# Compact set of invariants characterizing graph states of up to eight qubits


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The set of entanglement measures proposed by Hein, Eisert, and Briegel for $n$-qubit graph states [Phys. Rev. A **69**, 062311 (2004)] fails to distinguish between inequivalent classes under local Clifford operations if $n \geq 7$. On the other hand, the set of invariants proposed by van den Nest, Dehaene, and De Moor (VDD) [Phys. Rev. A **72**, 014307 (2005)] distinguishes between inequivalent classes, but contains too many invariants (more than $2 \times 10^{36}$ for $n=7$) to be practical. Here we solve the problem of deciding which entanglement class a graph state of $n \leq 8$ qubits belongs to by calculating some of the state's intrinsic properties. We show that four invariants related to those proposed by VDD are enough for distinguishing between all inequivalent classes with $n \leq 8$ qubits.


## I. INTRODUCTION

Graph states [1,2] are fundamental in quantum information, especially in quantum error correction [3–5] and measurement-based quantum computation [6]. Graph states also play a fundamental role in the study of entanglement. Two quantum states have the same entanglement if they are equivalent under stochastic local operations and classical communication (SLOCC). For $n=3$, there are six classes under SLOCC [7]. For $n \geq 4$ the number of classes under SLOCC is infinite and is specified by an exponentially increasing number of parameters. However, if we focus on graph states of $n<27$ qubits, then the discussion becomes simpler. On one hand, every two graph states which are SLOCC equivalent are also equivalent under local unitary (LU) operations [8]. On the other hand, previous results suggest that, for graph states of $n<27$ qubits, the notion of LU equivalence and local Clifford equivalence (LC equivalence) coincide. The "LU$\Leftrightarrow$LC conjecture" states that "every two LU-equivalent stabilizer states must also be LC equivalent." Ji *et al.* proved that the LU$\Leftrightarrow$LC conjecture is false [9]. However, the LU$\Leftrightarrow$LC is true for several classes of $n$ qubit graph states [10,11] and the simplest counterexamples to the conjecture are graph states of $n=27$ qubits [9]. Indeed, Ji *et al.* "believe that 27 is the smallest possible size of counterexamples of LU$\Leftrightarrow$LC." In this paper we assume that deciding whether or not two graph states of $n<27$ qubits have the same entanglement is equivalent to deciding whether or not they are LC equivalent.

The aim of this paper is to solve the following problem. Given an $n$-qubit graph state with $n<9$ qubits, decide which entanglement class it belongs to just by examining some of the state's intrinsic properties (i.e., without generating the whole LC class). The solution to this problem is of practical importance. If one needs to prepare a graph state $|G\rangle$ and knows that it belongs to one specific class, then one can prepare $|G\rangle$ by preparing the LC-equivalent state $|G'\rangle$ requiring the minimum number of entangling gates and the minimum preparation depth of that class (see [1,2,12]) and then transform $|G'\rangle$ into $|G\rangle$ by means of simple one-qubit unitary operations. The problem is that, so far, we do not know a simple set of invariants which distinguishes between all classes of entanglement, even for graph states with $n \leq 7$ qubits.

The classification of graph states' entanglement has been achieved, up to $n=7$ qubits, by Hein, Eisert, and Briegel (HEB) [1] (see also [2]) and has recently been extended to $n=8$ qubits [12]. The criteria for ordering the classes in [1,2,12] are based on several entanglement measures: the minimum number of two-qubit gates required for the preparation of a member of the class, the Schmidt measure for the $n$-partite split (which measures the genuine $n$-party entanglement of the class [13]), and the Schmidt ranks for all bipartite splits (or rank indexes [1,2]). The problem is that this set of entanglement measures fails to distinguish between inequivalent classes (i.e., between different types of entanglement). There is already an example of this problem in $n=7$: none of these entanglement measures allows us to distinguish between the classes 40, 42, and 43 in [1,2]. A similar problem occurs in $n=8$: none of these entanglement measures allows us to distinguish between classes 110 and 111, between classes 113 and 114, and between classes 116 and 117 in [12]. Therefore, we cannot use these invariants for deciding which entanglement class a given state belongs to. Reciprocally, if we have such a set of invariants, then we can use it to unambiguously label each of the classes.

Van den Nest, Dehaene, and De Moor (VDD) proposed a finite set of invariants that characterizes all classes [14]. However, already for $n=7$, this set has more than $2 \times 10^{36}$ invariants which are not explicitly calculated anywhere, so this set is not useful for classifying a given graph state. Indeed, VDD "believe that [their set of invariants] can be improved—if not for all stabilizer states then at least for some interesting subclasses of states" [14]. Moreover, they state that "it is likely that only [some] invariants need to be considered in order to recognize LC equivalence" [14] and that "it is not unlikely that there exist smaller complete lists of invariants which exhibit less redundancies" [14]. In this paper we show that, if $n \leq 8$, then four invariants are enough to recognize the type of entanglement.


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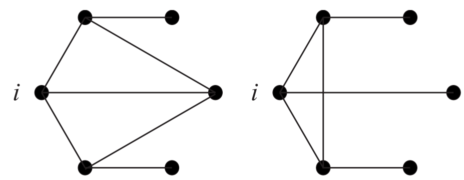


FIG. 1. Graphical effect of local complementation on qubit $i$. Local complementation on qubit $i$ on the graph on the left (right) leads to the graph on the right (left).

The paper is organized as follows. In Sec. II we introduce some basic concepts of the graph state formalism and review some of the results about the invariants proposed by VDD that will be useful in our discussion. In Sec. III we present our results and in Sec. IV our conclusions.

## II. BASIC CONCEPTS

### A. Stabilizer

The Pauli group $\mathcal{G}_n$ on $n$ qubits consists of all $4\times 4^n n$-fold tensor products of the form $M=\alpha_M M_1\otimes\cdots\otimes M_n$, where $\alpha_M\in\{\pm 1,\pm\imath\}$ is an overall phase factor and $M_i$ is either the $2\times 2$ identity matrix $\sigma_0=\mathbb{1}$ or one of the Pauli matrices $X=\sigma_x$, $Y=\sigma_y$, and $Z=\sigma_z$.

An $n$-qubit stabilizer $\mathcal{S}$ in the Pauli group is defined as an Abelian subgroup of $\mathcal{G}_n$ which does not contain the operator $-\mathbb{1}$ [15]. A stabilizer consists of $2^k$ Hermitian (therefore, they must have real overall phase factors $\pm 1$) $n$-qubit Pauli operators $s_i=\alpha_i M_1^{(i)}\otimes\cdots\otimes M_n^{(i)}\in\mathcal{G}_n, i=1,\ldots,2^k$ for some $k\leq n$. We will call the operators $s_i$ stabilizing operators.

In group theory, a set of elements $\{g_1,\ldots,g_l\}$ in a group $G$ is said to generate the group $G$ if every element of $G$ can be written as a product of elements from $\{g_1,\ldots,g_l\}$. The notation $G=\langle g_1,\ldots,g_l\rangle$ is commonly used to describe this fact, and the set $\{g_1,\ldots,g_l\}$ is called the generator of $G$. The generator of an $n$-qubit stabilizer $\mathcal{S}$ is a subset (not necessarily unique) $\gamma_S=\{g_1,\ldots,g_k\}$, consisting of $k\leq n$ independent stabilizing operators, such that $\mathcal{S}=\langle\gamma_S\rangle$. In this context, independent means that no product of the form $g_1^{a_1}\cdots g_k^{a_k}$, where $a_i\in\{0,1\}$ yields the identity except when all $a_i=0$. As a consequence, removing any operator $g_i$ from the generator makes the generated group smaller.

By definition, given a stabilizer $\mathcal{S}$, the stabilizing operators $s_i$ commute, so that they can be diagonalized simultaneously and, therefore, share a common set of eigenvectors that constitute a basis of the so-called vector space $V_{\mathcal{S}}$ stabilized by $\mathcal{S}$. The vector space $V_{\mathcal{S}}$ is of dimension $2^q$ when $|\gamma_S|=n-q$. Remarkably, if $|\mathcal{S}|=2^n$, then there exists a unique common eigenstate $|\psi\rangle$ on $n$ qubits with eigenvalue 1, such that $s_i|\psi\rangle=|\psi\rangle$ for every stabilizing operator $s_i\in\mathcal{S}$. Such a state $|\psi\rangle$ is called a stabilizer state because it is the only state that is fixed (stabilized) by every operator of the stabilizer $\mathcal{S}$.

Graph states are a special kind of stabilizer states (with $k=n$) associated with graphs. It has been demonstrated that every stabilizer state is equivalent under local complementation (defined below) to some (generally non unique) graph state [5].

TABLE I. Stabilizer and supports for the $|\mathrm{LC}_3\rangle$.

| Stabilizing operators | | Support | Weight |
|---|---|---|---|
| $XZ\mathbb{1}$ | $s_1=g_1$ | {1,2} | 2 |
| $ZXZ$ | $s_2=g_2$ | {1,2,3} | 3 |
| $\mathbb{1}ZX$ | $s_3=g_3$ | {2,3} | 2 |
| $\mathbb{1}\mathbb{1}\mathbb{1}$ | $s_4=g_1g_1$ | {∅} | 0 |
| $YYZ$ | $s_5=g_1g_2$ | {1,2,3} | 3 |
| $X\mathbb{1}X$ | $s_6=g_1g_3$ | {1,3} | 2 |
| $ZYY$ | $s_7=g_2g_3$ | {1,2,3} | 3 |
| $-YXY$ | $s_8=g_1g_2g_3$ | {1,2,3} | 3 |

TABLE II. Weight distribution for graph states up to six qubits.

| Graph state | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 3 | | | | |
| 2 | 1 | 0 | 3 | 4 | | | |
| 3 | 1 | 0 | 6 | 0 | 9 | | |
| 4 | 1 | 0 | 2 | 8 | 5 | | |
| 5 | 1 | 0 | 10 | 0 | 5 | 16 | |
| 6 | 1 | 0 | 4 | 6 | 11 | 10 | |
| 7 | 1 | 0 | 2 | 8 | 13 | 8 | |
| 8 | 1 | 0 | 0 | 10 | 15 | 6 | |
| 9 | 1 | 0 | 15 | 0 | 15 | 0 | 33 |
| 10 | 1 | 0 | 7 | 8 | 7 | 24 | 17 |
| 11 | 1 | 0 | 6 | 0 | 33 | 0 | 24 |
| 12 | 1 | 0 | 4 | 8 | 13 | 24 | 14 |
| 13 | 1 | 0 | 3 | 8 | 15 | 24 | 13 |
| 14 | 1 | 0 | 2 | 8 | 17 | 24 | 12 |
| 15 | 1 | 0 | 3 | 8 | 15 | 24 | 13 |
| 16 | 1 | 0 | 3 | 0 | 39 | 0 | 21 |
| 17 | 1 | 0 | 1 | 8 | 19 | 24 | 11 |
| 18 | 1 | 0 | 0 | 8 | 21 | 24 | 10 |
| 19 | 1 | 0 | 0 | 0 | 45 | 0 | 18 |

### B. Graph state

A $n$-qubit graph state $|G\rangle$ is a pure state associated to a graph $G(V,E)$ consisting of a set of $n$ vertices $V=\{1,\ldots,n\}$ and a set of edges $E$ connecting pairs of vertices, $E\subset V\times V$. Each vertex represents a qubit. The graph $G$ provides a mathematical characterization of $|G\rangle$. The graph state $|G\rangle$ associated to the graph $G$ is the unique $n$-qubit state fulfilling

$$g_i|G\rangle=|G\rangle,\quad \text{for}\quad i=1,\ldots,n, \tag{1}$$

where $g_i$ are the generators of the state's stabilizer group, defined as the set $\{s_j\}_{j=1}^{2^n}$ of all products of the generators. $g_i$ is the generator operator associated to the vertex $i$, defined by

$$g_i := X^{(i)}\otimes_{(i,j)\in E}Z^{(j)}, \tag{2}$$

where the product is extended to those vertices $j$ which are connected with $i$ and $X^{(i)}(Z^{(i)})$ denotes the Pauli matrix $\sigma_x(\sigma_z)$ acting on the $i$th qubit.

TABLE III. Invariants for the $n$-qubit graph states with $3 \le n \le 6$. Notation: value$_{\text{multiplicity}}$. The numeration of the classes is the one in [1,2].

| No. | Invariants |
|---|---|
| 1 | $0_2, 1_1, 3_1$ |
| 2 | $0_3, 1_4, 4_1$ |
| 3 | $0_8, 1_7, 9_1$ |
| 4 | $0_8, 1_3, 2_4, 5_1$ |
| 5 | $0_{15}, 1_{16}, 16_1$ |
| 6 | $0_{18}, 1_8, 2_3, 4_2, 10_1$ |
| 7 | $0_{17}, 1_7, 2_6, 5_1, 8_1$ |
| 8 | $0_{15}, 1_{11}, 3_5, 6_1$ |
| 9 | $0_{32}, 1_{31}, 33_1$ |
| 10 | $0_{38}, 1_{15}, 2_8, 8_2, 17_1$ |
| 11 | $0_{41}, 1_{16}, 4_6, 24_1$ |
| 12 | $0_{38}, 1_{14}, 2_7, 4_1, 5_2, 8_1, 14_1$ |
| 13 | $0_{42}, 1_6, 2_8, 4_6, 5_1, 13_1$ |
| 14 | $0_{37}, 1_{12}, 2_8, 3_4, 6_2, 12_1$ |
| 15 | $0_{42}, 1_{12}, 4_6, 5_3, 13_1$ |
| 16 | $0_{44}, 1_4, 2_{12}, 5_3, 21_1$ |
| 17 | $0_{34}, 1_{18}, 2_6, 3_1, 5_4, 11_1$ |
| 18 | $0_{33}, 1_{21}, 3_3, 4_6, 10_1$ |
| 19 | $0_{47}, 1_1, 3_{15}, 18_1$ |

### C. Local complementation

Two $n$-qubit states, $|\phi\rangle$ and $|\psi\rangle$, have the same $n$-partite entanglement if and only if there are $n$ one-qubit unitary transformations $U_i$, such that $|\phi\rangle = \otimes_{i=1}^{n} U_i |\psi\rangle$. If these one-qubit unitary transformations belong to the Clifford group, then the two states are said to be LC equivalent. VDD found that the successive application of a transformation with a simple graphical description is enough to generate the complete equivalence class of graph states under local unitary operations within the Clifford group (hereafter simply referred to as class or orbit) [16]. This simple transformation is local complementation.

On the stabilizer, local complementation on the qubit $i$ induces the map $Y^{(i)} \mapsto Z^{(i)}, Z^{(i)} \mapsto -Y^{(i)}$ on the qubit $i$ and the map $X^{(j)} \mapsto -Y^{(j)}, Y^{(j)} \mapsto X^{(j)}$ on the qubits $j$ connected with $i$ [2]. On the generators, local complementation on the qubit $i$ maps the generators $g_j^{\text{old}}$, with $j$ connected with $i$, to $g_j^{\text{new}} g_i^{\text{new}}$.

Graphically, local complementation on qubit $i$ acts as follows. Those vertices connected with $i$ which were connected from each other become disconnected from each other and vice versa. An example is in Fig. 1.

Using local complementation, one can generate the orbits of all LC-inequivalent $n$-qubit graph states. There are 45 orbits for $n \le 7$ [1,2] and 101 orbits for $n=8$ [12].

### D. Supports and LC invariants related to supports

Let $|\psi\rangle$ be a stabilizer state and $\mathcal{S}(|\psi\rangle)$ the corresponding stabilizer. Given a stabilizing operator $s_i = \alpha_i M_1^{(i)} \otimes \cdots \otimes M_n^{(i)}$, its support supp$(s_i)$ is the set of all $j \in \{1,\ldots,n\}$ such that $M_j^{(i)}$ differs from the identity. Therefore, the support of $s_i$ is the set of the labels of the qubits on which the action of the Pauli matrices is nontrivial (i.e., there is a $X$, $Y$, or $Z$ Pauli matrix acting on the qubit). Notice that the support is preserved under the maps induced on the stabilizer by local complementation (see Sec. II C).

Let $\omega \subseteq \{1,\ldots,n\}$ be the support of a stabilizing operator $s_i$, supp$(s_i)=\omega$. The weight of the operator $s_i$ is the cardinality of its support, $|\omega|$. The identity operator $\mathbb{1} \otimes \cdots \otimes \mathbb{1}$, which is always present in a stabilizer due to the underlying group structure, fulfills $\omega = \{\emptyset\}$ and, therefore, is of weight zero.

The set of operators $\{s_i\}_{i=1}^{2^k}$ of a stabilizer $\mathcal{S}$ can be classified into equivalence classes according to their supports, defining a partition in the stabilizer. We will say that two stabilizing operators $s_i$ and $s_j$ of $\mathcal{S}$ belong to the same equivalence class $[\omega]$ if they have the same support $\omega$, i.e., supp$(s_i)$=supp$(s_j)=\omega$. We denote by $A_\omega(|\psi\rangle)$ the number of elements (stabilizing operators) $s_i \in \mathcal{S}(|\psi\rangle)$ with supp$(s_i)=\omega$. In other words, $A_\omega(|\psi\rangle)$ is the cardinality of the equivalence class $[\omega]$. Since any graph state $|G\rangle$ is a special type of stabilizer state, these definitions can also be applied to them.

TABLE IV. Invariants for the seven-qubit graph states. Notation: value$_{\text{multiplicity}}$. The numeration of the classes is the one in [1,2].

| No. | Invariants |
|---|---|
| 20 | $0_{63}, 1_{64}, 64_1$ |
| 21 | $0_{78}, 1_{32}, 2_{15}, 16_2, 34_1$ |
| 22 | $0_{84}, 1_{32}, 4_8, 8_3, 40_1$ |
| 23 | $0_{77}, 1_{31}, 2_{15}, 8_3, 17_1, 26_1$ |
| 24 | $0_{87}, 1_{25}, 4_8, 5_6, 16_1, 25_1$ |
| 25 | $0_{92}, 1_{12}, 2_8, 4_7, 5_4, 8_4, 20_1$ |
| 26 | $0_{87}, 1_{16}, 2_{14}, 4_7, 8_1, 10_2, 28_1$ |
| 27 | $0_{80}, 1_{25}, 2_{11}, 3_3, 4_3, 5_3, 8_1, 14_1, 23_1$ |
| 28 | $0_{85}, 1_{15}, 2_{16}, 4_3, 6_7, 9_1, 18_1$ |
| 29 | $0_{87}, 1_{12}, 2_{15}, 4_9, 5_3, 13_1, 22_1$ |
| 30 | $0_{80}, 1_{21}, 2_{12}, 3_6, 4_1, 5_4, 8_3, 17_1$ |
| 31 | $0_{86}, 1_{28}, 4_3, 5_4, 8_6, 20_1$ |
| 32 | $0_{89}, 1_{12}, 2_{16}, 4_4, 5_4, 8_2, 32_1$ |
| 33 | $0_{72}, 1_{40}, 2_3, 3_4, 4_4, 9_4, 18_1$ |
| 34 | $0_{85}, 1_{14}, 2_{17}, 4_7, 5_1, 6_2, 13_1, 22_1$ |
| 35 | $0_{79}, 1_{25}, 2_{12}, 4_2, 5_6, 8_3, 17_1$ |
| 36 | $0_{86}, 1_{14}, 2_{17}, 4_4, 5_2, 6_2, 8_2, 26_1$ |
| 37 | $0_{80}, 1_{21}, 2_{12}, 3_8, 4_1, 5_2, 6_2, 12_1, 21_1$ |
| 38 | $0_{74}, 1_{32}, 2_8, 3_3, 4_5, 7_5, 16_1$ |
| 39 | $0_{77}, 1_{22}, 2_{16}, 3_5, 4_2, 7_5, 16_1$ |
| 40 | $0_{70}, 1_{36}, 2_7, 3_7, 6_7, 15_1$ |
| 41 | $0_{78}, 1_{22}, 2_{14}, 3_5, 4_3, 5_4, 11_1, 20_1$ |
| 42 | $0_{74}, 1_{26}, 2_{15}, 3_5, 6_7, 15_1$ |
| 43 | $0_{84}, 1_8, 2_{21}, 3_7, 6_7, 15_1$ |
| 44 | $0_{78}, 1_{24}, 2_3, 3_{15}, 4_6, 10_1, 19_1$ |
| 45 | $0_{83}, 1_{22}, 3_{10}, 4_{10}, 6_2, 24_1$ |

TABLE V. Invariants for the eight-qubit graph states. Notation: $\text{value}_{\text{multiplicity}}$. The numeration of the classes is the one in [12].

| No. | Invariants | No. | Invariants |
|---|---|---|---|
| 46 | $0_{128}, 1_{127}, 129_1$ | 97 | $0_{163}, 1_{44}, 2_{17}, 3_{14}, 4_5, 5_4, 7_2, 10_6, 22_1$ |
| 47 | $0_{158}, 1_{63}, 2_{32}, 32_2, 65_1$ | 98 | $0_{157}, 1_{55}, 2_{17}, 3_9, 4_4, 5_3, 6_3, 7_1, 9_4, 12_2, 24_1$ |
| 48 | $0_{173}, 1_{64}, 4_{15}, 16_3, 84_1$ | 99 | $0_{165}, 1_{40}, 2_{18}, 3_{17}, 4_4, 5_2, 6_2, 7_1, 9_4, 12_2, 24_1$ |
| 49 | $0_{158}, 1_{62}, 2_{31}, 16_1, 17_2, 32_1, 50_1$ | 100 | $0_{152}, 1_{59}, 2_{16}, 3_8, 4_{12}, 9_8, 21_1$ |
| 50 | $0_{176}, 1_{63}, 8_{16}, 65_1$ | 101 | $0_{168}, 1_{58}, 4_{18}, 8_3, 9_6, 12_2, 24_1$ |
| 51 | $0_{176}, 1_{56}, 4_7, 5_8, 8_7, 32_1, 44_1$ | 102 | $0_{177}, 1_{26}, 2_{26}, 3_4, 4_{11}, 5_2, 6_2, 8_6, 20_1, 32_1$ |
| 52 | $0_{192}, 1_{24}, 2_{16}, 4_8, 5_7, 8_4, 16_4, 37_1$ | 103 | $0_{174}, 1_{20}, 2_{40}, 3_9, 6_4, 7_2, 8_2, 12_4, 27_1$ |
| 53 | $0_{180}, 1_{30}, 2_{30}, 4_8, 8_2, 10_2, 16_2, 17_1, 49_1$ | 104 | $0_{200}, 1_{21}, 4_{24}, 5_6, 13_4, 57_1$ |
| 54 | $0_{163}, 1_{54}, 2_{22}, 3_8, 8_4, 9_2, 18_1, 24_1, 42_1$ | 105 | $0_{159}, 1_{58}, 2_{15}, 3_4, 4_{12}, 9_2, 10_4, 16_1, 34_1$ |
| 55 | $0_{185}, 1_{32}, 2_{16}, 4_{13}, 8_7, 20_2, 44_1$ | 106 | $0_{193}, 1_{19}, 2_{15}, 3_{12}, 6_{12}, 9_1, 12_3, 54_1$ |
| 56 | $0_{181}, 1_{30}, 2_{23}, 4_6, 6_7, 8_3, 9_2, 12_2, 18_1, 30_1$ | 107 | $0_{196}, 1_9, 2_{24}, 4_{12}, 5_4, 8_8, 13_2, 41_1$ |
| 57 | $0_{191}, 1_{32}, 2_9, 4_{16}, 10_6, 16_1, 66_1$ | 108 | $0_{180}, 1_{26}, 2_{26}, 3_4, 4_4, 6_{10}, 9_2, 12_3, 36_1$ |
| 58 | $0_{176}, 1_{49}, 4_{14}, 5_{14}, 20_2, 41_1$ | 109 | $0_{164}, 1_{40}, 2_{28}, 3_2, 4_8, 5_4, 6_1, 7_2, 10_6, 22_1$ |
| 59 | $0_{183}, 1_{28}, 2_{25}, 4_6, 5_2, 6_3, 8_2, 10_2, 13_2, 14_1, 16_1, 34_1$ | 110 | $0_{174}, 1_{32}, 2_{22}, 3_{13}, 4_4, 7_1, 8_2, 10_6, 11_1, 31_1$ |
| 60 | $0_{179}, 1_{32}, 2_{25}, 4_9, 6_3, 8_3, 10_1, 14_2, 20_1, 38_1$ | 111 | $0_{166}, 1_{40}, 2_{22}, 3_9, 4_9, 5_1, 6_1, 7_4, 10_1, 13_1, 16_1, 31_1$ |
| 61 | $0_{186}, 1_{24}, 2_{20}, 4_{10}, 5_8, 8_2, 10_4, 16_1, 40_1$ | 112 | $0_{168}, 1_{31}, 2_{32}, 3_9, 4_8, 5_1, 7_2, 13_4, 31_1$ |
| 62 | $0_{169}, 1_{46}, 2_{15}, 3_7, 4_5, 5_7, 8_1, 9_2, 12_1, 15_1, 18_1, 33_1$ | 113 | $0_{161}, 1_{46}, 2_{21}, 3_{10}, 4_4, 5_6, 6_1, 8_3, 11_2, 14_1, 26_1$ |
| 63 | $0_{175}, 1_{27}, 2_{31}, 3_4, 4_6, 6_2, 8_7, 9_1, 14_2, 26_1$ | 114 | $0_{158}, 1_{51}, 2_{20}, 3_{12}, 4_2, 5_3, 6_1, 7_2, 8_2, 11_4, 26_1$ |
| 64 | $0_{200}, 1_8, 2_{14}, 4_{18}, 5_6, 8_6, 13_1, 14_2, 29_1$ | 115 | $0_{164}, 1_{40}, 2_{28}, 3_2, 4_7, 5_6, 8_7, 14_1, 26_1$ |
| 65 | $0_{188}, 2_{28}, 4_{16}, 6_4, 9_2, 12_4, 33_1$ | 116 | $0_{161}, 1_{38}, 2_{37}, 3_7, 4_2, 6_1, 7_3, 10_6, 28_1$ |
| 66 | $0_{181}, 1_{20}, 2_{26}, 3_8, 4_8, 6_3, 7_4, 8_1, 10_3, 16_1, 28_1$ | 117 | $0_{161}, 1_{43}, 2_{23}, 3_{14}, 4_4, 5_3, 6_1, 7_2, 10_2, 13_2, 28_1$ |
| 67 | $0_{179}, 1_{24}, 2_{26}, 3_4, 4_8, 5_2, 6_8, 9_2, 12_1, 18_1, 30_1$ | 118 | $0_{155}, 1_{55}, 2_{12}, 3_{16}, 4_9, 9_8, 21_1$ |
| 68 | $0_{170}, 1_{35}, 2_{20}, 3_{12}, 4_7, 5_2, 7_4, 8_1, 10_2, 13_2, 25_1$ | 119 | $0_{152}, 1_{59}, 2_{16}, 3_{10}, 4_9, 6_1, 8_6, 11_2, 23_1$ |
| 69 | $0_{180}, 1_{54}, 4_8, 5_7, 16_4, 17_2, 37_1$ | 120 | $0_{160}, 1_{42}, 2_{29}, 3_3, 4_{12}, 6_1, 8_6, 11_2, 23_1$ |
| 70 | $0_{176}, 1_{62}, 8_{14}, 16_1, 17_2, 32_1$ | 121 | $0_{192}, 1_{25}, 4_{24}, 5_6, 8_8, 41_1$ |
| 71 | $0_{188}, 1_{22}, 2_{32}, 5_7, 8_4, 17_2, 69_1$ | 122 | $0_{176}, 1_{24}, 2_{24}, 3_6, 4_{16}, 6_1, 7_2, 10_6, 22_1$ |
| 72 | $0_{148}, 1_{84}, 2_8, 3_7, 8_4, 17_4, 35_1$ | 123 | $0_{190}, 1_{28}, 2_{12}, 3_1, 5_{16}, 8_6, 11_2, 51_1$ |
| 73 | $0_{185}, 1_{32}, 2_{15}, 4_{12}, 8_{10}, 16_1, 50_1$ | 124 | $0_{200}, 1_5, 2_{32}, 5_6, 8_8, 13_4, 41_1$ |
| 74 | $0_{178}, 1_{30}, 2_{26}, 4_9, 6_6, 8_3, 9_2, 24_1, 36_1$ | 125 | $0_{169}, 1_{35}, 2_{28}, 3_4, 4_4, 5_6, 6_4, 8_1, 9_2, 12_2, 33_1$ |
| 75 | $0_{166}, 1_{54}, 2_{14}, 4_6, 5_7, 8_4, 9_1, 14_2, 17_1, 29_1$ | 126 | $0_{170}, 1_{44}, 2_{14}, 3_6, 5_{12}, 8_6, 10_1, 11_2, 26_1$ |
| 76 | $0_{188}, 1_{26}, 2_{14}, 4_{14}, 5_2, 8_6, 9_2, 13_1, 14_2, 29_1$ | 127 | $0_{161}, 1_{48}, 2_{19}, 3_6, 4_9, 5_4, 7_6, 10_1, 16_1, 28_1$ |
| 77 | $0_{186}, 1_{28}, 2_{18}, 4_{12}, 5_4, 10_4, 14_2, 16_1, 40_1$ | 128 | $0_{161}, 1_{42}, 2_{33}, 3_3, 4_6, 6_1, 7_3, 10_6, 28_1$ |
| 78 | $0_{191}, 1_{24}, 2_{22}, 4_1, 5_8, 8_7, 14_2, 60_1$ | 129 | $0_{160}, 1_{50}, 2_{18}, 3_8, 4_9, 6_3, 7_2, 9_4, 12_1, 30_1$ |
| 79 | $0_{178}, 1_{32}, 2_{25}, 4_7, 6_6, 8_4, 12_3, 42_1$ | 130 | $0_{156}, 1_{52}, 2_{19}, 3_9, 4_{10}, 6_1, 8_6, 11_2, 23_1$ |
| 80 | $0_{166}, 1_{49}, 2_{15}, 3_6, 4_8, 5_6, 8_1, 9_1, 11_1, 14_1, 20_1, 35_1$ | 131 | $0_{152}, 1_{59}, 2_{16}, 3_{12}, 4_6, 6_2, 7_4, 10_4, 25_1$ |
| 81 | $0_{156}, 1_{70}, 2_3, 3_4, 4_{15}, 9_2, 10_1, 13_4, 28_1$ | 132 | $0_{156}, 1_{52}, 2_{16}, 3_{13}, 4_{10}, 7_5, 10_2, 13_1, 25_1$ |
| 82 | $0_{179}, 1_{27}, 2_{27}, 3_4, 4_6, 6_4, 8_2, 9_1, 12_5, 30_1$ | 133 | $0_{148}, 1_{69}, 2_{12}, 3_2, 4_{16}, 9_8, 21_1$ |
| 83 | $0_{179}, 1_{24}, 2_{26}, 3_4, 4_{10}, 5_2, 6_4, 8_2, 9_2, 12_1, 18_1, 30_1$ | 134 | $0_{188}, 1_{34}, 3_{20}, 6_3, 9_{10}, 54_1$ |
| 84 | $0_{165}, 1_{49}, 2_{14}, 3_6, 4_{10}, 7_6, 8_1, 10_2, 13_2, 25_1$ | 135 | $0_{166}, 1_{44}, 2_{20}, 3_3, 4_{12}, 8_{10}, 35_1$ |
| 85 | $0_{160}, 1_{56}, 2_{16}, 3_4, 4_{10}, 5_4, 8_1, 14_4, 32_1$ | 136 | $0_{191}, 1_{30}, 2_3, 3_3, 4_{12}, 7_{15}, 10_1, 48_1$ |
| 86 | $0_{190}, 1_{10}, 2_{30}, 4_{16}, 5_3, 9_2, 10_2, 16_2, 37_1$ | 137 | $0_{154}, 1_{51}, 2_{26}, 3_8, 4_6, 6_2, 7_4, 10_4, 25_1$ |
| 87 | $0_{200}, 1_9, 2_{16}, 4_{24}, 5_2, 13_4, 57_1$ | 138 | $0_{154}, 1_{51}, 2_{24}, 3_{14}, 4_1, 6_5, 9_6, 27_1$ |
| 88 | $0_{176}, 1_{28}, 2_{16}, 3_{14}, 4_{12}, 7_1, 8_4, 11_4, 23_1$ | 139 | $0_{183}, 1_{12}, 2_{31}, 3_{10}, 5_{10}, 6_6, 12_3, 30_1$ |
| 89 | $0_{174}, 1_{30}, 2_{32}, 4_4, 5_2, 6_6, 8_5, 14_2, 32_1$ | 140 | $0_{160}, 1_{36}, 2_{34}, 3_9, 4_8, 6_4, 9_2, 12_2, 27_1$ |
| 90 | $0_{175}, 1_{24}, 2_{33}, 3_4, 4_{10}, 6_2, 7_4, 8_1, 16_2, 34_1$ | 141 | $0_{212}, 1_1, 3_{14}, 6_{28}, 45_1$ |
| 91 | $0_{168}, 1_{28}, 2_{44}, 3_1, 4_2, 6_4, 7_2, 8_2, 12_4, 27_1,$ | 142 | $0_{184}, 1_{43}, 6_{28}, 45_1$ |
| 92 | $0_{175}, 1_{27}, 2_{31}, 3_4, 4_8, 6_2, 8_1, 9_1, 10_6, 34_1$ | 143 | $0_{179}, 1_{14}, 2_{35}, 3_{15}, 7_3, 8_8, 10_1, 32_1$ |
| 93 | $0_{170}, 1_{33}, 2_{26}, 3_9, 4_4, 5_2, 6_6, 7_1, 9_1, 12_2, 15_1, 27_1$ | 144 | $0_{172}, 1_9, 2_{56}, 3_6, 6_4, 8_8, 29_1$ |
| 94 | $0_{182}, 1_{20}, 2_{26}, 3_8, 4_{10}, 5_2, 8_3, 10_1, 13_2, 16_1, 34_1$ | 145 | $0_{188}, 1_{37}, 3_2, 6_{28}, 45_1$ |
| 95 | $0_{164}, 1_{41}, 2_{28}, 3_6, 4_4, 5_4, 6_2, 7_2, 11_2, 14_2, 29_1$ | 146 | $0_{164}, 1_{21}, 2_{56}, 3_2, 6_4, 8_8, 29_1$ |
| 96 | $0_{167}, 1_{40}, 2_{23}, 3_7, 4_5, 5_5, 7_1, 8_4, 11_2, 14_1, 29_1$ | | |

**E. Invariants of Van den Nest, Dehaene, and De Moor**

The following theorem is a key result obtained by VDD in [14] that presents a finite set of invariants which characterizes the LC equivalence class of any stabilizer state (i.e., functions that remain invariant under the action of all local Clifford transformations). We have chosen an adapted formulation of the theorem to group multiplication involving Pauli operators [see Eq. (3)], slightly different from VDD's original notation, which is based on the well-known equivalent formulation of the stabilizer formalism in terms of algebra over the field $\mathbb{F}_2$=GF(2), where arithmetic is performed modulo 2 and each stabilizing operator is identified with a $2n$-dimensional binary index operator.

*Theorem 1.* Let $|\gamma\rangle$ be a stabilizer state on $n$ qubits corresponding to a stabilizer $\mathcal{S}_{|\gamma\rangle}$. Let $r\in\mathbb{N}_0$ and consider subsets $\omega_k,\omega_{kl}\subseteq\{1,\dots,n\}$ for every $k,l\in\{1,\dots,r\}$, with $k<l$. Denote $\Omega:=(\omega_1,\omega_2,\dots,\omega_{12},\omega_{13},\dots)$ and let $\mathcal{T}^{\Omega}_{n,r}(|\gamma\rangle)$ be the set consisting of all tuples $(s_1,\dots,s_r)\in\mathcal{S}_{|\gamma\rangle}\cdots\mathcal{S}_{|\gamma\rangle}$ satisfying

$$\mathrm{supp}(s_k)=\omega_k,\;\;\mathrm{supp}(s_k s_l)=\omega_{kl}. \tag{3}$$

Then, (i) $|\mathcal{T}^{\Omega}_{n,r}(|\psi\rangle)|$ is LC invariant and (ii) the LC equivalence class of $|\psi\rangle$ is completely determined by the values of all invariants $|\mathcal{T}^{\Omega}_{n,n}(|\psi\rangle)|$ (i.e., where $r=n$).

VDD provide another family of support-related invariants, based on a second theorem with the same formulation than the one above, except for the substitution of conditions (3) by new constraints

$$\mathrm{supp}(s_k)\subseteq\omega_k,\;\;\mathrm{supp}(s_k s_l)\subseteq\omega_{kl}. \tag{4}$$

These new LC invariants are the dimensions of certain vector spaces and, in principle, are more manageable from a computational point of view because they involve the generator matrix of the stabilizer and rank calculation. Nevertheless, we will focus our attention on the first family of invariants, since they suffice to solve the problem we address in this paper with no extra computational effort. To resort to the second family would be justified in case we had to use invariants with a high $r$ value to achieve LC discrimination among graph states up to eight qubits. We refer the reader to Ref. [14] for a proof of Theorem 1 and the extension to the second family of LC invariants.

The invariants of Theorem 1 are the cardinalities of certain subsets $\mathcal{T}^{\Omega}_{n,r}(|\psi\rangle)$ of $\mathcal{S}_{|\psi\rangle}\cdots\mathcal{S}_{|\psi\rangle}$, which are defined in terms of simple constraints (3) on the supports of the stabilizing operators. VDD pointed out that, for $r=1$, these invariants count the number of operators in the stabilizer with a prescribed support. Therefore, fixing $r=1$, for every possible support $\omega_k\subseteq\{1,\dots,n\}$, there is an invariant

$$|\{s\in\mathcal{S}_{|\psi\rangle}|\mathrm{supp}(s)=\omega_k\}|. \tag{5}$$

That is, the invariants for $r=1$ are the $A_{\omega_k}(|\psi\rangle)$, i.e., the cardinalities of the equivalence classes $[\omega_k]$ of the stabilizer. The number of possible supports in a stabilizer of an $n$-qubit state is equal to $2^n$ and, therefore, there are $2^n$ VDD's invariants for $r=1$. Many of them could be equal to zero. In fact, when dealing specifically with graph states, it can be easily seen that $A_{\omega_k}(|\psi\rangle)=0$ when referred to supports fulfilling $|\omega_k|=1$ because stabilizing operators of weight 1 are not present in the stabilizer of a graph state due to the inherent connectivity of the graphs associated to the states that rules out isolated vertices.

On the other hand, VDD consider the invariants $A_{\omega_k}(|\psi\rangle)$ as "local versions" of the so-called weight distribution of a stabilizer, a concept frequently used in classical and quantum coding theory. For $r\geq 2$, the new series of invariants involve $r$-tuples of stabilizing operators and their corresponding supports and constitute a generalization of the weight distribution. Let us denote

$$A_d(|\psi\rangle)=\sum_{\omega,|\omega|=d}A_{\omega}(|\psi\rangle), \tag{6}$$

the number of stabilizing operators with weight equal to $d$. According to this notation, the weight distribution of a stabilizer is the $(n+1)$-tuple

$$W_{|\psi\rangle}=\{A_d(|\psi\rangle)\}_{d=0}^{n}. \tag{7}$$

In principle, $W_{|\psi\rangle}$ could be a compact way to present the whole information about the invariants $A_{\omega}(|\psi\rangle)$, i.e., VDD's invariants with $r=1$. This question will be addressed later.

In order to clarify the content of VDD's theorem, let us briefly discuss the way it works when applied to a particular graph state. We have chosen the three-qubit linear cluster state, $|\mathrm{LC}_3\rangle$, because of its simplicity, combined with a sufficient richness in the stabilizer structure. Table I shows the stabilizer of $|\mathrm{LC}_3\rangle$ with its eight stabilizing operators, $\{s_1,\dots,s_8\}$. Three of them ($s_1=g_1$, $s_2=g_2$, and $s_3=g_3$) constitute a generator. $|\mathrm{LC}_3\rangle$ is a three-qubit graph state, so there are $2^3=8$ possible supports (8 being the number of subsets in the set $\{1,2,3\}$):

$$\{\emptyset\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}. \tag{8}$$

VDD's invariants for $r=1$. In this case, $\Omega=(\omega_1)$. By $\Omega$, we denote each of all the possible ways to choose a single support $\omega_1$, so there are eight choices for $\Omega$, which are those listed in Eq. (8). Given a particular choice of $\Omega=(\omega_1)$, the set $\mathcal{T}^{\Omega}_{n,1}(|\mathrm{LC}_3\rangle)$ contains all stabilizing operators $s_1$ for the $|\mathrm{LC}_3\rangle$ fulfilling

$$\mathrm{supp}(s_1)=\omega_1, \tag{9}$$

so, as a matter of fact, $\mathcal{T}^{\Omega}_{n,1}(|\mathrm{LC}_3\rangle)$ is the equivalence class $[\omega_1]$ associated to the support $\omega_1$. Only five out of the eight possible supports are in fact present in the stabilizer of the $|\mathrm{LC}_3\rangle$ (see the column "Support" in Table I) and, therefore, we can distinguish between five nonempty equivalence classes $[\omega]$. According to VDD's theorem, the LC invariants for $r=1$, $|\mathcal{T}^{\Omega}_{n,1}(|\mathrm{LC}_3\rangle)|$, are the cardinalities $A_{\omega}(|\mathrm{LC}_3\rangle)$ of such equivalence classes $[\omega]$, namely,

$$A_{\{\emptyset\}}(|\mathrm{LC}_3\rangle)=1, \tag{10a}$$

$$A_{\{1\}}(|\mathrm{LC}_3\rangle)=0, \tag{10b}$$

$$A_{\{2\}}(|\mathrm{LC}_3\rangle)=0, \tag{10c}$$

$$A_{\{3\}}(|\mathrm{LC}_3\rangle)=0, \tag{10d}$$

$$A_{\{1,2\}}(|\mathrm{LC}_3\rangle) = 1, \tag{10e}$$

$$A_{\{1,3\}}(|\mathrm{LC}_3\rangle) = 1, \tag{10f}$$

$$A_{\{2,3\}}(|\mathrm{LC}_3\rangle) = 1, \tag{10g}$$

$$A_{\{1,2,3\}}(|\mathrm{LC}_3\rangle) = 4. \tag{10h}$$

VDD's invariants for $r=2$. In this case, $\Omega=(\omega_1,\omega_2;\omega_{12})$. By $\Omega$ we denote each of all the possible different ways to choose two supports $\omega_1$, $\omega_2$, and then a third support $\omega_{12}$. Let $M=2^n$ be the number of possible supports and $n$ being the number of qubits. On one hand, there are $\binom{M}{2}$ different combinations of two supports $(\omega_1,\omega_2)$, plus $M$ couples of the form $(\omega_1,\omega_2=\omega_1)$. On the other hand, there are $M$ possible choices for $\omega_{12}$. As a consequence, there are $M[M+\binom{M}{2}]$ ways to choose $\Omega$. For $n=3$, this number is 288. Given a particular choice of $\Omega=(\omega_1,\omega_2;\omega_{12})$, the set $\mathcal{T}^{\Omega}_{n,2}(|\mathrm{LC}_3\rangle)$ contains all the two-tuples of the stabilizing operators $(s_1,s_2)$ of the $|\mathrm{LC}_3\rangle$ fulfilling

$$\mathrm{supp}(s_1)=\omega_1,\ \ \mathrm{supp}(s_2)=\omega_2,\ \ \mathrm{supp}(s_1s_2)=\omega_{12}. \tag{11}$$

Many of these sets $\mathcal{T}^{\Omega}_{n,2}(|\mathrm{LC}_3\rangle)$ could be empty because the stabilizer fails to fulfill any of the conditions (11). The cardinalities of the 288 sets, $|\mathcal{T}^{\Omega}_{n,2}(|\mathrm{LC}_3\rangle)|$, are the VDD's invariants that we are interested in. For instance, if we choose $\Omega=(\omega_1,\omega_2;\omega_{12})$ such that

$$\omega_1=\{1,2\},\ \ \omega_2=\{1,2,3\},\ \ \omega_{12}=\{1,2,3\}, \tag{12}$$

then, according to the information in Table I, there is only one operator with support $\omega_1$, namely, $s_1$, and four operators with support $\omega_2$: $s_2$, $s_5$, $s_7$, and $s_8$. We obtain $\mathcal{T}^{\Omega}_{n,2}(|\mathrm{LC}_3\rangle)=\{(s_1,s_2),(s_1,s_5),(s_1,s_7),(s_1,s_8)\}$ because these four two-tuples verify conditions (11) and the value of the corresponding VDD's invariant is the cardinality of the set, $|\mathcal{T}^{\Omega}_{n,2}(|\mathrm{LC}_3\rangle)|=4$.

Another example. If we choose $\Omega$ such that $\omega_1=\{1,2\}$, $\omega_2=\{2,3\}$, and $\omega_{12}=\{1,2,3\}$, then the VDD's invariant is $|\mathcal{T}^{\Omega}_{n,2}(|\mathrm{LC}_3\rangle)|=0$ because the two-tuple $(s_1,s_3)$ defined by the supports $\omega_1$ and $\omega_2$ fulfills $s_1s_3=s_6$ and $s_6$ does not match with support $\omega_{12}$.

## III. RESULTS

The number of VDD's invariants for an $n$-qubit graph state grows very rapidly with $r$ (and, of course, with $n$). If $n=3$, there are eight invariants for $r=1$ and 288 invariants for $r=2$. For an eight-qubit graph state, there are 256 invariants for $r=1$ and 8 421 376 for $r=2$. Obviously, the problem of calculating all the VDD's invariants for graph states up to eight qubits becomes completely unfeasible if there are no restrictions on $r$. The total number of VDD's invariants for a given $n$-qubit graph state, and all possible values of $r$, is $M+\sum_{r=2}^{n}C'(M,r)C'(M,P)$, where $M=2^n$, $P=\binom{r}{2}$, and $C'(M,r)$ denotes the combinations with repetition of $M$ elements choose $r$. For $n=7$, this formula gives $2.18\times10^{36}$; for $n=8$, it gives $1.88\times10^{53}$.

How many of them are needed to distinguish between all LC equivalence classes? VDD stated that "the LC equivalence class of $|\psi\rangle$ is completely determined by the values of all invariants $|\mathcal{T}^{\Omega}_{n,n}(|\psi\rangle)|$ (i.e., where $r=n$)" [14]. However, this number [i.e., $C'(M,r)C'(M,P)$ with $r=n$] is still too large to be practical. For $n=7$ is $2.18\times10^{36}$ and for $n=8$ is $1.88\times10^{53}$ (i.e., most of the invariants correspond to the case $r=n$). We are interested in the minimum value of $r$ that yields a series of invariants sufficient to distinguish between all the 146 LC equivalence classes of graph states up to $n=8$ qubits. In Ref. [14], the authors point out that there are examples of equivalence classes in stabilizer states which are characterized by invariants of small $r$; for instance, those equivalent to GHZ states. In addition, they remark that a characterization based on small $r$ values could be feasible, at least for some interesting subclasses or subsets of stabilizer states. We have calculated the VDD's invariants for $r=1$ for the 146 LC equivalence classes of graph states with up to $n=8$ qubits. This implies calculating the cardinalities $A_{\omega}(|\psi\rangle)$ of the corresponding equivalence classes $[\omega]$ of the 146 representatives of the LC-equivalence classes, 30 060 invariants in total (since there are 1, 1, 2, 4, 11, 26, and 101 classes of two-, three-, four-, five-, six-, seven-, and eight-qubit graph states, respectively, and the number of VDD's invariants with $r=1$ is $2^n$ for each class). Our results confirm the conjecture that invariants with $r=1$ are enough for distinguishing between the 146 LC equivalence classes for graph states up to eight qubits. It is therefore unnecessary to resort to families of VDD's invariants $|\mathcal{T}^{\Omega}_{n,r}(|\psi\rangle)|$ with $r\geq2$.

Our goal is not to show the values of these 30 060 invariants but to compress all this information and construct simple invariants from it. However, in order to do it properly, some requirements should be fulfilled. (I) The compacted information must be unambiguous and easily readable. (II) The compacted information must be LC invariant. (III) The compacted information concerning different LC equivalence classes must still distinguish between any of them.

Following the comments of VDD in Ref. [14] about considering the invariants $A_{\omega}(|\psi\rangle)$ as "local versions" of the weight distribution $W_{|\psi\rangle}$ of a stabilizer, we have calculated $W_{|\psi\rangle}$ for the 146 LC classes of equivalence, according to definition (7). It can easily be seen that, if $A_{\omega}(|\psi\rangle)$ is LC invariant, then $W_{|\psi\rangle}$ is also LC invariant and permits a compact way to compress the information of the invariants $A_{\omega}(|\psi\rangle)$. Unfortunately, $W_{|\psi\rangle}$ is not able to distinguish between any two LC classes of equivalence. Table II shows that the weight distribution fails to distinguish between LC classes starting from $n=6$. Graph states with labels 13 and 15 in Refs. [1,2] have the same weight distribution and this degeneration increases as the number of qubits grows, as we have checked out calculating $W_{|\psi\rangle}$ for all graph states up to eight qubits.

Therefore, we must look for a way to compress the information about the invariants $A_{\omega}(|\psi\rangle)$, which satisfies (I)–(III). The fact that the stabilizing operators of a stabilizer can be classified into equivalence classes according to their supports (equivalence classes $[\omega]$), and that the cardinalities of such classes $[\omega]$ are the invariants $A_{\omega}(|\psi\rangle)$, leads us to introduce two definitions. Two classes $[\omega_1]$ and $[\omega_2]$ are equipotent if and only if both have the same cardinality, i.e., $A_{\omega_1}(|\psi\rangle)$

$=A_{\omega_2}(|\gamma\rangle)$, regardless of whether their stabilizing operators have different weights $|\omega_1| \neq |\omega_2|$ or not. It is clear that the number of equipotent equivalence classes $[\omega]$ for a given cardinality $A_\omega(|\gamma\rangle)$ is LC invariant. We will call it the $A_\omega$ multiplicity (or $A_\omega$ potency) and denote it by $M(A_\omega)$. For instance, if we take a look at the list of invariants $A_\omega(|\mathrm{LC}_3\rangle)$ [see Eqs. (10a)–(10h)] we find that the value 0 appears three times (so there are three equivalence classes $[\omega]$ with that cardinality) and then $M(0)=3$. Using this criterion, $M(1)=4$ and $M(4)=1$ for the $|\mathrm{LC}_3\rangle$.

If we tabulate the values of $A_\omega(|\gamma\rangle)$ together with the corresponding values of $M(A_\omega)$, we obtain a two-index compact information, which is LC invariant and, more importantly, LC discriminant, as required. The results are shown in Tables III–V.

In Table V we can see that four numbers are enough to distinguish between all classes of graph states with $n=8$ qubits: the multiplicities of the values 0, 1, 3, and 4. Indeed, in Tables III and IV we see that these four numbers are enough to distinguish between all classes of graph states with $n \leq 8$ qubits.

## IV. CONCLUSIONS

We have shown that, to decide which entanglement class a graph state of $n \leq 8$ qubits belongs to, it is enough to calculate four quantities. These four LC invariants characterize any LC class of $n \leq 8$ qubits.

This result solves a problem raised in the classification of graph states of $n \leq 8$ qubits developed in Refs. [1,2,12]. A compact set of invariants that characterize all inequivalent classes of graph states with a higher number of qubits can be obtained by applying the same strategy. This can be done numerically up to $n=12$, a number of qubits beyond the present experimental capability in the preparation of graph states [17].

We have also shown that the conjecture [18] that the list of LC invariants given in Eq. (4) is sufficient to characterize the LC equivalence classes of all stabilizer states, which is not true in general [1], is indeed true for graph states of $n \leq 8$ qubits. Moreover, we have shown that, for graph states of $n \leq 8$ qubits, the list of LC invariants given in Eq. (5), which is more restrictive than the list given in Eq. (4), is enough. This solves a problem suggested in [14], regarding the possibility of characterizing special subclasses of stabilizer states using subfamilies of invariants.

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