big) \quad \forall\, q_h \in Q_h.$$

(b) To find $\widetilde{\mathbf{u}}_h^{m+1} \in \mathbf{Y}_h$ such that, $\forall\, \mathbf{v}_h \in \mathbf{Y}_h$,

$$\Big(\frac{\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}_h^m}{k}, \mathbf{v}_h\Big) + c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h\Big) + \Big(\nabla \widetilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h\Big) + \Big(\nabla(2p_h^m - p_h^{m-1}), \mathbf{v}_h\Big) = \Big(\mathbf{f}^{m+1}, \mathbf{v}_h\Big).$$

Then, the computation for pressure and velocity is decoupled (for this, the scheme is called a pressure segregation method). In fact, (a) is a Poisson problem for the pressure and (b) is a linear convection-diffusion problem for the velocity (which is also decoupled by components of $\widetilde{\mathbf{u}}_h^{m+1}$).

Note that, we have to begin with a pressure p_h^{-1} (for $m = 0$), which has not sense. For this, either we have to begin with several auxiliary initial steps with another scheme or we have to begin with a first step with the scheme written as before, i.e., with $\widetilde{\mathbf{u}}_h^0$, p_h^0 y $u_h^0 = \widetilde{\mathbf{u}}_h^0$, beginning then with a approximation of the initial pressure.

It is easy to extend the results given in the previous Section about the continuous dependence for the semi-discrete in time scheme, to this fully discrete scheme. Indeed, from $(S_2)_{b,h}^{m+1}$ and (14), we have

$$|\widetilde{\mathbf{u}}_h^{m+1}|^2 = |\mathbf{u}_h^{m+1}|^2 + |k\,\nabla(p_h^{m+1} - p_h^m)|^2 \tag{15}$$

hence, in particular, $|\mathbf{u}_h^{m+1}| \le |\widetilde{\mathbf{u}}_h^{m+1}|$. From $(S_2)_{a,h}^{m+1}$

$$|k\,\nabla(p_h^{m+1} - p_h^m)|^2 = (\widetilde{\mathbf{u}}_h^{m+1}, k\,\nabla(p_h^{m+1} - p_h^m)) = (\widetilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m, k\,\nabla(p_h^{m+1} - p_h^m))$$

then

$$|\mathbf{u}_h^{m+1} - \widetilde{\mathbf{u}}_h^{m+1}| \le |\widetilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m|.$$

Moreover, using the antisymmetric property of $c(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \widetilde{\mathbf{u}}_h^{m+1})$, see (1), one can extend results of stability and convergence for the semidiscrete in time schemes to this fully discrete scheme, that is, the following estimates hold, for any $r < N$:

$$\begin{gathered} \|\mathbf{u}_h^{r+1}\|_{l^\infty(L^2)} + \|\widetilde{\mathbf{u}}_h^{r+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|k\,\nabla p_h^{r+1}\|_{l^\infty(L^2)} \le C \\ \sum_{m=0}^{r} |\widetilde{\mathbf{u}}_h^{m+1} - \mathbf{u}_h^m|^2 + \sum_{m=0}^{r} |\mathbf{u}_h^{m+1} - \widetilde{\mathbf{u}}_h^{m+1}|^2 \le C \end{gathered} \tag{16}$$

Indeed, by making $\Big((S_1)_h^{m+1}, 2\,k\,\widetilde{\mathbf{u}}_h^{m+1}\Big)$, using the fact that

$$2\,k\,(\nabla p_h^m, \widetilde{\mathbf{u}}_h^{m+1}) = 2(k\,\nabla p_h^m, k\,\nabla(p_h^{m+1} - p_h^m)),$$

and the equalities $(a-b)2a = a^2 - b^2 + (a-b)^2$ and $(a-b)2b = a^2 - b^2 - (a-b)^2$, we have

$$|\widetilde{\mathbf{u}}^{m+1}|^2 - |\mathbf{u}_h^m|^2 + |\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}_h^m|^2 + k\,\|\widetilde{\mathbf{u}}^{m+1}\|^2 + |k\,\nabla p_h^{m+1}|^2 - |k\,\nabla p_h^m|^2 - |k\,\nabla(p_h^{m+1} - p_h^m)|^2 \le k\,\|\mathbf{f}^{m+1}\|_{H^{-1}}^2. \tag{17}$$

Adding (17) and (15), we arrive at

$$|\widetilde{\mathbf{u}}^{m+1}|^2 - |\mathbf{u}_h^m|^2 + |\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}_h^m|^2 + |k\,\nabla p_h^{m+1}|^2 - |k\,\nabla p_h^m|^2 + k\,\|\widetilde{\mathbf{u}}_h^{m+1}\|^2 \le k\,\|\mathbf{f}^{m+1}\|_{H^{-1}}^2$$

Now, adding from $m = 0$ to r ($r < N$) in the above expression, we obtain the desired estimates (16).

2.2 Problems related to the space discrete errors

We will present an error analysis for the fully discrete scheme $(\widetilde{\mathbf{u}}_h^{m+1}, \mathbf{u}_h^{m+1}, p_h^{m+1})$ as an approximation of the semidiscrete scheme $(\widetilde{\mathbf{u}}^{m+1}, \mathbf{u}^{m+1}, p^{m+1})$. Consequently, we consider the following errors:

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \qquad \widetilde{\mathbf{e}}_d^{m+1} = \widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}_h^{m+1}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}$$

These errors can be decomposed as follows (splitting the discrete part and the interpolation one):

$$\mathbf{e}_d^{m+1} = \mathbf{e}_h^{m+1} + \mathbf{e}_i^{m+1}, \qquad \widetilde{\mathbf{e}}_d^{m+1} = \widetilde{\mathbf{e}}_h^{m+1} + \widetilde{\mathbf{e}}_i^{m+1}, \qquad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1}$$

being $\mathbf{e}_i$ interpolation errors and $\mathbf{e}_h$ space discrete errors, concretely

$$\mathbf{e}_h^{m+1} = K_{h,k}\mathbf{u}^{m+1} - \mathbf{u}_h^{m+1} \text{ and } \mathbf{e}_i^{m+1} = \mathbf{u}^{m+1} - K_{h,k}\mathbf{u}^{m+1},$$

$$\widetilde{\mathbf{e}}_h^{m+1} = I_h\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}_h^{m+1} \text{ and } \widetilde{\mathbf{e}}_i^{m+1} = \widetilde{\mathbf{u}}^{m+1} - I_h\widetilde{\mathbf{u}}^{m+1},$$

$$e_{p,h}^{m+1} = J_h p^{m+1} - p_h^{m+1} \text{ and } e_{p,i}^{m+1} = p^{m+1} - J_h p^{m+1}.$$

Remark 14 *From the equalities* $\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m)$ *and* $K_{h,k}\mathbf{u}^{m+1} = I_h\widetilde{\mathbf{u}}^{m+1} - k\,\nabla J_h(p^{m+1} - p^m)$*, one has*

$$\mathbf{e}_i^{m+1} = \widetilde{\mathbf{e}}_i^{m+1} - k\,\nabla(e_{p,i}^{m+1} - e_{p,i}^m). \tag{18}$$

In particular, by using this equality replacing m *for* $m-1$*, we get*

$$\frac{1}{k}(\widetilde{\mathbf{e}}_i^{m+1} - \mathbf{e}_i^m) = e_i(\delta_t\widetilde{\mathbf{u}}^{m+1}) + \nabla(e_{p,i}^m - e_{p,i}^{m-1}). \tag{19}$$

Moreover, owing to the choice of the interpolation operators I_h *and* J_h*,*

$$\Big(\mathbf{e}_i^{m+1}, \nabla q_h\Big) = \Big(\widetilde{\mathbf{e}}_i^{m+1}, \nabla q_h\Big) - k\,\Big(\nabla(e_{p,i}^{m+1} - e_{p,i}^m), \nabla q_h\Big) = 0, \quad \forall\, q_h \in Q_h. \tag{20}$$

On the other hand, from $\Big(\mathbf{u}_h^{m+1}, \nabla q_h\Big) = 0$*,* $\forall\, q_h \in Q_h$ *and* $\Big(\mathbf{u}^{m+1}, \nabla q_h\Big) = 0$*,* $\forall\, q_h \in H^1 \cap L_0^2$*, then*

$$\Big(\mathbf{e}_d^{m+1}, \nabla q_h\Big) = 0 \quad \forall q_h \in Q_h. \tag{21}$$

Finally, from (20) and (21), we arrive at

$$\Big(\mathbf{e}_h^{m+1}, \nabla q_h\Big) = 0 \quad \forall q_h \in Q_h.$$

Comparing $(S_1)^{m+1}, (S_2)^{m+1}$ and $(S_1)_h^{m+1}, (S_2)_{b,h}^{m+1}$, we have the following variational problems verified by the space errors $\widetilde{\mathbf{e}}_d^{m+1}$ and $(\mathbf{e}_d^{m+1}, e_{p,d}^{m+1})$ respectively:

$$\frac{1}{k}\Big(\widetilde{\mathbf{e}}_d^{m+1} - \mathbf{e}_d^m, \mathbf{v}_h\Big) + \Big(\nabla\,\widetilde{\mathbf{e}}_d^{m+1}, \nabla\,\mathbf{v}_h\Big) + \Big(\nabla e_{p,d}^m, \mathbf{v}_h\Big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \forall\,\mathbf{v}_h \in \mathbf{Y}_h,$$

$$\mathbf{e}_d^{m+1} = \widetilde{\mathbf{e}}_d^{m+1} - k\nabla(e_{p,d}^{m+1} - e_{p,d}^m),$$

where

$$\mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h\Big) - c\Big(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\Big) = -c\Big(\widetilde{\mathbf{e}}_d^m, \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\Big) - c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{e}}_d^{m+1}, \mathbf{v}_h\Big).$$

Then, splitting the error in the discrete and the interpolation parts and by using (18)-(19), we obtain

$$(E_1)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\, \widetilde{\mathbf{e}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla e_{p,h}^m, \mathbf{v}_h\Big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(e_i(\delta_t \widetilde{\mathbf{u}}^{m+1}), \mathbf{v}_h\Big) - \Big(\nabla \widetilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(\nabla(2\, e_{p,i}^m - e_{p,i}^{m-1}), \mathbf{v}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h \end{array}\right.$$

$$(E_2)_h^{m+1} \qquad \mathbf{e}_h^{m+1} = \widetilde{\mathbf{e}}_h^{m+1} - k\nabla(e_{p,h}^{m+1} - e_{p,h}^m).$$

Finally, adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$,

$$(E_3)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\, \widetilde{\mathbf{e}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla e_{p,h}^m, \mathbf{v}_h\Big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(e_i(\delta_t \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\Big) - \Big(\nabla \widetilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(\nabla(2\, e_{p,i}^m - e_{p,i}^{m-1}), \mathbf{v}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h. \end{array}\right.$$

2.3 $O(h)$ error estimates for $\widetilde{\mathbf{e}}_h^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and for $\mathbf{e}_h^{m+1}$ in $l^\infty(\mathbf{L}^2)$

Theorem 15 *We assume hypotheses of Theorem 6, $|\mathbf{e}_h^0| \le C\, h$ and $k|\nabla e_{p,h}^0| \le C\, h$. Then, the following error estimates hold*

$$\|\widetilde{\mathbf{e}}_h^{m+1}\|^2_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}_h^{m+1}\|^2_{l^\infty(\mathbf{L}^2)} + \|k\, \nabla e_{p,h}^{m+1}\|^2_{l^\infty(\mathbf{L}^2)} \le C\, h^2 \tag{22}$$

$$\|\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m\|^2_{l^2(\mathbf{L}^2)} \le C\, k\, h^2. \tag{23}$$

Proof: By making $\Big((E_1)_h^{m+1}, 2\, k\, \widetilde{\mathbf{e}}_h^{m+1}\Big)$ and using that $\Big(\nabla e_{p,h}^m, \mathbf{e}_h^{m+1}\Big) = 0$, hence

$$2\, k\, \Big(\nabla e_{p,h}^m, \widetilde{\mathbf{e}}_h^{m+1}\Big) = 2\Big(k\, \nabla e_{p,h}^m, k\, \nabla(e_{p,h}^{m+1} - e_{p,h}^m)\Big) = |k\, \nabla e_{p,h}^{m+1}|^2 - |k\, \nabla e_{p,h}^m|^2 - |k\, \nabla(e_{p,h}^{m+1} - e_{p,h}^m)|^2,$$

we arrive at

$$\begin{array}{l} |\widetilde{\mathbf{e}}_h^{m+1}|^2 - |\mathbf{e}_h^m|^2 + |\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m|^2 + 2\, k\, \|\widetilde{\mathbf{e}}_h^{m+1}\|^2 + |k\, \nabla e_{p,h}^{m+1}|^2 - |k\, \nabla e_{p,h}^m|^2 - |k\, \nabla(e_{p,h}^{m+1} - e_{p,h}^m)|^2 \\ = -2\, k\, \Big(e_i(\delta_t \widetilde{\mathbf{u}}^{m+1}), \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k\, \Big(\nabla \widetilde{\mathbf{e}}_i^{m+1}, \nabla \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k\, \Big(\nabla(2\, e_{p,i}^m - e_{p,i}^{m-1}), \widetilde{\mathbf{e}}_h^{m+1}\Big) \\ +2\, k\, c\Big(\widetilde{\mathbf{e}}_h^m, \widetilde{\mathbf{u}}^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) + 2\, k\, c\Big(\widetilde{\mathbf{e}}_i^m, \widetilde{\mathbf{u}}^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k\, c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{e}}_h^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) \\ -2\, k\, c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{e}}_i^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) := \sum_{i=1}^7 I_i \end{array} \tag{24}$$

We bound the RHS of (24) as follows:

$$I_1 \le \varepsilon\, k\, \|\widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k \|e_i(\delta_t \widetilde{\mathbf{u}}^{m+1})\|^2_{H^{-1}} \le \varepsilon\, k\, \|\widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, h^2\, k |\delta_t \widetilde{\mathbf{u}}^{m+1}|^2$$

$$I_2 \le \varepsilon\, k\, \|\widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, h^2\, k\, \|\widetilde{\mathbf{u}}^{m+1}\|^2_{H^2}$$

$$I_3 = 2\, k\, \Big((2\, e_{p,i}^m - e_{p,i}^{m-1}), \nabla \cdot \widetilde{\mathbf{e}}_h^{m+1}\Big) \le \varepsilon\, k\, \|\widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, h^2 (\|p^m\|^2 + \|p^{m-1}\|^2)$$

With respect to the nonlinear terms,

$$I_4 = 2\, k\, c\Big(\widetilde{\mathbf{e}}_h^m, \widetilde{\mathbf{u}}^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) \le C\, k |\widetilde{\mathbf{e}}_h^m|\, \|\widetilde{\mathbf{u}}^{m+1}\|_{W^{1,3}\cap L^\infty} \|\widetilde{\mathbf{e}}_h^{m+1}\| \le \varepsilon\, k\, \|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\, k\, |\widetilde{\mathbf{e}}_h^m|^2$$

$$I_5 = 2\,k\,c\Big(\widetilde{\mathbf{e}}_i^m, \widetilde{\mathbf{u}}^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) \le \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,|\widetilde{\mathbf{e}}_i^m|^2 \le \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,h^4\,k\,\|\widetilde{\mathbf{u}}^m\|^2_{H^2}$$

$$I_6 = 2\,k\,c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{e}}_h^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) = 0$$

$$\begin{aligned} I_7 = 2\,k\,c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{e}}_i^{m+1}, \widetilde{\mathbf{e}}_h^{m+1}\Big) &\le C\,k\,\|\widetilde{\mathbf{u}}_h^m\|^2\,\|\widetilde{\mathbf{e}}_i^{m+1}\|^2_{L^3} + \varepsilon\,k\|\widetilde{\mathbf{e}}_h^{m+1}\|^2 \\ &\le C\,k\,\|\widetilde{\mathbf{u}}_h^m\|^2\,|\widetilde{\mathbf{e}}_i^{m+1}|\,\|\widetilde{\mathbf{e}}_i^{m+1}\| + \varepsilon\,k\|\widetilde{\mathbf{e}}_h^{m+1}\|^2 \\ &\le C\,k\,h^3\,\|\widetilde{\mathbf{u}}_h^m\|^2\,\|\widetilde{\mathbf{u}}^{m+1}\|^2_{H^2} + \varepsilon\,k\|\widetilde{\mathbf{e}}_h^{m+1}\|^2 \\ &\le C\,k\,h^3\,\|\widetilde{\mathbf{u}}_h^m\|^2 + \varepsilon\,k\|\widetilde{\mathbf{e}}_h^{m+1}\|^2 \end{aligned}$$

On the other hand, from $(E_2)_h^{m+1}$ and the L^2-orthogonality property, we have

$$|\widetilde{\mathbf{e}}_h^{m+1}|^2 = |\mathbf{e}_h^{m+1}|^2 + |k\,(\nabla\,e_{p,h}^{m+1} - e_{p,h}^m)|^2 \tag{25}$$

Then, adding (24) and (25), the last term of the LHS of (24) is canceled and we obtain

$$\begin{aligned} &|\mathbf{e}_h^{m+1}|^2 - |\mathbf{e}_h^m|^2 + |\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m|^2 + k\,\|\widetilde{\mathbf{e}}_h^{m+1}\|^2 + |k\,\nabla e_{p,h}^{m+1}|^2 - |k\,\nabla e_{p,h}^m|^2 \\ &\le C\,k\,|\widetilde{\mathbf{e}}_h^m|^2 + C\,k\,h^2 + C\,k\,h^3\,\|\widetilde{\mathbf{u}}_h^m\|^2. \end{aligned}$$

Finally, taking into account (25), $|\widetilde{\mathbf{e}}_h^m|^2 = |\mathbf{e}_h^m|^2 + |k\,\nabla(e_{p,h}^m - e_{p,h}^{m-1})|^2 \le |\mathbf{e}_h^m|^2 + 2|k\,\nabla e_{p,h}^m|^2 + 2|k\,\nabla e_{p,h}^{m-1}|^2$, then adding from $m=0$ to r (with any $r < M$), since $k\sum\|\widetilde{\mathbf{u}}_h^m\|^2 \le C$, we can get (applying discrete Gromwall's Lemma):

$$|\mathbf{e}_h^{r+1}|^2 + |k\,\nabla e_{p,h}^{r+1}|^2 + \sum_{m=0}^{r}|\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m|^2 + k\,\sum_{m=0}^{r}\|\widetilde{\mathbf{e}}_h^{m+1}\|^2 \le C\Big(|\mathbf{e}_h^0|^2 + |k\,\nabla e_{p,h}^0|^2 + h^2\Big)$$

hence the estimates (22)-(23) hold. ■

As consequence of Theorem 15 and constraint (**H**), we can obtain stability estimates in regular norms:

- From (22) and constraint (**H**), we obtain in particular $|\nabla e_{p,h}^{m+1}| \le C$ hence, since $|\nabla e_{p,i}^{m+1}| \le C$ and $|\nabla p^{m+1}| \le C$, we arrive at
$$|\nabla p_h^{m+1}| \le C \tag{26}$$

- From (22) and constraint (**H**), one has in particular
$$\|\widetilde{\mathbf{u}}_h^{m+1}\|^2 \le C\Big(\|\widetilde{\mathbf{e}}_h^{m+1}\|^2 + \|\widetilde{\mathbf{e}}_i^{m+1}\|^2 + \|\widetilde{\mathbf{u}}^{m+1}\|^2\Big) \le C\left(\frac{h^2}{k} + h^2\,\|\widetilde{\mathbf{u}}^{m+1}\|^2_{H^2} + 1\right) \le C. \tag{27}$$

- Finally, from (23) and using again the constraint (**H**), one has
$$\frac{\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m}{k} \quad \text{is bounded in } l^\infty(\mathbf{L}^2).$$

hence, by using that $\dfrac{\widetilde{\mathbf{e}}_i^{m+1}-\mathbf{e}_i^m}{k}$ is bounded in $l^\infty(\mathbf{L}^2)$ and $\sum_m \left|\dfrac{\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m}{k}\right|^2 \le C$, we obtain

$$\frac{\widetilde{\mathbf{u}}_h^{m+1}-\mathbf{u}_h^m}{k} \text{ is bounded in } l^\infty(\mathbf{L}^2). \tag{28}$$

Previous estimates (26)-(28) will be used jointly with the following result, in order to improve the stability estimates for $\widetilde{\mathbf{u}}_h^{m+1}$.

Lemma 16 *[7] Let $\mathbf{g}\in\mathbf{L}^2$. If $\mathbf{u}_h\in\mathbf{Y}_h$ is the solution of the problem*

$$\Big(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h\Big)=\Big(\mathbf{g},\mathbf{v}_h\Big)\quad\forall\,\mathbf{v}_h\in\mathbf{Y}_h.$$

then

$$\|\mathbf{u}_h\|_{W^{1,6}}\le C\,|\mathbf{g}|.$$

Corollary 17 *Assuming hypotheses of Theorem 15, $(\mathbf{H})$ and $\|\mathbf{u}_h^0\|^2_{W^{1,6}}\le C_0$ (that is $\mathbf{u}_0\in\mathbf{W}^{1,6}$), one has*

$$\widetilde{\mathbf{u}}_h^{m+1} \textit{ is bounded in } l^\infty(\mathbf{W}^{1,6}). \tag{29}$$

Proof: Re-writting $(S_1)_h^{m+1}$ as :
$(S_1)_h^{m+1}$

$$\Big(\nabla\widetilde{\mathbf{u}}_h^{m+1},\nabla\mathbf{v}_h\Big) \;=\; -\frac{1}{k}\Big(\widetilde{\mathbf{u}}_h^{m+1}-\mathbf{u}_h^m,\mathbf{v}_h\Big)-c\Big(\widetilde{\mathbf{u}}_h^m,\widetilde{\mathbf{u}}_h^{m+1},\mathbf{v}_h\Big)+\Big(\mathbf{f}^{m+1},\mathbf{v}_h\Big)-\Big(\nabla p_h^m,\mathbf{v}_h\Big)$$

then, using the Lemma 16,

$$\|\widetilde{\mathbf{u}}_h^{m+1}\|^2_{W^{1,6}}\le C\left(\left|\frac{\widetilde{\mathbf{u}}_h^{m+1}-\mathbf{u}_h^m}{k}\right|^2+|\mathbf{f}^{m+1}|^2+\|\widetilde{\mathbf{u}}_h^m\|^2_{W^{1,3}\cap L^\infty}\,\|\widetilde{\mathbf{u}}_h^{m+1}\|^2+|\nabla p_h^m|^2\right)$$

Bounding

$$\|\widetilde{\mathbf{u}}_h^m\|^2_{W^{1,3}\cap L^\infty}\le\|\widetilde{\mathbf{u}}_h^m\|^{1-\delta}\,\|\widetilde{\mathbf{u}}_h^m\|^{1+\delta}_{W^{1,6}}\le C\,\|\widetilde{\mathbf{u}}_h^m\|^{1+\delta}_{W^{1,6}}\le\varepsilon\,\|\widetilde{\mathbf{u}}_h^m\|^2_{W^{1,6}}+C$$

and using (27), (28) and (26), we have

$$\|\widetilde{\mathbf{u}}_h^{m+1}\|^2_{W^{1,6}}\le\varepsilon\|\widetilde{\mathbf{u}}_h^m\|^2_{W^{1,6}}+C\left(|\delta_t\mathbf{u}_h^{m+1}|^2+|\nabla p_h^{m+1}|^2+|\mathbf{f}^{m+1}|^2+1\right)\le\varepsilon\|\widetilde{\mathbf{u}}_h^m\|^2_{W^{1,6}}+C_1.$$

Then, it suffices to choose ε small such that $\varepsilon\,C_0\le C_1$ and $\varepsilon\,2\,C_1\le C_1$, to arrive at

$$\|\widetilde{\mathbf{u}}_h^m\|_{W^{1,6}}\le 2\,C_1\quad\forall\,m,$$

which is the estimate of $\widetilde{\mathbf{u}}_h^{m+1}$ in $l^\infty(\mathbf{W}^{1,6})$. ■

2.4 $O(h)$ error estimates for $\delta_t \mathbf{e}_h^{m+1}$ in $l^\infty(\mathbf{L}^2)$, $\delta_t \widetilde{\mathbf{e}}_h^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and $(\widetilde{\mathbf{e}}_h^{m+1}, e_{p,d}^{m+1})$ in $l^\infty(\mathbf{H}^1 \times L^2)$

By making $\delta_t (E_1)_h^{m+1}$ and $\delta_t (E_2)_h^{m+1}$, one obtains ($\forall\, m \ge 1$):

$$\frac{1}{k}\Big(\delta_t \widetilde{\mathbf{e}}_d^{m+1} - \delta_t \mathbf{e}_d^m, \mathbf{v}_h\Big) + \Big(\nabla \delta_t \widetilde{\mathbf{e}}_d^{m+1}, \nabla\, \mathbf{v}_h\Big) - \Big(\delta_t \nabla e_{p,d}^m, \mathbf{v}_h\Big) = \delta_t\, \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h$$

where,

$$\delta_t \mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c\Big(\delta_t \widetilde{\mathbf{e}}_d^m,\, \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\Big) + c\Big(\delta_t \widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{e}}_d^{m+1}, \mathbf{v}_h\Big) + c\Big(\widetilde{\mathbf{e}}_d^{m-1}, \delta_t \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\Big) + c\Big(\widetilde{\mathbf{u}}_h^{m-1}, \delta_t \widetilde{\mathbf{e}}_d^{m+1}, \mathbf{v}_h\Big)$$

and

$$\delta_t \mathbf{e}_d^{m+1} = \delta_t \widetilde{\mathbf{e}}_d^{m+1} - k \nabla (\delta_t e_{p,d}^{m+1} - \delta_t e_{p,d}^m)$$

The following L^2-orthogonality property holds:

$$\Big(\delta_t \mathbf{e}_d^{m+1}, \nabla q_h\Big) = 0, \quad \forall\, q_h \in Q_h.$$

Then, due to the choice of interpolation operators,

$$(D_1)_h^{m+1} \quad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\delta_t \widetilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla \delta_t \widetilde{\mathbf{e}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla \delta_t e_{p,h}^m, \mathbf{v}_h\Big) = \delta_t\, \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(\mathbf{e}_i(\delta_t \delta_t \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}_h\Big) - \Big(\nabla \delta_t \widetilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(\nabla (2\, \delta_t e_{p,i}^m - \delta_t e_{p,i}^{m-1}), \mathbf{v}_h\Big) \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h \end{array} \right.$$

$$(D_2)_h^{m+1} \qquad \delta_t \mathbf{e}_h^{m+1} = \delta_t \widetilde{\mathbf{e}}_h^{m+1} - k \nabla (\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m)$$

and

$$\Big(\delta_t \mathbf{e}_h^{m+1}, \nabla q_h\Big) = 0, \quad \forall\, q_h \in Q_h.$$

Theorem 18 *Under the hypotheses of Theorems 10 and 15, assuming the hypothesis for the first step of the scheme*

$$|\delta_t \mathbf{e}_h^1| + |k\, \nabla \delta_t e_{p,h}^1| \le C\, h,$$

then

$$\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)} + \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)} + \|k\, \delta_t \nabla e_{p,h}^{m+1}\|_{l^\infty(\mathbf{L}^2)} \le C\, h. \tag{30}$$

Remark 19 *Reasoning with similar arguments as in [13], we can get $|\delta_t \mathbf{e}_h^1|^2 \le C\, h^2/k$ and, in particular, using* $(\mathbf{H})$*, one deduces $|\delta_t \mathbf{e}_h^1|^2 \le C\, h$.*

Proof: Since the initial estimate $|\delta_t \mathbf{e}_h^1| + |k\, \nabla \delta_t e_{p,h}^1| \le C\, h$ is assumed, it suffices to prove the generic estimate for $\delta_t \mathbf{e}_h^{m+1}$, $\delta_t \widetilde{\mathbf{e}}_h^{m+1}$ and $k\, \delta_t \nabla e_{p,h}^{m+1}$, for each $m \ge 1$.

Taking $2\, k\, \delta_t \widetilde{\mathbf{e}}_h^{m+1} \in \mathbf{Y}_h$ as test function in $(D_1)_h^{m+1}$ one has

$$\begin{array}{l} |\delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^m|^2 + |\delta_t \widetilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m|^2 + 2\, k \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 - 2\, k \Big(\delta_t e_{p,h}^m, \nabla \cdot \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) \\ = -2\, k \Big(\mathbf{e}_i(\delta_t \delta_t \widetilde{\mathbf{u}}^{m+1}), \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k \Big(\nabla \delta_t \widetilde{\mathbf{e}}_i^{m+1}, \nabla \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k \Big(\nabla (2\, \delta_t e_{p,i}^m - \delta_t e_{p,i}^{m-1}), \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) \\ +2\, \delta_t \mathbf{NL}_h^{m+1}(\delta_t \widetilde{\mathbf{e}}_h^{m+1}) := I_1 + I_2 + I_3 + I_4. \end{array} \tag{31}$$

We bound the RHS as:

$$I_1 \le \varepsilon\, k\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, h^2\, |\delta_t\delta_t \widetilde{\mathbf{u}}^{m+1}|^2$$

$$I_2 \le \varepsilon\, k\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, h^2 \|\delta_t \widetilde{\mathbf{u}}^{m+1}\|_{H^2}^2$$

$$I_3 \le \varepsilon\, k\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, h^2 (\|\delta_t p^m\|^2 + \|\delta_t p^{m-1}\|^2)$$

The nonlinear terms, for $m \ge 1$, are treated as follows:

$$\begin{aligned} I_4 &= 2\,k\,c\Big(\delta_t \widetilde{\mathbf{e}}_d^m,\, \widetilde{\mathbf{u}}^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) + 2\,k\,c\Big(\delta_t \widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{e}}_d^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) \\ &+ 2\,k\,c\Big(\widetilde{\mathbf{e}}_d^{m-1}, \delta_t \widetilde{\mathbf{u}}^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) + 2\,k\,c\,\Big(\widetilde{\mathbf{u}}_h^{m-1}, \delta_t \widetilde{\mathbf{e}}_d^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) := \sum_{i=1}^{4} J_i \end{aligned}$$

Bounding each J_i term:

$$J_1 = 2\,k\,c\Big(\delta_t \widetilde{\mathbf{e}}_h^m,\, \widetilde{\mathbf{u}}^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) + 2\,k\,c\Big(\delta_t \widetilde{\mathbf{e}}_i^m,\, \widetilde{\mathbf{u}}^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) = J_{11} + J_{12}$$

$$J_{11} \le \varepsilon\, k \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, \|\widetilde{\mathbf{u}}^{m+1}\|_{W^{1,3}\cap L^\infty}^2\, |\delta_t \widetilde{\mathbf{e}}_h^m|^2$$

$$J_{12} \le \varepsilon\, k \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, \|\widetilde{\mathbf{u}}^{m+1}\|^2\, |\delta_t \widetilde{\mathbf{e}}_i^m|^2 \le \varepsilon\, k \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, h^2\, \|\delta_t \widetilde{\mathbf{u}}^m\|^2$$

$$J_2 \le \varepsilon\, k\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k |\delta_t \widetilde{\mathbf{e}}_i^m|^2 + C\, k\, \|\widetilde{\mathbf{e}}_d^{m+1}\|_{W^{1,3}\cap L^\infty}^2\, |\delta_t \widetilde{\mathbf{e}}_h^m|^2$$

$$J_3 = 2\,k\,c\Big(\widetilde{\mathbf{e}}_h^{m-1}, \delta_t \widetilde{\mathbf{u}}^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) + 2\,k\,c\Big(\widetilde{\mathbf{e}}_i^{m-1}, \delta_t \widetilde{\mathbf{u}}^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) = J_{31} + J_{32}$$

$$J_{31} \le C\, k\, \|\widetilde{\mathbf{e}}_h^{m-1}\|\, \|\delta_t \widetilde{\mathbf{u}}^{m+1}\|_{L^3}\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\| \le \varepsilon\, k\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\, k\, \|\widetilde{\mathbf{e}}_h^{m-1}\|^2$$

$$J_{32} \le \varepsilon\, k\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, \|\widetilde{\mathbf{e}}_i^{m-1}\|^2 \le \varepsilon\, k\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\, k \|\delta_t \widetilde{\mathbf{u}}^{m+1}\|_{H^2}^2\, \|\widetilde{\mathbf{e}}_i^{m-1}\|^2$$

$$J_4 = 2\,k\,c\Big(\widetilde{\mathbf{u}}_h^{m-1}, \delta_t \widetilde{\mathbf{e}}_h^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) + 2\,k\,c\Big(\widetilde{\mathbf{u}}_h^{m-1}, \delta_t \widetilde{\mathbf{e}}_i^{m+1},\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) = J_{41} + J_{42}$$

$$J_{41} = 0$$

$$J_{42} \le C\, k\, \|\widetilde{\mathbf{u}}_h^{m-1}\|\, \|\delta_t \widetilde{\mathbf{e}}_i^{m+1}\|_{L^3}\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\| \le \varepsilon\, k\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\, k\, \|\widetilde{\mathbf{u}}_h^{m-1}\|^2\, h^3 \|\delta_t \widetilde{\mathbf{u}}^{m+1}\|_{H^2}^2$$

Here, we have used the error interpolation $\|\delta_t \widetilde{\mathbf{e}}_i^{m+1}\|_{L^3} \le C\, |\delta_t \widetilde{\mathbf{e}}_i^{m+1}|^{1/2}\, \|\delta_t \widetilde{\mathbf{e}}_i^{m+1}\|^{1/2} \le C\, h^{3/2} \|\delta_t \widetilde{\mathbf{u}}^{m+1}\|_{H^2}$.

On the other hand, from $(D_2)_h^{m+1}$ and the L^2-orthogonality, we have

$$|\delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2 = |\delta_t \mathbf{e}_h^{m+1}|^2 + |k\, \nabla(\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m)|^2 \tag{32}$$

Then, by adding (31) and (32), taking into account the above estimates and writting the term

$$\begin{aligned} 2\,k\,\Big(\nabla \delta_t e_{p,h}^m, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) &= 2\,k\,\Big(\nabla \delta_t e_{p,h}^m, k\, \nabla(\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m)\Big) \\ &= |k\, \nabla \delta_t e_{p,h}^{m+1}|^2 - |k\, \nabla \delta_t e_{p,h}^m|^2 - |k\, \nabla(\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m)|^2 \end{aligned}$$

$\Big($using that $\Big(\nabla\delta_t e_{p,h}^m, \delta_t \mathbf{e}_h^{m+1}\Big) = 0\Big)$, we obtain

$$|\delta_t \mathbf{e}_h^{m+1}|^2 - |\delta_t \mathbf{e}_h^m|^2 + |\delta_t \widetilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m|^2 + k\, \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + |k\, \nabla \delta_t e_{p,h}^{m+1}|^2 - |k\, \delta_t \nabla e_{p,h}^m|^2$$
$$\le C\, k\, \|\widetilde{\mathbf{e}}_d^{m+1}\|^2_{W^{1,3}\cap L^\infty} |\delta_t \widetilde{\mathbf{e}}_h^m|^2 + C\, k\, h^2 \le C\, k\, |\delta_t \widetilde{\mathbf{e}}_h^m|^2 + C\, k\, h^2$$

where in the last inequality the estimate $\|\widetilde{\mathbf{e}}_d^{m+1}\|^2_{W^{1,3}\cap L^\infty} \le C$ (due to $\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} \le C$ and (29)) has been used. Finally, taking into account (32),

$$|\delta_t \widetilde{\mathbf{e}}_h^m|^2 = |\delta_t \mathbf{e}_h^m|^2 + |k\, \nabla(\delta_t e_{p,h}^m - \delta_t e_{p,h}^{m-1})|^2 \le |\delta_t \mathbf{e}_h^m|^2 + 2|k\, \nabla \delta_t e_{p,h}^m|^2 + 2|k\, \delta_t \nabla e_{p,h}^{m-1}|^2,$$

therefore, by adding from $m = 1$ to r (with any $r < M$), the discrete Gromwall's Lemma can be applied, hence using the hypothesis $|\delta_t \mathbf{e}_h^1| \le C\, h$ and $|k\, \nabla\, \delta_t e_{p,h}^1| \le C\, h$, we get

$$|\delta_t \mathbf{e}_h^{r+1}|^2 + |k\, \nabla \delta_t e_{p,h}^{r+1}|^2 + \sum_{m=1}^{r} |\delta_t \widetilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m|^2 + k \sum_{m=1}^{r} \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 \le C\Big(|\delta_t \mathbf{e}_h^1|^2 + |k\, \nabla \delta_t e_{p,h}^1|^2 + h^2\Big)$$

and (30) hold. ■

Corollary 20 *Assuming hypotheses of Theorem 18, the following error estimates hold*

$$\|e_{p,h}^{m+1}\|_{l^\infty(L^2)} \le C\, h \quad \textit{and} \quad \|\widetilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(H^1)} \le C\, h.$$

Proof: Arguing as in the semi-discrete in time scheme, from the discrete inf-sup condition applied to $(E_3)_h^{m+1}$ and the already obtained bounds $\|\widetilde{\mathbf{e}}_h^{m+1}\|_{l^2(H^1)} \le C\, h$ and $\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(L^2} \le C\, h$, we have the estimate

$$\|e_{p,h}^m\|_{l^2(L^2)} \le C\, h. \tag{33}$$

On the other hand, by multiplying $(E_3)_h^{m+1}$ by $2\, k\, \delta_t \widetilde{\mathbf{e}}_h^{m+1}$ we have

$$|\nabla \widetilde{\mathbf{e}}_h^{m+1}|^2 - |\nabla \widetilde{\mathbf{e}}_h^m|^2 + |\nabla \widetilde{\mathbf{e}}_h^{m+1} - \nabla \widetilde{\mathbf{e}}_h^m|^2 = -2\, k\Big(\nabla e_{p,h}^{m+1}, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k\Big(\nabla e_{p,i}^{m+1}, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big)$$
$$-2\, k\Big(\delta_t \mathbf{e}_h^{m+1}, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k\Big(\delta_t \mathbf{e}_i^{m+1}, \delta_t \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k\, \Big(\nabla \widetilde{\mathbf{e}}_i^{m+1}, \delta_h \widetilde{\mathbf{e}}_h^{m+1}\Big) - 2\, k\, \mathbf{NL}_h^{m+1}(\delta_t \widetilde{\mathbf{e}}_h^{m+1})$$

Then we obtain

$$|\nabla \widetilde{\mathbf{e}}_h^{m+1}|^2 - |\nabla \widetilde{\mathbf{e}}_h^m|^2 + |\nabla \widetilde{\mathbf{e}}_h^{m+1} - \nabla \widetilde{\mathbf{e}}_h^m|^2 \le 2\, k\, |e_{p,h}^{m+1}|^2 + 2\ k\, |e_{p,i}^{m+1}|^2$$
$$+C\, k\, |\nabla \cdot \delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2 + C\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k\, |\delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2$$
$$+2\, k\, \|e_i(\delta_t \mathbf{u}^{m+1})\|^2_{H^{-1}} + 2\, k\, \|\widetilde{\mathbf{e}}_i^{m+1}\|^2 + 2\, k\, |\mathbf{NL}_h^{m+1}|^2$$

then

$$|\nabla \widetilde{\mathbf{e}}_h^{m+1}|^2 - |\nabla \widetilde{\mathbf{e}}_h^m|^2 + |\nabla \widetilde{\mathbf{e}}_h^{m+1} - \nabla \widetilde{\mathbf{e}}_h^m|^2 \le 2\, k\, |e_{p,h}^{m+1}|^2$$
$$+C\, k\, |\nabla \cdot \delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2 + C\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k\, |\delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2$$

$$+C\,k\,h^2\|p^{m+1}\|^2 + C\,k\,h^2|\delta_t \mathbf{u}^{m+1}|^2 + C\,k\,h^2\|\widetilde{\mathbf{u}}^{m+1}\|^2_{H^2} + 2\,k\,|\mathbf{NL}_h^{m+1}|^2$$

Bounding the last term of the right side hand as follows, taking into account (29),

$$2\,k\,\mathbf{NL}_h^{m+1}(\delta_t \widetilde{\mathbf{e}}_h^{m+1}) \leq \varepsilon\,k\|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\,k\,|\widetilde{\mathbf{e}}_h^{m+1}|^2 + C\,k\,|\widetilde{\mathbf{e}}_h^{m}|^2 + C\,k\,|\widetilde{\mathbf{e}}_i^{m+1}|^2 + C\,k|\widetilde{\mathbf{e}}_i^{m}|^2$$

$$\leq \varepsilon\,k\|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|^2 + C\,k\,|\widetilde{\mathbf{e}}_h^{m+1}|^2 + C\,k\,|\widetilde{\mathbf{e}}_h^{m}|^2 + C\,k\,h^2$$

we arrive at

$$|\nabla \widetilde{\mathbf{e}}_h^{m+1}|^2 - |\nabla \widetilde{\mathbf{e}}_h^{m}|^2 + |\nabla \widetilde{\mathbf{e}}_h^{m+1} - \nabla \widetilde{\mathbf{e}}_h^{m}|^2 \leq k\,|e_{p,h}^{m+1}|^2 + k|\nabla \cdot \delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2$$

$$+C\,k\,|\delta_t \mathbf{e}_h^{m+1}|^2 + k|\nabla\,\delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2 + C\,k\,|\delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2 + C\,k\,|\widetilde{\mathbf{e}}_h^{m+1}|^2 + C\,k|\widetilde{\mathbf{e}}_h^{m}|^2 + C\;k\;h^2$$

Adding from $m = 0$ to r we arrive at

$$|\nabla \widetilde{\mathbf{e}}_h^{r+1}|^2 \leq |\nabla \widetilde{\mathbf{e}}_h^{0}|^2 + C\,k \sum_{m=0}^{r} |e_{p,h}^{m+1}|^2 + C\,k \sum_{m=0}^{r} |\nabla\,\delta_t \widetilde{\mathbf{e}}_h^{m+1}|^2 + C\,k\,\sum_{m=0}^{r} |\delta_t \mathbf{e}_h^{m+1}|^2 + C\,h^2$$

Then, we obtain

$$\|\widetilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(H^1)} \leq C\,h \tag{34}$$

applying (33) and the estimates obtained in Theorems 15 and 18.

On the other hand, using again the discrete inf-sup condition and taking into account (34), we arrive at

$$\|e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq C\,h$$

and the proof is finished. ■

Remark 21 *Combining Theorem 12 and Corollary 20, the following error estimate for the total error holds*

$$\|\mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}_h^{m+1}\|_{l^\infty(H^1)} + \|p(t_{m+1}) - p_h^{m+1}\|_{l^\infty(L^2)} \leq C\,(k+h).$$

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Convergence and error estimates of a time fractional-step method for the Primitive Equations *

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Abstract

The purpose of this paper is the numerical analysis of a first order fractional-step scheme in time, using decomposition of the viscosity, and stable finite elements in space for a model of Primitive Equations of the Ocean. The aim of the paper is twofold. First, we prove that the scheme is unconditional stable and convergence towards weak solutions of the Primitive Equations and second, we will prove optimal error estimates for the velocity as well as for the pressure for a regular enough solution, provided $k \leq C\, h^2$ where k and h are respectively the time step and the mesh size. In both arguments, H^2 regularity of the hydrostatic Stokes problem and some inverse inequalities must be assumed.

Subject Classification 35Q35, 65M12, 65M15, 75D05

Keywords: Primitive Equations, finite elements, anisotropic estimates, splitting methods, stability, convergence, error estimates

Introduction

Assuming some simplifications (basically hydrostatic pressure and "the rigid lid" hypothesis), the $3D$ Navier-Stokes equations derive to the so-called "Primitive Equations" (also called the Hydrostatic Navier-Stokes equations). These equations are a general mathematical model in the field of geophysical fluids ([24, 28]). In particular, they describe the general circulation of the water in lakes and oceans [25]. For simplicity, we take constant density, Cartesian coordinates (x in the easterly direction, y in the northerly direction and z perpendicular to the surface of the Earth) and we assume that the effects due to temperature and salinity can be decoupled

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from the flow dynamics. Then, the model governed by the Primitive Equations is ([23, 24, 25]):

$$(P) \qquad \begin{cases} \partial_t \mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_{\mathbf{x}} p &= \mathbf{f} \quad \text{in } \Omega\times(0,T), \\ \partial_z p = -\rho\, g, \qquad \nabla\cdot\mathbf{U} &= 0 \quad \text{in } \Omega\times(0,T), \\ \mathbf{u} = 0 \;\text{ on } \Gamma_l\times(0,T), \quad \mathbf{u} = u_3 n_3 &= 0 \quad \text{on } \Gamma_b\times(0,T), \\ \nu\,\partial_z\mathbf{u} = \mathbf{g}_s, \quad u_3 &= 0 \quad \text{on } \Gamma_s\times(0,T), \\ \mathbf{u}_{|t=0} &= \mathbf{u}_0 \quad \text{in } \Omega, \end{cases}$$

where $\Omega = \{(\mathbf{x},z)\in \boldsymbol{R}^3 :\ \mathbf{x}=(x,y)\in S,\ -D(\mathbf{x})<z<0\}$ is the $3D$ domain filled by the water, with $S\subset \boldsymbol{R}^2$ the surface domain (a regular bounded $2D$ domain) and $D:\overline{S}\to \boldsymbol{R}_+$ (with $D>0$ in S) the function describing the bottom. Then, $\Gamma_s = \overline{S}\times\{0\}$ is the part of the boundary of Ω corresponding to the surface, $\Gamma_b = \{(\mathbf{x},-D(\mathbf{x})) :\ \mathbf{x}\in S\}$ corresponds to the bottom (with outwards normal vector $(\mathbf{n_x}, n_3)$) and $\Gamma_l = \{(\mathbf{x},z) :\ \mathbf{x}\in\partial S, -D(\mathbf{x})<z<0\}$ correspond to the lateral walls. Note that if $D\in C(\overline{S})$ then the bottom has no skips and the condition $u_3 n_3 = 0$ on $\Gamma_b\times(0,T)$ derives to $u_3=0$ on $\Gamma_b\times(0,T)$.

The unknowns of the problem are $\mathbf{U} = (\mathbf{u}, u_3) : \Omega\times(0,T)\to \boldsymbol{R}^3$ the 3D velocity field (with $\mathbf{u}=(u_1,u_2)$ the corresponding horizontal velocity and u_3 the vertical velocity) and p : $\Omega\times(0,T)\to\boldsymbol{R}$ the pressure.

Also, $\mathbf{b}(\mathbf{u}) = f\mathbf{u}^\perp$ represents the effect of the Coriolis Forces, with $\mathbf{u}^\perp = (-u_2,u_1)^t$ and $f = 2|w|\sin\theta$, where w is the angular velocity of the Earth and $\theta=\theta(y)$ is the latitude, $\rho\in\boldsymbol{R}_+$ is the water density (that it is assumed a positive constant), $g\in\boldsymbol{R}_+$ is the gravity acceleration (another positive constant), $\mathbf{f}:\Omega\times(0,T)\to\boldsymbol{R}^2$ is a field of external horizontal forces (depending for instance on the salinity and temperature) and $\mathbf{g}_s : \Gamma_s\times(0,T)\to\boldsymbol{R}^2$ represents the stress of the wind on the surface.

Finally, $\nabla = (\nabla_\mathbf{x}, \partial_z)^t$ stands for the three-dimensional gradient operator (with $\nabla_\mathbf{x} = (\partial_x,\partial_y)^t$ its horizontal component) y Δ stands for the three-dimensional Laplacian operator.

For simplicity, we have considered in (P) isotropic diffusion, which is written as $-\nu\Delta\mathbf{u}$, where $\nu>0$ is a viscosity coefficient. In general, due to the difference between the horizontal and vertical dimensions of the domain, it is usual to consider anisotropic (eddy) diffusion; for instance

$$-\nabla_\mathbf{x}\cdot(\nu_h\nabla_\mathbf{x}\mathbf{u}) - \nu_v\,\partial_z^2\mathbf{u}$$

where $\nu_h,\ \nu_v>0$ are the horizontal and vertical eddy coefficients respectively, being $\nu_v\ll\nu_h$ ([28]). The results of this paper can be easily extended to this case.

Now, we will give two reformulations of the problem (P) which will drive us two different spatial approximations.

If one defines $p_s(t;\mathbf{x}) = p(t;\mathbf{x},z) - \rho g z$, then $p_s : S \times (0,T) \to \boldsymbol{R}$ is a new unknown (defined only on the surface S), that it will be called the *surface pressure*.

Notice that the equation $\nabla \cdot \mathbf{U} = 0$ in $\Omega \times (0,T)$ and the boundary condition $u_3 = 0$ on $\Gamma_s \times (0,T)$ is equivalent to the explicit formula

$$u_3(t;\mathbf{x},z) = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{u}(t;\mathbf{x},s)\, ds \tag{1}$$

Moreover, one has the equality,

$$\int_{-D(\mathbf{x})}^0 \nabla \cdot \mathbf{U}\, dz = \nabla_{\mathbf{x}} \cdot \langle \mathbf{u} \rangle - (\mathbf{u}, u_3)(\mathbf{x}, -D(\mathbf{x})) \cdot (\nabla_{\mathbf{x}} D(\mathbf{x}), 1) = 0, \quad \text{in } S \times (0,T), \tag{2}$$

where $\langle \mathbf{u} \rangle$ denotes the total vertical flux of the horizontal velocity:

$$\langle \mathbf{u} \rangle (t;\mathbf{x}) = \int_{-D(\mathbf{x})}^0 \mathbf{u}(t;\mathbf{x},z)\, dz.$$

Therefore, since $(\nabla_{\mathbf{x}} D(\mathbf{x}), 1)$ is parallel to the normal vector $(\mathbf{n}_{\mathbf{x}}, n_3)$ on Γ_b, if we assume $\nabla \cdot \mathbf{U} = 0$ in $\Omega \times (0,T)$, the so-called slip condition $\mathbf{u} \cdot \mathbf{n}_{\mathbf{x}} + u_3 n_3 = 0$ on $\Gamma_b \times (0,T)$ is equivalent to the constraint $\nabla_{\mathbf{x}} \cdot \langle \mathbf{u} \rangle = 0$ in $S \times (0,T)$ ([23, 24, 25]).

Then, the problem (P) can be reformulated as the following *integro-differential problem*:

$$(Q) \qquad \left\{ \begin{array}{rcll} \partial_t \mathbf{u} + (\mathbf{U} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_{\mathbf{x}} p_s & = & \mathbf{f} & \text{in } \Omega \times (0,T), \\ \nabla_{\mathbf{x}} \cdot \langle \mathbf{u} \rangle & = & 0 & \text{in } S \times (0,T), \\ \nu \partial_z \mathbf{u} = \mathbf{g}_s \quad \text{on } \Gamma_s \times (0,T), \qquad \mathbf{u} & = & 0 & \text{on } (\Gamma_b \cup \Gamma_l) \times (0,T), \\ \mathbf{u}_{|t=0} & = & \mathbf{u}_0 & \text{in } \Omega, \end{array} \right.$$

where $\mathbf{U} = (\mathbf{u}, u_3)$ and the vertical velocity u_3 depends on the $\nabla_{\mathbf{x}} \cdot \mathbf{u}$ by the integral formula (1).

The existence of weak solutions $(\mathbf{u}, p_s)$ of the problem (Q) is well known, see Lions-Teman-Wang [25] and Lewandowski [23], always in domains with side-walls (i.e. $D \geq D_{min} > 0$ in $\overline{S}$). In these works, a compactness method is used to obtain the velocity $\mathbf{u}$ in a space with the restriction $\nabla \cdot \langle \mathbf{u} \rangle = 0$ and afterwards, the surface pressure p_s is obtained by means of a specific De Rham's lemma on the surface S. In domains without side-walls (i.e. when the depth function D can degenerate to zero), the existence of weak solutions $(\mathbf{u}, u_3, p)$ of (P) is obtained by an asymptotic limit applied to the Navier-Stokes equations with anisotropic viscosity when the ratio depth over horizontal diameter (of the domain) goes to zero; see Besson-Laydi [5] for the stationary case and Azerad-Guillén [1, 2] for the evolution one. The existence of a weak solution of the stationary problem related to (Q) in domains without side-walls is proved in Chacón-Guillén [9] by internal approximation arguments: a mixed (velocity-pressure) variational formulation is approximated by a conformed Finite Element method verifying the so-called "hydrostatic Inf-Sup condition", see [9]. Moreover, Ortegón in [27], obtains a generalization of De Rham's Lemma to general domains without side-walls.

Respect to the regularity results for the Primitive Equations, the existence of strong solutions (with $H^2(\Omega)$-regularity for the horizontal velocity) is treated by Ziane in [31] for the linear stationary problem associated to (Q). This result is extended in [19] to the linear evolution case. With respect to the nonlinear problem, existence and uniqueness of strong solution for $2D$ domains, global in time for small enough data or local in time for small enough depth, is demonstrated in [19]. The extension (and improvement) of this kind of results to $3D$ domains can be seen in [14]. Finally, assuming flat bottom and Neumann boundary condition on the bottom, the existence of global in time regular solutions without constraints is proved in [8]. In [22], this result is also obtained with Dirichlet boundary conditions on the bottom.

In order to give a fully differential formulation, we consider the following reformulation of problem (P), replacing the integral equation for u_3 given in (1) by a differential equation:

$$
(R)\quad \left\{
\begin{array}{rcll}
\partial_t \mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_{\mathbf{x}}\, p_s &=& \mathbf{f} & \text{in } \Omega\times(0,T),\\
\partial_z^2 u_3 + \partial_z \nabla_{\mathbf{x}}\cdot\mathbf{u} &=& 0 & \text{in } \Omega\times(0,T),\\
\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle &=& 0 & \text{in } S\times(0,T),\\
\nu\partial_z\mathbf{u} = \mathbf{g}_s \quad \text{on } \Gamma_s\times(0,T), \qquad \mathbf{u} &=& 0 & \text{on } (\Gamma_b\cup\Gamma_l)\times(0,T),\\
u_3 &=& 0 & \text{on } (\Gamma_s\cup\Gamma_b)\times(0,T).\\
\mathbf{u}_{|t=0} &=& \mathbf{u}_0 & \text{in } \Omega,
\end{array}
\right.
$$

This reformulation is based on the following equivalence: assuming the slip-condition on Γ_b, one has

$$
\left.\begin{array}{l}
\partial_z(\nabla\cdot\mathbf{U}) = 0 \quad \text{in } \Omega\\
\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle = 0 \quad \text{in } S
\end{array}\right\} \Leftrightarrow \nabla\cdot\mathbf{U} = 0 \text{ in } \Omega.
$$

Indeed, from the equation $\partial_z(\nabla\cdot\mathbf{U}) = 0$, one has $\nabla\cdot\mathbf{U} = g(\mathbf{x})$. Integrating in vertical this equality and using (2) (taking into account that $\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle = 0$ and the slip-condition $\mathbf{u}\cdot\mathbf{n_x} + u_3\, n_3 = 0$ on Γ_b holds), one has

$$
0 = \int_{-D(\mathbf{x})}^{0} g(\mathbf{x})\, dz = D(\mathbf{x})\, g(\mathbf{x}) \quad \text{in } S.
$$

Therefore $g \equiv 0$ in S and $\nabla\cdot\mathbf{U} = 0$. Conversely, since $\nabla\cdot\mathbf{U} = 0$ then $\partial_z(\nabla\cdot\mathbf{U}) = 0$. Moreover, $\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle = 0$ is deduced again from (2) integrating in vertical $\nabla\cdot\mathbf{U} = 0$, taking into account the slip-condition on Γ_b.

Notice that in (R) the vertical velocity u_3 is uniquely defined by the z-elliptic problem

$$
\partial_z^2 u_3 = -\partial_z\nabla_{\mathbf{x}}\cdot\mathbf{u} \quad \text{in } \Omega\times(0,T), \quad u_3|_{\Gamma_s\cup\Gamma_b} = 0. \tag{3}
$$

The goal of this paper is to design a numerical scheme for (R) based in a fractional-step time scheme (using decomposition of the viscosity) and Finite Element in space, obtaining at one hand unconditional stability and convergence towards a weak solution of (R) and on the other hand error estimates respect to a sufficiently regular solution of (R), under the constraint $k \le C\, h^2$. Basically, in every time step m, three subproblems must be solved. Given $(\mathbf{u}_h^m, p_{s,h}^m)$, firstly the vertical velocity $u_{3,h}^m$ is computed in function of $\nabla_{\mathbf{x}} \cdot \mathbf{u}_h^m$, afterwards we obtain an intermediate horizontal velocity $\mathbf{u}_h^{m+1/2}$ and finally we obtain $\mathbf{u}_h^{m+1}$ and $p_{s,h}^{m+1}$ by means of a linear Stokes type problem.

From the numerical analysis point of view, convergence of some Finite Element schemes for the stationary problem related to (Q), has been proved in [9], where the so-called *hydrostatic inf-sup* stability condition appear. To approximate the time-dependent problem, a stabilized scheme with finite elements was used by T. Chacón and D. Rodríguez in [10, 11], and R. Bermejo in [3] and R. Bermejo and P. Galán in [4] have used a semi-lagrangian projection scheme in time together with finite elements in space. On the other hand, some numerical analysis results of a time fractional-step scheme with decomposition of the viscosity for the transient Navier-Stokes Equations can be seen in [6, 7] and [15, 16, 17].

This paper is organized as follows. In Section 1, we give some preliminaries. In Section 2, we describe the fully discrete scheme, obtaining in Section 3 its stability and convergence as $(k,h) \to 0$ towards weak solutions of the problem (R). In Section 4, assuming the existence of a (unique) sufficiently regular solution of (R), we obtain error estimates for velocities as well as for the pressure. In this section, we present the regularity hypotheses for the solution which will appear latter and we describe the problems related to the space discrete errors in the first two subsections. Afterwards, we obtain error estimates, concretely, for $l = 1, 2$ (where l is the order of approximation of the finite element spaces), error estimates of order $O(\sqrt{k} + h^l)$ for the velocities $\mathbf{u}_h^{m+1/2}$ and $\mathbf{u}_h^{m+1}$, improved error estimates (of order $O(k + h^l)$) for the "end of step" velocity $\mathbf{u}_h^{m+1}$, error estimates of order $O(\sqrt{k} + h^l)$ for the discrete derivative of "end of step" velocity in $l^2(\mathbf{L}^2)$ (which drive to error estimates of order $O(\sqrt{k} + h^2)$ for the pressure $p_{s,h}^{m+1}$), error estimates of order $O(\sqrt{k} + h^2)$ for the discrete derivative of velocities $\mathbf{u}_h^{m+1/2}$ and $\mathbf{u}_h^{m+1}$ and improved error estimates (of order $O(k+h^2)$) for the discrete derivative of the "end of step" velocity $\mathbf{u}_h^{m+1}$ (these estimates drive to error estimates of order $O(k+h^2)$ for the pressure $p_{s,h}^{m+1}$). On the other hand, in Section 5 assuming a specific vertical structured grids, we obtain error estimates (of order $O(k + h^{l+1})$) for the "end of step" velocity $\mathbf{u}_h^{m+1}$ but with respect to weaker norms than above (which drives to error estimates of order $O(k + h^l)$ for $p_{s,h}$ in $l^2(\mathbf{L}^2)$). Finally, some comments about the treatment of the Coriolis term are given in Section 6.

In this paper, the following discrete Gronwall lemma will be frequently used (for a proof, see [21, p. 369]):

Lemma 1 *Let k, B and a_m, b_m, c_m, γ_m be nonnegative numbers.*

a) (Discrete Gronwall inequality) *We assume*

$$a_{r+1} + k\sum_{m=0}^{r} b_m \leq k\sum_{m=0}^{r}\gamma_m a_m + k\sum_{m=0}^{r} c_m + B \qquad \forall r \geq 0.$$

Then, one has

$$a_{r+1} + k\sum_{m=0}^{r} b_m \leq \exp\left(k\sum_{m=0}^{r}\gamma_m\right)\left\{k\sum_{m=0}^{r} c_m + B\right\} \qquad \forall r \geq 0.$$

b) (Generalised discrete Gronwall inequality) *We assume*

$$a_r + k\sum_{m=0}^{r} b_m \leq k\sum_{m=0}^{r}\gamma_m a_m + k\sum_{m=0}^{r} c_m + B \qquad \forall r \geq 0$$

such that $k\gamma_m < 1$ *for all* m*. Then, setting* $\sigma_m \equiv (1-k\gamma_m)^{-1}$*, one has*

$$a_r + k\sum_{m=0}^{r} b_m \leq \exp\left(k\sum_{m=0}^{r}\sigma_m\gamma_m\right)\left\{k\sum_{m=0}^{r} c_m + B\right\} \qquad \forall r \geq 0.$$

1 Preliminaries

1.1 Space of functions

To define the notion of weak solution of problem (R), we introduce the following Hilbert spaces:

$$\begin{aligned} H^1_{b,l}(\Omega) &= \{\mathbf{v} \in H^1(\Omega) \ / \ \mathbf{v}|_{\Gamma_b \cup \Gamma_l} = 0\}, \\ \mathbf{H} &= \{\mathbf{v} \in L^2(\Omega)^2 \ / \ \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle = 0 \text{ in } S, \ \langle \mathbf{v} \rangle \cdot \mathbf{n}_{\partial S} = 0\}, \\ \mathbf{V} &= \{\mathbf{v} \in H^1_{b,l}(\Omega)^2 \ / \ \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle = 0 \text{ in } S\}, \end{aligned}$$

being $\mathbf{n}_{\partial S}$ the normal exterior vector of ∂S. Observe that spaces $\mathbf{H}$ and $\mathbf{V}$ are the hydrostatic version of the classical spaces for the Navier-Stokes equations.

We denote $\mathbf{H}^1_{b,l}(\Omega) = H^1_{b,l}(\Omega)^2$, etc. The norm and scalar product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and $(\cdot,\cdot)$, whereas in $\mathbf{H}^1_{b,l}(\Omega)$ by $\|\cdot\|$ we denote the norm of the gradient in L^2, that is $\|u\| = |\nabla u|$. On the other hand, we denote $\mathbf{H}^{-1}_{b,l}(\Omega)$ and $H^{-1/2}(\Gamma_s)$ the dual spaces of $\mathbf{H}^1_{b,l}(\Omega)$ and $H^{1/2}(\Gamma_s)$ respectively, with duality products $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\Gamma_s}$.

The space for the surface pressure will be:

$$L^2_0(S) = \left\{q \in L^2(S) \ / \ \int_S q = 0\right\}.$$

The vertical velocity u_3 is obtained in function of $\nabla_{\mathbf{x}} \cdot \mathbf{u}$ by means of either the integral formulation (1) or the differential formulation (3). In this process, one has not L^2 regularity for the horizontal derivatives of u_3, so let us to define the (anisotropic) Hilbert space

$$H(\partial_z) = \{v \in L^2(\Omega) \ / \ \partial_z v \in L^2(\Omega)\}, \quad (\text{resp. } H^k(\partial_z) = \{v \in H^k(\Omega) \ / \ \partial_z v \in H^k(\Omega)\})$$

and $H_0(\partial_z) = \{v \in H(\partial_z) \ / \ v = 0 \text{ on } \Gamma_s \cup \Gamma_b\}$. The inner product in $H(\partial_z)$ is defined by $(v,w)_{H(\partial_z)} = \Big(v,w\Big) + \Big(\partial_z v, \partial_z w\Big)$ and in $H_0(\partial_z)$ is defined by $(v,w)_{H_0(\partial_z)} = \Big(\partial_z v, \partial_z w\Big)$, owing to a vertical Poincare inequality (see (4) below).

Notice that, given $\mathbf{u} \in \mathbf{H}^1_{b,l}(\Omega)$ then the weak solution u_3 of problem (3) is defined by:

$$u_3 \in H_0(\partial_z) \quad \text{such that} \quad (u_3, w)_{H_0(\partial_z)} = \Big(-\nabla_{\mathbf{x}} \cdot \mathbf{u}, \partial_z w\Big) \quad \forall\, w \in H_0(\partial_z).$$

That is, u_3 is the $H_0(\partial_z)$-projection of $-\int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{u}$.

Due to the loss of regularity of u_3 ($u_3 \in L^2$ but $u_3 \notin H^1$), the vertical convection term $u_3 \partial_z \mathbf{u}$ does not belong to $\mathbf{H}^{-1}_{b,l}(\Omega)$, therefore more regular test functions must be introduced in the variational formulation of (R). For instance, it suffices with $\mathbf{v} \in \mathbf{H}^1_{b,l}(\Omega)$ such that $\partial_z \mathbf{v} \in \mathbf{L}^3(\Omega)$, because in this case one has (see [9]):

$$\Big\langle (\mathbf{U}\cdot\nabla)\mathbf{u}, \mathbf{v}\Big\rangle_\Omega = -\int_\Omega (\mathbf{U}\cdot\nabla)\mathbf{v}\cdot\mathbf{u} < +\infty.$$

Another possibility is to assume $\mathbf{v} \in \mathbf{H}^1_{b,l}(\Omega) \cap \mathbf{L}^\infty(Q)$, and then $\displaystyle\int_\Omega (\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} < +\infty$.

As usual for fully discrete schemes, we will use the following skew-symmetric part of the trilinear form for the treatment of the convective terms: for each $\mathbf{U} \in \mathbf{H}^1_{b,l} \times H_0(\partial_z)$, $\mathbf{v} \in \mathbf{H}^1$, $\mathbf{w} \in \mathbf{H}^1$ with either $\mathbf{w} \in \mathbf{L}^\infty$ or $\partial_z \mathbf{w} \in \mathbf{L}^3$,

$$\begin{aligned} c\Big(\mathbf{U},\mathbf{v},\mathbf{w}\Big) &= \int_\Omega \Big\{(\mathbf{U}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}(\nabla\cdot\mathbf{U})\mathbf{v}\cdot\mathbf{w}\Big\} \qquad \text{if } \mathbf{w} \in \mathbf{L}^\infty \\ &= -\int_\Omega \Big\{(\mathbf{U}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{U})\mathbf{v}\cdot\mathbf{w}\Big\} \qquad \text{if } \partial_z\mathbf{w} \in \mathbf{L}^3. \end{aligned}$$

Obviously, $c(\mathbf{U},\mathbf{v},\mathbf{w}) = \displaystyle\int_\Omega (\mathbf{U}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}$ whether $\nabla\cdot\mathbf{U} = 0$. By simplicity, the vertical part of these trilinear forms will be denoted in the same manner, i.e.

$$c\Big(u_3,\mathbf{v},\mathbf{w}\Big) = \int_\Omega \Big\{u_3\,\partial_z\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}\partial_z u_3\,\mathbf{v}\cdot\mathbf{w}\Big\} = -\int_\Omega \Big\{u_3\,\partial_z\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}\partial_z u_3\,\mathbf{v}\cdot\mathbf{w}\Big\}$$

Previous equalities hold even for discrete spaces. Therefore, we can use in the sequel any of these two possibilities.

With previous definitions, we consider the following weak (mixed) variational formulation of (R):

To find $\mathbf{U} = (\mathbf{u}, u_3) \in L^2(0,T;\mathbf{V}\times H_0(\partial_z))$ with $\mathbf{u} \in L^\infty(0,T;\mathbf{H})$ and $p_s \in \mathcal{D}'(0,T;L^2_0(S))$ verifying:

$$(R)_w \qquad \begin{cases} \Big(\mathbf{u}_t,\mathbf{v}\Big) + c\Big(\mathbf{U},\mathbf{u},\mathbf{v}\Big) + \nu\Big(\nabla\mathbf{u},\nabla\mathbf{v}\Big) - \Big(p_s, \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\rangle\Big)_S \\ \qquad = \Big\langle \mathbf{f},\mathbf{v}\Big\rangle_\Omega + \Big\langle \mathbf{g}_s,\mathbf{v}\Big\rangle_{\Gamma_s}, \quad \forall\,\mathbf{v} \in \mathbf{V}\cap\mathbf{W}^{1,3}_{b,l}\cap\mathbf{L}^\infty, \\ \Big(\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle, q_s\Big)_S = 0, \quad \forall\, q_s \in L^2_0(S), \\ \Big(\partial_z\, u_3, \partial_z w\Big) = -\Big(\nabla_{\mathbf{x}}\cdot\mathbf{u}, \partial_z w\Big), \quad \forall\, w \in H_0(\partial_z). \end{cases}$$

Notice that in the above formulation, by simplicity, we do not consider the Coriolis term because this term does not add new difficulties. At the end, in Section 6, we analyze the more convenient form to introduce it in the scheme.

1.2 Some 3D anisotropic spaces and related estimates.

Given $p, q \in [1, +\infty]$, it will be said that a function u belongs to $L^q_z L^p_{\mathbf{x}}(\Omega)$ if:

$$u(\cdot, z) \in L^p(S_z) \quad \text{and} \quad \|u(\cdot, z)\|_{L^p(S_z)} \in L^q(-D_{\max}, 0),$$

where $S_z = \{\mathbf{x} \in S \ : \ (\mathbf{x}, z) \in \Omega\}$, and its norm is given by $\big\| \|u(\cdot, z)\|_{L^p(S_z)} \big\|_{L^q(-D_{\max},0)}$. The most useful norms that we will use in this paper are:

$$\|u\|_{L^2_z L^4_{\vec{x}}(\Omega)} = \left(\int_{-D_{\max}}^0 \|u(\cdot, z)\|^2_{L^4(S_z)} dz \right)^{1/2}$$

$$\|u\|_{L^\infty_z L^2_{\vec{x}}(\Omega)} = \sup_{z \in (-D_{\max}, 0)} \|u(\cdot, z)\|_{L^2(S_z)},$$

For sake of simplicity, we sometimes denote $L^q_z L^p_{\mathbf{x}}$ instead of $L^q_z L^p_{\mathbf{x}}(\Omega)$, and L^p instead of $L^p(\Omega)$, when there is no risk of confusion.

In a similar way, we define the spaces

$$H^1_z L^2_{\mathbf{x}} \equiv H^1(-D_{\max}, 0; L^2(S_z)), \quad L^2_z H^1_{\mathbf{x}} \equiv L^2(-D_{\max}, 0; H^1(S_z)).$$

Notice that $H^1_z L^2_{\mathbf{x}} = H(\partial_z)$.

Also, we will use frequently the following inequalities (see [14]):

- Horizontal Gagliardo-Nirenberg inequality (related to $2D$ subdomains):

$$\|u\|_{L^2_z L^4_{\mathbf{x}}} \le C\, |u|^{1/2}\, |\nabla_{\mathbf{x}} u|^{1/2} \quad \forall\, u \in L^2_z H^1_{\mathbf{x}} \ \text{ such that } u|_{\Gamma_b \cup \Gamma_l} = 0,$$
$$\|u\|_{L^2_z L^4_{\mathbf{x}}} \le C\, |u|^{1/2}\, \|u\|^{1/2} \quad \forall\, u \in H^1$$

- Vertical Poincaré Inequality (related to $1D$ subdomains):

$$|v| \le D_{\max}^{1/2}\, |\partial_z v|, \quad \forall\, v \in H^1_z L^2_{\mathbf{x}} \ \text{ such that } v|_{\Gamma_b} = 0 \ \text{ or } v|_{\Gamma_s} = 0. \tag{4}$$

- Vertical Gagliardo-Nirenberg inequality (related to $1D$ subdomains):

$$\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \le C\, (|v| + |v|^{1/2}\, |\partial_z v|^{1/2}), \quad \forall\, v \in H^1_z L^2_{\mathbf{x}}. \tag{5}$$

  Moreover, if $v|_{\Gamma_b} = 0$ or $v|_{\Gamma_s} = 0$, $\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \le C\, |v|^{1/2}\, |\partial_z v|^{1/2}$.

  In particular, from (4) and (5), one has

$$\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \le C\, |\partial_z v|, \quad \forall\, v \in H^1_z L^2_{\mathbf{x}} \ \text{ such that } v|_{\Gamma_b} = 0 \ \text{ or } v|_{\Gamma_s} = 0. \tag{6}$$

1.3 Finite element approximation.

We will consider a finite element approximation of problem $(R)_w$. We restrict ourselves to the case where the surface domain $S \subset \mathbb{R}^2$ has a polygonal boundary and the bottom function D is globally continuous and locally P_1, hence Ω is a particular $3D$ polyhedron. Moreover, the following hypothesis will be imposed about Ω:

(**H0**) Regularity of the Domain: Assume $\Omega \subset \mathbb{R}^3$ such that the Hydrostatic Stokes Problem has $\mathbf{H}^2(\Omega) \times H^1(S)$ regularity for (horizontal) velocity and pressure respectively. For this, the following hypothesis must be imposed (see [31]):

$$D \geq D_{\min} > 0 \quad \text{in } S.$$

Now, we discretize this domain. Let $\mathcal{T}_h(\Omega)$ a regular and quasi-uniform triangulation of Ω (with elements $K \in \mathcal{T}_h(\Omega)$) and consider $\mathcal{T}_h(S)$ its associated triangulation of S with elements $T \in \mathcal{T}_h(S)$. For simplicity, the same mesh size h is taken for both triangulations of S and Ω (i.e. isotropic meshes are considered). Moreover, for sake of generality, we do not impose any special structure of these triangulations. Normally, as we will see in Section 5, more specific properties can be obtained for vertical structured grids (see [9], [20] for more details about how to construct these vertical structured grids).

We consider three families of finite element spaces: $\mathbf{X}_h \subset \mathbf{H}^1_{b,l}(\Omega)$ for the horizontal velocity, $Y_h \subset H_0(\partial_z)$ for the vertical velocity and $Q_h \subset L^2_0(S)$ for the pressure. Functions of $\mathbf{X}_h$ are globally continuous, whereas functions in Y_h must be globally continuous only respect to vertical direction and Q_h could contain discontinuous functions.

The following properties are required about these finite element spaces:

(**H1**) The approximating spaces $\mathbf{X}_h$ and Q_h are required to satisfy the so called hydrostatic "$inf - sup$" condition ([9]): There exists $\beta > 0$ (independent of h) such that, for all $h > 0$,

$$\inf_{q_h \in Q_h \setminus \{0\}} \left(\sup_{\mathbf{v}_h \in \mathbf{X}_h \setminus \{0\}} \frac{\Big(q_h, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h \rangle\Big)_S}{(|\nabla_{\mathbf{x}} \mathbf{v}_h| + \|\partial_z \mathbf{v}_h\|_{L^3}) \, \|q_h\|_{L^2(S)}} \right) \geq \beta.$$

(**H2**) The following inverse inequalities hold: for each $\mathbf{u}_h \in \mathbf{X}_h$,

$$\|\mathbf{u}_h\|_{L^2_z L^4_{\mathbf{x}}} \leq C\, h^{-1/2} |\mathbf{u}_h|, \quad \|\mathbf{u}_h\|_{L^\infty_z L^4_{\mathbf{x}}} \leq C\, h^{-1/2} \|\mathbf{u}_h\|_{L^\infty_z L^2_{\mathbf{x}}}, \quad \|\mathbf{u}_h\|_{L^2_z L^\infty_{\mathbf{x}}} \leq C\, h^{-1} |\mathbf{u}_h|$$

$$\|\mathbf{u}_h\|_{L^3} \leq C\, h^{-1/2} |\mathbf{u}_h|, \quad \|\mathbf{u}_h\| \leq C\, h^{-1} |\mathbf{u}_h|, \quad \|\mathbf{u}_h\|_{W^{1,6}} \leq C\, h^{-1} \|\mathbf{u}_h\|$$

(**H3**) The approximation properties $O(h^l)$ (for $l = 1$ or 2):

$$h^{-1} |\mathbf{v} - I_h \mathbf{v}| + \|\mathbf{v} - I_h \mathbf{v}\| \leq C\, h^l \, \|\mathbf{v}\|_{\mathbf{H}^{l+1}} \qquad \forall\, \mathbf{v} \in \mathbf{H}^{l+1}(\Omega) \cap \mathbf{V},$$

$$|\mathbf{v} - I_h\mathbf{v}| \le C\, h^l\, \|\mathbf{v}\|_{\mathbf{H}^l} \quad \forall\, \mathbf{v} \in \mathbf{H}^l \cap \mathbf{V},$$
$$\|q - J_h q\|_{L^2(S)} \le C\, h^l\, \|q\|_{H^l(S)} \qquad \forall\, q \in H^l(S) \cap L^2_0(S),$$
$$\|v_3 - K_h v_3\|_{H(\partial_z)} \le C\, h^l\, \|v_3\|_{H^{l+1}} \quad \forall\, v_3 \in H^{l+1}(\Omega) \cap H_0(\partial_z), \tag{7}$$

where $(I_h, J_h) : \mathbf{V} \times L^2_0(S) \to \mathbf{X}_h \times Q_h$ are the global interpolation operators defined as:

$$(I_h\mathbf{v}, J_h q) \in \mathbf{X}_h \times Q_h : \begin{cases} \Big(\nabla(I_h\mathbf{v} - \mathbf{v}), \nabla\, \mathbf{v}_h\Big) - \Big(J_h q - q, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h \rangle\Big)_S = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h, \\ \Big(\nabla_{\mathbf{x}} \cdot \langle I_h \mathbf{v} \rangle, q_h\Big)_S = 0 \quad \forall\, q_h \in Q_h, \end{cases}$$

and $K_h : H_0(\partial_z) \to Y_h$ is the operator defined as:

$$K_h v_3 \in Y_h \ : \ \Big(\partial_z(K_h v_3 - v_3), \partial_z y_h\Big) = 0 \quad \forall\, y_h \in Y_h.$$

There are some possibilities to define the discrete spaces $(\mathbf{X}_h, Y_h, Q_h)$ verifying **(H1)-(H3)**. For instance, to approximate the pressure, we can consider

$$Q_h = \{q_h \in C^0(\overline{S}) : q_h|_T \in \boldsymbol{P}_1(T), \forall\, T \in \mathcal{T}_h(S)\} \cap L^2_0(S).$$

To choice $(\mathbf{X}_h, Y_h)$ there are at least two possibilities ([9], [20]):

1. (*Taylor-Hood*) $(l = 2)$

$$\begin{aligned} \mathbf{X}_h &= \{v_h \in C^0(\overline{\Omega}) : \mathbf{v}_h|_K \in \boldsymbol{P}_2(K), \forall\, K \in \mathcal{T}_h(\Omega)\}^2 \cap \mathbf{H}^1_{b,l}(\Omega), \\ Y_h &= \{y_h \in C^0(\overline{\Omega}) : y_h|_K \in \boldsymbol{P}_2(K), \forall\, K \in \mathcal{T}_h(\Omega)\} \cap H_0(\partial_z), \end{aligned}$$

2. (*Mini-element by tetrahedron*) $(l = 1)$. We define $\mathcal{P}(K) = \boldsymbol{P}_1(K) \oplus \alpha_K \lambda_1\lambda_2\lambda_3\lambda_4$ with $\alpha_K \in \mathbb{R}$ and $\lambda_i \in \boldsymbol{P}_1(K)$ such that $\lambda_i(a_j) = \delta_{ij}$, being a_j the vertices of the tetrahedron K. Then, we consider

$$\begin{aligned} \mathbf{X}_h &= \{u_h \in C^0(\overline{\Omega}) : \mathbf{u}_h|_K \in \mathcal{P}(K), \forall K \in \mathcal{T}_h\}^2 \cap \mathbf{H}^1_{b,l}(\Omega), \\ Y_h &= \{y_h \in C^0(\overline{\Omega}) : y_h|_K \in \boldsymbol{P}_1(K), \forall\, K \in \mathcal{T}_h(\Omega)\} \cap H_0(\partial_z). \end{aligned}$$

For vertical structured meshes formed by prims (see Section 5), there are other possibilities for $\mathbf{X}_h$, considering a bubble by prism or a bubble by each column of vertical prisms (see [20]). It is important to recall that these latter possibilities are not stables for the Navier-Stokes case. Also in the case of right prims, other possibilities for Y_h are possible, by means of tensorial products in horizontal and vertical, as for instance [18]:

- $\boldsymbol{P}_0(\mathbf{x}) \otimes \boldsymbol{P}_1(z)$ and C^0_z (for $l = 1$),
- $\boldsymbol{P}_1(\mathbf{x}) \otimes \boldsymbol{P}_2(z)$ continuous and C^1_z (for $l = 2$).

2 Description of the scheme

The time interval $[0,T]$ is divided into M subintervals, for instance of equal length $k = T/M$, considering the partition of $[0,T]$, $\{t_m = m\,k\}_{m=0}^{M}$.

In general, a discrete scheme is an iterative scheme in time, where in each step m, given $\{(\mathbf{f}^m, \mathbf{g}_s^m)\}_{m=1}^{M}$ some approximations of data $(\mathbf{f}, \mathbf{g}_s)$ in $t = t_m$, a sequence $\{(\mathbf{u}_h^m, u_{3,h}^m, p_{s,h}^m)\}_m$ will be computed, which pretends to be an approximation to a solution $(\mathbf{u}, u_3, p_s)$ of (R) at the instant $t = t_m$.

We are going to present the time fractional-step scheme, splitting the three main difficulties of the problem (R):

- the computation of the vertical velocity,
- the non linear convective terms, $(\mathbf{U}\cdot\nabla)\mathbf{u}$ (in particular, the vertical convection $u_3\partial_z\mathbf{u}$ is less regular than in the Navier-Stokes case),
- the restriction $\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle = 0$ in $S\times(0,T)$.

Indeed, given $(\mathbf{u}_h^m, p_{s,h}^m)$, firstly the vertical velocity $u_{3,h}^m$ is computed in function of $\nabla_{\mathbf{x}}\cdot\mathbf{u}_h^m$, afterwards we obtain an intermediate horizontal velocity $\mathbf{u}_h^{m+1/2}$ using convective terms but not the restriction of divergence type, and finally we obtain $\mathbf{u}_h^{m+1}$ and $p_{s,h}^{m+1}$ by means of a linear Stokes type problem considering the restriction $\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}_h^{m+1}\rangle = 0$ (the diffusion terms will appear in both cases, so the method is called with decomposition of the viscosity).

The fully discrete scheme remains as follows:

Initialization: Let $\mathbf{u}_h^0 \in \mathbf{X}_h$ be an approximation of $\mathbf{u}_0$.

Step of time $m+1$:

Sub-step 0: Given $\mathbf{u}_h^m \in \mathbf{X}_h$, to compute $u_{3,h}^m \in Y_h$ such that, for all $v_{3,h}\in Y_h$

$$(S_0)_h^m \qquad \Big(\partial_z u_{3,h}^m, \partial_z v_{3,h}\Big) = -\Big(\nabla_{\mathbf{x}}\cdot\mathbf{u}_h^m, \partial_z v_{3,h}\Big),$$

Sub-step 1: Given $\mathbf{U}_h^m = (\mathbf{u}_h^m, u_{3,h}^m) \in \mathbf{X}_h\times Y_h$, to compute $\mathbf{u}_h^{m+1/2}\in\mathbf{X}_h$ such that, for all $\mathbf{v}_h\in\mathbf{X}_h$

$$(S_1)_h^{m+1} \qquad \left\{\begin{array}{l} \dfrac{1}{k}\Big(\mathbf{u}_h^{m+1/2}-\mathbf{u}_h^m, \mathbf{v}_h\Big) + c\Big(\mathbf{U}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h\Big) + \Big(\nabla\,\mathbf{u}_h^{m+1/2}, \nabla\,\mathbf{v}_h\Big) \\ = \Big\langle \mathbf{f}^{m+1}, \mathbf{v}_h\Big\rangle_\Omega + \Big\langle \mathbf{g}_s^{m+1}, \mathbf{v}_h\Big\rangle_{\Gamma_s} \end{array}\right.$$

Sub-step 2: Given $\mathbf{u}_h^{m+1/2}\in\mathbf{X}_h$, to compute $(\mathbf{u}_h^{m+1}, p_h^{m+1})\in\mathbf{X}_h\times Q_h$, such that for all $(\mathbf{v}_h, q_h)\in\mathbf{X}_h\times Q_h$,

$(S_2)_h^{m+1}$

$$\left\{\begin{array}{l} \dfrac{1}{k}\Big(\mathbf{u}_h^{m+1}-\mathbf{u}_h^{m+1/2}, \mathbf{v}_h\Big) + \Big(\nabla(\mathbf{u}_h^{m+1}-\mathbf{u}_h^{m+1/2}), \nabla\mathbf{v}_h\Big) - \Big(p_h^{m+1}, \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}_h\rangle\Big)_S = 0, \\ \Big(\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}_h^{m+1}\rangle, q_h\Big)_S = 0. \end{array}\right.$$

In our case (unstructured grids), $\langle \mathbf{u} \rangle$ is not easy to be computed. Then in order to do the effective implementation of the scheme, is better to write the integral in the $2D$ surface S of $(S_2)_h^{m+1}$ as integrals in the $3D$ domain Ω. This is,

$$-\Big(p_s, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle\Big)_S = -\Big(p_s, \nabla_{\mathbf{x}} \cdot \mathbf{v}\Big) \quad \text{(or} \quad = \Big(\nabla_{\mathbf{x}} p_s, \mathbf{v}\Big) \quad \text{when pressures are continuous)}$$

$$\Big(\nabla_{\mathbf{x}} \cdot \langle \mathbf{u} \rangle, q_s\Big)_S = \Big(\nabla_{\mathbf{x}} \cdot \mathbf{u}, q_s\Big) \quad \text{or} \quad = -\Big(\mathbf{u}, \nabla_{\mathbf{x}} q_s\Big).$$

However, to do the numerical analysis, we can follow with the integrals in S.

In the sub-step 0, a linear z-elliptic problem must be computed. In the sub-step 1, a decoupled linear convection-diffusion scheme must be computed, whereas the sub-step 2 can be seen as a (generalized) Hydrostatic Stokes problem, which is well-defined imposing **(H1)**.

Notice that, adding $(S_1)_h^{m+1}$ and $(S_2)_h^{m+1}$, we get for each $\mathbf{v}_h \in \mathbf{X}_h$:

$$(S)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\mathbf{u}_h^{m+1} - \mathbf{u}_h^m, \mathbf{v}_h\Big) + c\Big(\mathbf{U}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h\Big) + \Big(\nabla \mathbf{u}_h^{m+1}, \nabla \mathbf{v}_h\Big) \\ -\Big(p_h^{m+1}, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h \rangle\Big)_S = \Big\langle \mathbf{f}^{m+1}, \mathbf{v}_h \Big\rangle_\Omega + \Big\langle \mathbf{g}_s^m, \mathbf{v}_h \Big\rangle_{\Gamma_s} \end{array} \right.$$

This formulation will be used to prove the convergence of the scheme.

3 Unconditional stability and convergence.

In this Section, we are going to study stability properties of these schemes and convergence towards a weak solution of the continuous problem (R). For this, we will obtain some a priori estimates (stability) that let us make a pass to the limit (convergence), where compactness results must be applied to "control" the limit in the convective terms.

Fixed the (uniform) partition of $[0, T]$ of diameter $k = T/M$: $\{t_m = mk\}_{m=0}^M$, for a given vector $u = (u^m)_{m=0}^M$ with $u^m \in X$ (X being a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^{M} \|u^m\|_X^2\right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\dots,M} \|u^m\|_X$$

In this section, we only consider the weak regularity on the data

$$(\mathbf{WR}) \qquad \mathbf{f} \in L^2(0, T; \mathbf{H}_{b,l}^{-1}(\Omega)), \quad \mathbf{g}_s \in L^2(0, T; \mathbf{H}^{-1/2}(\Gamma_s)) \quad \text{and} \quad \mathbf{u}_0 \in \mathbf{H},$$

and we choose

$$\mathbf{f}^{m+1} = \frac{1}{k}\int_{t_m}^{t_{m+1}} \mathbf{f}(t)\, dt, \quad \mathbf{g}_s^{m+1} = \frac{1}{k}\int_{t_m}^{t_{m+1}} \mathbf{g}_s(t)\, dt.$$

Lemma 2 *(Stability) Assume* (**WR**) *and* (**H1**). *If* $(\mathbf{u}_h^0)$ *bounded in* $\mathbf{L}^2$, *then the following estimates hold:*

$$\|\mathbf{u}_h^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|\mathbf{u}_h^{m+1/2}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C, \tag{8}$$

$$\|\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}\|_{l^2(L^2)} + \|\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^{m}\|_{l^2(L^2)} \le C\, k^{1/2}, \tag{9}$$

$$\|u_{3,h}^{m+1}\|_{l^2(H(\partial_z)\cap L^\infty_z L^2_{\mathbf{x}})} \le C. \tag{10}$$

Proof. Estimates (8) and (9) are obtained making

$$\sum_{m=0}^{r} \Big((S_1)_h^{m+1}, \mathbf{u}_h^{m+1/2}\Big) + \Big((S_2)_h^{m+1}, \mathbf{u}_h^{m+1}\Big), \quad \forall\, r = 1, \cdots, M-1,$$

and using that

$$c\Big(\mathbf{U}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{u}_h^{m+1/2}\Big) = 0.$$

On the other hand, taking $\partial_z u_{3,h}^{m+1}$ as test functions in Sub-step 0, one has $|\partial_z u_{3,h}^{m+1}| \le |\nabla_{\mathbf{x}}\cdot \mathbf{u}_h^{m+1}|$. Therefore, (10) is a consequence of previous estimate (8) and inequality (4). ∎

Now, we define the following sequences of functions (defined for all $t \in [0,T]$)):

- ⋄ $\mathbf{u}_{k,h}^{(i)} : [0,T] \to \mathbf{H}^1_{b,l}(\Omega)$, such that $\mathbf{u}_{k,h}^{(i)}(t) = \mathbf{u}_h^{m+i/2}$ if $t \in (t_m, t_{m+1}]$, $i = 0,1,2$.
- ⋄ $u_{3,k,h}^{(0)} : [0,T] \to L^2(\Omega)$, such that $u_{3,k,h}^{(0)}(t) = u_{3,h}^m$ if $t \in (t_m, t_{m+1}]$.
- ⋄ $\mathbf{u}_{k,h} : [0,T] \to \mathbf{V}$, continuous, linear by subintervals and $\mathbf{u}_{k,h}(t_m) = \mathbf{u}_h^m$.

Theorem 3 (Convergence) *Assume* (**WR**) *and* (**H0**)-(**H2**), *then there exists a subsequence* (k', h') *of* (k,h), *with* $(k',h') \downarrow 0$, *and a weak solution* $\mathbf{U} = (\mathbf{u}, u_3)$ *of* (R) *in* $(0,T)$, *such that:* $(\mathbf{u}_{k',h'}^{(i)})$ *(for each* $i = 0,1,2$*) and* $(\mathbf{u}_{k',h'})$ *converge to* $\mathbf{u}$ *strongly in* $L^2(0,T;\mathbf{L}^2(\Omega))$, *weakly-star in* $L^\infty(0,T;\mathbf{L}^2(\Omega))$ *and weakly in* $L^2(0,T;\mathbf{H}^1_{b,l}(\Omega))$, *whereas* $(u_{3,k',h'}^{(0)})$ *converges to* u_3 *weakly in* $L^2(0,T;H_0(\partial_z))$.

Proof. Owing to definition of the functions $\mathbf{u}_{k,h}^{(i)}, u_{3,k,h}^{(0)}$ and $\mathbf{u}_{k,h}$, Lemma 2 says:

$$(\mathbf{u}_{k,h}^{(i)})_{k,h} \text{ and } (\mathbf{u}_{k,h})_{k,h} \ \text{ are bounded in } \ L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^1_{b,l}), \quad \forall\, i = 0,1,2,$$

$$(u_{3,k,h}^{(0)})_{k,h} \ \text{ is bounded in } \ L^2(H_0(\partial_z)). \tag{11}$$

On the other hand, from (9), there exists $C = C(\nu, \mathbf{u}_0, \mathbf{f}, \mathbf{g}_s) > 0$ such that, $\forall\, i,j = 0,1,2$,

$$\|\mathbf{u}_{k,h}^{(i)} - \mathbf{u}_{k,h}^{(j)}\|^2_{L^2(\mathbf{L}^2)} \le C\, k, \quad \|\mathbf{u}_{k,h}^{(i)} - \mathbf{u}_{k,h}\|^2_{L^2(\mathbf{L}^2)} \le C\, k. \tag{12}$$

Therefore, there exist subsequences of $(\mathbf{u}^{(i)}_{k,h})_{k,h}$ and $(\mathbf{u}_{k,h})_{k,h}$ (denoted in the same way) and a limit function $\mathbf{u}$ (thanks to (12), the uniqueness of the limits $\mathbf{u}^{(i)} = \mathbf{u}$, for all $i = 0, 1, 2$ hold), verifying the following weak convergences as $(h, k) \to 0$

$$(\mathbf{u}^{(i)}_{k,h})_{k,h} \to \mathbf{u}, \quad (\mathbf{u}_{k,h})_{k,h} \to \mathbf{u} \quad \text{in} \begin{cases} L^2(0,T;\mathbf{H}^1_{b,l}(\Omega)) - \text{weak} \\ L^\infty(0,T;\mathbf{L}^2(\Omega)) - \text{weak}* \end{cases}$$

Moreover, from (11)

$$u^{(0)}_{3,k,h} \to u_3 \quad \text{in } L^2(H_0(\partial_z)).$$

Finally, $(S)^{m+1}_h$ can be rewritten (eliminating the pressure) as follows:

$$\Big(\partial_t \mathbf{u}_{k,h}, \mathbf{v}_h\Big) + c\Big(\mathbf{U}^{(0)}_{k,h}, \mathbf{u}^{(1)}_{k,h}, \mathbf{v}_h\Big) + \Big(\nabla \mathbf{u}^{(2)}_{k,h}, \nabla \mathbf{v}_h\Big) = \Big(\mathbf{f}^{m+1}, \mathbf{v}_h\Big) + \Big(\mathbf{g}^{m+1}_s, \mathbf{v}_h\Big)_{\Gamma_s} \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h \cap \mathbf{V}, \tag{13}$$

On the other hand, $(S_0)^m_h$ is rewritten as

$$\Big(\partial_z u^{(0)}_{3,k,h}, \partial_z w_h\Big) = -\Big(\nabla_{\mathbf{x}} \cdot \mathbf{u}^{(0)}_{k,h}, \partial_z w_h\Big) \quad \forall\, w_h \in Y_h \tag{14}$$

To take limits in (13), we need for instance compactness of $(\mathbf{u}^{(1)}_{k,h})_{k,h}$ in $L^2(\mathbf{L}^2(\Omega))$. But, owing to (12), it suffices to obtain compactness of $(\mathbf{u}^{(2)}_{k,h})_{k,h}$ in $L^2(\mathbf{L}^2(\Omega))$. Assuming this compactness, writing (13)-(14) in (k', h'), the pass to the limit when $(k', h') \to 0$ can be realized by a standard way, concluding that $(\mathbf{u}, u_3)$ is a weak solution of the continuous problem (R).

Therefore, it suffices to demonstrate the compactness of $(\mathbf{u}^{(2)}_{k,h})_{k,h}$. For this, let us denote

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h \;/\; \Big(\nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h \rangle, q_h\Big)_S = 0, \forall q_h \in Q_h\}$$

and $A_h^{-1} : \mathbf{V}_h \to \mathbf{V}_h$ is the inverse of the discrete "hydrostatic" Stokes operator, that is, given $\mathbf{u}_h \in \mathbf{V}_h$, $A_h^{-1}\mathbf{u}_h$ is the weak solution of the problem:

$$A_h^{-1}\mathbf{u}_h \in \mathbf{V}_h \text{ such that } \Big(\nabla A_h^{-1}\mathbf{u}_h, \nabla \mathbf{v}_h\Big) = \Big(\mathbf{u}_h, \mathbf{v}_h\Big) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h \tag{15}$$

In particular, taking $\mathbf{v}_h = \mathbf{u}_h \in \mathbf{V}_h$ in (15), one has

$$|\mathbf{u}_h|^2 = \Big(\nabla A_h^{-1}\mathbf{u}_h, \nabla \mathbf{u}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h$$

and taking $\mathbf{v}_h = A_h^{-1}\mathbf{u}_h \in \mathbf{V}_h$ in (15),

$$|\nabla A_h^{-1}\mathbf{u}_h|^2 = \Big(\mathbf{u}_h, A_h^{-1}\mathbf{u}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h. \tag{16}$$

Moreover, $|\nabla A_h^{-1}\mathbf{u}_h|$ and $\|\mathbf{u}_h\|_{V_h'}$ are equivalent norms. Indeed,

$$|\nabla A_h^{-1}\mathbf{u}_h|^2 = \Big(\mathbf{u}_h, A_h^{-1}\mathbf{u}_h\Big) \le C\, \|\mathbf{u}_h\|_{V_h'}\, |\nabla A_h^{-1}\mathbf{u}_h|,$$

hence

$$|\nabla A_h^{-1}\mathbf{u}_h| \le C\, \|\mathbf{u}_h\|_{V_h'}.$$

To obtain the inverse bound, we take $\mathbf{v}_h \in \mathbf{V}_h$ in (15), then

$$\Big(\mathbf{u}_h, \mathbf{v}_h\Big) = \Big(\nabla A_h^{-1}\mathbf{u}_h, \nabla \mathbf{v}_h\Big) \le |\nabla A_h^{-1}\mathbf{u}_h|\,|\nabla \mathbf{v}_h| \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

hence

$$\|\mathbf{u}_h\|_{\mathbf{V}_h'} \le |\nabla\, A_h^{-1}\, \mathbf{u}_h|.$$

Now, to obtain compactness of $\mathbf{u}_{k,h}^{(2)}$ we follow an argument of [13]. First of all, we prove the following result

Lemma 4 *Under hypothesis of Theorem 3, one has*

$$\int_0^{T-\delta} \|\mathbf{u}_{k,h}^{(2)}(t+\delta) - \mathbf{u}_{k,h}^{(2)}(t)\|^2_{V_h'}\, dt \le C\,\delta, \quad \forall\, \delta : 0 < \delta < T, \tag{17}$$

where $C > 0$ depends only on the data.

Proof. Since $\mathbf{u}_{k,h}^{(2)}$ is a piecewise constant function, it suffices to suppose that δ is proportional to time step k, i.e., $\delta = r\,k$ for any $r = 1, ..., M$. Then, to obtain (17), it suffices to prove

$$k \sum_{n=0}^{M-r} \|\mathbf{u}_h^{n+r} - \mathbf{u}_h^n\|^2_{V_h'} \le C(r\,k), \quad \forall\, r = 1, \dots, M. \tag{18}$$

Multiplying the equation of $(S)_h^{m+1}$ by $k\mathbf{v}_h$ for any $\mathbf{v}_h \in \mathbf{V}_h$ and adding from $m = n$ to $n-1+r$, we have

$$\begin{aligned} \Big(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n, \mathbf{v}_h\Big) &= -k \sum_{m=n}^{n-1+r} c\Big(\mathbf{U}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h\Big) - k \sum_{m=n}^{n-1+r} \Big(\nabla \mathbf{u}_h^{m+1}, \nabla \mathbf{v}_h\Big) \\ &+ \; k \sum_{m=n}^{n-1+r} \Big\{ \Big(p_h^{m+1}, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h\rangle\Big)_S + \Big\langle \mathbf{f}^{m+1}, \mathbf{v}_h\Big\rangle_\Omega + \Big\langle \mathbf{g}_s^m, \mathbf{v}_h\Big\rangle_{\Gamma_s} \Big\} \end{aligned} \tag{19}$$

Now, taking as test function $\mathbf{v}_h = kA_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)$ in (19), using (16) and adding from $n = 0$ to $M - r$, we get

$$\begin{aligned} k \textstyle\sum_{n=0}^{M-r} \|\mathbf{u}_h^{n+r} - \mathbf{u}_h^n\|^2_{V_h'} &= -k^2 \textstyle\sum_{n=0}^{M-r} \sum_{m=n}^{n-1+r} c\Big(\mathbf{U}_h^m, \mathbf{u}_h^{m+1/2}, A_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\Big) \\ &- \; k^2 \textstyle\sum_{n=0}^{M-r} \sum_{m=n}^{n-1+r} \Big(\nabla \mathbf{u}_h^{m+1}, \nabla\, A_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\Big) \\ &+ \; k^2 \sum_{m=n}^{n-1+r} \Big\{ \Big\langle \mathbf{f}^{m+1}, A_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\Big\rangle_\Omega + \Big\langle \mathbf{g}_s^{m+1}, A_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\Big\rangle_{\Gamma_s} \Big\} \\ &= \; J_1 + J_2 + J_3 \end{aligned}$$

Now, we have to bound the RHS. The bound for J_3 is rather standard. Since J_2 is easier to bound than J_1, we only analyze the more complicate term of J_1:

$$\begin{aligned} c\Big(u_{3,h}^m, \mathbf{u}_h^{m+1/2}, A_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\Big) &= -\frac{1}{2}\Big(\partial_z u_{3,h}^m\, \mathbf{u}_h^{m+1/2}, A_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\Big) \\ &- \; \Big(u_{3,h}^m\, \mathbf{u}_h^{m+1/2}, \partial_z A_h^{-1}(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\Big) = I_1 + I_2 \end{aligned}$$

Since I_1 is easier to bound than I_2 (in fact, the I_1 term is a classical isotropic term which appear in the Navier-Stokes case, see [16]), we only bound I_2 as follows:

$$I_2 \le \|u^m_{3,h}\|_{L^\infty_z L^2_{\mathbf{x}}} \|\mathbf{u}^{m+1/2}_h\|_{L^2_z L^6_{\mathbf{x}}} \|\partial_z A_h^{-1}(\mathbf{u}^{n+r}_h - \mathbf{u}^n_h)\|_{L^2_z L^3_{\mathbf{x}}} \le C\,\|\mathbf{u}^m_h\|\,\|\mathbf{u}^{m+1/2}_h\|\,\|A_h^{-1}(\mathbf{u}^{n+r}_h - \mathbf{u}^n_h)\|_{W^{1,3}}$$

Then, it suffices with the following stability inequality for A_h^{-1}:

$$\|A_h^{-1}\mathbf{v}_h\|_{W^{1,3}} \le C\,|\mathbf{v}_h| \qquad \forall\,\mathbf{v}_h \in \mathbf{V}_h$$

In fact, the following Lemma hold (see Appendix for a proof):

Lemma 5 *Assuming* **(H0)** *and that the inverse inequality $\|\mathbf{v}_h\|_{W^{1,6}} \le C\frac{1}{h}\|\mathbf{v}_h\|_{H^1}$ holds, then*

$$\|A_h^{-1}\mathbf{v}_h\|_{W^{1,6}} \le C\,|\mathbf{v}_h|, \quad \forall\,\mathbf{v}_h \in \mathbf{V}_h.$$

In particular, $\|A^{-1}\mathbf{v}_h\|_{W^{1,3}} \le C|\mathbf{v}_h|$ hence the following bound holds:

$$J_1 \le C\,k^2 \sum_{n=0}^{M-r} \sum_{m=n}^{n-1+r} \|\mathbf{u}^m_h\|\,\|\mathbf{u}^{m+1/2}_h\|\,|\mathbf{u}^{n+r}_h - \mathbf{u}^n_h|$$

Then, applying Fubini's discrete rule, we obtain

$$J_1 \le C\,k^2 \sum_{m=0}^{M-1} \|\mathbf{u}^m_h\|\,\|\mathbf{u}^{m+1}_h\| \sum_{n=\overline{m-r+1}}^{\overline{m}} |\mathbf{u}^{n+r}_h - \mathbf{u}^n_h|$$

where

$$\overline{m} = \begin{cases} 0 & \text{if } m < 0 \\ m & \text{if } 0 \le m \le M-r \\ M-r & \text{if } m > M-r \end{cases}$$

Since $|\overline{m} - \overline{m-r+1}| \le r$, then $\sum_{n=\overline{m-r+1}}^{\overline{m}} |\mathbf{u}^{n+r}_h - \mathbf{u}^n_h| \le C\,r$. Finally, since $k\sum_{m=0}^{M-1} \|\mathbf{u}^m_h\|\,\|\mathbf{u}^{m+1}_h\| \le C$, one arrives at $J_1 \le C\,(r\,k)$. On the other hand, one also has $J_2 + J_3 \le C\,(r\,k)$ and the proof of Lemma 4 is finished. ■

One observes that the fractional derivative in time for the discrete velocities has been bounded in the norm $\mathbf{V}'_h$, which moves with respect to the space parameter h. But, the compactness results (see for instance J. Simon [29]) does not work in these conditions. Therefore, we will use the already cited argument of [13] in order to find a fixed norm where the time fractional derivative can be bounded. For this, we consider the orthogonal projections

$$R_h : \mathbf{V}_h \to \mathbf{V} \text{ defined as } \Big(\nabla(R_h\mathbf{v}_h - \mathbf{v}_h), \nabla\mathbf{w}\Big) = 0, \quad \forall\,\mathbf{w} \in \mathbf{V}.$$

The operator R_h has the following properties (following similar arguments as in [13]):

$$\|R_h\mathbf{u}_h\|_{H^1} \le \|\mathbf{u}_h\|_{H^1} \quad \text{(continuous dependency in } H^1\text{)},$$

$$|R_h \mathbf{u}_h - \mathbf{u}_h| \le C\, h \|\nabla_{\mathbf{x}} \cdot \langle \mathbf{u}_h \rangle\|_{L^2(S)} \quad \text{(error estimate in } L^2)$$

and

$$\|R_h \mathbf{u}_h\|_{V'} \le \|\mathbf{u}_h\|_{V_h'} + C\, h.$$

For the second estimate the H^2 regularity of the Stokes hydrostatic problem with second member $R_h\mathbf{u}_h - \mathbf{u}_h$ is used, and for the last estimate, it uses the orthogonal projector $P_h : \mathbf{V} \to \mathbf{V}_h$ defined as $\Big(P_h \mathbf{v} - \mathbf{v}, \mathbf{v}_h\Big) = 0, \ \forall\, \mathbf{v}_h \in \mathbf{V}_h$ (see [13] for more details).

From here, using (18), one has

$$k \sum_{n=0}^{M-r} \|R_h(\mathbf{u}_h^{n+r} - \mathbf{u}_h^n)\|_{V'}^2 \le C\, k \sum_{n=0}^{M-r} \|\mathbf{u}_h^{n+r} - \mathbf{u}_h^n\|_{V_h'}^2 + C\, h \le C(r\, k + h).$$

The above inequality can be written as

$$\int_0^{T-\delta} \|R_h\, \mathbf{u}_{h,k}^{(2)}(t+\delta) - R_h\, \mathbf{u}_{h,k}^{(2)}(t)\|_{V'}^2\, dt \le C\,(\delta + h)$$

Now, we can apply a compactness result (by perturbations) due to P. Azérad and F. Guillén-González ([2]), obtaining that $R_h \mathbf{u}_{k,h}^{(2)} \to \mathbf{u}$ in $L^2(0,T;\mathbf{L}^2)$-strong. From here, one has $\mathbf{u}_{k,h}^{(2)} \to \mathbf{u}$ in $L^2(0,T;\mathbf{L}^2)$-strong (see [13] for more details).

The proof of Theorem 3 is finished. ∎

4 Error estimates

In this section, we will obtain error estimates (for the velocity as well as for the pressure) with respect to a sufficiently regular solution $\{\mathbf{u}, u_3, p_s\}$ of the problem (R).

Concretely, for k small enough, we begin obtaining for $l = 1, 2$ (where l is the order of approximation of finite element spaces), error estimates of order $O(\sqrt{k} + h^l)$ for the velocities $\mathbf{u}_h^{m+1/2}$ and $\mathbf{u}_h^{m+1}$. Afterwards, we improve to error estimates (of order $O(k+h^l)$) for the "end of step" velocity $\mathbf{u}_h^{m+1}$ and we obtain error estimates of order $O(\sqrt{k}+h^l)$ for the discrete derivative of "end of step" velocity in $l^2(\mathbf{L}^2)$ (which drives to error estimates of order $O(\sqrt{k} + h^2)$ for the pressure $p_{s,h}^{m+1}$). Moreover, for $l = 2$, we get error estimates of order $O(\sqrt{k} + h^2)$ for the discrete derivative of velocities $\mathbf{u}_h^{m+1/2}$ and $\mathbf{u}_h^{m+1}$ and error estimates of order $O(k + h^2)$ for the discrete derivative of the "end of step" velocity $\mathbf{u}_h^{m+1}$, which drives to order $O(k + h^2)$ for the pressure).

The following constraint between the time step size k and the mesh size h will be assumed in order to obtain these optimal error estimates $O(k + h)$:

(**H**) There exists a constant $\alpha > 0$ (independent of k and h) such that $k \le \alpha\, h^2$.

4.1 Regularity hypotheses

To obtain error estimates, the following regularity hypotheses for the solution $(\mathbf{u}, p_s)$ will appear:

- To obtain order $O(k^{1/2}+h)$ for errors in velocities in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ we assume:

(R1) $$\mathbf{U}\in L^\infty(\mathbf{H}^{l+1}),\quad \mathbf{u}_t\in L^2(\mathbf{H}^l)\quad p_s\in L^2(H^l),\quad \sqrt{t}\,\mathbf{u}_{tt}\in L^2(\mathbf{H}^{-1}_{b,l})$$

- To obtain order $O(k+h)$ in end of step velocity error in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$, we also impose

(R2) $$\mathbf{u}_t\in L^2(\mathbf{H}^{l+1})\qquad \mathbf{u}_{tt}\in L^2(\mathbf{H}^{-1})$$

- To get order $O(k^{1/2}+h)$ for time discrete derivative of end of step velocity in $l^2(\mathbf{L}^2)$, for end-of-step velocity error in $l^\infty(\mathbf{H}^1)$ and for pressure error in $l^2(L^2(S))$, we assume

(R3) $$\mathbf{U}_t\in L^2(\mathbf{H}^1)\quad \mathbf{u}_{tt}\in L^2(\mathbf{L}^2),$$

- To get order $O(k^{1/2}+h^2)$ for time discrete derivatives of velocities in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$, we assume:

(R4)
$$\partial_t p_s\in L^2(H^1)\cap L^2(H^l),\ \mathbf{U}_t\in L^\infty(\mathbf{L}^3)\cap L^2(\mathbf{H}^{l+1})\quad \mathbf{u}_t\in L^\infty(\mathbf{H}^2),\ \mathbf{u}_{tt}\in L^2(\mathbf{H}^l),$$
$$\mathbf{U}_{tt}\in L^2(\mathbf{L}^2),\ \sqrt{t}\mathbf{u}_{ttt}\in L^2(\mathbf{H}^{-1}_{b,l})$$

- To obtain order $O(k+h^2)$ for time discrete derivative of end of step velocity in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$, we assume:

(R5) $$\mathbf{U}_t\in L^\infty(\mathbf{H}^2),\quad \mathbf{u}_{ttt}\in L^2(\mathbf{V}')$$

- Previous regularity hypotheses yields to order $O(k+h)$ for pressure error in $l^2(L^2(S))$.

4.2 Problems related to the space discrete errors

We will present an error analysis for the fully discrete scheme $(\mathbf{u}_h^{m+1/2},\mathbf{u}_h^{m+1},p_h^{m+1})$ as an approximation of $(\mathbf{u}(t_{m+1}),\mathbf{u}(t_{m+1}),p(t_{m+1})$. Consequently, we consider the following errors:

$$\mathbf{e}^{m+1/2}=\mathbf{u}(t_{m+1})-\mathbf{u}_h^{m+1/2},\qquad \mathbf{e}^{m+1}=\mathbf{u}(t_{m+1})-\mathbf{u}_h^{m+1},\qquad e_p^{m+1}=p_s(t_{m+1})-p_h^{m+1}$$
$$e_3^{m+1}=u_3(t_{m+1})-u_{3,h}^{m+1},\qquad \mathbf{E}^{m+1}=(\mathbf{e}^{m+1},e_3^{m+1}).$$

These errors can be decomposed as follows (splitting interpolation and discrete parts):

$$\mathbf{e}^{m+1/2}=\mathbf{e}_i^{m+1}+\mathbf{e}_h^{m+1/2},\qquad \mathbf{e}^{m+1}=\mathbf{e}_i^{m+1}+\mathbf{e}_h^{m+1},\qquad e_p^{m+1}=e_{p,i}^{m+1}+e_{p,h}^{m+1}$$
$$e_3^{m+1}=e_{3,i}^{m+1}+e_{3,h}^{m+1},\qquad \mathbf{E}^{m+1}=\mathbf{E}_h^{m+1}+\mathbf{E}_i^{m+1}$$

Concretely
$$\mathbf{e}_i^{m+1}=\mathbf{u}(t_{m+1})-I_h\mathbf{u}(t_{m+1})\quad\text{and}\quad \mathbf{e}_h^{m+1}=I_h\mathbf{u}(t_{m+1})-\mathbf{u}_h^{m+1}$$

$$\mathbf{e}_h^{m+1/2} = I_h\mathbf{u}(t_{m+1}) - \mathbf{u}_h^{m+1/2}$$

$$e_{p,i}^{m+1} = p(t_{m+1}) - J_h p(t_{m+1}) \quad \text{and} \quad e_{p,h}^{m+1} = J_h p(t_{m+1}) - p_h^{m+1}$$

$$u_{3,i}^{m+1} = u_3(t_{m+1}) - K_h u_3(t_{m+1}) \quad \text{and} \quad e_{3,h}^{m+1} = K_h u_3(t_{m+1}) - u_{3,h}^{m+1}$$

$$\mathbf{E}_i^{m+1} = (\mathbf{e}_i^{m+1}, e_{3,i}^{m+1}) \quad \text{and} \quad \mathbf{E}_h^{m+1} = (\mathbf{e}_h^{m+1}, e_{3,h}^{m+1})$$

Using the following development with integral rest of a function $\phi = \phi(t)$:

$$\phi(t+k) - \phi(t) = \phi'(t+k)\,k - \int_t^{t+k} (s-t)\phi''(s)\,ds,$$

and the variational problem $(R)_w$ verified for an exact solution $(\mathbf{u}, p_s)$ in $t = t_{m+1}$, one has:

$(R)_w^{m+1}$

$$\begin{cases} \Big(\dfrac{1}{k}(\mathbf{u}(t_{m+1}) - \mathbf{u}(t_m)), \mathbf{v}\Big) + c\Big(\mathbf{U}(t_m), \mathbf{u}(t_{m+1}), \mathbf{v}\Big) + \Big(\nabla \mathbf{u}(t_{m+1}), \nabla \mathbf{v}\Big) - \Big(p_s(t_{m+1}), \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}\rangle\Big)_S \\ = \Big\langle \mathbf{f}(t_{m+1}), \mathbf{v}\Big\rangle_\Omega + \Big\langle \mathbf{g}_s(t_{m+1}), \mathbf{v}\Big\rangle_{\Gamma_s} + \Big(\mathcal{E}^{m+1}, \mathbf{v}\Big), \quad \forall\, \mathbf{v} \in \mathbf{W}_{b,l}^{1,3} \cap \mathbf{L}^\infty \\ \Big(\nabla_{\mathbf{x}} \cdot \langle \mathbf{u}(t_{m+1})\rangle, q\Big)_S = 0, \quad \forall\, q \in L_0^2(S), \\ \Big(\partial_z\, u_3(t_{m+1}), \partial_z v_3\Big) = -\Big(\nabla_{\mathbf{x}} \cdot \mathbf{u}(t_{m+1}), \partial_z v_3\Big), \quad \forall\, v_3 \in H_0(\partial_z) \end{cases}$$

where $\mathcal{E}^{m+1} = -\dfrac{1}{k}\displaystyle\int_{t_m}^{t_{m+1}} (t - t_m)\,\mathbf{u}_{tt}(t)\,dt - \Big(\int_{t_m}^{t_{m+1}} \mathbf{U}_t \cdot \nabla\Big)\,\mathbf{u}(t_{m+1})$ is the consistency error.

For simplicity, we assume $\mathbf{f} \in C([0,T]; \mathbf{H}_{b,l}^{-1})$ and $\mathbf{g}_s \in C([0,T]; \mathbf{H}^{-1/2}(\Gamma_s))$ and we choose

$$\mathbf{f}^{m+1} = \mathbf{f}(t_{m+1}) \quad \text{and} \quad \mathbf{g}_s^{m+1} = \mathbf{g}_s(t_{m+1}),$$

therefore, the data errors $\mathbf{f}(t_{m+1}) - \mathbf{f}^{m+1}$ and $\mathbf{g}_s(t_{m+1}) - \mathbf{g}_s^{m+1}$ vanish.

Comparing $(R)_w^{m+1}$ with $(S_0)_h^m$ and $(S_1)_h^{m+1}$, the following variational problems for the space error $e_{3,h}^m$ and $\mathbf{e}_h^{m+1/2}$ hold:

$(E_0)_h^m$
$$\Big(\partial_z e_{3,h}^m, \partial_z v_{3,h}\Big) = -\Big(\nabla_{\mathbf{x}} \cdot (\mathbf{e}_h^m + \mathbf{e}_i^m), \partial_z v_{3,h}\Big) \quad \forall v_{3,h} \in Y_h$$

$(E_1)_h^{m+1}$
$$\begin{cases} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\, \mathbf{e}_h^{m+1/2}, \nabla\, \mathbf{v}_h\Big) - \Big(p_s(t_{m+1}), \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h\rangle\Big)_S \\ = \mathbf{NL}^{m+1}(\mathbf{v}_h) + \Big(\mathcal{E}^{m+1}, \mathbf{v}_h\Big) \\ -\Big(\delta_t \mathbf{e}_i^{m+1}, \mathbf{v}_h\Big) - \Big(\nabla \mathbf{e}_i^{m+1}, \nabla \mathbf{v}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h \end{cases}$$

where $\delta_t \mathbf{e}_i^{m+1} = \dfrac{1}{k}\Big(\mathbf{e}_i^{m+1} - \mathbf{e}_i^m\Big)$ and

$$\mathbf{NL}^{m+1}(\mathbf{v}_h) = -c\Big(\mathbf{E}_h^m + \mathbf{E}_i^m, \mathbf{u}(t_{m+1}), \mathbf{v}_h\Big) - c\Big(\mathbf{U}_h^m, \mathbf{e}_h^{m+1/2} + \mathbf{e}_i^{m+1}, \mathbf{v}_h\Big)$$

On the other hand, adding and substrating $I_h\mathbf{u}(t_{m+1})$ to $(S_2)_h^{m+1}$ one has for each $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$

$(E_2)_h^{m+1}$
$$\begin{cases} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}, \mathbf{v}_h\Big) + \Big(\nabla(\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}), \nabla \mathbf{v}_h\Big) = -\Big(p_h^{m+1}, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h\rangle\Big)_S \\ \Big(\nabla_{\mathbf{x}} \cdot \langle \mathbf{e}_h^{m+1}\rangle, q_h\Big)_S = 0, \end{cases}$$

Due to the choice of the projector K_h, $\Big(\partial_z e_{3,i}^{m+1}, \partial_z w_h\Big) = 0$. Since the same discrete space for $\mathbf{u}_h^{m+1/2}$ and $\mathbf{u}_h^{m+1}$ has been chosen, the interpolation error depending on $\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m$ in $(E_1)_h^{m+1}$ is $\mathbf{e}_i^{m+1} - \mathbf{e}_i^m$ and the interpolation error depending on $\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}$ in $(E_2)_h^{m+1}$ is zero. Finally, since $\Big(\nabla_{\mathbf{x}} \cdot \langle I_h \mathbf{u}(t_{m+1}) \rangle, q_h\Big)_S = 0$, the corresponding interpolation error $\Big(\nabla_{\mathbf{x}} \cdot \langle \mathbf{e}_i^{m+1} \rangle, q_h\Big)_S = 0$.

Adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$, one arrives at:

$$(E)_h^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}\Big(\mathbf{e}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla \mathbf{e}_h^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(e_{p,h}^{m+1}, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h \rangle\Big)_S \\ \quad = -\dfrac{1}{k}\Big(\mathbf{e}_i^{m+1} - \mathbf{e}_i^m, \mathbf{v}_h\Big) + \mathbf{NL}^{m+1}(\mathbf{v}_h) + \Big(\mathcal{E}^{m+1}, \mathbf{v}_h\Big), \quad \forall \mathbf{v}_h \in \mathbf{X}_h \\ \Big(\nabla_{\mathbf{x}} \cdot \langle \mathbf{e}_h^{m+1} \rangle, q_h\Big)_S = 0, \quad \forall q_h \in Q_h \end{array} \right.$$

Due to the choice for the interpolation operator (I_h, J_h) related to the hydrostatic Stokes problem, the interpolation error $\Big(\nabla \mathbf{e}_i^{m+1}, \nabla \mathbf{v}_h\Big) + \Big(e_{p,i}^{m+1}, \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}_h \rangle\Big)_S = 0$ and does not appear in $(E)_h^{m+1}$.

From $(E_0)_h^m$, the following estimate holds:

$$|\partial_z e_{3,h}^m| \le C(|\nabla_{\mathbf{x}} \cdot \mathbf{e}_h^m| + |\nabla_{\mathbf{x}} \cdot \mathbf{e}_i^m|). \tag{20}$$

Therefore, by using inequality (6),

$$\|e_{3,h}^m\|^2_{L_z^\infty L_{\mathbf{x}}^2} \le C(\|\mathbf{e}_h^m\| + \|\mathbf{e}_i^m\|). \tag{21}$$

On the other hand, the approximation property (7) reads:

$$|\partial_z e_{3,i}^m| \le C\, h^l\, \|u_3(t_m)\|_{H^{l+1}} \le C\, h^l. \tag{22}$$

(here, we use the regularity $u_3 \in L^\infty(0,T;H^{l+1})$). Therefore, by using again (6),

$$\|e_{3,i}^m\|_{L_z^\infty L_{\mathbf{x}}^2} \le C\, h^l. \tag{23}$$

4.3 $O(\sqrt{k} + h^l)$ error estimates for both velocities in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$

Theorem 6 *We assume* **(H0)**-**(H3)**, **(R1)** *and* $|\mathbf{e}_h^0| \le C\, h^l$. *Then, for any k small enough, the following error estimates hold*

$$\|\mathbf{e}_h^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,(\sqrt{k} + h^l) \tag{24}$$

$$\|\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m\|_{l^2(\mathbf{L}^2)} + \|\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}\|_{l^2(\mathbf{L}^2)} \le C\,\sqrt{k}(\sqrt{k} + h^l). \tag{25}$$

Notice that in this result, the constraint (**H**) on parameters (k,h) is not necessary although k small enough must be imposed.

Proof. The main idea is to make $2\,k \displaystyle\sum_{m=0}^{M-1} \Big\{ \Big((E_1)_h^{m+1}, \mathbf{e}_h^{m+1/2}\Big) + \Big((E_2)_h^{m+1}, \mathbf{e}_h^{m+1}\Big) \Big\}$.

In fact, making $2\,k\left((E_1)_h^{m+1},\mathbf{e}_h^{m+1/2}\right)$ and taking into account that $c\Big(\mathbf{U}_h^m,\mathbf{e}_h^{m+1/2},\mathbf{e}_h^{m+1/2}\Big)=0$, we arrive at

$$
\begin{aligned}
&|\mathbf{e}_h^{m+1/2}|^2-|\mathbf{e}_h^m|^2+|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m|^2+2\,k\,\|\mathbf{e}_h^{m+1/2}\|^2\\
=\;& 2\,k\left(p_s(t_{m+1}),\nabla_{\mathbf{x}}\cdot\mathbf{e}_h^{m+1/2}\right)+2\,k\,c\Big(\mathbf{E}_h^m+\mathbf{E}_i^m,\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1/2}\Big) \qquad (26)\\
+\;& 2\,k\,c\Big(\mathbf{U}_h^m,\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1/2}\Big)+2\,k\left(\mathcal{E}^{m+1}-\delta_t\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1/2}\right)-\left(\nabla\mathbf{e}_i^{m+1},\nabla\mathbf{e}_h^{m+1/2}\right)=\sum_{i=1}^{5}I_1
\end{aligned}
$$

We bound only the more difficult terms of the RHS of (26):

$$
\begin{aligned}
I_1 &= 2\,k\Big(p_s(t_{m+1}),\nabla_{\mathbf{x}}\cdot\mathbf{e}_h^{m+1/2}\Big)=2\,k\left(p_s(t_{m+1}),\nabla_{\mathbf{x}}\cdot(\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m)\right)+2\,k\left(e_{p,i}^{m+1},\nabla_{\mathbf{x}}\cdot\mathbf{e}_h^m\right)\\
&= -2\,k\left(\nabla_{\mathbf{x}}p_s(t_{m+1}),\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m\right)+2\,k\left(e_{p,i}^{m+1},\nabla_{\mathbf{x}}\cdot\mathbf{e}_h^m\right)\\
&\le \varepsilon|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^m|^2+C\,k^2\|p_s(t_{m+1})\|^2+\varepsilon\,k\|\mathbf{e}_h^m\|^2+C\,k\,h^{2l}\|p_s(t_{m+1})\|^2_{H^l(S)}
\end{aligned}
$$

(here we have used that $\left(J_hp_s(t_{m+1}),\nabla_{\mathbf{x}}\cdot\mathbf{e}_h^m\right)=0$),

$$
I_4\sim 2\,k\left(\delta_t\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1/2}\right)=2\,k\left(\mathbf{e}_i(\delta_t\mathbf{u}(t_{m+1})),\mathbf{e}_h^{m+1/2}\right)\le\varepsilon\,k\,\|\mathbf{e}_h^{m+1/2}\|^2+C\,h^{2l}\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^2_{H^l}
$$

$$
I_2\sim 2\,k\,c\Big(e_{3,h}^m,\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1/2}\Big)=2\,k\left(e_{3,h}^m\,\partial_z\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1/2}\right)+k\left(\partial_z e_{3,h}^m\,\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1/2}\right):=L_1+L_2
$$

$$
\begin{aligned}
L_1 &\le 2\,k\,\|e_{3,h}^m\|_{L_z^\infty L_{\mathbf{x}}^2}\|\partial_z\mathbf{u}(t_{m+1})\|_{L_z^2L_{\mathbf{x}}^4}\|\mathbf{e}_h^{m+1/2}\|_{L_z^2L_{\mathbf{x}}^4}\\
&\le C\,k\,(\|\mathbf{e}_h^m\|+\|\mathbf{e}_i^m\|)|\mathbf{e}_h^{m+1/2}|^{1/2}\|\mathbf{e}_h^{m+1/2}\|^{1/2}\\
&\le \varepsilon\,k\,\|\mathbf{e}_h^m\|^2+C\,k\,h^{2l}+\varepsilon\,k\,\|\mathbf{e}_h^{m+1/2}\|^2+C\,k\,|\mathbf{e}_h^{m+1/2}|^2
\end{aligned}
$$

(here, we have used that $\mathbf{u}\in L^\infty(\mathbf{H}^{l+1})$ and (21))

$$
\begin{aligned}
L_2 &\le k\,|\partial_z e_{3,h}^m|\,\|\mathbf{u}(t_{m+1})\|_{L^\infty}\,|\mathbf{e}_h^{m+1/2}|\le\varepsilon\,k\,(\|\mathbf{e}_h^m\|^2+\|\mathbf{e}_i^m\|^2)+C\,k\,|\mathbf{e}_h^{m+1/2}|^2\\
&\le \varepsilon\,k\,\|\mathbf{e}_h^m\|^2+C\,k\,|\mathbf{e}_h^{m+1/2}|^2+C\,k\,h^{2l}
\end{aligned}
$$

(here, we have used again that $\mathbf{u}\in L^\infty(\mathbf{H}^2)$ and (20)).

By a similar way, the interpolation part can be bounded as follows:

$$
\begin{aligned}
I_2\sim 2\,k\,c\Big(e_{3,i}^m,\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1/2}\Big) &= 2\,k\Big(e_{3,i}^m\,\partial_z\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1/2}\Big)+k\left(\partial_z e_{3,i}^m\,\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1/2}\right)\\
&\le C\,k\,\|e_{3,i}^m\|_{H(\partial_z)}\,\|\mathbf{u}(t_{m+1})\|_{H^2}\|\mathbf{e}_h^{m+1/2}\|\le\varepsilon\,k\,\|\mathbf{e}_h^{m+1/2}\|^2+C\,k\,h^{2l}
\end{aligned}
$$

(here, we have used that $\mathbf{u}\in L^\infty(\mathbf{H}^2)$, $u_3\in L^\infty(\mathbf{H}^{l+1})$ and (22))

$$
I_3=2\,k\,c\Big(\mathbf{U}_h^m,\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1/2}\Big)=2\,k\,c\Big(\mathbf{E}^m,\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1/2}\Big)-2\,k\,c\Big(\mathbf{U}(t_m),\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1/2}\Big)
$$

and their more difficult terms can be bounded as follows:

$$
\begin{aligned}
2\,k\,c\Big(e^m_{3,h}+e^m_{3,i},\mathbf{e}^{m+1}_i,\mathbf{e}^{m+1/2}_h\Big) &= 2\,k\Big((e^m_{3,h}+e^m_{3,i})\,\partial_z\mathbf{e}^{m+1}_i,\mathbf{e}^{m+1/2}_h\Big)\\
&+ k\Big((\partial_z e^m_{3,h}+\partial_z e^m_{3,i})\,\mathbf{e}^{m+1}_i,\mathbf{e}^{m+1/2}_h\Big)\\
&\le 2\,k\|e^m_{3,h}+e^m_{3,i}\|_{L^\infty_z L^2_{\mathbf{x}}}\|\partial_z\mathbf{e}^{m+1}_i\|_{L^2_z L^2_{\mathbf{x}}}\|\mathbf{e}^{m+1/2}_h\|_{L^2_z L^\infty_{\mathbf{x}}}\\
&+ k\|\partial_z\, e^m_{3,h}+\partial_z e^m_{3,i}\|_{L^2}\|\mathbf{e}^{m+1}_i\|_{L^6}\|\mathbf{e}^{m+1/2}_h\|_{L^3}\\
&\le C\,k\,(\|\mathbf{e}^m_h\|+\|\mathbf{e}^m_i\|+h^l)\|\mathbf{e}^{m+1}_i\|\,(h^{-1}+h^{-1/2})|\mathbf{e}^{m+1/2}_h|\\
&\le C\,k\,(\|\mathbf{e}^m_h\|+h^l)h^l\,h^{-1}|\mathbf{e}^{m+1/2}_h|\\
&\le \varepsilon\,k\,\|\mathbf{e}^m_h\|^2+C\,k\,h^{2l}+C\,k\,|\mathbf{e}^{m+1/2}_h|^2
\end{aligned}
$$

(here we have used (20), (21) and the inverse inequalities $\|\mathbf{e}^{m+1/2}_h\|_{L^2_z L^\infty_{\mathbf{x}}}\le C\,h^{-1}|\mathbf{e}^{m+1/2}_h|$ and $\|\mathbf{e}^{m+1/2}_h\|_{L^3}\le C\,h^{-1/2}|\mathbf{e}^{m+1/2}_h|$).

On the other hand,

$$
2\,k\,c\Big(u_3(t_m),\mathbf{e}^{m+1}_i,\mathbf{e}^{m+1/2}_h\Big)\le\varepsilon\,k\,\|\mathbf{e}^{m+1/2}_h\|^2+C\,k\,h^{2l}
$$

Therefore, applying previous estimates to (26) and making $|\mathbf{e}^{m+1/2}_h|^2\le 2(|\mathbf{e}^m_h|^2+|\mathbf{e}^{m+1/2}_h-\mathbf{e}^m_h|^2)$, we get

$$
\begin{aligned}
&|\mathbf{e}^{m+1/2}_h|^2-|\mathbf{e}^m_h|^2+|\mathbf{e}^{m+1/2}_h-\mathbf{e}^m_h|^2+2\,k\,\|\mathbf{e}^{m+1/2}_h\|^2\\
&\le C\,k(|\mathbf{e}^m_h|^2+|\mathbf{e}^{m+1/2}_h-\mathbf{e}^m_h|^2)+2\,k\left(\mathcal{E}^{m+1},\mathbf{e}^{m+1/2}_h\right)+\varepsilon\,k\|\mathbf{e}^{m+1/2}_h\|^2\\
&+\varepsilon\,k\,\|\mathbf{e}^m_h\|^2+C\,k^2+C\,k\,h^{2l}
\end{aligned}
\tag{27}
$$

On the other hand, making $2\,k\Big((E_2)^{m+1}_h,\mathbf{e}^{m+1}_h\Big)$, we arrive at

$$
|\mathbf{e}^{m+1}_h|^2-|\mathbf{e}^{m+1/2}_h|^2+|\mathbf{e}^{m+1}_h-\mathbf{e}^{m+1/2}_h|^2+k\left\{\|\mathbf{e}^{m+1}_h\|^2-\|\mathbf{e}^{m+1/2}_h\|^2+\|\mathbf{e}^{m+1}_h-\mathbf{e}^{m+1/2}_h\|^2\right\}=0. \tag{28}
$$

Adding (27) and (28) from $m=0$ to r (with any $r<M$) and choosing ε and k small enough, one has

$$
\begin{aligned}
&|\mathbf{e}^{r+1}_h|^2+\sum_{m=0}^{r}\left(|\mathbf{e}^{m+1}_h-\mathbf{e}^{m+1/2}_h|^2+\frac{1}{2}|\mathbf{e}^{m+1/2}_h-\mathbf{e}^m_h|^2\right)\\
&+k\sum_{m=0}^{r}\left(\frac{1}{2}\|\mathbf{e}^{m+1}_h\|^2+\|\mathbf{e}^{m+1}_h-\mathbf{e}^{m+1/2}_h\|^2+\frac{1}{2}\|\mathbf{e}^{m+1/2}_h\|^2\right)\le C\,k\sum_{m=0}^{r}|\mathbf{e}^m_h|^2+C(k+h^{2l})
\end{aligned}
$$

Therefore, applying discrete Gromwall's Lemma, we can get (24) and (25). ■

4.4 $O(k+h^l)$ error estimates for $\mathbf{e}^{m+1}_h$ in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$.

Theorem 7 *We assume hypotheses of Theorem 6,* (**R2**) *and* (**H**)*. Then, the following error estimate holds*

$$
\|\mathbf{e}^{m+1}_h\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}\le C\,(k+h^l). \tag{29}
$$

Note that, from (20) and (29), we also have

$$\|e_{3,h}^{m+1}\|_{l^2(H(\partial_z))} \le C\,(k+h^l).$$

Proof. The main idea is to make $2\,k\sum_{m=0}^{M-1}\Big((E)_h^{m+1},\mathbf{e}_h^{m+1}\Big)$.

In fact, making $2\,k\Big((E)_h^{m+1},\mathbf{e}_h^{m+1}\Big)$, the pressure term vanish, and we arrive at

$$\begin{aligned}
& |\mathbf{e}_h^{m+1}|^2-|\mathbf{e}_h^m|^2+|\mathbf{e}_h^{m+1}-\mathbf{e}_h^m|^2+2\,k\,\|\mathbf{e}_h^{m+1}\|^2\\
= \; & -2\,k\Big(\delta_t\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1}\Big)-2\,k\,c\Big(\mathbf{E}_h^m,\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1}\Big)-2\,k\,c\Big(\mathbf{E}_i^m,\mathbf{u}(t_{m+1}),\mathbf{e}_h^{m+1}\Big)\\
- \; & 2\,k\,c\Big(\mathbf{U}_h^m,\mathbf{e}_h^{m+1/2},\mathbf{e}_h^{m+1}\Big)-2\,k\,c\Big(\mathbf{U}_h^m,\mathbf{e}_i^{m+1/2},\mathbf{e}_h^{m+1}\Big)+2\,k\Big(\mathcal{E}^{m+1},\mathbf{e}_h^{m+1}\Big):=\sum_{i=1}^{6}I_i
\end{aligned}$$

We bound of similar way as in Theorem 6 the terms of the RHS:

$$I_1=-2\,k\Big(\mathbf{e}_i(\delta_t\mathbf{u}(t_{m+1})),\mathbf{e}_h^{m+1}\Big)\le \varepsilon\,k\,\|\mathbf{e}_h^{m+1}\|^2+C\,h^{2l}\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|_{H^{l+1}}^2.$$

The vertical part of I_5 is bounded as

$$\begin{aligned}
I_5\sim 2\,k\,c\Big(e_{3,h}^m,\mathbf{e}_i^{m+1},\mathbf{e}_h^{m+1}\Big) &\le C\,\|e_{3,h}^m\|_{L_z^\infty L_{\mathbf{x}}^2}\|\mathbf{e}_i^{m+1}\|_{H^1}\|\mathbf{e}_h^{m+1}\|_{L_z^2L_{\mathbf{x}}^\infty}\le C\,\|\mathbf{e}_h^m\|\,h\,h^{-1}|\mathbf{e}_h^{m+1}|\\
&\le \varepsilon\,k\,\|\mathbf{e}_h^m\|^2+C\,k(|\mathbf{e}_h^{m+1}-\mathbf{e}_h^m|^2+|\mathbf{e}_h^m|^2)
\end{aligned}$$

Now, the term $I_4=c\Big(\mathbf{U}_h^m,\mathbf{e}_h^{m+1/2},\mathbf{e}_h^{m+1}\Big)\neq 0$, but using that $c\Big(\mathbf{U}_h^m,\mathbf{e}_h^{m+1},\mathbf{e}_h^{m+1}\Big)=0$, this term I_4 is decomposed as

$$I_4=-2\,k\,c\Big(\mathbf{E}^m,\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1},\mathbf{e}_h^{m+1}\Big)+2\,k\,c\Big(\mathbf{U}(t_m),\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1},\mathbf{e}_h^{m+1}\Big):=J_1+J_2.$$

The more complicate terms to bound are the vertical parts:

$$J_1\sim 2\,k\Big((e_{3,h}^m+e_{3,i}^m)\,(\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}),\partial_z\mathbf{e}_h^{m+1}\Big)+2\,k\Big(\partial_z(e_{3,h}^m+e_{3,i}^m)\,(\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}),\mathbf{e}_h^{m+1}\Big)=J_{1,1}+J_{1,2}.$$

Since $J_{1,2}$ is easier to bound than $J_{1,1}$, we only bound $J_{1,1}$:

$$\begin{aligned}
J_{1,1} &\le 2\,k\,\|e_{3,h}^m+e_{3,i}^m\|_{L_z^\infty L_{\mathbf{x}}^2}\|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}\|_{L_z^2L_{\mathbf{x}}^\infty}\|\partial_z\mathbf{e}_h^{m+1}\|_{L_z^2L_{\mathbf{x}}^2}\\
&\le C\,k\,(\|\mathbf{e}_h^m\|+\|\mathbf{e}_i^m\|+h^l\,\|u_3(t_m)\|_{H^{l+1}})\,h^{-1}|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}|\,\|\mathbf{e}_h^{m+1}\|\\
&\le C\,k\,(h^{-2}\,|\mathbf{e}_h^m|+1)\,|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}|\,\|\mathbf{e}_h^{m+1}\|\\
&\le \varepsilon\;k\,\|\mathbf{e}_h^{m+1}\|^2+C\,k\,h^{-4}|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}|^2\,|\mathbf{e}_h^m|^2+C\,k\,|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}|^2
\end{aligned}$$

Here, we have used (21) and (23) and the inverse inequalities

$$\|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}\|_{L_z^2\,L_{\mathbf{x}}^\infty}\le C\,h^{-1}|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}|\quad\text{and}\quad \|\mathbf{e}_h^m\|\le C\,h^{-1}|\mathbf{e}_h^m|.$$

Moreover, J_2 can be bounded as

$$J_2\le \varepsilon\;k\,\|\mathbf{e}_h^{m+1}\|^2+C\,k\,|\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}|^2$$

Finally, adding from $m=0$ to r (with any $r<M$) and taking into account the estimate

$$C\,k\,h^{-4}\sum_m |\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m+1}|^2 \le C\,h^{-2}(k+h^{2l}) \le C,$$

(where estimate (25) of Theorem 6 and (**H**) have been applied) we can apply the discrete Gromwall's Lemma, obtaining the desired estimates. ∎

4.5 $O(\sqrt{k}+h^l)$ error estimates for $\delta_t\mathbf{e}_h^{m+1}$ in $l^2(\mathbf{L}^2)$ and for $\mathbf{e}_h^{m+1}$ in $l^\infty(\mathbf{H}^1)$

We will use the following notations for the discrete derivatives of errors

$$\delta_t\mathbf{e}_h^{m+1}=\frac{\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m}}{k}, \qquad \delta_t\mathbf{e}_h^{m+1/2}=\frac{\mathbf{e}_h^{m+1/2}-\mathbf{e}_h^{m-1/2}}{k}.$$

Theorem 8 *Assume hypotheses of Theorem 6, $\|\mathbf{e}_h^0\|\le C\,h^l$, additional regularity hypothesis* (**R3**) *and the constraints on (k,h) given in* (**H**). *Then, the following error estimate holds*

$$\|\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{H}^1)}+\|\delta_t\mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\,(\sqrt{k}+h^l).$$

Proof. Taking $2\,k\Big((E)_h^{m+1},\delta_t\mathbf{e}_h^{m+1}\Big)$ the pressure term vanish, and we arrive at

$$\begin{aligned}
&\|\mathbf{e}_h^{m+1}\|^2-\|\mathbf{e}_h^{m}\|^2+\|\mathbf{e}_h^{m+1}-\mathbf{e}_h^{m}\|^2+2\,k\,|\delta_t\mathbf{e}_h^{m+1}|^2\\
=\;& -2\,k\Big(\delta_t\mathbf{e}_i^{m+1},\delta_t\mathbf{e}_h^{m+1}\Big)-2\,k\,c\Big(\mathbf{E}_h^m,\mathbf{u}(t_{m+1}),\delta_t\mathbf{e}_h^{m+1}\Big)-2\,k\,c\Big(\mathbf{E}_i^m,\mathbf{u}(t_{m+1}),\delta_t\mathbf{e}_h^{m+1}\Big)\\
-\;& 2\,k\,c\Big(\mathbf{U}_h^m,\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)-2\,k\,c\Big(\mathbf{U}_h^m,\mathbf{e}_i^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)+2\,k\Big(\mathcal{E}^{m+1},\delta_t\mathbf{e}_h^{m+1}\Big):=\sum_{i=1}^6 I_i
\end{aligned}$$

We must bound the I_i terms of the RHS,

$$I_1=-2\,k\Big(\mathbf{e}_i(\delta_t\mathbf{u}(t_{m+1})),\delta_t\mathbf{e}_h^{m+1}\Big)\le \varepsilon\,k\,|\delta_t\mathbf{e}_h^{m+1}|^2+C\,h^{2l}\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|_{H^l}^2.$$

Now, the term $I_4=-2\,k\,c\Big(\mathbf{U}_h^m,\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)$ does not vanish,

$$I_4=2\,k\,c\Big(\mathbf{E}^m,\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)-2\,k\,c\Big(\mathbf{U}(t_m),\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big):=J_1+J_2.$$

The more complicate terms to bound are the vertical parts of J_1 and J_2:

$$J_1\sim 2\,k\Big((e_{3,h}^m+e_{3,i}^m)\,\partial_z\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)+k\Big(\partial_z(e_{3,h}^m+e_{3,i}^m)\,\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)=J_{1,1}+J_{1,2}.$$

$$\begin{aligned}
J_{1,1}\;&\le\; 2\,k\,\|e_{3,h}^m+e_{3,i}^m\|_{L_z^\infty L_{\mathbf{x}}^2}\|\partial_z\mathbf{e}_h^{m+1/2}\|_{L_z^2L_{\mathbf{x}}^\infty}\|\delta_t\mathbf{e}_h^{m+1}\|_{L_z^2L_{\mathbf{x}}^2}\\
&\le\; C\,k\,(\|\mathbf{e}_h^m\|+h^l\|\mathbf{u}(t_m)\|_{H^{l+1}}+h^l\|u_3(t_m)\|_{H^{l+1}})\,h^{-1}\|\mathbf{e}_h^{m+1/2}\|\,|\delta_t\mathbf{e}_h^{m+1}|\\
&\le\; \varepsilon\;k\,|\delta_t\mathbf{e}_h^{m+1}|^2+C\,k\,h^{-2}\|\mathbf{e}_h^{m+1/2}\|^2\|\mathbf{e}_h^m\|^2+C\,k\,\|\mathbf{e}_h^{m+1/2}\|^2
\end{aligned}$$

(here, we have used (21) and (23) and the inverse inequality $\|\partial_z \mathbf{e}_h^{m+1/2}\|_{L^2_z L^\infty_{\mathbf{x}}} \le C\, h^{-1} \|\mathbf{e}_h^{m+1/2}\|$),

$$
\begin{aligned}
J_{1,2} &\le k\, \|\partial_z(e^m_{3,h} + e^m_{3,i})\|_{L^2} \|\mathbf{e}_h^{m+1/2}\|_{L^\infty} \|\delta_t \mathbf{e}_h^{m+1}\|_{L^2} \le k\, (\|\mathbf{e}_h^m\| + h^l)\, h^{-1/2} \|\mathbf{e}_h^{m+1/2}\|\, |\delta_t \mathbf{e}_h^{m+1}| \\
&\le \varepsilon\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k\, h^{-1} \|\mathbf{e}_h^{m+1/2}\|^2 \|\mathbf{e}_h^m\|^2 + C\, k\, \|\mathbf{e}_h^{m+1/2}\|^2
\end{aligned}
$$

(here, we have used the inverse inequality $\|\mathbf{e}_h^{m+1/2}\|_{L^\infty} \le C\, h^{-1/2} \|\mathbf{e}_h^{m+1/2}\|$),

$$
J_2 \le C\, k\, \|\mathbf{e}_h^{m+1/2}\|\, |\delta_t \mathbf{e}_h^{m+1}| \le \varepsilon\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k\, \|\mathbf{e}_h^{m+1/2}\|^2
$$

On the other hand, the vertical part of I_2 and I_3 are bounded as

$$
I_2 \le C\, k \Big(\|\mathbf{e}_h^m\| + h^l \Big) \|\mathbf{u}(t_{m+1})\|_{H^3} |\delta_t \mathbf{e}_h^{m+1}| \le \varepsilon\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k \|\mathbf{e}_h^m\|^2 + C\, k\, h^{2l}
$$

$$
I_3 \le \|e^m_{3,i}\|_{H(\partial_z)}\, \|\mathbf{u}(t_{m+1})\|_{H^3}\, |\delta_t \mathbf{e}_h^{m+1}| \le \varepsilon\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k\, h^{2l}
$$

We write I_5 as

$$
I_5 = 2\, k\, c\Big(\mathbf{E}^m, \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1}\Big) - 2\, k\, c\Big(\mathbf{U}(t_m), \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1}\Big)
$$

and we bound the vertical part

$$
2\, k \Big((e^m_{3,h} + e^m_{3,i}) \partial_z \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1} \Big) - k \Big(\partial_z (e^m_{3,h} + e^m_{3,i}) \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1} \Big) = K_1 + K_2
$$

as follows

$$
\begin{aligned}
K_1 &\le 2\, k\, \|e^m_{3,h} + e^m_{3,i}\|_{L^\infty_z L^2_{\mathbf{x}}} \|\partial_z \mathbf{e}_i^{m+1}\|_{L^2_z L^2_{\mathbf{x}}} \|\delta_t \mathbf{e}_h^{m+1}\|_{L^2_z L^\infty_{\mathbf{x}}} \\
&\le C\, k\, (\|\mathbf{e}_h^m\| + \|\mathbf{e}_i^m\| + h^l)\, h^l \|\mathbf{u}(t_{m+1})\|_{H^{l+1}}\, h^{-1}\, |\delta_t \mathbf{e}_h^{m+1}| \\
&\le \varepsilon\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k\, \|\mathbf{e}_h^m\|^2 + C\, k\, h^{2l}
\end{aligned}
$$

$$
\begin{aligned}
K_2 &\le k\, \|\partial_z(e^m_{3,h} + e^m_{3,i})\|_{L^2}\, \|\mathbf{e}_i^{m+1/2}\|_{L^6}\, \|\delta_t \mathbf{e}_h^{m+1}\|_{L^3} \\
&\le C\, k\, (\|\mathbf{e}_h^m\| + \|\mathbf{e}_i^m\| + h^l)\, h^l \|\mathbf{u}(t_{m+1})\|_{H^{l+1}}\, h^{-1/2}\, |\delta_t \mathbf{e}_h^{m+1}| \\
&\le \varepsilon\, k\, |\delta_t \mathbf{e}_h^{m+1}|^2 + C\, k\, h\, \|\mathbf{e}_h^m\|^2 + C\, k\, h^{2l}
\end{aligned}
$$

Finally, taking into account the above estimates and adding from $m = 0$ to r (with any $r < M$), since owing to estimates of Theorem 6 and (**H**),

$$
k\, h^{-2} \sum_m \|\mathbf{e}_h^{m+1/2}\|^2 \le C\, h^{-2}(k + h^{2l}) \le C, \quad (l \ge 1),
$$

we can apply the discrete Gromwall's Lemma obtaining the desired estimates. ∎

4.6 $O(\sqrt{k}+h^l)$ error estimates for $e_{p,h}^{m+1}$ in $l^2(\mathbf{L}^2)$

Corollary 9 *Assuming hypotheses of Theorem 8, one has*

$$\|e_{p,h}^{m+1}\|_{l^2(L^2)} \le C\,(\sqrt{k}+h^l).$$

The proof is rather standard, starting from estimates of previous Theorems and applying the hydrostatic *Inf-Sup* condition **(H1)**.

Finally, since the constraint imposed is $k \le C\,h^2$, it is important to remark that since the order obtained at this moment is $O(\sqrt{k}+h^l) = O(h+h^l)$, then this order is optimal for $O(h)$ approximation ($l=1$). In the next Section, we study an argument to arrive at optimal order for the case $l=2$.

4.7 An alternative way for $O(h^2)$ approximation ($l=2$)

4.7.1 $O(\sqrt{k}+h^2)$ error estimates for $\delta_t \mathbf{e}_h^{m+1}$ and $\delta_t \mathbf{e}_h^{m+1/2}$ in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$.

Making $\delta_t(E_1)_h^{m+1}$ and $\delta_t(E_2)_h^{m+1}$ for each $m \ge 1$, one obtains $\forall\,\mathbf{v}_h \in \mathbf{X}_h$:

$(D_1)_h^{m+1}$

$$\left\{\begin{array}{l} \dfrac{1}{k}\Big(\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m},\mathbf{v}_h\Big)+\Big(\nabla\delta_t\mathbf{e}_h^{m+1/2},\nabla\,\mathbf{v}_h\Big)-\Big(\delta_t p_s(t_{m+1}),\nabla_{\mathbf{x}}\cdot\langle\mathbf{v}_h\rangle\Big)_S \\ \quad =\Big(\delta_t\mathcal{E}^{m+1},\mathbf{v}_h\Big)+\delta_t\mathbf{NL}_h^{m+1}(\mathbf{v}_h)-\dfrac{1}{k}\Big((\delta_t\mathbf{e}_i^{m+1}-\delta_t\mathbf{e}_i^{m}),\mathbf{v}_h\Big)-\Big(\nabla\,\delta_t\mathbf{e}_i^{m+1},\nabla\,\mathbf{v}_h\Big)\end{array}\right.$$

where,

$$\begin{aligned}\delta_t\mathbf{NL}_h^{m+1}(\mathbf{v}_h) &= -c\Big(\delta_t\mathbf{E}^m,\mathbf{u}(t_{m+1}),\mathbf{v}_h\Big)-c\Big(\delta_t\mathbf{U}_h^m,\mathbf{e}^{m+1/2},\mathbf{v}_h\Big)\\ &\quad -c\Big(\mathbf{E}^{m-1},\delta_t\mathbf{u}(t_{m+1}),\mathbf{v}_h\Big)-c\Big(\mathbf{U}_h^{m-1},\delta_t\mathbf{e}^{m+1/2},\mathbf{v}_h\Big)\end{aligned}$$

and, for all $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times Q_h$,

$(D_2)_h^{m+1}$

$$\left\{\begin{array}{l}\dfrac{1}{k}\Big(\delta_t\mathbf{e}_h^{m+1}-\delta_t\mathbf{e}_h^{m+1/2},\mathbf{v}_h\Big)+\Big(\nabla(\delta_t\mathbf{e}_h^{m+1}-\delta_t\mathbf{e}_h^{m+1/2}),\nabla\mathbf{v}_h\Big)=-\Big(\delta_t p_{s,h}^{m+1},\nabla_{\mathbf{x}}\cdot\langle\mathbf{v}_h\rangle\Big)_S\\ \Big(\nabla_{\mathbf{x}}\cdot\langle\delta_t\mathbf{e}_h^{m+1}\rangle,q_h\Big)_S=0.\end{array}\right.$$

Finally, adding $(D_1)_h^{m+1}$ and $(D_2)_h^{m+1}$ we obtain, for all $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times Q_h$:

$$(D_3)_h^{m+1}\qquad\left\{\begin{array}{l}\dfrac{1}{k}\Big(\delta_t\mathbf{e}_h^{m+1}-\delta_t\mathbf{e}_h^{m},\mathbf{v}_h\Big)+\Big(\nabla\,\delta_t\mathbf{e}_h^{m+1},\nabla\,\mathbf{v}_h\Big)+\Big(\delta_t e_{p,h}^{m+1},\nabla_{\mathbf{x}}\cdot\langle\mathbf{v}_h\rangle\Big)_S\\ \quad=\Big(\delta_t\mathcal{E}^{m+1},\mathbf{v}_h\Big)+\delta_t\mathbf{NL}_h^{m+1}(\mathbf{v}_h)-\dfrac{1}{k}\Big(\delta_t\mathbf{e}_i^{m+1}-\delta_t\mathbf{e}_i^{m},\mathbf{v}_h\Big)\\ \Big(\nabla_{\mathbf{x}}\cdot\langle\delta_t\mathbf{e}_h^{m+1}\rangle,q_h\Big)_S=0.\end{array}\right.$$

Theorem 10 *Under the hypotheses of Theorem 7 for $l=2$,* **(R4)** *and assuming the following hypothesis for the first step of the scheme*

$$|\delta_t\mathbf{e}_h^{1/2}|\le C\,(\sqrt{k}+h^2),$$

then

$$\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_h^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,(\sqrt{k}+h^2),$$
$$\|\delta_t \mathbf{e}_h^{m+1} - \delta_t \mathbf{e}_h^{m+1/2}\|_{l^2(\mathbf{L}^2)} + \|\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m}\|_{l^2(\mathbf{L}^2)} \le C\,\sqrt{k}\,(\sqrt{k}+h^2),$$

Proof. Since the initial estimate $|\delta_t \mathbf{e}_h^1| \le C\,(\sqrt{k}+h^2)$ has been assumed, it is suffices to prove the generic estimate for $\delta_t \mathbf{e}_h^{m+1}$ and $\delta_t \mathbf{e}_h^{m+1/2}$, for each $m \ge 1$.

Taking $2\,k\,\delta_t \mathbf{e}_h^{m+1/2} \in \mathbf{X}_h$ as test function in $(D_1)_h^{m+1}$, one has

$$\begin{aligned}
&|\delta_t \mathbf{e}_h^{m+1/2}|^2 - |\delta_t \mathbf{e}_h^{m}|^2 + |\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m}|^2 + 2\;k\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 \\
&= -2\,k\left(\frac{1}{k}(\delta_t \mathbf{e}_i^{m+1} - \delta_t \mathbf{e}_i^{m}), \delta_t \mathbf{e}_h^{m+1/2}\right) - 2\,k\left(\nabla\,\delta_t \mathbf{e}_i^{m+1}, \nabla \delta_t \mathbf{e}_h^{m+1/2}\right) \\
&+2\,k\Big(\delta_t p_s(t_{m+1}), \nabla_{\mathbf{x}}\cdot\langle \delta_t \mathbf{e}_h^{m+1/2}\rangle\Big) + 2\,k\Big(\delta_t \mathcal{E}^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\Big) + 2\,k\delta_t \mathbf{NL}_h^{m+1}(\delta_t \mathbf{e}_h^{m+1/2}) \\
&:= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{30}$$

We bound the RHS of (30) as in Theorem 6 (recalling that approximation $O(h^2)$ is assumed)

$$I_1 \le \varepsilon\,k\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,h^4 \int_{t_m}^{t_{m+2}} \|\mathbf{u}_{tt}\|_{H^2}^2$$

$$I_2 \le \varepsilon\,k\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,h^4 \int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|_{H^3}^2$$

$$I_3 \le \varepsilon|\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m}|^2 + C\,k^2\|\delta_t p_s(t_{m+1})\|_{H^1(S)}^2 + \varepsilon\,k\|\delta_t \mathbf{e}_h^{m}\|^2 + C\,k\,h^4\|\delta_t p_s(t_{m+1})\|_{H^2}^2$$

The bound of I_4 depending on the consistency error is not problematic.

Now, we bound the more complicate terms of I_5, again as in the proof of Theorem 6:

$$\begin{aligned}
&2\,k\,c\Big(\delta_t e_{3,h}^m, \mathbf{u}(t_{m+1}), \delta_t \mathbf{e}_h^{m+1/2}\Big) \le \varepsilon\,k(\|\delta_t \mathbf{e}_h^m\|^2 + \|\delta_t \mathbf{e}_i^m\|^2) + \varepsilon\,k\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,k\,|\delta_t \mathbf{e}_h^{m+1/2}|^2 \\
&\le \varepsilon\,k\,\|\delta_t \mathbf{e}_h^m\|^2 + C\,h^4 \int_{t_m}^{t_{m+1}} \|\mathbf{u}_t\|_{H^3}^2 + \varepsilon\,k\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,k\,|\delta_t \mathbf{e}_h^{m+1/2}|^2
\end{aligned}$$

$$\begin{aligned}
2\,k\,c\Big(\delta_t e_{3,i}^m, \mathbf{u}(t_{m+1}), \delta_t \mathbf{e}_h^{m+1/2}\Big) \quad &\le \quad \varepsilon\,k\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,k\,\|\delta_t e_{3,i}^m\|_{H(\partial_z)}^2 \\
&\le \quad \varepsilon\,k\,\|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\,h^4 \int_{t_m}^{t_{m+1}} \|\partial_t u_3\|_{H^3}^2
\end{aligned}$$

$$\begin{aligned}
&2\,k\,c\Big(\delta_t e_{3,h}^m + \delta_t e_{3,i}^m, \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\Big) \le \varepsilon\,k\left(\|\delta_t \mathbf{e}_h^m\|^2 + \|\delta_t \mathbf{e}_i^m\|^2 + \|\delta_t e_{3,i}^m\|_{H(\partial_z)}^2\right) + C\,k\,|\delta_t \mathbf{e}_h^{m+1/2}|^2 \\
&\le \varepsilon\,k\,\|\delta_t \mathbf{e}_h^m\|^2 + C\,h^4 \int_{t_m}^{t_{m+1}} (\|\mathbf{u}_t\|_{H^3}^2 + \|\partial_t u_3\|_{H^3}^2) + C\,k\,|\delta_t \mathbf{e}_h^{m+1/2}|^2
\end{aligned}$$

$$\begin{aligned}
2\,k\,c\Big(\mathbf{E}^{m-1}, \delta_t \mathbf{u}(t_{m+1}), \delta_t \mathbf{e}_h^{m+1/2}\Big) &\sim 2\,k\left(e_{3,h}^{m-1} + e_{3,i}^{m-1}, \partial_z \delta_t \mathbf{u}(t_{m+1}), \delta_t \mathbf{e}_h^{m+1}\right) \\
&+k\left(\partial_z(e_{3,h}^{m-1} + e_{3,i}^{m-1}), \delta_t \mathbf{u}(t_{m+1}), \delta_t \mathbf{e}_h^{m+1}\right) = L_1 + L_2
\end{aligned}$$

Since L_2 is easier to bound than L_1, we only bound L_1:

$$L_1 \leq \|e_{3,h}^{m-1} + e_{3,i}^{m-1}\|_{L_z^\infty L_{\mathbf{x}}^2} \|\partial_z \delta_t \mathbf{u}(t_{m+1})\|_{L_z^2 L_{\mathbf{x}}^4} \|\delta_t \mathbf{e}_h^{m+1/2}\|_{L_z^2 L_{\mathbf{x}}^4}$$
$$\leq \varepsilon\, k\, \|\mathbf{e}_h^m\|^2 + C\, k\, h^{2l} + \varepsilon\, k\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\, k\, |\delta_t \mathbf{e}_h^{m+1/2}|^2$$

On the other hand, we bound the other terms of I_5 which have not similar terms in the proof of Theorem 6:

$$c\Big(\delta_t \mathbf{U}_h^m, \mathbf{e}_h^{m+1/2}, \delta_t \mathbf{e}_h^{m+1/2}\Big) = -c\Big(\delta_t \mathbf{E}^m, \mathbf{e}_h^{m+1/2}, \delta_t \mathbf{e}_h^{m+1/2}\Big) + c\Big(\delta_t \mathbf{U}(t_m), \mathbf{e}_h^{m+1/2}, \delta_t \mathbf{e}_h^{m+1/2}\Big).$$

The second term of the RHS is bounded by $\varepsilon\, k\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\, k\, |\mathbf{e}_h^{m+1/2}|^2$. With respect to the first term on the RHS, the more complicate term to bound is the vertical part:

$$-2\, k\, c\Big(\delta_t e_3^m, \mathbf{e}_h^{m+1/2}, \delta_t \mathbf{e}_h^{m+1/2}\Big)$$
$$= -2\, k\Big((\delta_t e_{3,h}^m + \delta_t e_{3,i}^m)\, \mathbf{e}_h^{m+1/2}, \partial_z \delta_t \mathbf{e}_h^{m+1/2}\Big) + k\Big(\partial_z(\delta_t e_{3,h}^m + \delta_t e_{3,i}^m)\, \mathbf{e}_h^{m+1/2}, \delta_t \mathbf{e}_h^{m+1/2}\Big) = J_1 + J_2$$

Since J_2 is easier to bound than J_1, we only bound J_1:

$$\begin{aligned}
J_1 &\leq 2\, k \|\delta_t e_{3,h}^m + \delta_t e_{3,i}^m\|_{L_z^\infty L_{\mathbf{x}}^2} \|\mathbf{e}_h^{m+1/2}\|_{L_z^2 L_{\mathbf{x}}^\infty} \|\partial_z\, \delta_t \mathbf{e}_h^{m+1/2}\|_{L_z^2 L_{\mathbf{x}}^2} \\
&\leq C\, k\Big(\|\delta_t e_h^m\| + \|\delta_t e_i^m\| + \frac{h^2}{k}\int_{t_{m-1}}^{t_m} \|\partial_t u_3\|_{H^3}\Big) \frac{1}{h} |\mathbf{e}_h^{m+1/2}|\, \|\delta_t \mathbf{e}_h^{m+1/2}\| \\
&\leq C\, k\, \Big(\frac{1}{h}|\delta_t \mathbf{e}_h^m| + \frac{h^2}{k}\int_{t_{m-1}}^{t_m} (\|\mathbf{u}_t\|_{H^3} + \|\partial_t u_3\|_{H^3})\Big) \frac{1}{h}\, |\mathbf{e}_h^{m+1/2}|\, \|\delta_t \mathbf{e}_h^{m+1/2}\| \\
&\leq \varepsilon\, k\, \|\delta_t \mathbf{e}_h^{m+1/2}\|^2 + C\, k\, \frac{1}{h^4}\Big(|\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m|^2 + |\mathbf{e}_h^m|^2\Big) |\delta_t \mathbf{e}_h^m|^2 \\
&+ C\, h^2 \left(\int_{t_{m-1}}^{t_m} \|\mathbf{U}_t\|_{H^3}^2\right) \Big(|\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^m|^2 + |\mathbf{e}_h^m|^2\Big)
\end{aligned}$$

here, we have used the inverse inequalities $\|\mathbf{v}_h\|_{H_{\mathbf{x}}^1} \leq C\, h^{-1}\, \|\mathbf{v}_h\|_{L_{\mathbf{x}}^2}$ and $\|\mathbf{v}_h\|_{L_{\mathbf{x}}^\infty} \leq C\, h^{-1}\, \|\mathbf{v}_h\|_{L_{\mathbf{x}}^2}$.

Notice that to apply later the discrete Gromwall's Lemma using the term $C\, k\, \frac{1}{h^4}\, |\mathbf{e}_h^m|^2\, |\delta_t \mathbf{e}_h^m|^2$, it is necessary that $k\, \frac{1}{h^4}\, \sum_m |\mathbf{e}_h^m|^2 \leq C$ and this is true for $l = 2$. In the last section, we will see a way to obtain the estimates for $l = 1$, in the case of vertical structured grids.

On the other hand, we have to bound

$$\begin{aligned}
c\Big(\mathbf{U}_h^{m-1}, \delta_t \mathbf{e}^{m+1/2}, \delta_t \mathbf{e}_h^{m+1/2}\Big) &= c\Big(\mathbf{U}_h^{m-1}, \delta_t \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\Big) \\
&= -c\Big(\mathbf{E}^{m-1}, \delta_t \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\Big) + c\Big(\mathbf{U}(t_{m-1}), \delta_t \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\Big)
\end{aligned}$$

(here we have used that $c\Big(\mathbf{U}_h^{m-1}, \delta_t \mathbf{e}_h^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\Big) = 0$). The more complicate term is the vertical part:

$$2\, k\, c\Big(e_3^{m-1}, \delta_t \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_i^{m+1}\Big)$$
$$= 2\, k\Big((e_{3,h}^{m-1} + e_{3,i}^{m-1})\, \delta_t \mathbf{e}_i^{m+1}, \partial_z\, \delta_t \mathbf{e}_h^{m+1/2}\Big) - k\Big(\partial_z(e_{3,h}^{m-1} + e_{3,i}^{m-1})\, \delta_t \mathbf{e}_i^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}\Big) = K_1 + K_2$$

Since K_2 is easier to bound than K_1, we only bound K_1:

$$\begin{array}{rcl} K_1 & \leq & 2\,k\,\|e_{3,h}^{m-1}+e_{3,i}^{m-1}\|_{L_z^\infty L_{\mathbf{x}}^2}\|\delta_t\mathbf{e}_i^{m+1}\|_{L_z^2L_{\mathbf{x}}^2}\|\partial_z\delta_t\mathbf{e}_h^{m+1}\|_{L_z^2L_{\mathbf{x}}^\infty} \\ & \leq & C\,k\,(\|\mathbf{e}_h^{m-1}\|+h^2\,\|\mathbf{U}(t_{m-1})\|_{H^3})\,h^2k^{-1}(\displaystyle\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t(t_{m+1})\|_{H^2})\,h^{-1}\,\|\delta_t\mathbf{e}_h^{m+1}\| \\ & \leq & \varepsilon\;k\,\|\delta_t\mathbf{e}_h^{m+1}\|^2+C\,h^2\displaystyle\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t(t_{m+1})\|_{H^2}^2(\|\mathbf{e}_h^{m-1}\|^2+h^2) \end{array}$$

On the other hand, making $2\,k\left((D_2)_h^m,\delta_t\mathbf{e}_h^{m+1}\right)$, we arrive at

$$\begin{array}{l} |\delta_t\mathbf{e}_h^{m+1}|^2-|\delta_t\mathbf{e}_h^{m+1/2}|^2+|\delta_t\mathbf{e}_h^{m+1}-\delta_t\mathbf{e}_h^{m+1/2}|^2 \\ +k\left\{\|\delta_t\mathbf{e}_h^{m+1}\|^2-\|\delta_t\mathbf{e}_h^{m+1/2}\|^2+\|\delta_t\mathbf{e}_h^{m+1}-\delta_t\mathbf{e}_h^{m+1/2}\|^2\right\}=0 \end{array} \tag{31}$$

Reasoning as in Theorem 6, adding (30) and (31) from $m=0$ to r (with any $r<M$), taking into account the previous estimates and ε and k small enough, we can apply the discrete Gromwall's Lemma obtaining the desired estimates. ■

4.7.2 $O(k+h^2))$ error estimates for $\delta_t\mathbf{e}_h^{m+1}$ in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$

Theorem 11 *Under the hypotheses of Theorem 10 and* **(R5)**, *assuming the following hypothesis for the first step of the scheme*

$$|\delta_t\mathbf{e}_h^1|\leq C\,(k+h^2),$$

then

$$\|\delta_t\mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}\leq C\,(k+h^2).$$

Proof. The main idea is to make $2\,k\left((D_3)_h^{m+1},\delta_t\mathbf{e}_h^{m+1}\right)$. Now, the pressure term vanish but the term $c\Big(\mathbf{U}_h^{m-1},\delta_t\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)\neq 0$ and can be decomposed as:

$$\begin{array}{l} 2\,k\,c\Big(\mathbf{U}_h^{m-1},\delta_t\mathbf{e}_h^{m+1/2},\delta_t\mathbf{e}_h^{m+1}\Big)=2\,k\,c\Big(\mathbf{U}_h^{m-1},\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1},\delta_t\mathbf{e}_h^{m+1}\Big) \\ =-2\,k\,c\Big(\mathbf{E}^{m-1},\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1},\delta_t\mathbf{e}_h^{m+1}\Big)+2\,k\,c\Big(\mathbf{U}(t_{m-1}),\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1},\delta_t\mathbf{e}_h^{m+1}\Big)=I_1+I_2. \end{array}$$

The more complicate term in I_1 is the vertical part:

$$\begin{array}{l} k\left((e_{3,h}^{m-1}+e_{3,i}^{m-1})\,(\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1}),\partial_z\delta_t\mathbf{e}_h^{m+1}\right) \\ \leq k\|e_{3,h}^{m-1}+e_{3,i}^{m-1}\|_{L_z^\infty L_{\mathbf{x}}^2}\|\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1}\|_{L_z^2L_{\mathbf{x}}^\infty}\|\partial_z\delta_t\mathbf{e}_h^{m+1}\|_{L_z^2L_{\mathbf{x}}^2} \\ \leq \dfrac{k}{h^2}|\mathbf{e}_h^{m-1}|\,|\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1}|\,\|\delta_t\mathbf{e}_h^{m+1}\|+\dfrac{k}{h}\|\mathbf{e}_i^{m-1}\|\,|\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1}|\,\|\delta_t\mathbf{e}_h^{m+1}\| \\ +C\dfrac{k}{h}\,\|e_{3,i}^{m-1}\|_{L_z^\infty L_{\mathbf{x}}^2}\,|\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1}|\,\|\delta_t\mathbf{e}_h^{m+1}\| \\ \leq\varepsilon\,k\,\|\delta_t\mathbf{e}_h^{m+1}\|^2+C\,\dfrac{k}{h^4}|\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1}|^2\;|\mathbf{e}_h^{m-1}|^2 \\ +C\,k\,h^2\,|\delta_t\mathbf{e}_h^{m+1/2}-\delta_t\mathbf{e}_h^{m+1}|^2 \end{array}$$

(here, we have used the inverse inequalities $\|\mathbf{v}\|_{L^\infty_{\mathbf{x}}} \le C\,h^{-1}\,|\mathbf{v}|_{L^2_{\mathbf{x}}}$ and $\|\mathbf{v}\| \le C\,h^{-1}\,|\mathbf{v}|$ and the estimates (21) and (23)). Since $|\mathbf{e}_h^{m-1}|^2 \le C\,(k^2+h^4)$, adding the third term of the RHS of the previous inequality,

$$k\,h^{-4}\sum_m |\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m+1}|^2 |\mathbf{e}_h^{m-1}|^2 \le k\,h^{-4}(k+h^4)(k^2+h^4) \le C\,(k^2+h^4),$$

where (**H**) and estimates of Theorem 10 have been used.

The more complicate term in I_2 is the vertical part:

$$2\,k\,c\Big(u_3(t_{m-1}), \delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m+1}, \delta_t \mathbf{e}_h^{m+1}\Big) \le \varepsilon\,k\,\|\delta_t \mathbf{e}_h^{m+1}\|^2 + C\,k\,|\delta_t \mathbf{e}_h^{m+1/2} - \delta_t \mathbf{e}_h^{m+1}|^2$$

Adding from $m = 0$ to r (with any $r < M$), we can apply the discrete Gromwall's Lemma obtaining the desired estimates. ■

Again, from estimates of previous Theorems and applying the hydrostatic *Inf-Sup* condition **(H1)**, we arrive at the following optimal error estimates for the pressure.

Corollary 12 *Assuming hypotheses of Theorem 11, one has*

$$\|e_{p,h}^{m+1}\|_{l^2(L^2)} \le C\,(k+h^2).$$

Since the constraint **(H)** impose $k \le C\,h^2$, the order obtained is $O(k+h^2) = O(h^2)$ whence the time and space approximations are of the same order.

Notice that the previous estimate for the pressure is not obtained in norm $l^\infty(L^2)$, due to the convection term depending on the intermediate error $\mathbf{e}_h^{m+1/2}$ which has not optimal approximation in $l^\infty(\mathbf{L}^2)$, only in $l^2(\mathbf{L}^2)$.

5 Vertical structured meshes

In this section, we will approximate the problem (Q) using appropriate vertical structured grids of finite elements. We consider the variational formulation of (Q) verified for an exact solution $(\mathbf{u}, p_s)$ at $t = t_{m+1}$:

$$(Q)_w^{m+1} \qquad \left\{ \begin{array}{l} \Big(\dfrac{1}{k}(\mathbf{u}(t_{m+1}) - \mathbf{u}(t_m)), \mathbf{v}\Big) + c\Big(\mathbf{U}(t_m), \mathbf{u}(t_{m+1}), \mathbf{v}\Big) + \Big(\nabla \mathbf{u}(t_{m+1}), \nabla \mathbf{v}\Big) \\ -\Big(p_s(t_{m+1}), \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle\Big)_S = \Big\langle \mathbf{f}(t_{m+1}), \mathbf{v}\Big\rangle_\Omega + \Big\langle \mathbf{g}_s(t_{m+1}), \mathbf{v}\Big\rangle_{\Gamma_s} \\ +\Big(\mathcal{E}^{m+1}, \mathbf{v}\Big), \quad \forall\,\mathbf{v} \in \mathbf{W}_{b,l}^{1,3} \cap \mathbf{L}^\infty \\ \Big(\nabla_{\mathbf{x}} \cdot \langle \mathbf{u}(t_{m+1}) \rangle, q\Big)_S = 0, \quad \forall\,q \in L^2_0(S) \end{array} \right.$$

where

$$u_3(t_m; \mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{u}(t_m; \mathbf{x}, s)\,ds.$$

To discretize the domain, we consider $\mathcal{T}_h(\Omega)$ a regular and quasi-uniform triangulation of Ω (with elements $K \in \mathcal{T}_h(\Omega)$) and $\mathcal{T}_h(S)$ its associated triangulation of S (with elements $T \in \mathcal{T}_h(S)$). In this case of vertical structured grids, more specific properties can be obtained. More about how to construct vertical structured meshes can be seen in [9], [10], [11] by using the so-called *iso-σ layers* or in [18] by using a $\boldsymbol{P}_0$ approximation on the bottom.

We consider three families of finite element spaces: $\mathbf{X}_h \subset \mathbf{H}^1_{b,l}(\Omega)$ for the horizontal velocity, $Y_h \subset H(\partial_z)$ for the vertical velocity and $Q_h \subset L^2_0(S)$ for the pressure. Functions of $\mathbf{X}_h$ are globally continuous, whereas functions in Y_h must be globally continuous only respect to vertical direction and Q_h could be discontinuous functions.

For vertical structured grids formed by prims, there are several possibilities to choose $\mathbf{X}_h$. For instance, considering a bubble by each tetrahedron (subdividing the prims) as can be seen in [9], or a bubble by prism or a bubble by each column of vertical prisms as in [20]. Notice that these possibilities are not stables for the Navier-Stokes case. On the other hand, in the case of right prims, some possibilities to choose Y_h are [18]:

- $\boldsymbol{P}_0(\mathbf{x}) \otimes \boldsymbol{P}_1(z)$ and z-continuous (for $l = 1$),
- $\boldsymbol{P}_1(\mathbf{x}) \otimes \boldsymbol{P}_2(z)$ continuous and z-C^1 (for $l = 2$).

Finally, for these structured meshes formed by vertical prims, it is easy to compute the vertical integrals of the variational formulation of (Q).

5.1 $O(k + h^{l+1})$ for $\mathbf{e}_h^{m+1}$ in $l^2(\mathbf{L}^2)$

To obtain the results of this subsection, we change the computation of the vertical velocity, which now will be computed as

$$u^m_{3,h}(\mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{u}^m_h(\mathbf{x}, s)\, ds.$$

Accordingly, the vertical interpolation operator can be defined as

$$K_h u_3(\mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}} \cdot I_h \mathbf{u}(\mathbf{x}, s)\, ds,$$

hence $e^m_{3,i} = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{e}^m_i$.

Theorem 13 *Assuming hypotheses of Theorem 7, and $\|A_h^{-1}\mathbf{e}_h^0\| \le C\, h^3$ (recall that A_h is the hydrostatic Stokes operator defined in (15)), one has, for each k small enough,*

$$\|\mathbf{e}_h^{m+1}\|_{l^2(\mathbf{L}^2)} \le C(k + h^{l+1}). \qquad (32)$$

Proof. Taking $\mathbf{v}_h = A_h^{-1}\mathbf{e}_h^{m+1} \in \mathbf{V}_h$ in $(E)_h^{m+1}$, we obtain

$$\|A_h^{-1}\,\mathbf{e}_h^{m+1}\|^2 - \|A_h^{-1}\,\mathbf{e}_h^{m}\|^2 + \|A_h^{-1}\,\mathbf{e}_h^{m+1} - A_h^{-1}\,\mathbf{e}_h^{m}\|^2 + 2\,k|\mathbf{e}_h^{m+1}|^2$$
$$\le 2\,k\,\mathbf{NL}_h^{m+1}(A_h^{-1}\,\mathbf{e}_h^{m+1}) - 2\,k\Big(\delta_t\mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) + 2\,k\Big(\mathcal{E}^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) := I_1 + I_2 + I_3.$$

We bound the more complicate terms of I_1.

$$2\,k\,c\Big(u_{3,h}^m, \mathbf{e}_h^{m+1/2}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) = -2\,k\,c\Big(e_{3,h}^m, \mathbf{e}_h^{m+1/2}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) + 2\,k\,c\Big(u_3(t_m), \mathbf{e}_h^{m+1/2}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) = L_1 + L_2$$

$$\begin{aligned}
L_1 &= k\left(\partial_z e_{3,h}^m\,\mathbf{e}_h^{m+1/2}, A_h^{-1}\mathbf{e}_h^{m+1}\right) - 2\,k\left(e_{3,h}^m\,\mathbf{e}_h^{m+1/2}, \partial_z A_h^{-1}\mathbf{e}_h^{m+1}\right)\\
&\le k\left(\|\partial_z e_{3,h}^m\|_{L^2}\|\mathbf{e}_h^{m+1/2}\|_{L^3}\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|_{L^6} + \|e_{3,h}^m\|_{L_z^\infty L_{\mathbf{x}}^2}\,\|\mathbf{e}_h^{m+1/2}\|_{L_z^2 L_{\mathbf{x}}^\infty}\,\|\partial_z A_h^{-1}\mathbf{e}_h^{m+1}\|_{L_z^2 L_{\mathbf{x}}^2}\right)\\
&\le C\,k\left(\|\mathbf{e}_h^m\| + \|\mathbf{e}_i^m\|\right)(h^{-1/2} + h^{-1})\left(|\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}| + |\mathbf{e}_h^{m+1}|\right)\|A_h^{-1}\mathbf{e}_h^{m+1}\|\\
&\le \varepsilon\,k\,|\mathbf{e}_h^{m+1}|^2 + \varepsilon\,k|\mathbf{e}_h^{m+1} - \mathbf{e}_h^{m+1/2}|^2 + C\frac{k}{h^2}\left(\|\mathbf{e}_h^m\|^2 + h^{2l}\right)\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2
\end{aligned}$$

(here, we use the inverse inequalities $\|\cdot\|_{L^3} \le h^{-1/2}|\cdot|$ in $3D$ and $\|\cdot\|_{L_{\mathbf{x}}^\infty} \le h^{-1}\|\cdot\|_{L_{\mathbf{x}}^2}$ in $2D$).

$$\begin{aligned}
L_2 &\le C\,k\left(\|u_3(t_m)\|_{L^\infty} + \|\partial_z u_3(t_m)\|_{L^3}\right)\left(|\mathbf{e}_h^{m+1}| + |\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^{m+1}|\right)\|A_h^{-1}\mathbf{e}_h^{m+1}\|\\
&\le \varepsilon\,k\left(|\mathbf{e}_h^{m+1}|^2 + |\mathbf{e}_h^{m+1/2} - \mathbf{e}_h^{m+1}|^2\right) + C\,k\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2
\end{aligned}$$

The vertical part of $2\,k\,c\Big(\mathbf{U}_h^m, \mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big)$ is

$$2\,k\,c\Big(u_{3,h}^m, \mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) = 2\,k\,c\Big(e_{3,h}^m + e_{3,i}^m, \mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) - 2\,k\,c\Big(u_3(t_m), \mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big)$$

and its more complicate term is

$$2\,k\,c\Big(e_{3,h}^m, \mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) = 2\,k\left(e_{3,h}^m\,\mathbf{e}_i^{m+1}, \partial_z A_h^{-1}\mathbf{e}_h^{m+1}\right) + k\left(\partial_z e_{3,h}^m\,\mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\right) = N_1 + N_2$$

and we bound

$$\begin{aligned}
N_1 &\le k\,\|e_{3,h}^m\|_{L_z^\infty L_{\mathbf{x}}^2}\,\|\mathbf{e}_i^{m+1}\|_{L_z^2 L_{\mathbf{x}}^2}\,\|\partial_z A_h^{-1}\mathbf{e}_h^{m+1}\|_{L^2 L_{\mathbf{x}}^\infty} \le C\,k\,\|\mathbf{e}_h^m\|\,|\mathbf{e}_i^{m+1}|\,h^{-1}\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|\\
&\le C\,\frac{k}{h^2}\|\mathbf{e}_h^m\|^2\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2 + C\,k\,h^{2(l+1)}
\end{aligned}$$

$$\begin{aligned}
N_2 &\le k\,\|\nabla_{\mathbf{x}}\cdot\mathbf{e}_h^m\|_{L^3}\|\mathbf{e}_i^{m+1}\|_{L^2}\|A_h^{-1}\mathbf{e}_h^{m+1}\|_{L^6} \le C\,k\,h^{-1/2}\|\mathbf{e}_h^m\|\,|\mathbf{e}_i^{m+1}|\|A_h^{-1}\mathbf{e}_h^{m+1}\|\\
&\le C\,\frac{k}{h}\|\mathbf{e}_h^m\|^2\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2 + C\,k\,h^{2(l+1)}
\end{aligned}$$

Finally, by a similar way, we bound

$$\begin{aligned}
2\,k\,c\Big(e_{3,i}^m, \mathbf{e}_i^{m+1}, A_h^{-1}\mathbf{e}_h^{m+1}\Big) &\le C\,k\left(\|\mathbf{e}_i^m\|\,|\mathbf{e}_i^{m+1}|\,h^{-1}\|A_h^{-1}\mathbf{e}_h^{m+1}\| + h^{-1/2}\|\mathbf{e}_i^m\|\,|\mathbf{e}_i^{m+1}|\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|\right)\\
&\le C\,k\,|\mathbf{e}_i^{m+1}|\,\|A_h^{-1}\mathbf{e}_h^{m+1}\| \le C\,k\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2 + C\,k\,h^{2(l+1)}
\end{aligned}$$

Other conflictive term of I_1 is:

$$2\,k\,c\Big(e^m_{3,h},\mathbf{u}(t_{m+1}),A_h^{-1}\mathbf{e}_h^{m+1}\Big)=2\,k\,\Big(e^m_{3,h}\,\partial_z\mathbf{u}(t_{m+1}),A_h^{-1}\mathbf{e}_h^{m+1}\Big)+k\,\Big(\partial_z e^m_{3,h}\,\mathbf{u}(t_{m+1}),A_h^{-1}\mathbf{e}_h^{m+1}\Big)=J_1+J_2$$

To bound J_1, it is necessary the explicit expression of $e^m_{3,h}=\int_z^0\nabla_{\mathbf{x}}\cdot\mathbf{e}^m_h=\nabla_{\mathbf{x}}\cdot\int_z^0\mathbf{e}^m_h$ which let us to integrate by parts,

$$\begin{aligned}
J_1 &= k\,\Big(\nabla_{\mathbf{x}}\cdot(\int_z^0\mathbf{e}^m_h)\,\partial_z\mathbf{u}(t_{m+1}),A_h^{-1}\mathbf{e}_h^{m+1}\Big)\\
&= -k\,\Big(\int_z^0\mathbf{e}^m_h,\nabla_{\mathbf{x}}\partial_z\mathbf{u}(t_{m+1})\,A_h^{-1}\mathbf{e}_h^{m+1}+\partial_z\mathbf{u}(t_{m+1})\,\nabla_{\mathbf{x}}A_h^{-1}\mathbf{e}_h^{m+1}\Big)\\
&\le k\,\|\int_z^0\mathbf{e}^m_h\|_{L^\infty_zL^2_{\mathbf{x}}}\Big(\|\nabla_{\mathbf{x}}\partial_z\mathbf{u}(t_{m+1})\|_{L^{6/5}_zL^3_{\mathbf{x}}}\|A_h^{-1}\mathbf{e}_h^{m+1}\|_{L^6_zL^6_{\mathbf{x}}}+\|\partial_z\mathbf{u}(t_{m+1})\|_{L^2_zL^\infty_{\mathbf{x}}}\|\nabla_{\mathbf{x}}A_h^{-1}\mathbf{e}_h^{m+1}\|_{L^2_zL^2_{\mathbf{x}}}\Big)\\
&\le \varepsilon\,k\,|\mathbf{e}^m_h|^2+C\,k\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2
\end{aligned}$$

In a similar way, by using that $\partial_z e^m_{3,h}=-\nabla_{\mathbf{x}}\cdot\mathbf{e}^m_h$,

$$\begin{aligned}
J_2 &\le k\,|\mathbf{e}^m_h|\Big(\|\nabla_{\mathbf{x}}\mathbf{u}(t_{m+1})\|_{L^3}\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|_{L^6}+\|\mathbf{u}(t_{m+1})\|_{L^\infty}|\nabla_{\mathbf{x}}A_h^{-1}\mathbf{e}_h^{m+1}|\Big)\\
&\le \varepsilon\,k\,|\mathbf{e}^m_h|^2+C\,k\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2
\end{aligned}$$

On the other hand,

$$\begin{aligned}
I_2 &= -2\,k\Big(\delta_t\mathbf{e}_i^{m+1},A_h^{-1}\,\mathbf{e}_h^{m+1}\Big)\le C\,k\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2+\varepsilon\,k|\mathbf{e}_i(\delta_t\mathbf{u}^{m+1})|^2\\
&\le C\,k\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2+\varepsilon\,k\,h^{2(l+1)}\|\delta_t\mathbf{u}^{m+1}\|^2_{H^{l+1}}\\
&\le C\,k\,\|A_h^{-1}\mathbf{e}_h^{m+1}\|^2+\varepsilon\,h^{2(l+1)}\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^2_{H^{l+1}}.
\end{aligned}$$

Adding from $m=0$ to r (with any $r<M$), since thanks to (**H**) one has $\frac{1}{h^2}\,\|\mathbf{e}^m_h\|^2\le C$ and $k\sum_m|\mathbf{e}_h^{m+1/2}-\mathbf{e}^m_h|^2\le k(k+h^{2l})\le C(k^2+h^{2(l+1)})$, we can apply the generalized discrete Gromwall's Lemma obtaining the desired estimates, for k small enough. ■

5.2 $O(k+h^l)$ for $e_{p,h}$ in $l^2(L^2)$

Owing to the improvement obtained in the above subsection, now we can prove the error estimates obtained in Theorems 10 and 11 for $l=1$.

Indeed, in the proof of Theorem 10, we can apply the discrete Gromwall's Lemma, since the term $C\,k\,\dfrac{1}{h^4}\displaystyle\sum_{m=0}^r|\mathbf{e}^m_h|^2\,|\delta_t\mathbf{e}^m_h|^2$ appear, and now for $l=1$, owing to (32), we have

$$k\,\frac{1}{h^4}\sum_{m=0}^r|\mathbf{e}^m_h|^2\le C.$$

The rest of the proof is similar. Consequently, one has

Corollary 14 *Assuming hypotheses of Theorem 13 and Theorem 11, one obtains*

$$\|e^{m+1}_{p,h}\|_{l^2(L^2)}\le C\,(k+h^l).$$

6 Scheme with Coriolis term

Looking at the results obtained in previous sections, the more convenient forms to introduce the Coriolis term in the scheme are the following (the Coriolis term is always refereed at the end-of-step velocity $\mathbf{u}^{m+1}$ because this is the better approximation in this scheme):

- To consider in $(S_2)_h^{m+1}$ the implicit term $\mathbf{b}(\mathbf{u}_h^{m+1})$. This term gives optimal stability, because $(\mathbf{b}(\mathbf{u}_h^{m+1}), \mathbf{u}_h^{m+1}) = 0$, however, with respect to the error estimates is a conflictive term, because it introduces the extra-term $\mathbf{b}(\mathbf{u}_h^{m+1})$ in $(E_2)_h^{m+1}$, which implies that we can not obtain the $O(k^{1/2})$ error estimates with similar arguments.

- To consider in $(S_1)_h^{m+1}$ the explicit term $\mathbf{b}(\mathbf{u}_h^m)$. This term introduces the extra-term $\Big(\mathbf{b}(\mathbf{u}_h^m), \mathbf{u}_h^{m+1/2}\Big) = \Big(\mathbf{b}(\mathbf{u}_h^m), \mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m\Big)$ in the stability estimates, which produces in the stability estimates an artificial exponential bound in time. Indeed, bounding as follows,

$$\Big(\mathbf{b}(\mathbf{u}_h^m), \mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m\Big) \le C\,|\mathbf{u}_h^m|^2 + \varepsilon|\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m|^2$$

  and applying the discrete Gronwall inequality in the proof of the stability, the new exponential bound appears. With respect to the error estimates, this term does not add new difficulties.

  On the other hand, now the computation for velocity $\mathbf{u}_h^{m+1/2}$ follows decoupled.

- To consider the following Coriolis correction scheme: we consider in $(S_1)_h^{m+1}$ the explicit term $\mathbf{b}(\mathbf{u}_h^m)$ and in $(S_2)_h^{m+1}$ a correction term $\mathbf{b}(\mathbf{u}_h^{m+1} - \mathbf{u}_h^m)$. This correction works well with respect to the consistency and stability of the scheme. Moreover, now again $(S_2)_h^{m+1}$ can be written as a projection step (written respect to $\mathbf{u}_h^{m+1} - \mathbf{u}_h^m$). Finally, this correction term is not conflictive respect to the error estimates.

  On the other hand, this option is better that the first one with respect to the implementation and the error estimates, and is a little better than the second option for stability and consistency and is a little worse for the implementation.

Appendix

Proof [of Lemma 5]:

We consider $A^{-1}\mathbf{v} \in \mathbf{V}$ the solution of the hydrostatic Stokes Problem with second member $\mathbf{v} \in \mathbf{V}$. This solution verifies (see [12] for the Stokes case)

$$\|A^{-1}\mathbf{v}_h - A_h^{-1}\mathbf{v}_h\| \le C\,h\,|\mathbf{v}_h| \tag{33}$$

On the other hand, we decompose $A_h^{-1}\mathbf{v}_h = A_h^{-1}\mathbf{v}_h - \widetilde{I}_h\, A^{-1}\mathbf{v}_h + \widetilde{I}_h\, A^{-1}\mathbf{v}_h$, where $\widetilde{I}_h$ is an interpolator with respect to $\mathbf{X}_h$. Then,

$$\|A_h^{-1}\mathbf{v}_h\|_{W^{1,6}} \le \|A_h^{-1}\mathbf{v}_h - \widetilde{I}_h A^{-1}\mathbf{v}_h\|_{W^{1,6}} + \|\widetilde{I}_h\, A^{-1}\mathbf{v}_h\|_{W^{1,6}}$$

We bound the RHS as follows, using in particular hypothesis **(H0)**,

$$\|\widetilde{I}_h A^{-1}\mathbf{v}_h\|_{W^{1,6}} \le C\, \|A^{-1}\mathbf{v}_h\|_{W^{1,6}} \le C\|A^{-1}\mathbf{v}_h\|_{H^2} \le C\,|\mathbf{v}_h|$$

(here the stability property $\|\widetilde{I}_h\mathbf{v}_h\|_{W^{1,6}} \le C\|\mathbf{v}_h\|_{W^{1,6}}$ is applied)

$$\|A_h^{-1}\mathbf{v}_h - \widetilde{I}_h A^{-1}\mathbf{v}_h\|_{W^{1,6}} \le \frac{C}{h}\|A_h^{-1}\mathbf{v}_h - \widetilde{I}_h A^{-1}\mathbf{v}_h\| \le \frac{C}{h}\Big(\|A_h^{-1}\mathbf{v}_h - A^{-1}\mathbf{v}_h\| + \|A^{-1}\mathbf{v}_h - \widetilde{I}_h A^{-1}\mathbf{v}_h\|\Big)$$

(here we have used the inverse inequality $\|\mathbf{v}_h\|_{W^{1,6}} \le C\, h^{-1}\|\mathbf{v}_h\|$ for each $\mathbf{v}_h \in \mathbf{X}_h$).

Finally, applying (33) and the error interpolation inequality $\|A^{-1}\mathbf{v}_h - \widetilde{I}_h\, A^{-1}\mathbf{v}_h\| \le C\, h\, \|A^{-1}\mathbf{v}_h\|_{H^2} \le C\, h|\mathbf{v}_h|$, we arrive at $\|A_h^{-1}\mathbf{v}_h - \widetilde{I}_h\, A^{-1}\mathbf{v}_h\|_{W^{1,6}} \le C\,|\mathbf{v}_h|$. Therefore, we conclude

$$\|A_h^{-1}\mathbf{v}_h\|_{W^{1,6}} \le C\,|\mathbf{v}_h|.$$

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Numerical analysis of an incremental pressure scheme in time for the Primitive Equations *

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Abstract

The purpose of this paper is the numerical analysis of a first order scheme in time, using a projection method with incremental pressure, for the model of Primitive Equations of the Ocean. First, we will prove unconditional stability and convergence towards a weak solution of the continuous problem. Afterwards, for a regular enough solution, we will prove optimal error estimates for the velocity as well as for the pressure.

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Introduction

Assuming some simplifications (basically hydrostatic pressure and "the rigid lid" hypothesis), the $3D$ Navier-Stokes equations derive to the so-called "Primitive Equations" (or the Navier-Stokes equations with hydrostatic pressure). These equations are a general mathematical model in the field of geophysical fluids ([20, 23]). In particular, they describe the general circulation of the water in lakes and oceans [21]. For simplicity, we take constant density, Cartesian coordinates (x in the easterly direction, y in the northerly direction and z perpendicular to the surface of the Earth) and we assume that the effects due to temperature and salinity can be decoupled from the flow dynamics.

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Let us consider $\Omega = \{(\mathbf{x}, z) \in I\!R^3 \;/\; \mathbf{x} = (x, y) \in S,\; -D(\mathbf{x}) < z < 0\}$ the $3D$ domain filled by the water, with $S \subset I\!R^2$ the surface domain (a regular bounded $2D$ domain) and $D : \overline{S} \to I\!R_+$ (with $D > 0$ in S) the function describing the bottom. Then, $\Gamma_s = \overline{S} \times \{0\}$ is the part of the boundary of Ω corresponding to the surface, $\Gamma_b = \{(\mathbf{x}, -D(\mathbf{x})) :\; \mathbf{x} \in S\}$ corresponds to the bottom (with outwards normal vector $(\mathbf{n_x}, n_3)$) and $\Gamma_l = \{(\mathbf{x}, z) :\; \mathbf{x} \in \partial S, -D(\mathbf{x}) < z < 0\}$ correspond to the lateral walls.

The unknowns of the problem are $\mathbf{U} = (\mathbf{u}, u_3) : \Omega \times (0, T) \to I\!R^3$ the 3D velocity field (with $\mathbf{u} = (u_1, u_2)$ the corresponding horizontal velocity and u_3 the vertical velocity) and $p_s : S \times (0, T) \to I\!R$ a potential function which is defined only on the surface S, that it will be called the surface pressure.

Then, the differential model governed by the Primitive Equations can be written as (ver [20, 21]):

$$
(EP) \qquad \left\{
\begin{array}{rcll}
\partial_t \mathbf{u} + (\mathbf{U} \cdot \nabla)\mathbf{u} - \nu\, \Delta \mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_H\, p_s & = & \mathbf{f} & \text{en } \Omega \times (0, T), \\
u_3(t; \mathbf{x}, z) = \displaystyle\int_z^0 \nabla_H \cdot \mathbf{u}(t; \mathbf{x}, s)\, ds \qquad \nabla_H \cdot \langle \mathbf{u} \rangle & = & 0 & \text{en } S \times (0, T), \\
\mathbf{u}|_{\Gamma_b \cup \Gamma_l} = 0, \quad \nu \partial_z \mathbf{u}|_{\Gamma_s} = \mathbf{g}_s, \quad \mathbf{u}_{|t=0} & = & \mathbf{u}_0 & \text{en } \Omega,
\end{array}
\right.
$$

where $\langle \mathbf{u} \rangle (t; \mathbf{x}) = \int_{-D(\mathbf{x})}^0 \mathbf{u}(t; \mathbf{x}, z)\, dz$.

Also, $\mathbf{b}(\mathbf{u}) = f \mathbf{u}^\perp$ represents the effect of the Coriolis Forces, with $\mathbf{u}^\perp = (-u_2, u_1)^t$ and $f = 2|w| \sin\theta$, where w is the angular velocity of the Earth and $\theta = \theta(y)$ is the latitude, $\rho \in I\!R_+$ is the water density (that it is assumed a positive constant), $g \in I\!R_+$ is the gravity acceleration (another positive constant), $\mathbf{f} : \Omega \times (0, T) \to I\!R^2$ is a field of external horizontal forces (depending for instance on the salinity and temperature) and $\mathbf{g}_s : \Gamma_s \times (0, T) \to I\!R^2$ represents the stress of the wind on the surface.

Finally, $\nabla = (\nabla_H, \partial_z)^t$ stands for the threedimensional gradient operator (with $\nabla_H = (\partial_x, \partial_y)^t$ its horizontal component) y Δ stands for the threedimensional Laplacian operator.

For simplicity, we have considered in (P) isotropic diffusion, which is written as $-\nu \Delta \mathbf{u}$, where $\nu > 0$ is a viscosity coefficient. In general, due to the difference between the horizontal and vertical dimensions of the domain, it is usual to consider anisotropic (eddy) diffusion; for instance

$$-\nabla_H \cdot (\nu_h \nabla_H \mathbf{u}) - \nu_v\, \partial_z^2 \mathbf{u}$$

where $\nu_h,\; \nu_v > 0$ are the horizontal and vertical eddy coefficients respectively, being $\nu_v \ll \nu_h$ ([23]). The results of this paper can be easily extended to this case.

Let us now to present some mathematical results about the problem (EP). The existence of a weak solution $(\mathbf{u}, p_s)$ of the problem (EP) is well known, see Lewandowski [19] and Lions-Temam-Wang [21], always in domains with side-walls (i.e. $D \geq D_{min} > 0$ in $\overline{S}$). In these works, a compactness method is used to obtain the velocity $\mathbf{u}$ in a space with the restriction $\nabla \cdot \langle \mathbf{u} \rangle = 0$ and afterwards, the surface pressure p_s is obtained, by a specific De Rham's lemma on the surface S. In domains without side-walls (i.e. when the depth function D can degenerate to zero), the existence of a weak solution $(\mathbf{u}, u_3, p_s)$ of (P) is obtained by an asymptotic limit applied to the Navier-Stokes equations with anisotropic viscosity when the ratio depth over horizontal diameter (of the domain) goes to zero; see Besson-Laydi [6] for the stationary case and Azerad-Guillén [1, 2] for the evolution one. Finally, the existence of a weak solution of (EP) in domains without side-walls can be proved by internal approximation arguments: a mixed (velocity-pressure) variational formulation of the stationary problem is approximated by a conformed Finite Element method in Chacón-Guillén [9]. In particular, the result of convergence of the present work (Theorem 3) can be also interpreted as a proof of the existence of a weak solution of (EP), in domains without side-walls. On the other hand, F. Ortegón in [22], obtains a generalization of De Rham's Lemma to more general domains without side-walls.

Existence and uniqueness of a strong solution of the linear stationary problem associated to (EP) is treated by Ziane in [29]. This result is extended in [17] to the linear evolution case. With respect to the nonlinear problem in bidimensional domains, in [17] is demonstrated existence and uniqueness of a strong solution, global in time for small enough data or local in time for small enough depth. The extension (and improvement) of this kind of results to tridimensional domains can be seen in [13]. Finally, assuming flat bottom and Neumann boundary condition on the bottom, the existence of global in time regular solutions without constraints is proved in [8]. In [18], this result is also obtained with Dirichlet boundary conditions on the bottom.

Approximations to the Primitive Equations Model were presented by R. Bermejo in [4] and R. Bermejo and P. Galán in [5], using a semi-lagrangian projection scheme in time together with finite elements method in space.

From the numerical analysis point of view, convergence of some Finite Element approximations for the stationary problem, has been proved in [9].

We present an approximation to the problem (EP) using a time projection scheme. Projection methods are becoming widely used in the context of Navier-Stokes equations, where these methods split the convection-diffusion to the incompressible constraint.

The origin of projection methods is generally credited to the works of Chorin [10] and Temam [27]. They developed a two-step scheme where the second step is a free divergence projection step. The convergence of this projection method, was proved in [28] for the time discrete scheme and in [11] for a fully discrete scheme associated to a problem with periodic boundary conditions. More recently, error estimates for projection methods have been obtained (see [25], [26] for time discrete schemes and see [12] for a fully discrete scheme). Basically, for the so-called Chorin-Temam projection scheme, one has time error estimates of order $O(k^{1/2})$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$

and of order $O(k)$ in $l^2(\mathbf{L}^2)$ for both velocities and order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure, improving to order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the intermediate velocity and order $O(k)$ in $l^2(L^2)$ for the pressure, for a modified projection scheme (called incremental pressure or Van-Kan scheme) where a pressure correction term is added in the projection step. In [12], optimal error estimates (for a mixed velocity-pressure formulation) are obtained under the constraint $k^2 \le \alpha h$ in the three-dimensional case, whereas that in [16], the same optimal error estimates hold under the constraint $h \le \alpha k$, for a pressure segregation method to solve the projection step.

On the other hand, these projection methods, without and with pressure correction, can be observed as pressure segregation methods in the sense that the problems for velocity and pressure can be decoupled. In [3], for a fully discreted scheme, using a Poisson equation for the pressure, the authors obtain the convergence and error estimates for the non-incremental method, proving $O(k^{1/2} + h)$ error estimate for the pressure, without to be necessary the inf-sup condition for the approximating spaces and imposing the constraint $\alpha\, h^2 \le k \le \beta\, h^2$.

The work of the present paper follows the line of [16] about a fully discrete scheme for the Navier-Stokes problem, but now for the Primitive Equations. For simplicity, we only consider here the time discrete scheme. Firstly, we will prove unconditional stability and convergence towards a weak solution. Afterwards, we will prove optimal error estimates for the velocity and for the pressure for a regular enough solution.

This paper is organized as follows. In Section 1, we describe the time discrete scheme of projection type. This scheme is a linear scheme where the convection is treated semi-implicitly. Basically, in every time step m, three subproblems must be solved. Firstly, an approximation u_3^m for the vertical velocity at $t = t_m$ is computed, afterwards an approximation of the horizontal velocity $\mathbf{u}$ at $t = t_{m+1}$ is computed (that it will be called $\widetilde{\mathbf{u}}^{m+1}$) and finally, we calculate a final approximation $\mathbf{u}^{m+1}$ for the horizontal velocity and an approximation p_s^{m+1} for the pressure p_s in $t = t_{m+1}$. Moreover, we prove the stability and convergence of the scheme towards a weak solution of the problem (EP).

In Section 2, assuming the existence of a sufficiently regular solution of (EP), we obtain different error estimates for the velocities (intermediate and end-of-step horizontal velocity and for the vertical one) as well as for the pressure. We begin obtaining $O(k)$-error estimates for both velocities $\widetilde{\mathbf{u}}^{m+1}$ and $\mathbf{u}^{m+1}$, after that we obtain $O(k)$ error estimates for the time discrete derivative of the velocities and finally, we conclude obtaining $O(k)$ error estimates for the pressure p_s^{m+1}. Finally, some comments about the treatment of the Coriolis term are given in Section 3.

A part of the results of this paper has already been announced in [15].

1 Description, stability and convergence

1.1 Spaces of funtions

To define the notion of weak solution of problem (EP), we introduce the following Hilbert spaces:

$$\begin{aligned} H^1_{b,l}(\Omega) &= \{v \in H^1(\Omega) \ / \ v|_{\Gamma_b \cup \Gamma_l} = 0\}, \\ \mathbf{H} &= \{\mathbf{v} \in L^2(\Omega)^2 \ / \ \nabla_H \cdot \langle \mathbf{v} \rangle = 0 \text{ in } S, \ \langle \mathbf{v} \rangle \cdot \mathbf{n}_{\partial S} = 0\}, \\ \mathbf{V} &= \{\mathbf{v} \in H^1_{b,l}(\Omega)^2 \ / \ \nabla_H \cdot \langle \mathbf{v} \rangle = 0 \text{ in } S\}, \end{aligned}$$

being $\mathbf{n}_{\partial S}$ the normal exterior vector of ∂S.

We denote $\mathbf{H}^1_{b,l}(\Omega) = H^1_{b,l}(\Omega)^2$, etc. The norm and scalar product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and $(\cdot,\cdot)$, whereas in $\mathbf{H}^1_{b,l}(\Omega)$ by $\|\cdot\|$ we denote the norm of the gradient. On the other hand, we denote $\mathbf{H}^{-1}_{b,l}(\Omega)$ and $H^{-1/2}(\Gamma_s)$ the dual spaces of $\mathbf{H}^1_{b,l}(\Omega)$ and $H^{1/2}(\Gamma_s)$ respectively, with duality products $\langle \cdot,\cdot \rangle_\Omega$ and $\langle \cdot,\cdot \rangle_{\Gamma_s}$.

The space for the surface pressure will be:

$$L^2_0(S) = \left\{ q \in L^2(S) \ / \ \int_S q = 0 \right\}.$$

The vertical velocity u_3 will be obtained in function of $\nabla_H \cdot \mathbf{u}$. In this process, one has not L^2 regularity for the horizontal derivatives of u_3, so let us to define the (anisotropic) Hilbert space

$$H(\partial_z) = \{v \in L^2(\Omega) \ / \ \partial_z v \in L^2(\Omega)\}, \quad (\text{resp. } H^k(\partial_z) = \{v \in H^k(\Omega) \ / \ \partial_z v \in H^k(\Omega)\})$$

$$H_s(\partial_z) = \{v \in H(\partial_z) \ / \ v = 0 \ \text{ on } \ \Gamma_s\}, \quad (\text{resp. } H_0(\partial_z) = \{v \in H(\partial_z) \ / \ v = 0 \text{ on } \Gamma_s \cup \Gamma_b\}).$$

Due to this loss of regularity of u_3 ($u_3 \in L^2$ but $u_3 \notin H^1$), the vertical convection term $u_3 \partial_z \mathbf{u}$ does not belong to $\mathbf{H}^{-1}_{b,l}(\Omega)$, therefore more regular test functions must be introduced in the variational formulation of (EP). For instance, it suffices with $\mathbf{v} \in \mathbf{H}^1_{b,l}(\Omega)$ such that $\partial_z \mathbf{v} \in \mathbf{L}^3(\Omega)$, because in this case one has (see [9]):

$$\Big\langle (\mathbf{U}\cdot\nabla)\mathbf{u}, \mathbf{v} \Big\rangle_\Omega = -\Big((\mathbf{U}\cdot\nabla)\mathbf{v}, \mathbf{u} \Big) < +\infty.$$

Another possibility is to assume $\mathbf{v} \in \mathbf{H}^1_{b,l}(\Omega) \cap \mathbf{L}^\infty(Q)$, and then $\Big((\mathbf{U}\cdot\nabla)\mathbf{u}, \mathbf{v} \Big) < +\infty$.

In the sequel, we will use the following skew-symmetric form of the convective term (usual for the space discrete schemes), for each $\mathbf{U} \in \mathbf{H}^1_{b,l} \times H_s(\partial_z)$, $\mathbf{v} \in \mathbf{H}^1_{b,l}$, $\mathbf{w} \in \mathbf{H}^1$ with $\mathbf{w} \in \mathbf{L}^\infty$ or $\partial_z \mathbf{w} \in \mathbf{L}^3$,

$$C(\mathbf{U}, \mathbf{v}) = (\mathbf{U}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{U})\,\mathbf{v}$$

and

$$\begin{aligned} c(\mathbf{U},\mathbf{v},\mathbf{w}) &= \int_\Omega C(\mathbf{U},\mathbf{v})\cdot\mathbf{w} = \int_\Omega \Big\{(\mathbf{U}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}(\nabla\cdot\mathbf{U})\mathbf{v}\cdot\mathbf{w}\Big\}, \quad \text{if } \mathbf{w}\in\mathbf{L}^\infty,\\ &\le C\,\|\mathbf{U}\|_{\mathbf{H}^1_{b,l}\times H_s(\partial_z)}\|\mathbf{v}\|_{H^1}\|\mathbf{w}\|_{L^\infty} \end{aligned}$$

or equivalently

$$\begin{aligned} c(\mathbf{U},\mathbf{v},\mathbf{w}) &= -\int_\Omega \Big\{(\mathbf{U}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{U})\mathbf{v}\cdot\mathbf{w}\Big\} \quad \text{if } \partial_z\mathbf{w}\in\mathbf{L}^3.\\ &\le C\,\|\mathbf{U}\|_{\mathbf{H}^1_{b,l}\times H_s(\partial_z)}\|\mathbf{v}\|_{H^1}\|\mathbf{w}\|_{W^{1,3}} \end{aligned}$$

Previous equalities hold even in the fully discrete case, hence we can use, in the sequel, any of these two possibilities. Obviously, $c(\mathbf{U},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{U}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}$ whether $\nabla\cdot\mathbf{U}=0$. The trilinear form $c(\cdot,\cdot,\cdot)$ verifies $c(\mathbf{U},\mathbf{v},\mathbf{v})=0$, for each $\mathbf{U}\in\mathbf{H}^1_{b,l}\times H_s(\partial_z)$ and $\mathbf{v}\in\mathbf{H}^1_{b,l}$. By simplicity, the vertical part of these trilinear forms will be denoted in the same manner, i.e.

$$c\Big(u_3,\mathbf{v},\mathbf{w}\Big) = \int_\Omega \Big\{u_3\,\partial_z\mathbf{v}\cdot\mathbf{w} + \frac{1}{2}\partial_z u_3\,\mathbf{v}\cdot\mathbf{w}\Big\} = -\int_\Omega \Big\{u_3\,\partial_z\mathbf{w}\cdot\mathbf{v} + \frac{1}{2}\partial_z u_3\,\mathbf{v}\cdot\mathbf{w}\Big\}$$

1.2 Description of the scheme

The time interval $[0,T]$ is divided into M subintervals, for simplicity of equal length $k=T/M$, considering the partition of $[0,T]$, $\{t_m = m\,k\}_{m=0}^M$.

In general, a time discrete scheme is an iterative scheme, where in each step m, given $\{(\mathbf{f}^m,\mathbf{g}_s^m)\}_{m=1}^M$ some approximations of data $(\mathbf{f},\mathbf{g}_s)$ at the instant $t=t_m$, a sequence $\{(\mathbf{u}_h^m, u_{3,h}^m, p_{s,h}^m)\}_m$ will be computed, which pretends to be an approximation to a solution $(\mathbf{u},u_3,p_s)$ of (EP) at $t=t_m$.

We are going to present an incremental pressure projection scheme, splitting the three main difficulties of the problem (EP):

- the non linear convective terms $(\mathbf{U}\cdot\nabla)\mathbf{u}$. In particular, the vertical convection $u_3\partial_z\mathbf{u}$ is less regular than in the Navier-Stokes case.
- the restriction $\nabla_H\cdot\langle\mathbf{u}\rangle = 0$ in $S\times(0,T)$,
- the computation of the vertical velocity.

By simplicity, we do not consider the Coriolis term because this term does not add new difficulties. At the end, in Section 3, we analyze the more convenient form for introduce it.

Indeed, given $(\mathbf{u}^m,p_s^m)$, firstly the vertical velocity $\widetilde{u}_3^m$ is computed in function of $\nabla_H\cdot\widetilde{\mathbf{u}}^m$, afterwards we obtain an intermediate horizontal velocity $\widetilde{\mathbf{u}}^{m+1}$ using the convective and the diffusion terms but not the restriction of divergence type, and finally we obtain $\mathbf{u}^{m+1}$ and p_s^{m+1} by means of a linear mixed problem considering the restriction $\nabla_H\cdot\langle\mathbf{u}^{m+1}\rangle = 0$. Moreover, since a pressure correction projection scheme will be considered, an explicit pressure term is introduced in the intermediate horizontal velocity problem and an implicit pressure correction term is considered in the mixed problem for the end-of-step velocity and pressure problem.

Concretely, the time discrete scheme is described as follows:

Initialization: Let $\widetilde{\mathbf{u}}^0 = \mathbf{u}(0)$ and let p_s^0 be given. To take $\mathbf{u}^0 = \widetilde{\mathbf{u}}^0$.

Sub-step 0 : Given $\widetilde{\mathbf{u}}^m$, to compute $\widetilde{u}_3^m$ as

$$(S_0)^m \qquad \widetilde{u}_3^m(\mathbf{x}, z) = \int_z^0 \nabla_H \cdot \widetilde{\mathbf{u}}^m(\mathbf{x}, s)\, ds.$$

Sub-step 1 : Given $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$, $\widetilde{u}_3^m$ and p_s^m, to find $\widetilde{\mathbf{u}}^{m+1} : \Omega \to I\!R^2$ solution of

$$(S_1)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}) - \nu \Delta \widetilde{\mathbf{u}}^{m+1} + \nabla_H p_s^m &=& \mathbf{f}^{m+1}, \\ \nu \partial_z \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_s} = \mathbf{g}_s^{m+1}, \quad \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_b \cup \Gamma_l} &=& 0, \end{array} \right.$$

where $\widetilde{\mathbf{U}}^m = (\widetilde{\mathbf{u}}^m, \widetilde{u}_3^m)$.

Sub-step 2 : Given $\widetilde{\mathbf{u}}^{m+1}$, to find $\mathbf{u}^{m+1} : \Omega \to I\!R^2$ and $p_s^{m+1} : S \to I\!R^2$ solution of

$$(S_2)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}) + \nabla_H (p_s^{m+1} - p_s^m) &=& 0 \quad \text{in } \Omega, \\ \nabla_H \cdot \langle \mathbf{u}^{m+1} \rangle = 0 \quad \text{in } S, \quad \langle \mathbf{u}^{m+1} \rangle \cdot \mathbf{n}|_{\partial S} &=& 0. \end{array} \right.$$

Since in this section we are only going to consider the weak regularity on the data, we take

$$\mathbf{f}^{m+1} = \frac{1}{k} \int_{t_m}^{t_{m+1}} \mathbf{f}(t)\, dt \quad \text{and} \quad \mathbf{g}_s^{m+1} = \frac{1}{k} \int_{t_m}^{t_{m+1}} \mathbf{g}_s(t)\, dt.$$

Notice that the vertical velocity furnished in Sub-step 0 does not satisfies exactly the boundary condition on the bottom, i.e. in general $\widetilde{u}_3^m|_{\Gamma_b} \neq 0$, due to the corresponding horizontal velocity $\widetilde{\mathbf{u}}^m$ has not verifies the constraint of free divergence type, that is $\nabla_H \cdot \langle \widetilde{\mathbf{u}}^m \rangle \neq 0$ in S.

On the other hand, we have written the convection term in Sub-step 2 in a linear and semi-implicit form. Moreover, the use of the antisymmetric form $c(\cdot, \cdot, \cdot)$ will not be strictly necessary for this time discrete scheme, (because $\nabla \cdot \widetilde{\mathbf{U}}^m = 0$ and although $\widetilde{u}_3^m|_{\Gamma_b} \neq 0$ but $\widetilde{\mathbf{u}}|_{\Gamma_b}^{m+1} = 0$, then after to do integration by parts, the boundary terms do not appear), but we prefer consider it because this antisymmetric form must be used in any fully discrete scheme.

Remark 1 *Adding $(S_1)^{m+1}$ and $(S_2)^{m+1}$, we get*

$$(S_3)^{m+1} \qquad \left\{ \begin{array}{rcl} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}) - \nu \Delta \widetilde{\mathbf{u}}^{m+1} + \nabla_H p_s^{m+1} &=& \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \nu \partial_z \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_s} = \mathbf{g}_s^{m+1}, \quad \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_b} = 0, \quad \nabla_H \cdot \langle \mathbf{u}^{m+1} \rangle &=& 0 \quad \text{in } S. \end{array} \right.$$

$(S_3)^{m+1}$ can be viewed as consistence relations, because if we could demonstrate that $\widetilde{\mathbf{u}}^{m+1}$ and $\mathbf{u}^{m+1}$ converge to the same limit function $\mathbf{u}$ and the convergence is sufficiently strong, taking

limits in $(S_3)^{m+1}$, we will find that $\mathbf{u}$ is a solution of the continuous problem (EP). Moreover, since both $\widetilde{\mathbf{u}}^{m+1}$ and $\mathbf{u}^{m+1}$ will be approximations to $\mathbf{u}(t_{m+1})$, then $(S_3)^{m+1}$ says us that the viscosity terms is taken in an implicit way and the convection term in a "semi-implicit" way.

Effective implementation of the scheme.

The introduction of the end-of-step velocity $\mathbf{u}^m$ is not necessary and the computation of $\widetilde{\mathbf{u}}^{m+1}$ and p_s^{m+1} can be decoupled. In fact, each time step of this scheme is computed as follows: Given $(p_s^{m-1}, p_s^m, \widetilde{\mathbf{u}}^m)$,

(0) to find $\widetilde{u}_3^m$ solving $(S_0)^m$,

(1) to find $\widetilde{\mathbf{u}}^{m+1}$ solving the convection-diffusion problem:

$$(\widetilde{S})^{m+1} \quad \begin{cases} \dfrac{\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}^m}{k} + C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta \widetilde{\mathbf{u}}^{m+1} + \nabla_H (2\, p_s^m - p_s^{m-1}) = \mathbf{f}^{m+1} & \text{in } \Omega, \\ \nu \partial_z \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_s} = \mathbf{g}_s^{m+1}, \quad \widetilde{\mathbf{u}}^{m+1}|_{\Gamma_b \cup \Gamma_l} = 0, \end{cases}$$

(2) to find p_s^{m+1} solving the elliptic $2D$ problem

$$(E)^{m+1} \qquad \begin{cases} k \nabla_H \cdot (D\, \nabla_H (p_s^{m+1} - p_s^m)) = \nabla_H \cdot \langle \widetilde{\mathbf{u}}^{m+1} \rangle & \text{in } S \\ k\, D \nabla_H (p_s^{m+1} - p_s^m) \cdot \mathbf{n}_{\partial S} = 0 & \text{on } \partial S. \end{cases}$$

The problem $(\widetilde{S})^{m+1}$ has been obtained by writing $\mathbf{u}^m = \widetilde{\mathbf{u}}^m - k \nabla_H (p_s^m - p_s^{m-1})$ in $(S_1)^{m+1}$.

On the other hand, integrating vertically $(S_2)^{m+1}$ between $z = -D(\mathbf{x})$ and $z = 0$, we obtain the system

$$\langle \mathbf{u}^{m+1} \rangle - \langle \widetilde{\mathbf{u}}^{m+1} \rangle + k\ D\ \nabla_H (p_s^{m+1} - p_s^m) = 0 \qquad \text{in } S.$$

Taking horizontal divergence and multiplying by $\mathbf{n}_{\partial S}$ (the normal exterior vector to ∂S), the unknown $\mathbf{u}^{m+1}$ is eliminated and we arrive at the Neumann elliptic problem $(E)^{m+1}$ for the pressure p_s^{m+1}:

Hence, the computation for the pressure and velocity is decoupled, so the method becomes a pressure segregation scheme. In fact, $(\widetilde{S})^{m+1}$ is a linear convection-diffusion problem for the velocity and $(E)^{m+1}$ is a elliptic problem for the pressure.

Only for the numerical analysis, we would to introduce the projected velocity $\mathbf{u}^{m+1}$ in function of p_s^m, $\widetilde{\mathbf{u}}^{m+1}$ and p_s^{m+1}, by

$$\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k \nabla_H (p_s^{m+1} - p_s^m) \quad \text{ in } \Omega. \tag{1}$$

Notice that, to initialize the scheme we must known $\widetilde{\mathbf{u}}^0$, p^0 and p^{-1} . Then, we would have to begin with a pressure p^{-1}, which has not sense. For this, either we have to begin with several auxiliary initial steps with another scheme, and taking these preliminary steps as initial data for our fractional step algorithm at subsequent time steps, or we have to begin with a first step with the scheme written as before, i.e., with $\widetilde{\mathbf{u}}^0$, p_s^0 and $u^0 = \widetilde{\mathbf{u}}^0$, beginning then with a approximation of the initial pressure. This type of problems for the initialization of the scheme is inherent to the incremental pressure schemes.

1.3 Variational formulations, stability and convergence of the scheme

We define

$$\begin{aligned} \mathbf{W}^{1,3}_{b,l}(\Omega) &= \{\mathbf{v} \in W^{1,3}(\Omega)^2 \;/\; \mathbf{v}|_{\Gamma_b \cup \Gamma_l} = 0\}, \\ \mathbf{V}_3 &= \{\mathbf{v} \in \mathbf{W}^{1,3}_{b,l}(\Omega) \;/\; \nabla_H \cdot \langle \mathbf{v} \rangle = 0 \text{ in } S\}. \end{aligned}$$

Variational formulation of (EP)**:**

Given $\mathbf{f} \in L^2(0,T;\mathbf{H}_b^{-1}(\Omega)), \mathbf{g}_s \in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$ and $\mathbf{u_0} \in \mathbf{H}$:
To find $\mathbf{U} = (\mathbf{u}, u_3) \in (L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})) \times L^2(0,T;H(\partial_z))$ such that,

$$(EP)_v \begin{cases} \langle \partial_t \mathbf{u}(t), \mathbf{w} \rangle_\Omega - c(\mathbf{U}(t), \mathbf{w}, \mathbf{u}(t)) + \nu\,(\nabla \mathbf{u}(t), \nabla \mathbf{w}) \\ \qquad = \langle \mathbf{f}(t), \mathbf{w} \rangle_\Omega + \langle \mathbf{g}_s(t), \mathbf{w} \rangle_{\Gamma_s} \quad \forall\, \mathbf{w} \in \mathbf{V}_3, \quad \text{a.e. } t \in (0,T), \\ u_3(t;\mathbf{x},z) = \displaystyle\int_z^0 \nabla_H \cdot \mathbf{u}(t;\mathbf{x},s)\,ds \quad \text{a.e. } t \in (0,T),\ (\mathbf{x},z) \in \Omega \end{cases}$$

A solution of the previous problem $\mathbf{U} = (\mathbf{u}, u_3)$, is called a weak solution of (EP). Notice that, $(EP)_v$ is a "non-hilbertian" formulation, because test functions are more regular than the solution. Moreover, by virtue of a De Rham's Lemma for hydrostatic case [20], if $\mathbf{U}$ verifies $(EP)_v$, then there exists $p_s \in \mathcal{D}'(0,T;L^2_0(S))$ such that $(\mathbf{U}, p_s)$ is a distributional solution of (EP).

Now, we give the variational formulations of $(S_1)^{m+1}$-$(S_2)^{m+1}$ and its equivalent variational formulations of $(\widetilde{S})^{m+1}$-$(E)^{m+1}$. To assure this equivalence, the pressure has to be H^1 functions. In fact, although the pressure is defined in S, in order to obtain the regularity, it will be more simple to consider it as z-independent function defined in the whole domain Ω.

Variational formulation of $(S_1)^{m+1}$**:**

Given $p_s^m \in H^1(\Omega) \cap L^2_0(\Omega)$, $\widetilde{\mathbf{u}}^m \in \mathbf{H}^1_{b,l}(\Omega)$ and $\mathbf{u}^m \in \mathbf{H}$, to find $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}^1_{b,l}(\Omega)$ such as,

$$(S_1)_v^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m, \mathbf{w}) - c(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}, \mathbf{w}) + \nu\,(\nabla \widetilde{\mathbf{u}}^{m+1}, \nabla \mathbf{w}) + (\nabla_H p_s^m, \mathbf{w}) \\ = \langle \mathbf{f}^{m+1}, \mathbf{w} \rangle_\Omega + \langle \mathbf{g}_s^{m+1}, \mathbf{w} \rangle_{\Gamma_s} \quad \forall\, \mathbf{w} \in \mathbf{W}^{1,3}_{b,l}(\Omega). \end{cases}$$

Variational formulation of $(S_2)^{m+1}$**:**

Given $p_s^m \in H^1(\Omega) \cap L^2_0(\Omega)$ and $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}^1_{b,l}(\Omega)$, to find $\mathbf{u}^{m+1} \in \mathbf{L}^2(\Omega)$ and $p_s^{m+1} \in H^1(\Omega) \cap L^2_0(\Omega)$ such as,

$$(S_2)_v^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}) + (\nabla_H (p_s^{m+1} - p_s^m), \mathbf{v})_\Omega = 0 \quad \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega) \\ (\mathbf{u}^{m+1}, \nabla_H q_s) = 0 \quad \forall\, q_s \in H^1(\Omega) \cap L^2_0(\Omega). \end{cases}$$

In particular, $\mathbf{u}^{m+1} \in \mathbf{H}$. In fact, $\mathbf{u}^{m+1}$ is the projection of $\widetilde{\mathbf{u}}^{m+1}$ on $\mathbf{H}$.

Variational formulation of $(E)^{m+1}$**:**

Given $p_s^m \in H^1(\Omega) \cap L_0^2(\Omega)$ and $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}_{b,l}^1(\Omega)$, to find $p_s^{m+1} \in H^1(\Omega) \cap L_0^2(\Omega)$ such as

$$(E)_v^{m+1} \qquad k\Big(D\,\nabla_H(p_s^{m+1} - p_s^m), \nabla_H q_s\Big) = \Big(\widetilde{\mathbf{u}}^{m+1}, \nabla_H q_s\Big) \quad \forall\, q_s \in H^1(\Omega) \cap L_0^2(\Omega).$$

It is easy to deduce that $(S_2)^{m+1}$ is equivalent to obtain firstly p_s^{m+1} as solution of the elliptic problem $(E)_v^{m+1}$ and, after that, to obtain $\mathbf{u}^{m+1}$ from (1).

Variational formulation of $(S_3)^{m+1}$

Adding $(S_1)_v^{m+1}$ and $(S_2)_v^{m+1}$, we get the variational formulation of $(S_3)^{m+1}$:

$$(S_3)_v^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m, \mathbf{v}) + c(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}) + \nu(\nabla\widetilde{\mathbf{u}}^{m+1}, \nabla\mathbf{v}) + (\nabla_H p_s^{m+1}, \mathbf{v}) \\ = \langle \mathbf{f}^{m+1}, \mathbf{v}\rangle_\Omega + \langle \mathbf{g}_s^{m+1}, \mathbf{v}\rangle_{\Gamma_s} \quad \forall\, \mathbf{v} \in \mathbf{W}_{b,l}^{1,3}(\Omega), \\ (\mathbf{u}^{m+1}, \nabla_H q_s) = 0 \quad \forall\, q_s \in H^1(\Omega) \cap L_0^2(\Omega). \end{array} \right.$$

Now, we are going to study stability properties of these schemes and convergence towards a weak solution of the continuous problem (EP). For this, first we will obtain some a priori (stability) estimates and a posterior pass to the limit (convergence), where compactness results must be applied to "control" the limit of the convective terms.

We will obtain stability estimates for $(\widetilde{\mathbf{u}}^{m+1})$ and $(\mathbf{u}^{m+1})$ in $L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega))$, with the corresponding estimates for $(\widetilde{u}_3^m)$ in $L^2(0,T;H(\partial_z))$.

With respect to the projection step $(S_2)_v^{m+1}$, the following result holds

Lemma 1 *(Existence, uniqueness and continuous dependence of* $(S_2)_v^{m+1}$*).*

a) *Continuous dependence in* L^2**.** *The problem* $(S_2)_v^{m+1}$ *has a unique solution* $(\mathbf{u}^{m+1}, p_s^{m+1}) \in \mathbf{H} \times (H^1(\Omega) \cap L_0^2(\Omega))$*. Moreover,*

$$|\mathbf{u}^{m+1}| \le |\widetilde{\mathbf{u}}^{m+1}| \quad \textit{and} \quad |\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}| \le |\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|.$$

Moreover, using the orthogonality property $\Big(\mathbf{u}^{m+1}, \nabla_H q_s\Big) = 0 \quad \forall$ *z-independent function* $q_s \in H^1(\Omega)$*, we have*

$$|\widetilde{\mathbf{u}}^{m+1}|^2 = |\mathbf{u}^{m+1}|^2 + |k\nabla_H(p_s^{m+1} - p_s^m)|^2.$$

b) *Continuous dependence in* H^1**.** *Assuming* $S \in C^3$ *and* $D \in W^{1,\infty}(S)$ *(together with hypotheses assumed in [7]) and either* $D \ge D_{min} > 0$ *in* S *or* $D > 0$ *in* S *and* S *simply connected. If* $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}_{b,l}^1(\Omega)$*, then* $(\mathbf{u}^{m+1}, p_s^{m+1}) \in \mathbf{H}^1(\Omega) \cap H^2(S)$*. Moreover, there exists* $C = C(\Omega, D) > 0$ *such that*

$$\|\mathbf{u}^{m+1}\|_{H^1(\Omega)} \le C\,\|\widetilde{\mathbf{u}}^{m+1}\|. \tag{2}$$

Proof. The proof is based on the equivalence between $(S_2)_v^{m+1}$ and to obtain firstly p_s^{m+1} as solution of the elliptic problem $(E)_v^{m+1}$ and, after that, to obtain $\mathbf{u}^{m+1}$ from (1).

The proof of **a)** is easy, taking into account that $\mathbf{u}^{m+1}$ is the projection of $\widetilde{\mathbf{u}}^{m+1}$ on $\mathbf{H}$, hence in particular $|\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}| = \min_{\mathbf{u}\in\mathbf{H}} |\mathbf{u}-\widetilde{\mathbf{u}}^{m+1}|$, and the weak regularity of the elliptic problem $(E)^{m+1}$, which implies that $D^{1/2}\nabla_H(p_s^{m+1}-p_s^m) \in L^2(S)$, that is $\nabla_H(p_s^{m+1}-p_s^m) \in L^2(\Omega)$. In fact, we can obtain the proof by using the orthogonality property:

$$\Big(\mathbf{u}^{m+1}, \nabla_H q_s\Big) = 0 \quad \forall\, z\text{-independent function } q_s \in H^1(\Omega).$$

Indeed, by multiplying $(S_2)^{m+1}$ by $\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}$, we get

$$\begin{aligned} |\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}|^2 &= \Big(k\,\nabla_H(p_s^{m+1}-p_s^m), \mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}\Big) = \Big(k\,\nabla_H(p_s^{m+1}-p_s^m), \mathbf{u}^{m}-\widetilde{\mathbf{u}}^{m+1}\Big) \\ &\le |k\,\nabla_H(p^{m+1}-p^m)|\,|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m| \le |\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}|\,|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m|, \end{aligned}$$

hence we get $|\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}| \le |\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m|$.

The proof of **b)** will be divided into two cases, depending on the domains with or without lateral walls.

1. Case with lateral walls (i.e. $D \ge D_{min} > 0$ in S).

Since $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}^1_{b,l}(\Omega)$, then $\nabla_H\langle\widetilde{\mathbf{u}}^{m+1}\rangle = \langle\nabla_H\widetilde{\mathbf{u}}^{m+1}\rangle$. In particular, $\langle\widetilde{\mathbf{u}}^{m+1}\rangle \in \mathbf{H}^1_0(S)$ and $\nabla_H\cdot\langle\widetilde{\mathbf{u}}^{m+1}\rangle \in L^2_0(S)$. Then, since $D \in W^{1,\infty}(S)$ and $D \ge D_{min} > 0$ in S, we can apply the $H^2(S)$-regularity of the elliptic problem $(E)^{m+1}$ and to deduce that $p_s^{m+1}-p_s^m \in H^2(S)$ and it verifies

$$k\,\|\nabla_H(p_s^{m+1}-p_s^m)\|_{H^1(S)} \le C\,\|\nabla_H\cdot\langle\widetilde{\mathbf{u}}^{m+1}\rangle\|_{L^2(S)} \le C\|\widetilde{\mathbf{u}}^{m+1}\|,$$

for a constant $C = C(\Omega, D) > 0$. Then, from (1), $\mathbf{u}^{m+1} \in \mathbf{H}^1(\Omega)$ and

$$\begin{aligned} \|\mathbf{u}^{m+1}\|_{H^1(\Omega)} &\le \|\widetilde{\mathbf{u}}^{m+1}\|_{H^1(\Omega)} + k\,\|\nabla_H(p_s^{m+1}-p_s^m)\|_{H^1(\Omega)} \\ &= \|\widetilde{\mathbf{u}}^{m+1}\|_{H^1(\Omega)} + k\,\|D^{1/2}\nabla_H(p_s^{m+1}-p_s^m)\|_{H^1(S)} \le C\,\|\widetilde{\mathbf{u}}^{m+1}\| \end{aligned}$$

arriving at (2).

2. General case (i.e. $D > 0$ in S and S simply connected).

Firstly, we prove that the Neumann elliptic problem $(E)^{m+1}$ verified for the pressure p_s^{m+1}, can be rewritten as a Dirichlet problem, for a "potential" function Ψ.

Indeed, $(E)^{m+1}$ can be written as

$$\nabla_H\cdot\Big(k\,D\,\nabla_H(p_s^{m+1}-p_s^m) - \langle\widetilde{\mathbf{u}}^{m+1}\rangle\Big) = 0 \quad \text{in } S.$$

Then, there exists a function $\Psi : S \to I\!R$ such as

$$\nabla_H^\perp\Psi = k\,D\,\nabla_H(p_s^{m+1}-p_s^m) - \langle\widetilde{\mathbf{u}}^{m+1}\rangle \quad \text{in } S,$$

where $\nabla_H^\perp = (-\partial_y, \partial_x)^t$. Then, one has

$$\nabla_H p_s^{m+1} = \nabla_H p_s^m + \frac{1}{k}\frac{1}{D}\Big(\nabla_H^\perp\Psi + \langle\widetilde{\mathbf{u}}^{m+1}\rangle\Big) \quad \text{in } S. \tag{3}$$

Now, taking the rotational operator ($\nabla_H^\perp \cdot$) in the above equation, we arrive at

$$\nabla_H \cdot \Big(\frac{1}{D}\,\nabla_H \Psi\Big) = -\nabla_H^\perp \cdot \Big(\frac{1}{D}\langle \widetilde{\mathbf{u}}^{m+1}\rangle\Big) \quad \text{in } S.$$

On the other hand, the boundary condition on ∂S of $(E)^{m+1}$ can be written as:

$$\nabla_H^\perp \Psi \cdot \mathbf{n}_{\partial S} = \nabla_H \Psi \cdot \tau_{\partial S} = 0,$$

where $\tau_{\partial S}$ is the tangential vector to ∂S. Hence, since S is simply connected, this condition becomes the Dirichlet condition $\Psi = 0$ on ∂S.

In conclusion, the problem $(E)^{m+1}$ can be written as the following Dirichlet elliptic problem for the function Ψ:

$$\left\{ \begin{array}{rcl} \nabla_H \cdot \Big(\dfrac{1}{D}\,\nabla_H \Psi\Big) & = & -\nabla_H^\perp \cdot \Big(\dfrac{1}{D}\langle \widetilde{\mathbf{u}}^{m+1}\rangle\Big) \quad \text{in } S, \\ \Psi & = & 0 \quad \text{on } \partial S. \end{array} \right. \tag{4}$$

By using the weighted weak regularity of the problem (4), let us see that if $\nabla_H p_s^m \in \mathbf{L}^2(\Omega)$ and $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{L}^2(\Omega)$ then $\nabla_H p_s^{m+1} \in \mathbf{L}^2(\Omega)$ and

$$k\,\|\nabla_H (p_s^{m+1} - p_s^m)\|_{\mathbf{L}^2(\Omega)} \leq C\,\|\widetilde{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2(\Omega)}. \tag{5}$$

Indeed, as $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{L}^2(\Omega)$ one has $D^{-1/2}\,\langle \widetilde{\mathbf{u}}^{m+1}\rangle \in L^2(S)$. Then, the weak solution Ψ of (4) verifies (see [7]):

$$\Psi \in L^2(S), \quad D^{-1/2}\nabla_H \Psi \in L^2(S) \quad \text{and} \quad \|D^{-1}\,\nabla_H \Psi\|_{L^2(\Omega)} \leq C \|\widetilde{\mathbf{u}}^{m+1}\|_{L^2(\Omega)}.$$

With this regularity, from (3) one has $\nabla_H(p_s^{m+1} - p_s^m) \in \mathbf{L}^2(\Omega)$ and the inequality (5) holds.

On the other hand, by using the weighted strong regularity of the problem (4), let us see that if $\nabla_H p_s^m \in \mathbf{H}^1(\Omega)$ and $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}^1(\Omega)$ then $\nabla_H p_s^{m+1} \in \mathbf{H}^1(\Omega)$ and it verifies

$$k\,\|\nabla_H (p_s^{m+1} - p_s^m)\|_{\mathbf{H}^1(\Omega)} \leq C\,\|\widetilde{\mathbf{u}}^{m+1}\|_{\mathbf{H}^1(\Omega)}$$

(whence $\mathbf{u}^{m+1} \in \mathbf{H}^1(\Omega)$ and (2) hold). Indeed, since $\widetilde{\mathbf{u}}^{m+1} \in \mathbf{H}^1(\Omega)$, in particular

$$D^{1/2}\nabla_H^\perp \cdot (\frac{1}{D}\langle \widetilde{\mathbf{u}}^{m+1}\rangle) \in L^2(S).$$

Then, by using a weighted H^2-regularity result of the problem (4) (see [7]), one has

$$D^{1/2}\nabla_H(\frac{1}{D}\nabla_H \Psi) \in L^2(S).$$

Then, from (3), $D^{1/2}\nabla_H(\nabla_H(p_s^{m+1} - p_s^m)) \in L^2(S)$, i.e. $\nabla_H(\nabla_H(p_s^{m+1} - p_s^m)) \in L^2(\Omega)$. ■

Lemma 2 (Stability) *Under assumptions of Lemma 1, let* $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}_{b,l}(\Omega))$, $\mathbf{g}_s \in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma_s))$ *and* $\mathbf{u}_0 \in \mathbf{V}$. *If* $|k\nabla_H p_s^0| \le C_0$, *then there exists* $C = C(C_0,\nu,\mathbf{f},\mathbf{g}_s,\Omega) > 0$ *such that,*

$$
\begin{aligned}
&|\widetilde{\mathbf{u}}^{r+1}|^2 + |\mathbf{u}^{r+1}|^2 \le C, \qquad \forall\, r = 0,...,M-1\\
&k\sum_{m=0}^{M-1}\Big\{\|\widetilde{\mathbf{u}}^{m+1}\|^2 + \|\mathbf{u}^{m+1}\|^2\Big\} \le C,\\
&\sum_{m=0}^{M-1}\Big\{|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m|^2 + |\mathbf{u}^{m+1}-\widetilde{\mathbf{u}}^{m+1}|^2\Big\} \le C
\end{aligned}
\tag{6}
$$

Remark:

With respect to the estimates for the vertical velocity $\widetilde{u}_3^m$, using that $\partial_z \widetilde{u}_3^m = -\nabla_H \cdot \widetilde{\mathbf{u}}^m$ and the inequality (16), there exists a constant $C = C(C_0,\nu,\mathbf{f},\mathbf{g}_s,\Omega) > 0$, such as

$$
k\sum_{m=0}^{M-1}\|\widetilde{u}_3^m\|^2_{H(\partial_z)} \le C.
$$

Proof. Multiplying $(S_1)^{m+1}$ by $2\,k\,\widetilde{\mathbf{u}}^{m+1}$ and using the equality $(a-b)2a = a^2 - b^2 + (a-b)^2$, we get:

$$
\begin{aligned}
&|\widetilde{\mathbf{u}}^{m+1}|^2 - |\mathbf{u}^m|^2 + |\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m|^2 + \nu\, k\|\widetilde{\mathbf{u}}^{m+1}\|^2 + 2\,k(\nabla_H p_s^m, \widetilde{\mathbf{u}}^{m+1})\\
&\le \frac{2k}{\nu}\{\|\mathbf{f}^{m+1}\|^2_{H^{-1}_{b,l}} + \|\mathbf{g}_s^{m+1}\|^2_{H^{-1/2}(\Gamma_s)}\}.
\end{aligned}
\tag{7}
$$

On the other hand, multiplying $(S_2)^{m+1}$ by $k\left(\mathbf{u}^{m+1} + \widetilde{\mathbf{u}}^{m+1} + k\nabla_H(p_s^{m+1}+p_s^m)\right)$ and using the orthogonality properties $\left(\mathbf{u}^{m+1}, \nabla_H p_s^{m+1}\right) = 0 = \left(\mathbf{u}^{m+1}, \nabla_H p_s^m\right)$, we get:

$$
|\mathbf{u}^{m+1}|^2 - |\widetilde{\mathbf{u}}^{m+1}|^2 + |k\nabla_H p_s^{m+1}|^2 - |k\nabla_H p_s^m|^2 - 2\,k\Big(\nabla_H p_s^m, \widetilde{\mathbf{u}}^{m+1}\Big) = 0 \tag{8}
$$

Making now $\sum_{m=0}^{r}\{(7)+(8)\}$, the pressure term $\left(\nabla_H p_s^m, \widetilde{\mathbf{u}}^{m+1}\right)$ cancel, obtaining

$$
\begin{aligned}
&|\mathbf{u}^{r+1}|^2 + |k\nabla_H p_s^{r+1}|^2 + \sum_{m=0}^{r}|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m|^2 + k\,\nu\sum_{m=0}^{r}\|\widetilde{\mathbf{u}}^{m+1}\|^2\\
&\le |\mathbf{u}^0|^2 + |k\nabla_H p_s^0|^2 + \frac{2\,k}{\nu}\sum_{m=0}^{r}\{\|\mathbf{f}^{m+1}\|^2_{H^{-1}_{b,l}} + \|\mathbf{g}_s^{m+1}\|^2_{H^{-1/2}(\Gamma_s)}\}
\end{aligned}
$$

Then we obtain the stability of $(\mathbf{u}^{m+1})$ and $(k\nabla_H p_s^{m+1})$ in $l^\infty(0,T;\mathbf{L}^2(\Omega))$, of $(\widetilde{\mathbf{u}}^{m+1})$ in $l^2(0,T;\mathbf{H}^1(\Omega))$ and the estimate $\sum_{m=0}^{M-1}|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^m|^2 \le C$.

Finally, by using Lemma 1 the estimates of $(\widetilde{\mathbf{u}}^{m+1})$ in $l^\infty(\mathbf{L}^2)$, of $(\mathbf{u}^{m+1})$ in $l^2(0,T;\mathbf{H}^1(\Omega))$ and the estimate $\sum_{m=0}^{M-1}|\widetilde{\mathbf{u}}^{m+1}-\mathbf{u}^{m+1}|^2 \le C$ hold. ■

Now, we define $\mathbf{u}_k^{(1)}$, $\widetilde{\mathbf{u}}_k^{(1)}$, $\mathbf{u}_k^{(0)}$, $\widetilde{\mathbf{u}}_k^{(0)}$, $\widetilde{u}_{3,k}^{(0)}$ the functions defined in $[0,T]$, constant by subintervals, equal to $\mathbf{u}^{m+1}$, $\widetilde{\mathbf{u}}^{m+1}$, $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$ and $\widetilde{u}_3^m$ in $(t_m, t_{m+1}]$, respectively. On the other hand, $\mathbf{u}_k$ is the continuous function in $[0,T]$, linear by subintervals and such that $\mathbf{u}_k(t_m) = \mathbf{u}^m$.

Theorem 3 (Convergence) *Under hypothesis of Lemma 2, there exists a subsequence (k') of (k), with $k' \downarrow 0$, and a weak solution $\mathbf{U} = (\mathbf{u}, u_3)$ of (EP) in $(0,T)$, such as: $\widetilde{\mathbf{u}}^{(1)}_{k'}$, $\mathbf{u}^{(1)}_{k'}$, $\widetilde{\mathbf{u}}^{(0)}_{k'}$, $\mathbf{u}^{(0)}_{k'}$ and $\mathbf{u}_{k'}$ converge to $\mathbf{u}$ weakly-$*$ in $L^\infty(0,T;\mathbf{L}^2(\Omega))$, weakly in $L^2(0,T;\mathbf{H}^1(\Omega))$ and strongly in $L^2(0,T;\mathbf{L}^2(\Omega))$, whereas that $\widetilde{u}^{(0)}_{3,k'}$ converges to u_3 weakly in $L^2(0,T;H(\partial_z))$.*

By definition of the functions $\mathbf{u}^{(1)}_k$, $\widetilde{\mathbf{u}}^{(1)}_k$, $\mathbf{u}^{(0)}_k$, $\widetilde{\mathbf{u}}^{(0)}_k$, $u^{(0)}_{3,k}$ and $\mathbf{u}_k$, Lemma 2 has the following interpretation:

$$(\widetilde{\mathbf{u}}^{(1)}_k)_k,\ (\mathbf{u}^{(1)}_k)_k,\ (\widetilde{\mathbf{u}}^{(0)}_k)_k,\ (\mathbf{u}^{(0)}_k)_k \text{ and } (\mathbf{u}_k)_k \ \text{ are bounded in } \ L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^1), \tag{9}$$

$$(\widetilde{u}^{(0)}_{3,k})_k \text{ is bounded in } \ L^2(H(\partial_z)), \tag{10}$$

On the other hand, from (6), there exists $C = C(\nu, \mathbf{u}_0, \mathbf{f}, \mathbf{g}_s, \Omega) > 0$ such that,

$$\|\mathbf{u}^{(1)}_k - \widetilde{\mathbf{u}}^{(1)}_k\|^2_{L^2(\mathbf{L}^2)} \le C\,k \quad \text{and} \quad \|\widetilde{\mathbf{u}}^{(1)}_k - \mathbf{u}^{(0)}_k\|^2_{L^2(\mathbf{L}^2)} \le C\,k. \tag{11}$$

In particular,

$$\|\mathbf{u}^{(1)}_k - \mathbf{u}_k\|^2_{L^2(\mathbf{L}^2)} \le \|\mathbf{u}^{(1)}_k - \mathbf{u}^{(0)}_k\|^2_{L^2(\mathbf{L}^2)} \le C\,k, \tag{12}$$

and finally

$$\|\widetilde{\mathbf{u}}^{(1)}_k - \mathbf{u}_k\|^2_{L^2(\mathbf{L}^2)} \le C\,k. \tag{13}$$

On the other hand, the variational formulation of $(S_3)^{m+1}_v$ can be interpreted as follows

$$\left\{ \begin{array}{l} (\partial_t \mathbf{u}_k, \mathbf{v})_\Omega + c(\widetilde{\mathbf{U}}^{(0)}_k, \widetilde{\mathbf{u}}^{(1)}_k, \mathbf{v}) + \nu(\nabla \widetilde{\mathbf{u}}^{(1)}_k, \nabla \mathbf{v}) \\ = \langle \mathbf{f}_k, \mathbf{v}\rangle_\Omega + \langle \mathbf{g}_{s,k}, \mathbf{v}\rangle_{\Gamma_s}, \quad \forall\, \mathbf{v} \in \mathbf{W}^{1,3}_{b,l}(\Omega) \cap \mathbf{H}, \end{array} \right. \tag{14}$$

where test functions in $\mathbf{H}$ have been taken to eliminate the pressure term. Here, $\mathbf{f}_k$ and $\mathbf{g}_{s,k}$ are the functions defined in $[0,T]$, constant by subintervals, equal to $\mathbf{f}^{m+1}$ and $\mathbf{g}^{m+1}_s$ in $(t_m, t_{m+1}]$ respectively.

If we want to take limits in (14), we need for instance compactness of $(\widetilde{\mathbf{u}}^{(1)}_k)_k$ in $L^2(\mathbf{L}^2(\Omega))$ to control the limit of the convective terms. Taking into account (11), it suffices to obtain compactness of $(\mathbf{u}_k)_k$. For this, firstly, we will prove the following result

Lemma 4 *The sequence $(\partial_t \mathbf{u}_k)_k$ is bounded in $L^1(0,T;(\mathbf{W}^{1,3}_{b,l}(\Omega) \cap \mathbf{H})')$, with $(\mathbf{W}^{1,3}_{b,l}(\Omega) \cap \mathbf{H})'$ being the dual space of $\mathbf{W}^{1,3}_{b,l}(\Omega) \cap \mathbf{H}$.*

Proof. From (14), we have

$$\langle \partial_t \mathbf{u}_k(t), \mathbf{v}\rangle = \langle h_k(t), \mathbf{v}\rangle \quad \text{a.e. } t \in (0,T), \quad \forall\, \mathbf{v} \in \mathbf{W}^{1,3}_{b,l}(\Omega) \cap \mathbf{H},$$

where $\langle h_k(t), \mathbf{v}\rangle = c(\widetilde{\mathbf{U}}^{(0)}_k, \widetilde{\mathbf{u}}^{(1)}_k, \mathbf{v}) - \nu(\!(\widetilde{\mathbf{u}}^{(1)}_k, \mathbf{v})\!) + \langle \mathbf{f}_k, \mathbf{v}\rangle_\Omega + \langle \mathbf{g}_{s,k}, \mathbf{v}\rangle_{\Gamma_s}$ Then,

$$\begin{aligned} \langle h_k(t), \mathbf{v}\rangle \ &\le\ \left\{ \nu\,|\nabla \widetilde{\mathbf{u}}^{(1)}_k(t)| + \|\mathbf{f}_k(t)\|_{\mathbf{H}^{-1}_{b,l}} + \|\mathbf{g}_{s,k}(t)\|_{\mathbf{H}^{-1/2}(\Gamma_s)} \right\} \|\mathbf{v}\| \\ &+\ \|\widetilde{\mathbf{U}}^{(0)}_k(t)\|_{\mathbf{H}^1 \times H(\partial_z)} \|\mathbf{u}^{(1)}_k(t)\|\, \|\mathbf{v}\|_{\mathbf{W}^{1,3}}, \end{aligned}$$

hence we obtain the conclusion of this lemma. ■

From Lemma 4 and the estimate of $(\mathbf{u}_k)_k$ in $L^2(\mathbf{H}^1 \cap \mathbf{H})$, it suffices to apply a compactness result of Aubin-Lions type, with the spaces $\mathbf{H}^1 \cap \mathbf{H} \hookrightarrow \mathbf{H} \hookrightarrow (\mathbf{W}^{1,3}_{b,l}(\Omega) \cap \mathbf{H})'$ (where all these embeddings are dense and compact), to obtain the following compactness result:

Lemma 5 *The sequence $(\mathbf{u}_k)_k$ is relatively compact in $L^2(0, T; \mathbf{H})$.*

We are now in position to demonstrate the convergence result.

Proof of Theorem 3: From estimates (9) and (10), there exists a subsequence $(k') \subset (k)$, with $k' \to 0$ and there exist $\widetilde{\mathbf{u}}^{(1)}$ and $\widetilde{\mathbf{u}}^{(0)} \in L^2(\mathbf{H}^1_{b,l}) \cap L^\infty(\mathbf{L}^2)$, $\mathbf{u}^{(1)}$, $\mathbf{u}^{(0)}$ and $\mathbf{u} \in L^2(\mathbf{H}^1) \cap L^\infty(\mathbf{H})$, with a vertical velocity $u_3 \in L^2(H(\partial_z))$, such that $\widetilde{\mathbf{u}}^{(1)}_{k'}$, $\mathbf{u}^{(1)}_{k'}$, $\widetilde{\mathbf{u}}^{(0)}_{k'}$, $\mathbf{u}^{(0)}_{k'}$ and $\mathbf{u}_{k'}$ converge to $\widetilde{\mathbf{u}}^{(1)}$, $\mathbf{u}^{(1)}$, $\widetilde{\mathbf{u}}^{(0)}$, $\mathbf{u}^{(0)}$ and $\mathbf{u}$ weakly-$\star$ in $L^\infty(0, T; \mathbf{L}^2(\Omega))$ and weakly in $L^2(0, T; \mathbf{H}^1(\Omega))$, while $\widetilde{u}^{(0)}_{3,k'}$ converge to u_3 weakly in $L^2(0, T; H(\partial_z))$.

Moreover, due to the compactness result given in Lemma 5, there exists a subsequence (again denoted by k') such that

$$\mathbf{u}_{k'} \to \mathbf{u} \quad \text{strongly in } L^2(\mathbf{H}).$$

Then, from (11), (12) and (13) and the uniqueness of the limits, we have that all these limit functions $\widetilde{\mathbf{u}}^{(1)}$, $\mathbf{u}^{(1)}$, $\widetilde{\mathbf{u}}^{(0)}$, $\mathbf{u}^{(0)}$ and $\mathbf{u}$ are the same function which belongs to $L^2(\mathbf{V}) \cap L^\infty(\mathbf{H})$. Moreover, since $\partial_z u_3 = -\nabla_H \cdot \widetilde{\mathbf{u}}^{(0)} = -\nabla_H \cdot \mathbf{u}$ and $\nabla_H \cdot \langle \mathbf{u} \rangle = 0$ then $u_3|_{\Gamma_b \cup \Gamma_l} = 0$ and $u_3 \in L^2(0, T; H_0(\partial_z))$. Finally, we also get

$$\widetilde{\mathbf{u}}^{(1)}_{k'} \to \mathbf{u} \quad \text{strongly in } L^2(\mathbf{L}^2).$$

Then, writing (14) in k', the pass to the limit as $k' \to 0$ can be realized by a standard way, concluding that $(\mathbf{u}, u_3)$ is a weak solution of the continuous problem (EP). □

2 Error estimates

In this section, we will obtain optimal error estimates (for the velocity as well as for the pressure) with respect to a sufficiently regular solution $\{\mathbf{u}, u_3, p_s\}$ of the problem (EP).

The results that we will obtain in this section, can be considered as extensions to the Primitive Equations of the results obtained in [16] for Navier-Stokes Equations.

For this, we introduce the following notations for the errors in $t = t_{m+1}$:

$$\begin{aligned} \widetilde{\mathbf{e}}^{m+1} &= \mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}^{m+1}, & \mathbf{e}^{m+1} &= \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1} \\ \widetilde{e}^{m+1}_3 &= u_3(t_{m+1}) - \widetilde{u}^{m+1}_3, & e^{m+1}_3 &= u_3(t_{m+1}) - u^{m+1}_3 \\ e^{m+1}_{p,s} &= p_s(t_{m+1}) - p^{m+1}_s. \end{aligned}$$

and for the discrete in time derivatives

$$\delta_t p^{m+1} = \frac{p^{m+1} - p^m}{k} \qquad \delta_t p(t_{m+1}) = \frac{p(t_{m+1}) - p(t_m)}{k}.$$

Differential problems associated to the errors:

For simplicity, we assume $\mathbf{f} \in C([0,T]; \mathbf{H}_{b,l}^{-1})$ and $\mathbf{g}_s \in C([0,T]; \mathbf{H}^{-1/2}(\Gamma_s))$, hence we can choose

$$\mathbf{f}^{m+1} = \mathbf{f}(t_{m+1}) \quad \text{ and } \quad \mathbf{g}_s^{m+1} = \mathbf{g}_s(t_{m+1}).$$

Therefore the data errors $\mathbf{f}(t_{m+1}) - \mathbf{f}^{m+1}$ and $\mathbf{g}_s(t_{m+1}) - \mathbf{g}_s^{m+1}$ vanish; error estimates depending on the data can be seen in [24], for parabolic (linear) problems.

Comparing (EP) at $t = t_{m+1}$ and $(S_1)^{m+1}$, it is easy to arrive at (see [16], [25] for the Navier-Stokes case):

$$(E_1)^{m+1} \left\{ \begin{array}{l} \dfrac{1}{k}(\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m) - \nu \Delta \widetilde{\mathbf{e}}^{m+1} + \nabla_H (e_{p,s}^m + k\, \delta_t p_s(t_{m+1})) = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \ \text{ in } \Omega \\ \nu\, \partial_z \widetilde{\mathbf{e}}^{m+1}|_{\Gamma_s} = 0, \quad \widetilde{\mathbf{e}}^{m+1}|_{\Gamma_b \cup \Gamma_l} = 0, \end{array} \right.$$

where

$$\mathcal{E}^{m+1} := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1} = -\frac{1}{k} \int_{t_m}^{t_{m+1}} (t - t_m)\, \partial_{tt}^2 \mathbf{u}(t)\, dt - \left(\Big(\int_{t_m}^{t_{m+1}} \partial_t \mathbf{U} \Big) \cdot \nabla \right) \mathbf{u}(t_{m+1})$$

$$\mathbf{NL}^{m+1} = -C\Big((\widetilde{\mathbf{e}}^m, \widetilde{e}_3^m), \mathbf{u}(t_{m+1}) \Big) - C(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{e}}^{m+1})$$

On the other hand, adding and subtracting the terms $\mathbf{u}(t_{m+1})$, $p_s(t_{m+1})$ and $p_s(t_m)$ in $(S_2)^{m+1}$,

$$(E_2)^{m+1} \left\{ \begin{array}{rcl} \dfrac{1}{k}(\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}) + \nabla_H \Big(e_{p,s}^{m+1} - e_{p,s}^m - k\, \delta_t p_s(t_{m+1}) \Big) & = & 0 \ \text{ in } \Omega \\ \nabla_H \cdot \langle \mathbf{e}^{m+1} \rangle = 0 \ \text{ in } S, \qquad \langle \mathbf{e}^{m+1} \rangle \cdot \mathbf{n}_{\partial S} & = & 0 \ \text{ on } \partial S. \end{array} \right.$$

Again, the problem $(E_2)^{m+1}$ can be decomposed into two problems as follows:

$$(E_2)_a^{m+1} \qquad \left\{ \begin{array}{l} k\, \nabla_H \cdot (D\, \nabla_H (e_{p,s}^{m+1} - e_{p,s}^m - k\, \delta_t p(t_{m+1})) = \nabla \cdot \langle \widetilde{\mathbf{e}}^{m+1} \rangle \quad \text{in } S, \\ k\, \nabla_H (e_{p,s}^{m+1} - e_{p,s}^m - k\, \delta_t p_s(t_{m+1})) \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}, \end{array} \right.$$

and

$$(E_2)_b^{m+1} \qquad \mathbf{e}^{m+1} = \widetilde{\mathbf{e}}^{m+1} - k\, \nabla (e_{p,s}^{m+1} - e_{p,s}^m - k\, \delta_t p_s(t_{m+1})) \quad \text{in } \Omega.$$

Regularity hypotheses:

In the sequel, we will assume the following regularity hypothesis on Ω:

(H0) $\Omega \subset I\!R^3$ such that the $\mathbf{H}^2(\Omega)$-regularity for the Poisson problem related to $(E_1)^{m+1}$:

$$-\Delta \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad \partial_z \mathbf{u}|_{\Gamma_s} = \mathbf{g}, \quad \mathbf{u}|_{\Gamma_b \cup \Gamma_l} = 0$$

hold.

In order to obtain the different error estimates, the following regularity hypotheses for the (unique) solution $(\mathbf{u}, p)$ of (EP) will appear:

(H1) $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}), \quad p_s \in L^\infty(H^1), \quad \partial_t p_s \in L^2(H^1), \quad \mathbf{u}_t \in L^2(\mathbf{H}^1), \quad \mathbf{u}_{tt} \in L^2(\mathbf{H}^{-1}_{b,l})$

(H2) $\partial_{tt} p_s \in L^2(H^1), \quad \mathbf{u}_t \in L^\infty(\mathbf{W}^{1,3}), \quad \mathbf{u}_{tt} \in L^2(\mathbf{H}^1), \quad \mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1}_{b,l})$

(H3) $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1}_{b,l})$

Remark 2 *Unfortunately, as in Navier-Stokes case, in order to obtain* **(H1)**-**(H3)***, it is necessary to assume hypotheses which imply non local compatibility conditions for the data.*

In this section, by C we will denote different constants, always independent of k.

With a similar reasoning to the proof of the Lemma 1, now for $(E_2)^{m+1}$, we can obtain the following result:

Lemma 6 *(Continuous dependence of the errors).*
a) *Continuous dependence with respect to L^2. If $\widetilde{\mathbf{e}}^{m+1} \in L^2(\Omega)$, then $\mathbf{e}^{m+1} \in \mathbf{H}$. Moreover,*

$$|\mathbf{e}^{m+1}| \le |\widetilde{\mathbf{e}}^{m+1}| \quad \textit{and} \quad |\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}| \le |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|.$$

and

$$|\widetilde{\mathbf{e}}^{m+1}|^2 = |\mathbf{e}^{m+1}|^2 + |k\ \nabla_H (e^{m+1}_{p,s} - e^m_{p,s} - k\,\delta_t p_s(t_{m+1}))|^2 = |\mathbf{e}^{m+1}|^2 + |\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}|^2.$$

b) *Continuous dependence with respect to H^1. Assume $S \in C^3$ and $D \in W^{1,\infty}(S)$. If $\widetilde{\mathbf{e}}^{m+1} \in \mathbf{H}^1_{b,l}(\Omega)$, then $\mathbf{e}^{m+1} \in \mathbf{H}^1(\Omega)$. Moreover, there exists $C = C(\Omega, D) > 0$ such that*

$$\|\mathbf{e}^{m+1}\|_{H^1(\Omega)} \le C\,\|\widetilde{\mathbf{e}}^{m+1}\|.$$

2.1 Some 3D anisotropic spaces and related estimates.

To obtain optimal error estimates, we will use anysotropic spaces with their estimates. In this sense, given $p, q \in [1, +\infty]$, it will be said that a function u belongs to $L^q_z L^p_{\mathbf{x}}(\Omega)$ if:

$$u(\cdot, z) \in L^p(S_z) \quad \text{and} \quad \|u(\cdot, z)\|_{L^p(S_z)} \in L^q(-D_{\max}, 0),$$

where $S_z = \{\mathbf{x} \in S \ : \ (\mathbf{x}, z) \in \Omega\}$, and its norm is given by $\left\| \|u(\cdot,z)\|_{L^p(S_z)} \right\|_{L^q(-D_{\max},0)}$. The most useful norms that we will use in this paper are:

$$\|u\|_{L^2_z L^4_{\mathbf{x}}(\Omega)} = \left(\int_{-D_{\max}}^0 \|u(\cdot,z)\|^2_{L^4(S_z)} dz \right)^{1/2}$$

$$\|u\|_{L^\infty_z L^2_{\mathbf{x}}(\Omega)} = \sup_{z\in(-D_{\max},0)} \|u(\cdot,z)\|_{L^2(S_z)},$$

For sake of simplicity, we denote $L^q_z L^p_{\mathbf{x}}$ instead of $L^q_z L^p_{\mathbf{x}}(\Omega)$, and L^p instead of $L^p(\Omega)$, when there is no risk of confusion.

In a similar way, we define spaces

$$H^1_z L^2_{\mathbf{x}} \equiv H^1(-D_{max}, 0; L^2(S_z)), \quad L^2_z H^1_{\mathbf{x}} \equiv L^2(-D_{max}, 0; H^1(S_z)).$$

Notice that $H^1_z L^2_{\mathbf{x}} = H(\partial_z)$.

Also, we use the following anisotropic inequalities (see [13]):

- Horizontal Gagliardo-Nirenberg inequality (related to $2D$ subdomains):

$$\|u\|_{L^2_z\, L^4_{\mathbf{x}}} \le C\, |u|^{1/2}\, |\nabla_H u|^{1/2} \quad \forall\, u \in L^2_z H^1_{\mathbf{x}} \quad \text{such that} \quad u|_{\Gamma_b\cup\Gamma_l} = 0,$$

$$\|u\|_{L^2_z\, L^4_{\mathbf{x}}} \le C\, |u|^{1/2}\, \|u\|^{1/2}_{H^1(\Omega)} \quad \forall\, u \in H^1(\Omega). \tag{15}$$

- Vertical Poincaré Inequality (related to $1D$ subdomains):

$$|v| \le D^{1/2}_{max}\, |\partial_z v|, \quad \forall\, v \in H^1_z L^2_{\mathbf{x}} \quad \text{such that} \quad v|_{\Gamma_s} = 0. \tag{16}$$

- Vertical Gagliardo-Nirenberg inequality (related to $1D$ subdomains):

$$\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \le C\, (|v| + |v|^{1/2}\, |\partial_z v|^{1/2}), \quad \forall\, v \in H^1_z L^2_{\mathbf{x}}. \tag{17}$$

  Moreover, if $v|_{\Gamma_s} = 0$, one has $\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \le C\, |v|^{1/2}\, |\partial_z v|^{1/2}$.

  In particular, from (16) and (17), one has

$$\|v\|_{L^\infty_z L^2_{\mathbf{x}}} \le C\, |\partial_z v|, \quad \forall\, v \in H^1_z L^2_{\mathbf{x}} \quad \text{such that} \quad v|_{\Gamma_s} = 0. \tag{18}$$

2.2 $O(k)$-error estimates for the velocities in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$

Theorem 7 *Assuming* **(H1)** *and* $|\nabla_H e^0_{p,s}| \le C$. *Then, for* k *small enough,*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\, k.$$

Moreover,

$$\|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m\|_{l^2(\mathbf{L}^2)} \le C\, k^{3/2}.$$

Proof. We follow a similar argument to the Navier-Stokes case [16], but in this case the problem is more singular.
Multiplying $(E_1)^{m+1}$ by $2\,k\,\widetilde{\mathbf{e}}^{m+1}$, integrating in Ω:

$$\begin{aligned}&|\widetilde{\mathbf{e}}^{m+1}|^2 - |\mathbf{e}^m|^2 + |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + 2\;k\|\widetilde{\mathbf{e}}^{m+1}\|^2 + 2\,k\left(\nabla_H(e^m_{p,s} + k\,\delta_t p_s(t_{m+1})), \widetilde{\mathbf{e}}^{m+1}\right)\\ &\leq 2\,k\,\langle \mathcal{E}^{m+1}, \widetilde{\mathbf{e}}^{m+1}\rangle + 2\,k(\mathbf{NL}^{m+1}, \widetilde{\mathbf{e}}^{m+1})\end{aligned} \tag{19}$$

We bound the horizontal terms on the RHS as in [16].

$$2\,k\,\langle \mathcal{E}_1^{m+1}, \widetilde{\mathbf{e}}^{m+1}\rangle \;\;\leq\;\; \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k^2 \int_{t_m}^{t_{m+1}} \|\partial^2_{tt}\mathbf{u}\|^2_{\mathbf{H}^{-1}_{b,l}}\,dt$$

$$2\,k\left(\mathcal{E}_2^{m+1}, \widetilde{\mathbf{e}}^{m+1}\right) = 2\,k\left((\int_{t_m}^{t_{m+1}} \partial_t\mathbf{u}\cdot\nabla_H)\mathbf{u}(t_{m+1}), \widetilde{\mathbf{e}}^{m+1}\right) + 2\,k\left((\int_{t_m}^{t_{m+1}} \partial_t u_3\cdot\partial_z)\mathbf{u}(t_{m+1}), \widetilde{\mathbf{e}}^{m+1}\right)$$

$$\leq \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k^2\,\|\nabla_H\mathbf{u}(t_{m+1})\|^2_{L^3}\int_{t_m}^{t_{m+1}} |\partial_t\,\mathbf{u}|^2 + \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + k^2\,\|\mathbf{u}(t_{m+1})\|^2_{H^2}\int_{t_m}^{t_{m+1}} \|\partial_t\mathbf{u}\|^2$$

With respect to the convective terms, taking into account that $c(\widetilde{\mathbf{U}}^m, \widetilde{\mathbf{e}}^{m+1}, \widetilde{\mathbf{e}}^{m+1}) = 0$,

$$\begin{aligned}&2\,k\,(\mathbf{NL}^{m+1}, \widetilde{\mathbf{e}}^{m+1}) = 2\,k\,c(\widetilde{\mathbf{e}}^m + \widetilde{e}^m_3, \mathbf{u}(t_{m+1}), \widetilde{\mathbf{e}}^{m+1})\\ &= 2\,k\,c(\widetilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1}), \widetilde{\mathbf{e}}^{m+1}) + 2\,k\,c(\widetilde{e}^m_3, \mathbf{u}(t_{m+1}), \widetilde{\mathbf{e}}^{m+1}) = I_1 + I_2\end{aligned}$$

$$I_1 \leq \varepsilon\,k\,\|\widetilde{\mathbf{e}}^m\|^2 + C\,k\,\|\mathbf{u}(t_{m+1})\|^2_{L^\infty\cap W^{1,3}}|\widetilde{\mathbf{e}}^{m+1}|^2$$

With respect to I_2, we apply (15) for $\widetilde{\mathbf{e}}^{m+1}$ and (18) for $\widetilde{e}^m_3$,

$$\begin{aligned} I_2 &= 2\,k\left(\partial_z\widetilde{e}^m_3, \mathbf{u}(t_{m+1})\cdot\widetilde{\mathbf{e}}^{m+1}\right) + 2\,k\left(\widetilde{e}^m_3, \partial_z\mathbf{u}(t_{m+1})\cdot\widetilde{\mathbf{e}}^{m+1}\right)\\ &\leq |\partial_z\widetilde{e}^m_3|\,|\widetilde{\mathbf{e}}^{m+1}|\,\|\mathbf{u}(t_{m+1})\|_{L^\infty} + \|\widetilde{e}^m_3\|_{L^\infty_z L^2_{\mathbf{x}}}\,\|\widetilde{\mathbf{e}}^{m+1}\|_{L^2_z L^4_{\mathbf{x}}}\,\|\partial_z\mathbf{u}(t_{m+1})\|_{L^2_z L^4_{\mathbf{x}}}\\ &\leq \varepsilon\,k\,\|\widetilde{\mathbf{e}}^m\|^2 + \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + \frac{C}{\nu}\,k\left\{\|\mathbf{u}(t_{m+1})\|^2_{L^\infty} + \|\mathbf{u}(t_{m+1})\|^4_{H^2}\right\}|\widetilde{\mathbf{e}}^{m+1}|^2\\ &\leq \varepsilon\,k\,\|\widetilde{\mathbf{e}}^m\|^2 + \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,|\widetilde{\mathbf{e}}^{m+1}|^2\end{aligned}$$

Taking into account these estimates in (19), we arrive at

$$\begin{aligned}&|\widetilde{\mathbf{e}}^{m+1}|^2 - |\mathbf{e}^m|^2 + |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + 2\,\nu\,k\|\widetilde{\mathbf{e}}^{m+1}\|^2 + 2\,k\left(\nabla_H(e^m_{p,s} + k\,\delta_t p_s(t_{m+1})), \widetilde{\mathbf{e}}^{m+1}\right)\\ &\leq \varepsilon\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + \varepsilon\,k\,\|\widetilde{\mathbf{e}}^m\|^2 + C\,k^2\int_{t_m}^{t_{m+1}} \|\partial^2_{tt}\mathbf{u}\|^2_{\mathbf{H}^{-1}_{b,l}}\,dt + C\,k^2\,\|\nabla_H\mathbf{u}(t_{m+1})\|^2_{L^3}\int_{t_m}^{t_{m+1}} |\partial_t\,\mathbf{u}|^2\\ &\quad + k^2\,\|\mathbf{u}\|^2_{H^2}\int_{t_m}^{t_{m+1}} \|\partial_t\mathbf{u}\|^2 + C\,k\,|\widetilde{\mathbf{e}}^{m+1}|^2\end{aligned} \tag{20}$$

On the other hand, multiplying $(E_2)^{m+1}$ by $k\,(\mathbf{e}^{m+1} + \widetilde{\mathbf{e}}^{m+1}) + k^2\left(\nabla_H e^{m+1}_{p,s} + \nabla_H e^m_{p,s}\right)$, using that $(\mathbf{e}^{m+1}, \nabla e^{m+1}_{p,s}) = 0 = (\mathbf{e}^{m+1}, \nabla_H e^m_{p,s}) = (\mathbf{e}^{m+1}, \nabla_H\delta_t p_s(t_{m+1}))$ and taking into account

$(E_2)_b^{m+1}$ we obtain (as in [16]):

$$
\begin{aligned}
&|\mathbf{e}^{m+1}|^2 - |\widetilde{\mathbf{e}}^{m+1}|^2 + |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + \Big(|k\,\nabla_H e_{p,s}^{m+1}|^2 - |k\,\nabla_H e_{p,s}^m|^2 \Big) - 2\,k\,(\widetilde{\mathbf{e}}^{m+1}, \nabla_H e_{p,s}^m) \\
&\qquad = k^2\,(\widetilde{\mathbf{e}}^{m+1}, \nabla_H \delta_t p_s(t_{m+1})) + k^3 \Big(\nabla_H \delta_t p_s(t_{m+1}), \nabla_H e_{p,s}^{m+1} + \nabla_H e_{p,s}^m \Big) \\
&\quad \le \varepsilon\,|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + C\,k\,|k\,\nabla_H e_{p,s}^{m+1}|^2 + C\,k\,|k\,\nabla_H e_{p,s}^m|^2 + C\,k^3\,|\nabla_H \delta_t p_s(t_{m+1}))|^2
\end{aligned} \tag{21}
$$

Adding up $\sum_{m=0}^r \{(20)_m + (21)_m\}$, the term $2\,k(\widetilde{\mathbf{e}}^{m+1}, \nabla e_{p,s}^m)$ vanish and taking into account the estimate $|\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}| \le |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|$ and taking ε small enough, we obtain

$$
\begin{aligned}
&|\mathbf{e}^{r+1}|^2 + |k\,\nabla_H e_p^{r+1}|^2 + \tfrac{1}{2} \sum_{m=0}^r \Big|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + k \sum_{m=0}^r \|\widetilde{\mathbf{e}}^{m+1}\|^2 \\
&\le |\mathbf{e}^0|^2 + |k\,\nabla_H e_p^0|^2 + k \sum_{m=0}^r |k\,\nabla_H e_p^m|^2 \\
&+ C\,k \sum_{m=0}^r \Big(|\widetilde{\mathbf{e}}^{m+1}|^2 + |k\,\nabla_H e_{p,s}^{m+1}|^2 \Big) + C\,k^2 \Big(\|\mathbf{u}_{tt}\|^2_{L^2(\mathbf{H}_{b,l}^{-1})} + \|\mathbf{u}_t\|^2_{L^2(H^1)} + \|\nabla_H \partial_t p_s\|^2_{L^2(L^2)} \Big)
\end{aligned} \tag{22}
$$

Now, bounding in (22) as follows

$$
C\,k \sum_{m=0}^r |\widetilde{\mathbf{e}}^{m+1}|^2 \le C\,k \sum_{m=0}^r |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + C\,k \sum_{m=0}^r |\mathbf{e}^m|^2,
$$

$$
\begin{aligned}
C\,k \sum_{m=0}^r |k\nabla_H e_{p,s}^{m+1}|^2 &\le C\,k \sum_{m=0}^r |\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}|^2 + C\,k^3 \sum_{m=0}^r |\nabla_H \delta_t p_s(t_{m+1}))|^2 + C\,k \sum_{m=0}^r |k\nabla_H e_{p,s}^m|^2 \\
&\le C\,k \sum_{m=0}^r |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|^2 + C\,k^2 \|\nabla_H \partial_t p_s\|^2_{L^2(L^2)} + C\,k \sum_{m=0}^r |k\nabla_H e_{p,s}^m|^2
\end{aligned}
$$

taking k small enough such that $2\,C\,k \le 1/2$, and applying the Gronwall's discrete inequality, we arrive at

$$
\|\mathbf{e}^{m+1}\|_{l^\infty(L^2)} \le C\,k, \quad \|\widetilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \le C\,k \quad \text{and} \quad \|e_{p,s}^{m+1}\|_{l^\infty(H^1)} \le C.
$$

Finally, the estimates $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)} \le C\,k$ and $\|\mathbf{e}^{m+1}\|_{l^2(H^1)} \le C\,k$ are obtained from the Lemma 6.

■

Lemma 8 *Under hypotheses of Theorem 7 and for k small enough, the following estimates hold*

$$
\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C, \qquad \forall m.
$$

Proof. From the H^2-regularity of the Poisson problem $(E_1)^{m+1}$, one has

$$
\|\widetilde{\mathbf{e}}^{m+1}\|^2_{H^2} \le C \left(\Big| \frac{\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m}{k} \Big|^2 + |\nabla e_p^m|^2 + k^2 |\nabla\,\delta_t p(t_{m+1})|^2 + |\mathcal{E}^{m+1}|^2 + |\mathbf{NL}^{m+1}|^2 \right). \tag{23}
$$

Taking into account that $|\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m| \le C\,k$ and $\|e_{p,s}^{m+1}\|_{l^\infty(H^1)} \le C$ owing to Theorem 7, the more complicate term to bound of the RHS of (23) is the vertical part of $|\mathbf{NL}^{m+1}|^2$:

We bound this term as follows:

$$|\widetilde{e}_3^m\,\partial_z \widetilde{\mathbf{e}}^{m+1}|^2 \le \|\widetilde{e}_3^m\|^2_{L_z^\infty L_{\mathbf{x}}^4}\,\|\partial_z \widetilde{\mathbf{e}}^{m+1}\|^2_{L_z^2\,L_{\mathbf{x}}^4} \le \|\widetilde{\mathbf{e}}^m\|\,\|\widetilde{\mathbf{e}}^{m+1}\|\,\|\widetilde{\mathbf{e}}^m\|_{H^2}\,\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C\,k\|\widetilde{\mathbf{e}}^m\|_{H^2}\,\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2}$$

(here, we have used that $\|\widetilde{\mathbf{e}}^m\|\,\|\widetilde{\mathbf{e}}^{m+1}\| \le C\,k$). Then, we have

$$\frac{3}{4}\,\|\widetilde{\mathbf{e}}^{m+1}\|^2_{H^2} \le C_1\,k^2\,\|\widetilde{e}^m\|^2_{H^2} + C_2 \quad \forall m.$$

If k small enough is such that $C_1\,k^2 \le \frac{1}{4}$, then

$$\frac{3}{4}\|\widetilde{\mathbf{e}}^{m+1}\|^2_{H^2} \le \frac{1}{4}\,\|\widetilde{\mathbf{e}}^m\|^2 + C_2, \quad \forall m.$$

From here, we can proof that

$$\|\widetilde{\mathbf{e}}^{m+1}\|^2_{H^2} \le 2\,C_2 \quad \forall m$$

Indeed, for $m = 0$, since $\widetilde{\mathbf{e}}^0 = 0$, we have

$$\frac{3}{4}\|\widetilde{\mathbf{e}}^1\|^2_{H^2} \le C_2$$

then

$$\|\widetilde{\mathbf{e}}^1\|^2_{H^2} \le \frac{4}{3}\,C_2 \le 2\,C_2.$$

For $m = 1$

$$\frac{3}{4}\|\widetilde{\mathbf{e}}^2\|^2_{H^2} \le \frac{1}{4}\,\|\widetilde{\mathbf{e}}^1\|^2_{H^2} + C_2 \le \frac{1}{2}\,C_2 + C_2 = \frac{3}{2}\,C_2,$$

then

$$\|\widetilde{\mathbf{e}}^2\|^2_{H^2} \le 2\,C_2.$$

And so on.

Consequently, it is sufficient to choose k verifying $C_1\,k^2 \le \frac{1}{4}$. ∎

As a consequence of the previous lemma, one has

$$\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} \le C \qquad \forall m.$$

2.3 $O(k)$-error estimates for $(\widetilde{\mathbf{e}}^{m+1}, e_{p,s}^{m+1})$ in $l^\infty(\mathbf{H}^1 \times L^2)$

Firstly, we are going to obtain error estimates for discrete time derivative of velocity. Afterwards, we will obtain the $O(k)$ optimal error estimates for the pressure.

Lemma 9 *(Continuous dependence of discrete derivatives)*

$$|\delta_t \mathbf{e}^{m+1}| \le |\delta_t \widetilde{\mathbf{e}}^{m+1}|, \qquad |\delta_t \mathbf{e}^{m+1} - \delta_t \widetilde{\mathbf{e}}^{m+1}| \le |\delta_t \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^{m}|.$$

Moreover, there exists $C = C(\Omega) > 0$ such as

$$\|\delta_t \mathbf{e}^{m+1}\| \le C\, \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|.$$

and

$$|\delta_t \widetilde{\mathbf{e}}^{m+1}|^2 = |\delta_t \mathbf{e}^{m+1}|^2 + |\delta_t \mathbf{e}^{m+1} - \delta_t \widetilde{\mathbf{e}}^{m+1}|^2$$

The proof of this Lemma follows the same lines given in [16] for the Navier-Stokes case and in Lemma 1 for Primitive Equations problem.

Theorem 10 *Assuming hypotheses of Theorem 7,* (**H2**) *and the following constraint on the initial approximation*

$$|\delta_t \mathbf{e}^1| + |k\, \nabla_H \delta_t e_p^1| \le C\, k,$$

then one obtains

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)} + \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C\, k$$

for each k small enough.

Proof. By making $\delta_t(E_1)^{m+1}$ and $\delta_t(E_2)^{m+1}$, one obtains $\forall m$,

$$(D_1)^{m+1} \quad \left\{ \; \frac{\delta_t \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m}{k} - \Delta \delta_t \widetilde{\mathbf{e}}^{m+1} + \nabla_H(\delta_t e_{p,s}^m + k\, \delta_t \delta_t p_s(t_{m+1})) = \delta_t \mathcal{E}^{m+1} + \delta_t \mathbf{NL}^{m+1} \right.$$

where $\delta_t \delta_t p_s(t_{m+1}) = \dfrac{1}{k}(\delta_t p_s(t_{m+1}) - \delta_t p_s(t_m))$, and

$$(D_2)^{m+1} \qquad \left\{ \; \frac{\delta_t \mathbf{e}^{m+1} - \delta_t \widetilde{\mathbf{e}}^{m+1}}{k} + \nabla_H\Big(\delta_t e_{p,s}^{m+1} - \delta_t e_{p,s}^m - k\, \delta_t \delta_t p_s(t_{m+1})\Big) = 0. \right.$$

The proof follows similar lines of [16].

Multiplying $(D_1)^{m+1}$ by $2\, k\, \delta_t \widetilde{\mathbf{e}}^{m+1}$, we obtain:

$$|\delta_t \widetilde{\mathbf{e}}^{m+1}|^2 - |\delta_t \mathbf{e}^m|^2 + |\delta_t\, \widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m|^2 + 2\, \nu\, k \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2 + 2\, k \Big(\nabla_H(\delta_t e_{p,s}^m + k\, \delta_t \delta_t p_s(t_{m+1}), \delta_t \widetilde{\mathbf{e}}^{m+1}\Big)$$

$$= 2\, k \left(\delta_t \mathcal{E}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\right) + 2\, k \left(\delta_t \mathbf{NL}^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}\right)$$

We bound the RHS as follows

$$2\, k(\delta_t \mathcal{E}_1^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}) \le \varepsilon\, k \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|^2 + C\, k^2 \int_{t_{m-1}}^{t_{m+1}} \|\mathbf{u}_{ttt}\|^2_{\mathbf{H}^{-1}_{b,l}}$$

$$2\, k(\delta_t \mathcal{E}_2^{m+1}, \delta_t \widetilde{\mathbf{e}}^{m+1}) = 2\, k(\delta_t \mathbf{U}(t_{m+1}) \cdot \nabla(\mathbf{u}(t_{m+1}) - \mathbf{u}(t_m)), \delta_t\, \widetilde{\mathbf{e}}^{m+1})$$

$$+2\,k((\delta_t\mathbf{U}(t_{m+1})-\delta_t\mathbf{U}(t_m))\cdot\nabla\mathbf{u}(t_m),\delta_t\widetilde{\mathbf{e}}^{m+1})=I_1+I_2$$

$$I_1\le\varepsilon k\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2+C\,k\|\delta_t\mathbf{U}(t_{m+1})\|^2_{L^3}\,\|\int_{t_m}^{t_{m+1}}\partial_t\mathbf{u}\|^2_{L^6}\le\varepsilon k\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2+Ck^2\|\mathbf{u}_t\|^2_{L^\infty(\mathbf{W}^{1,3})}\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^2$$

$$I_2\le\varepsilon\,k\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2+C\,k\,\|\mathbf{u}(t_m)\|^2_{W^{1,3}}\left|\frac{1}{k}\Big(\int_{t_m}^{t_{m+1}}\mathbf{U}_t-\int_{t_{m-1}}^{t_m}\mathbf{U}_t\Big)\right|^2\le\varepsilon\,k\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2+C\,k^2\int_{t_{m-1}}^{t_{m+1}}\|\mathbf{u}_{tt}\|^2$$

Now, we bound the non-linear terms:

$$2\,k\left(\delta_t\mathbf{NL}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\right)=2\,k\,c\Big((\delta_t\widetilde{\mathbf{e}}^m,\delta_t\widetilde{e}_3^m),\mathbf{u}(t_{m+1}),\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)$$
$$+2\,k\,c\Big(\delta_t\widetilde{\mathbf{U}}^m,\widetilde{\mathbf{e}}^{m+1},\delta_t\,\widetilde{\mathbf{e}}^{m+1}\Big)+2\,k\,c\Big((\widetilde{\mathbf{e}}^{m-1},\widetilde{e}_3^{m-1}),\delta_t\mathbf{u}(t_{m+1}),\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)$$
$$+2\,k\,c\Big(\widetilde{\mathbf{U}}^{m-1},\delta_t\widetilde{\mathbf{e}}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)=\sum_{i=1}^4L_i$$

$$L_1=2\,k\,c\Big(\delta_t\widetilde{\mathbf{e}}^m,\mathbf{u}(t_{m+1}),\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)+2\,k\,c\Big(\delta_t\widetilde{e}_3^m,\mathbf{u}(t_{m+1}),\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)=L_{11}+L_{12}$$
$$L_{11}\le\varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^m\|^2+C\,k\,\|\mathbf{u}(t_{m+1})\|^2_{W^{1,3}\cap L^\infty}\,|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2$$

$$L_{12}\le\varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^m\|^2+C\,k\,|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2$$

(reasoning as in Theorem 7)

$$L_2=2\,k\,c\Big((\delta_t\widetilde{\mathbf{e}}^m,\delta_t\widetilde{e}_3^m),\widetilde{\mathbf{e}}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)+2\,k\,c\Big(\delta_t\mathbf{U}(t_m),\widetilde{\mathbf{e}}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)=L_{21}+L_{22}$$

$$L_{21}=2\,k\,c\Big(\delta_t\widetilde{\mathbf{e}}^m,\widetilde{\mathbf{e}}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)+2\,k\,c\Big(\delta_t\widetilde{e}_3^m,\widetilde{\mathbf{e}}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\Big)=L_{211}+L_{212}$$

$$L_{211}\le\varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^m\|^2+C\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2_{W^{1,3}\cap L^\infty}\,|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2$$

$$L_{212}=2\,k\left(\partial_z\delta_t\widetilde{e}_3^m,\widetilde{\mathbf{e}}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\right)+2\,k\left(\delta_t\widetilde{e}_3^m,\partial_z\widetilde{\mathbf{e}}^{m+1},\delta_t\widetilde{\mathbf{e}}^{m+1}\right)=A+B$$

$$A\le\varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^m\|^2+C\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2_{L^\infty}\,|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2$$

$$B\le\|\delta_t\widetilde{e}_3^m\|_{L^\infty_zL^2_x}\,\|\partial_z\widetilde{\mathbf{e}}^{m+1}\|_{L^2_zL^4_x}\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|_{L^2_zL^4_x}\le\varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^m\|^2+C\,k\,|\partial_z\widetilde{\mathbf{e}}^{m+1}|\,\|\partial_z\widetilde{\mathbf{e}}^{m+1}\|\,|\delta_t\widetilde{\mathbf{e}}^{m+1}|\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|$$
$$\le\varepsilon\,k\|\delta_t\widetilde{\mathbf{e}}^m\|^2+\varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2+C\,k^2|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2$$

(here we have used that $\|\widetilde{\mathbf{e}}^{m+1}\| \le C\,k^{1/2}$ and $\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C \;\forall m$)

$$L_{22} = 2\,k\,c\Big(\delta_t \mathbf{u}(t_m), \widetilde{\mathbf{e}}^{m+1}, \delta_t\widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\,c\Big(\delta_t u_3(t_m), \widetilde{\mathbf{e}}^{m+1}, \delta_t\widetilde{\mathbf{e}}^{m+1}\Big) = L_{221} + L_{222}$$

$$L_{221} \le \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\mathbf{u}_t\|_{L^\infty(H^1)}\,\|\widetilde{\mathbf{e}}^{m+1}\|^2$$

$$\begin{aligned} L_{222} &= 2\,k\,\Big(\partial_z\delta_t u_3(t_m), \widetilde{\mathbf{e}}^{m+1}, \delta_t\widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\,\Big(\delta_t u_3(t_m), \partial_z\widetilde{\mathbf{e}}^{m+1}, \delta_t\widetilde{\mathbf{e}}^{m+1}\Big) \\ &\le \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\mathbf{u}_t\|^2\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\mathbf{u}_t\|^2_{W^{1,3}}\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 \end{aligned}$$

$$L_3 = 2\,k\,c\Big(\widetilde{\mathbf{e}}^{m-1}, \delta_t\mathbf{u}(t_{m+1}), \delta_t\widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k\,c\Big(\widetilde{e}_3^{m-1}, \delta_t\mathbf{u}(t_{m+1}), \delta_t\widetilde{\mathbf{e}}^{m+1}\Big) = L_{31} + L_{32}$$

and by a similar way

$$L_{31} \le \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\mathbf{u}_t\|^2_{L^3}\,\|\widetilde{\mathbf{e}}^{m-1}\|^2$$

$$L_{32} \le \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\mathbf{u}_t\|^2_{W^{1,3}}\,\|\widetilde{\mathbf{e}}^{m-1}\|^2$$

Finally $L_4 = 0$.

Taking into account the above estimates, we arrive at

$$\begin{aligned} &|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 - |\delta_t\mathbf{e}^m|^2 + |\delta_t\,\widetilde{\mathbf{e}}^{m+1} - \delta_t\mathbf{e}^m|^2 + 2\,\nu\,k\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + 2\,k\Big((\nabla_H(\delta_t e^m_{p,s} + k\,\delta_t\delta_t p_s(t_{m+1})), \delta_t\widetilde{\mathbf{e}}^{m+1}\Big) \\ &\le \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2 + \varepsilon\,k\,\|\delta_t\widetilde{\mathbf{e}}^m\|^2 + C\,k|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 + C\,k\,\|\widetilde{\mathbf{e}}^{m+1}\|^2 + C\,k\,\|\widetilde{\mathbf{e}}^{m-1}\|^2 \\ &+C\,k^2\,\int_{t_{m-1}}^{t_{m+1}}\|\mathbf{u}_{ttt}\|^2_{\mathbf{H}^{-1}_{b,l}} + C\,k^2\,\int_{t_m}^{t_{m+1}}\|\mathbf{u}_t\|^3 + C\,k^2\,\int_{t_{m-1}}^{t_{m+1}}\|\mathbf{u}_{tt}\|^2 \end{aligned} \tag{24}$$

On the other hand, multiplying $(D_2)^{m+1}$ by $k(\delta_t\mathbf{e}^{m+1} + \delta_t\widetilde{\mathbf{e}}^{m+1}) + k^2\,(\nabla_H\delta_t e^{m+1}_{p,s} + \nabla_H\delta_t e^m_{p,s})$ obtenemos

$$\begin{aligned} &|\delta_t\mathbf{e}^{m+1}|^2 - |\delta_t\,\widetilde{\mathbf{e}}^{m+1}|^2 + |k\,\nabla_H\delta_t e^{m+1}_{p,s}|^2 - |k\,\nabla_H\delta_t e^m_{p,s}|^2 - 2\,k\,(\delta_t\widetilde{\mathbf{e}}^{m+1}, \nabla_H\delta_t e^m_{p,s}) \\ &= k^2\,\Big(\delta_t\widetilde{\mathbf{e}}^{m+1}, \nabla_H\delta_t\delta_t p_s(t_{m+1})\Big) + k^3\,\Big(\nabla_H\delta_t e^{m+1}_{p,s} + \nabla_H\delta_t e^m_{p,s}, \nabla_H\delta_t\delta_t p_s(t_{m+1})\Big) \end{aligned} \tag{25}$$

Now, making $\displaystyle\sum_{m=1}^{r}\{(24) + (25)\}$, taking into account the previous estimates, we arrive at

$$|\delta_t\mathbf{e}^{r+1}|^2 + |k\,\nabla_H\delta_t e^{r+1}_{p,s}|^2 + \sum_{m=1}^{r}\frac{1}{2}|\delta_t\widetilde{\mathbf{e}}^{m+1} - \delta_t\mathbf{e}^m|^2 + k\,\sum_{m=1}^{r}\|\delta_t\widetilde{\mathbf{e}}^{m+1}\|^2$$

$$\le |\delta_t\mathbf{e}^1|^2 + |k\,\nabla_H\delta_t e^1_{p,s}|^2 + C\,k\,\sum_{m=1}^{r}\{|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 + |k\,\nabla_H\delta_t e^{m+1}_{p,s}|^2\} + C\,k\,\sum_{m=1}^{r}|k\,\nabla_H\delta_t e^m_{p,s}|^2 + C\,k^2$$

Finally, reasoning as in Theorem 7 and applying the Gronwall's discrete inequality, we obtain

$$\|\delta_t\mathbf{e}^{m+1}\|_{l^\infty(L^2)} \le C\,k \quad\text{and}\quad \|\delta_t\widetilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \le C\,k$$

Finally, we have using the triangular inequality and Lemma 9, we arrive at

$$\|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)} \leq C\,k \quad \text{and} \quad |\delta_t \mathbf{e}^{m+1}\|_{l^2(H^1)} \leq C\,k$$

■

Theorem 11 *Under hypothesis of Theorem 10 and* (**H3**)*, the following error estimates hold*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} + \|e_{p,s}^{m+1}\|_{l^\infty(L^2)} \leq C\,k.$$

Proof: The proof follows the same lines of [16].

Adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$, we arrive at:

$$(E_3)^{m+1}\left\{\begin{array}{l} \delta_t \mathbf{e}^{m+1} - \nu\Delta\widetilde{\mathbf{e}}^{m+1} + \nabla_H e_{p,s}^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \\ \nabla_H\cdot\langle \mathbf{e}^{m+1}\rangle = 0 \ \text{ in } S, \qquad \langle \mathbf{e}^{m+1}\rangle\cdot \mathbf{n}_{\partial S} = 0 \text{ on } \partial S. \end{array}\right.$$

Then, from Theorem 10 and the continuous inf-sup condition applied to $(E_3)^{m+1}$, we can deduce the estimate $\|e_{p,s}^{m}\|_{l^2(L^2)} \leq C\,k$, taking into account that $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^2(H^1)} \leq C\,k$ and $\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)} \leq C\ k$.

On the other hand, rewritten $(E_3)^{m+1}$ as

$$-\Delta\widetilde{\mathbf{e}}^{m+1} = -\delta_t \mathbf{e}^{m+1} - \nabla_H e_{p,s}^{m+1} + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1}, \qquad \widetilde{\mathbf{e}}^{m+1}|_{\partial\Omega} = 0,$$

multiplying by $2\,k\,\delta_t \widetilde{\mathbf{e}}^{m+1}$, we obtain

$$|\nabla\widetilde{\mathbf{e}}^{m+1}|^2 - |\nabla\widetilde{\mathbf{e}}^{m}|^2 + |\nabla(\widetilde{\mathbf{e}}^{m+1} - \widetilde{\mathbf{e}}^{m})|^2$$

$$= -2\,k(\nabla e_{p,s}^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1}) - 2\,k\Big(\delta_t \mathbf{e}^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1}\Big) + 2\,k(\mathcal{E}^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1}) + 2\,k(\mathbf{NL}^{m+1}, \delta_t\,\widetilde{\mathbf{e}}^{m+1})$$

$$\leq k\,|e_{p,s}^{m+1}|^2 + k\,|\nabla\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 + C\,k\,|\delta_t \mathbf{e}^{m+1}|^2 + C\,k\,|\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 + 2\,k\,|\mathcal{E}^{m+1}|^2 + 2\,k\,|\mathbf{NL}^{m+1}|^2$$

Now, we bound the two last terms on the right side hand as follows

$$|\mathcal{E}^{m+1}|^2 \leq C\,k^2$$

$$|\mathbf{NL}^{m+1}|^2 \leq C(\|\widetilde{\mathbf{e}}^{m}\|^2 + \|\widetilde{\mathbf{e}}^{m+1}\|^2)$$

where in the second inequality, we have used $\|\widetilde{\mathbf{u}}^m\|_{L^\infty\cap W^{1,3}} \leq C$ and $\|\mathbf{u}(t_{m+1})\|_{L^\infty\cap W^{1,3}} \leq C$.

Adding from $m=0$ to r we obtain

$$|\nabla\widetilde{\mathbf{e}}^{r+1}|^2 + \sum_{m=0}^{r} |\nabla(\widetilde{\mathbf{e}}^{m+1} - \widetilde{\mathbf{e}}^{m})|^2 \leq |\nabla\widetilde{\mathbf{e}}^{0}|^2 + C\,k^2$$

$$+C\,k\sum_{m=0}^{r}\Big(|e_{p,s}^{m+1}|^2 + |\nabla\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 + |\delta_t \mathbf{e}^{m+1}|^2 + |\delta_t\widetilde{\mathbf{e}}^{m+1}|^2 + \|\widetilde{\mathbf{e}}^{m}\|^2 + \|\widetilde{\mathbf{e}}^{m+1}\|^2\Big)$$

hence, we arrive at $\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} \leq C\,k$ applying the estimates of precedent results.

Finally, by applying again the inf-sup condition and using that $\|\delta_t \mathbf{e}^{m+1}\|_{H^{-1}} \leq C\,|\delta_t \mathbf{e}^{m+1}| \leq C\,k$ and $\|\widetilde{\mathbf{e}}^{m+1}\| \leq C\,k$, we arrive at

$$\|e_{p,s}^{m+1}\|_{l^\infty(L^2)} \leq C\,k.$$

3 Scheme with Coriolis term

Looking at the results obtained in previous sections, the more convenient forms to introduce the Coriolis term in the scheme are the following (the Coriolis term is always refereed at the velocity $\widetilde{\mathbf{u}}^{m+1}$ because this is the better approximation in the projection scheme):

- Either to consider in $(S_1)^{m+1}$ the implicit term $\mathbf{b}(\widetilde{\mathbf{u}}^{m+1})$.

  This term does not introduce any extra-term in the stability estimates, because $(\mathbf{b}(\widetilde{\mathbf{u}}^{m+1}), \widetilde{\mathbf{u}}^{m+1}) = 0$, hence the scheme verifies the same type of energy's inequality that in the continuous case. However, this term couples the two equations of convection-diffusion for the horizontal velocity $\widetilde{\mathbf{u}}^{m+1}$.

- Or to consider in $(S_1)^{m+1}$ the explicit term $\mathbf{b}(\widetilde{\mathbf{u}}^m)$.

  This term introduces the extra-term $\Big(\mathbf{b}(\widetilde{\mathbf{u}}^m), \widetilde{\mathbf{u}}^{m+1}\Big) = \Big(\mathbf{b}(\widetilde{\mathbf{u}}^m), \widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}^m\Big)$ in the stability estimates. Then, bounding it as follows,

  $$\Big(\mathbf{b}(\widetilde{\mathbf{u}}^m), \widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}^m\Big) \le |\widetilde{\mathbf{u}}^m|\,|\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}^m| \le C\,|\widetilde{\mathbf{u}}^m|^2 + \varepsilon\,|\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m|^2 + \varepsilon\,|\mathbf{u}^m - \widetilde{\mathbf{u}}^m|^2,$$

  and using Lemma 1 and the discrete Gronwall inequality in the proof of Lemma 2, this produces an artificial exponential bound in time which does not appear in the energy's inequality of the continuous problem.

  On the other hand, now the main advantage is that the computation of two equations for $\widetilde{\mathbf{u}}^{m+1}$ follows decoupled.

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