Programa de Doctorado "Matemáticas"

PhD Dissertation

Reduced Basis Method applied to the Smagorinsky Turbulence Model

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En memoria de mi abuelo Pedro.
A mis padres y a mi hermano, siempre luchando.
A todos los que creyeron en mi.

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Resumen

Esta tesis se enmarca en la resolución numérica de modelos que simulan el comportamiento de flujos turbulentos mediante técnicas de orden reducido. La resolución numérica de este tipo de flujos es compleja y costosa, a la par que necesaria para el diseño óptimo de edificios eco-eficientes, entre otras muchas aplicaciones en ingeniería y arquitectura. La incorporación de técnicas de Orden Reducido es clave para la reducción en órdenes de magnitud el tiempo y coste computacional en la resolución numérica de estos problemas.

En esta tesis utilizaremos el modelo de Smagorinsky, un modelo de turbulencia de tipo LES (Large Eddy Simulation) basado en las ecuaciones de Navier-Stokes que permite la resolución de flujo turbulento sin la necesidad de mallas muy finas. Aún así, el coste computacional es elevado, sobre todo en casos 3D.

Dentro de la modelización de orden reducido, existen varias técnicas que permiten la obtención del modelo reducido. En esta tesis nos centramos en el desarrollo de métodos de Bases Reducidas (BR). Usaremos de forma auxiliar la “Proper Orthogonal Decomposition” (POD). Para la obtención del modelo reducido, es necesario el cálculo del error entre el modelo aproximado y el modelo reducido mediante el método BR, lo cuál puede llegar a ser muy costoso y complejo. Es por ello que se estudia la deducción de estimadores *a posteriori* que permiten estimar este error y cuyo cálculo sea más rápido.

El modelo de Smagorinsky es no lineal ya que deriva de las ecuaciones de Navier-Stokes y el desarrollo de estimadores para problemas no lineales requiere la utilización de técnicas matemáticas adaptadas.

Para un modelo estacionario no lineal, encontramos estimadores basados en la teoría Brezzi-Rappaz-Raviart (BRR) de aproximación de ramas no singulares de problemas no lineales paramétricos. Es una teoría esencialmente basada sobre el teorema de la función implícita. Este estimador ha sido ya elaborado para flujos estacionarios, aunque solo se ha aplicado a flujos 2D. Por lo que, comenzamos aplicando este estimador a un caso 3D, obteniendo grandes reducciones en cuanto a tiempo y coste computacional.

También aplicamos este estimador a un problema realista de diseño de claustros orientado a la optimización del confort térmico en la planta baja. Los parámetros a considerar para el problema son la altura y anchura del pasillo que rodea el claustro. Conseguimos obtener una respuesta óptima analizando 625 posibilidades en 16 minutos.

Uno de los principales desafíos que abordamos es extender la obtención de estimadores *a posteriori* a problemas no estacionarios. Para empezar, es necesario realizar estudios previos sobre estimaciones *a priori* que involucren a la velocidad y presión. Así, somos capaces de desarrollar un estimador *a posteriori* asumiendo ciertas hipótesis.

Por otro lado, desarrollamos una alternativa basándonos en la teoría de turbulencia en equilibrio estadístico de Kolmogórov, por la cuál sabemos que existe una cascada de energía que se propaga desde los grandes torbellinos hacia los más pequeños disipando la energía en ellos gracias a la viscosidad. Esta cascada genera un espectro de energía inercial con una forma determinada que utilizamos como estimador. Validamos dicho estimador con un test académico, utilizando una estrategia POD+Greedy, obteniendo resultados similares utilizando el estimador y el error real cometido.

Abstract

This PhD dissertation addresses the numerical simulation of models that simulate the behavior of turbulent flows through reduced-order techniques. The numerical simulation of this kind of flows is complex and pricey, as well as necessary to the Eco-efficient building optimized design, among many others applications in engineering and architecture. The integration of reduced-order techniques is the key to reducing by orders of magnitude the time and computational cost in the numerical simulation of these problems.

In this dissertation, we use the Smagorinsky model, a Large Eddy Simulation (LES) model based upon the Navier-Stokes equations that allows for the resolution of turbulent flows with coarser meshes. Even then, the computational cost is high, especially in 3D cases.

With respect to the Reduced Order Model (ROM), there exist some techniques to obtain the ROM. In this dissertation, we focus on the development of Reduced Basis (RB) method. Upon an ancillary basis, we shall use Proper Orthogonal Decomposition (POD). For ROM obtainment, it is necessary to compute the error between the approximated model and the reduced model through the RB method, which could become very expensive and complex. This is the reason to study *a posteriori* estimates to estimate the error and which computation is faster.

The Smagorinsky model is non-linear since it is derived by the Navier-Stokes equations and the estimator development for non-linear problems requires the use of adapted mathematical techniques.

For steady non-linear models, we find estimates based on the Brezzi-Rappaz-Raviart (BRR) theory of non-singular branches approximation of parametric non-linear problems. It is a theory essentially based upon the Implicit function theorem. This estimator has already been developed for steady flows, although it has only be applied to 2D flows. Therefore, we shall start applying this estimator to the 3D case, obtaining large reductions in time and computational cost.

We also apply this estimator to a realistic design problem for a cloister focused on the thermal comfort optimization of the ground floor. The parameters to consider for the problem are the height and width of the corridors around the cloister. We obtain an optimal answer by analyzing 625 possibilities in 16 minutes.

One of the main challenges addressed in this dissertation is to extend the *a posteriori* estimator to unsteady problems. To start, it is necessary to analyze previous studies on *a priori* estimations that involve velocity and pressure. Thus, we are able to develop an *a posteriori* estimator under some hypothesis.

Furthermore, we develop an alternative based Kolmogórov's theory of statistical equilibrium, based on the existence of an energy cascade that spreads the energy from large eddies to the smallest ones, dissipating the energy thanks to viscosity. This cascade generates an inertial energy spectrum with a determined shape that we use as the estimator. We have validated this estimator with an academic test, using a POD+Greedy strategy. We obtain similar results using the estimator and the committed real error.

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Notation and Abbreviations

P	number of parameters.
$\boldsymbol{\mu}$	parameter vector in $\mathbb{R}^P$.
$\mathcal{D}$	parameter set containing the ranges of the parameters.
d	dimension of the problem with 2 or 3 as possible values.
Ω	open bounded connected region with Lipschit boundary.
$\Gamma = \Gamma_D \cup \Gamma_N$ . .	boundary of Ω formed by Γ_D and Γ_N associated to Dirichlet and Neumann boundary conditions.
$T_f, I_f = (0, T_f)$.	final time greater than 0 and time domain.
$Q_T = I_f \times \Omega$ . .	time-spatial domain.
Δt	time step.
$\mathcal{T}_h$	a uniformly regular triangulation of Ω with $h > 0$ related to the mesh diameter.
$\mathbf{u}, \mathbf{v}, \mathbf{z}, \mathbf{w}$	velocity field.
p, q, r	pressure per mass density.
θ	temperature.
ν_t, κ_t	eddy viscosity and diffusivity.
$h_K, h_{\max}, h_{\min}$ . .	diameter of element $K \in \mathcal{T}_h$, the maximum and minimum.
Φ	map necessary for geometric parameter problems.
ω, σ	the width and height of the corridor in chapter 2 and 3.
γ_h, γ_N	continuity factors.
β_h, β_N	stability factors.
ρ_T	Lipschitz constant.
Δ_N	*a posteriori* error bound estimator.
BC	Boundary Condition.

PDE . . . . . . Partial Differential Equation.

NS . . . . Navier-Stokes.

LES . . . . Large Eddy Simulation.

FE . . . . Finite Element.

EIM . . . . . . Empirical Interpolation Method.

RBF . . . . . . Radial Basis Function.

ROM . . . . . . Reduced-Order Model.

RB . . . . Reduced Basis.

POD . . . . . . Proper Orthogonal Decomposition.

BRR . . . . . . Brezzi-Rappaz-Raviart.

Variable	Name	SI units
ρ	mass density	kg/m^3
μ	dynamical viscosity	kg/ms
ν	kinematic viscosity	m^2/s
κ	thermal diffusivity	m^2/s
c_p	specific heat capacity	J/kgK
k	thermal conductivity	W/mK
α	convective heat transfer	W/m^2K
β	thermal expansion ($1/\theta$ for ideal gases)	K^{-1}
$\mathbf{u}, \mathbf{v}, \mathbf{z}, \mathbf{w}$	velocity field	m/s
$E = \frac{1}{2}\lvert\mathbf{u}\rvert^2$	kinetic energy per mass density	m^2/s^2
$\varepsilon = 2\nu\lvert D\mathbf{u}\rvert^2$	dissipation per mass density	m^2/s^3
$\nabla\mathbf{u}$	deformation tensor	s^{-1}
$D\mathbf{u} = (1/2)(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$	symmetric deformation tensor	s^{-1}
$\mathbf{f}$	source term per mass unity	m/s^2
p, q, r	pressure per mass density	m^2/s^2
U	characteristic velocity	m/s
L	characteristic length	m

Table 1: Table of coefficients and variables.

Introduction

Nowadays, many of the problems in the fields of physics and engineering can be characterized by Partial Differential Equations (PDEs). At the same time, PDEs involve parameters that change from a resolution to another, modifying the results. This derives in the resolution of highly complex parametric systems, such as evolutionary and inverse problems, data assimilation in non-linear PDEs and optimal design of complex devices, among others.

Along this dissertation, we focus on Computational Fluid Dynamics, which is a branch of fluid mechanics. These flows frequently depend on parameters such that the ratio between high and width of a building to study its aerodynamics, the attack angle of an airfoil, or governing physical parameters such that viscosity, density, thermal conductivity, etc.

PDEs are solved applying mathematical methods of numerical approximation. Design problems or fluid variability calls for the solution of PDEs on many occasions, changing the parameters, what can become very computationally expense. Even the use of the use of High Performance Computation (HPC) could take a considerable time. Hence, the need of techniques to develop Reduced Order Models (ROM) whose resolution is expected to be much faster and with less resources.

The main purpose of the ROM method is to reduce by orders of magnitude the computational cost of numerical simulation of parameterized partial differential equations. This method is able to catch the dominant characteristics of the problem accepting some admissible errors. This method drastically reduces the calculation and allows to carry out realistic simulations in acceptable computing time without the need of HPC.

The ROMs are constructed in an off-line phase, which requires solving the full-order model (FOM) for several well-chosen cases. This information is treated to capture the most important one, using appropriate techniques, and to build the ROM that is run with highly reduced computational time in an on-line phase. Some basic references about ROM are the works of Quarteroni et al. [38], Schilders et al. [45] and Chinesta et al. [13].

There exists various kinds of ROMs to solve parametric PDEs. In this dissertation, we shall mainly develop Reduced Basis (RB) methods to compute parametric turbulent flows. We shall also use the Proper Orthogonal Decomposition (POD) as an auxiliary ROMs. These techniques will be introduced in Chapter 1.

The use of RB method requires the design of *a posteriori* error bound estimators, which is the main purpose of this dissertation. The RB method is built by means of the greedy algorithm, which looks for an optimal basis function to add to the current RB space. The estimator usually is based upon the dual norm of the residual that allows certifying the method. For linear PDEs, the procedure to obtain an *a posteriori* error bound estimator is well-known, applying continuity and coercivity results to the error (between full order and reduced order solutions) equation (*cf.* [38]. For non-linear PDEs, the *a posteriori* error bound estimator can be developed following the Brezzi-Rappaz-Raviart (BRR) theory (*cf.* [8]) for approximation of regular branches of solutions of non-linear PDEs.

The POD is the most used technique for the tensorized approximation of bi-parametric functions, in this case, the full order solution of evolution PDEs. This technique looks for minimizing the mean square error between the full order solution and the reduced solution over all possible subspace of the "snapshots" data of given dimension. This space is formed by the orthogonal functions (POD basis) that are obtained by singular value decomposition (SVD) of a suitable correlation matrix. Once the reduced space is built, a Galerkin projection of the governing PDEs can be employed to obtain a low-order dynamical system, in to the parameter window used to create the POD basis. The resulting low-order model is named standard POD-Galerkin ROM.

As we mentioned before, we are interested in the application of ROM to turbulent flows. Most of the phenomena which carry on a difficulty in the numerical resolution are turbulence flows. This phenomenon is characterized by chaotic, multi-scale dynamics, both in space and time. The governing PDEs are the Navier-Stokes (NS) equations whose resolution becomes unaffordable for many real flows of interest. Using the mesh size and time steps to accurately solve a single flow past a 3D obstacle with well-developed turbulence would require dozens of HPC nodes (*cf.* [12]).

To alleviate this, alternative models have been considered, such as Large Eddy Simulation (LES) models. The LES models are based upon the Kolmogórov's theory of statistical equilibrium, introduced in 1941 by Andrey Kolmogórov. This theory describes how energy is transferred from large to smaller eddies, (cf. [27, 28]). This is at high Reynolds number (a physical parameter that triggers turbulence), the flow exhibits an energy cascade: large scale eddies are broken down into smaller and smaller eddies until the scales are fine enough so that viscous forces can dissipate their energy. LES model strategy is based upon to solve large grid scales and to approximate small scales by an eddy diffusion term in the NS equations, which involves the mixing length of the sub-grid eddies, which is proportional to the grid size.

Even the computation of this kind of models is quite complex. Most of the turbulence effects are genuinely 3D (mainly due to vortex stretching triggered by inertial effects) do not admit a simplification to 2D. The solution of parametric turbulent flows would call for the resolution of this model multiple times, which should be computed in affordable times. At this point, we dip into ROM for the LES model, in particular, for the Smagorinsky model. This model was introduced by Joseph Smagorinsky in 1963 (see [46]), adding an eddy viscosity term to the averaged Navier-Stokes equations. The primary objective of this dissertation is to build *a posteriori* error bounds for the finite element - reduced order solution of the Smagorinsky turbulence model.

The use of BRR theory for the steady Smagorinsky model was developed by E. Delgado in his PhD dissertation [14]. The estimator has been validated for steady 2D problems only, therefore, as a first contact with this estimator, in this thesis, we validate the estimator with a numerical test in 3D considering the Reynolds number as the parameter for the problem. The results are shown in Section 1.5, in which we consider the lid-driven cavity problem for relatively low Reynolds numbers, for which the flow is steady. The outcomes are promising reducing the computational time from almost 2 hours to over 2 seconds, with speed-ups around 3 000.

The remaining thesis is divided into two parts. In the first part, we seek for the application of *a posteriori* error bound estimator to a realistic problem of multiparameter design. In the second part, we seek to extend to the unsteady Smagorinsky model the techniques to construct RB solvers developed for steady problems. First, it is necessary to develop *a priori* estimates for the unsteady Smagorinsky model to prove well-possednes. Subsequently, we develop an *a posteriori* error estimator following the BRR theory. Lastly, we seek for an *a posteriori* estimate based on the statistical equilibrium turbulence theory developed by Andréi Kolmogórov in 1941, and we apply it to an academic test.

Part I

The use of courtyards in the Andalusian architecture goes back to the Roman and Muslim culture. They were used as a communal space, however, the courtyards have many others utilities. Nowadays, we can find this kind of structure in many buildings and their purpose goes from providing light into buildings to create a fresh environment in the center of the building to refrigerate the rest of it. The air movement will renew the air inside the courtyard, which will lower the temperature with respect to the outside and so the building temperature (*cf.* [16, 32]).

Cloisters are a kind of courtyard widespread in old buildings such as cathedrals and monasteries. In Figure 1, an example of this cloister is shown. We are interested in the optimal thermal comfort design of the cloister, studying the dynamics of the air that flows into the courtyard by forced convection model, as a first approximation to our problem.

Figure 1: Merced Convent, placed in Córdoba (Spain).

For the building design, it is interesting to know the distribution of temperature along the courtyard to select the best thermal comfort design. We consider two parameters, the width and the height of the corridor around the cloister.

The resolution of this kind of flow could take a considerable computing time. This is the reason to consider to apply the Reduced Basis (RB) Method that we develop along Chapters 2 and 3.

As a first approximation, we consider the fluid dynamics only in 2D. We develop the RB problem in Chapter 2 building an *a posteriori* error bound estimator following the BRR theory for velocity and pressure. In Chapter 3, we add the temperature to the problem using a forced convection model. We build the RB problem developing an *a posteriori* error bound estimator for temperature, following now standard techniques since the problem is linear.

We are able to obtain the distribution of velocity, pressure, and temperature in barely 1.5 second in stead of 3.5 minutes with errors more than admissible.

Taking into account that our parameters are geometrical and the computational time for the ROM, it makes sense to apply our RB model to a design problem. We study which is the best combination of both geometrical parameters so that the temperature in the ground floor is the closest to the comfort one, in order to optimize the comfort properties of our cloister.

We design a functional to minimize the temperature in the ground floor and we computed the value of this functional for 625 pairs of parameters in less than 16 minutes. We obtain an optimum design that optimizes the thermal comfort in the lower part (to be occupied by people) of the cloister.

Part II

The main goal of this part is to build RB based upon error estimators for the unsteady Smagorinsky turbulence model.

We start this part studying *a priori* estimates for velocity and pressure for the time-space discrete Smagorinsky model in Chapter 4. We base this chapter on the results obtained by Volker John in Chapter 6 [25], where he developed *a priori* estimates for finite element solution of LES models. We obtain *a priori* estimates over appropriate space-time Sobolev spaces using inverse inequalities. This results are the key to apply the BRR theory in the next chapter.

Secondly, in Chapter 5, we are able to apply the BRR theory for the development of *a posteriori* error bound estimator, not without finding difficulties. The key is the definition of the norm related to the problem. Intuition could say that we need a norm relating the velocity derivative in time, deformation and pressure. However, we can not ensure the inf-sup condition considering this norm. This is why we consider a norm related to velocity and pressure, we prove well-posedness depending on the time step chosen, the inf-sup condition is achieved, and we obtain the *a posteriori* error bound estimator. Unfortunately, the computation of the Stability Factor, necessary for the computation of the *a posteriori* estimator, is not low-cost, and it is necessary the implementation of further techniques to reduce the complexity.

Finally, we look for an alternative, and we dip into the statistical equilibrium Kolmogórov theory. We explain this theory along Chapter 6, explaining the concept of energy spectrum and how to compute it. Lastly, we introduce the *a posteriori* error estimator using the energy spectrum. The continuous problem should achieve an specific energy spectrum, since the full-order model is intended to be a good approximation of the continuous problem, it should also achieve this energy spectrum in the resolved part of the inertial spectrum. The key is to use the error committed in this energy spectrum by the RB solution to select the new basis functions by the Greedy algorithm.

To validate the estimate, we develop an academic test in which we compare the use of the Kolmogórov estimator with the use of the error between the full order and reduced order solution to develop the RB problem. The test is based on a problem for which we obtain a velocity field that satisfies the $-5/3$ law, and we build a RB problem using the POD+Greedy strategy. The number of bases and the error are similar to those obtained using the true error as "error estimator", which indicates that the estimate works in this case.

In this case, the speed-ups are around 20, which is more than admissible as using the exact error does not provide a further speed-up improvement.

FreeFem++ and post processing.

All numerical test have been coded in FreeFem++ v. 4.8 (*cf.* [23]). Due to the complexity of the numerical test, we use parallel computation using:

- PETSc package (Portable, Extensible Toolkit for Scientific Computation). This package is embedded in FreeFem++ and allows us to solve variational problems.

  For that purpose, it is necessary a domain partition depending on the processor number, which can be boarded by commands defined in the `macro_ddm.idp` file, included in FreeFem++. Then, we define the FE matrices and we solve the problem using the PETSc package, obtaining a local solution in each partition. Finally, we recover the solution defined in the complete domain.

  This process has been automated using macros.

- MPI (Message Passing Interface). Again, we have commands that allow us the transmission of information between processors thanks to `macro_ddm.idp` file.

  We use these commands to compute independent loops in parallel, necessary for example, for the construction of the RB matrices. The main idea is to assign the computation of some loop iterations to each processor and finally share the result within the processors. We can apply this technique to the construction of matrices since these involve two loops with independent iterations.

Every offline phase along this dissertation has been performed in the cluster *Anonimus2021* which allows a High Performance Computation (HPC). This cluster is composed by 18 nodes, two of them dedicated to computation on GPUs. The rest of the nodes are composed by CPUs *AMD EPYC 7542*.

For the figures presented along this dissertation, we have used MATLAB 2016b (*cf.* [48]) and Paraview v. 5.9 (*cf.* [3])

1
Basics of Reduced Basis modelling

In this chapter, we introduce basic results that will be used along this dissertation. We introduce a general notation for Partial Differential Equations (PDEs) in Section 1.1, the Reduced Order Model (ROM) for PDEs in Section 1.2. We introduce the Empirical Interpolation Method (EIM) for the reduced linearization of some terms with respect to the parameter in Section 1.3. Finally, we show a numerical test, application of the Reduced Basis Method to the 3D lid-driven cavity flow. This test is an extension to the numerical test performed by E. Delgado *et al.* in [11]. In this paper, E. Delgado *et al.* develop a technique to construct *a posteriori* error estimator necessary for the Reduced Basis Method, and they validate the estimator for the 2D case. We apply this theory to the 3D case in Section 1.4.

1.1 Parametric Partial Differential Equations

We shall consider in this dissertation parametric PDEs. Let us denote by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P)$ the parameter vector where P is the total number of parameters. We note that each μ_i for $i = 1, \ldots, P$ can describe a property in the model, such as viscosity, density, thermal conductivity, etc. (physical parameters) or it can parameterize the shape of the domain (geometric parameters). Let us denote by $\mathcal{D} \subset \mathbb{R}^P$ the parametric set that will contain all possible parameter values. Finally, let us denote by $\Omega = \Omega(\boldsymbol{\mu}) \in \mathbb{R}^d$, $d = 2, 3$ the spatial domain and $I_f = (0, T_f)$ with $T_f > 0$ the time domain, Γ the boundary of Ω and $Q_T = I_f \times \Omega$.

Then, for all $\boldsymbol{\mu} \in \mathcal{D}$, we consider the following parabolic equation

$$\partial_t u(\boldsymbol{\mu}) + L(\boldsymbol{\mu}) u(\boldsymbol{\mu}) = f(\boldsymbol{\mu}), \quad \text{in } Q_T \tag{1.1}$$

where $f(\boldsymbol{\mu}) = f(t, \mathbf{x}; \boldsymbol{\mu})$ is a given function, $L(\boldsymbol{\mu}) = L(\mathbf{x}; \boldsymbol{\mu})$ is a generic elliptic operator acting on the unknown $u(\boldsymbol{\mu}) = u(t, \mathbf{x}; \boldsymbol{\mu})$.

Additionally, we consider

$$u(\mathbf{x}, 0; \boldsymbol{\mu}) = u^0(\mathbf{x}; \boldsymbol{\mu}),\ \mathbf{x} \in \Omega,\ \boldsymbol{\mu} \in \mathcal{D} \tag{1.2}$$

as the initial condition and

$$\begin{aligned} u(t, \mathbf{x}; \boldsymbol{\mu}) &= u_D(t, \mathbf{x}; \boldsymbol{\mu}),\ \mathbf{x} \in \Gamma_D, t \in I_f,\ \boldsymbol{\mu} \in \mathcal{D} \\ \frac{\partial u}{\partial \mathbf{n}}(t, \mathbf{x}; \boldsymbol{\mu}) &= u_N(t, \mathbf{x}; \boldsymbol{\mu}),\ \mathbf{x} \in \Gamma_N, t \in I_f,\ \boldsymbol{\mu} \in \mathcal{D} \end{aligned} \tag{1.3}$$

as the boundary conditions where u^0, u_D and u_N are given functions and $\Gamma_D \cup \Gamma_N = \partial\Omega$ with $\mathring{\Gamma}_D \cap \mathring{\Gamma}_N = \emptyset$.

In order to solve this problem numerically, we obtain a weak formulation of problem (1.1). Let $X = X(\Omega)$ be a suitable Hilbert space endowed with the X-norm. Then for all $t \in I_f$ and $\boldsymbol{\mu} \in \mathcal{D}$, we consider the problem

$$\begin{cases} \text{Find } u(t; \boldsymbol{\mu}) \in X \text{ such that} \\ (\partial_t u(t; \boldsymbol{\mu}), v)_\Omega + A(u(t; \boldsymbol{\mu}), v; \boldsymbol{\mu}) = (f(t; \boldsymbol{\mu}), v)_\Omega \quad \forall v \in X \end{cases} \tag{1.4}$$

where $A(\cdot, \cdot; \boldsymbol{\mu}) : X \times X \to \mathbb{R}$ is a bilinear operator associated to the $L(\boldsymbol{\mu})$ operator and the Neumann boundary condition for all $\boldsymbol{\mu} \in \mathcal{D}$. For simplicity, we have supposed homogeneous Dirichlet boundary conditions.

A sufficient condition for the existence and uniqueness of the solution of problem (1.4) for all $\boldsymbol{\mu} \in \mathcal{D}$ is that

- the bilinear form $A(\cdot, \cdot; \boldsymbol{\mu})$ is continuous and weakly coercive, that is

  $$|A(u, v; \boldsymbol{\mu})| \leq \gamma \|u\|_X \|v\|_X,\ A(v, v; \boldsymbol{\mu}) + \lambda \|v\|^2_{L^2(\Omega)} \geq \beta \|v\|^2_X, \quad \forall u, v \in X$$

  for some $\lambda \geq 0$ and $\beta > 0$;

- $u^0(\boldsymbol{\mu}) \in L^2(\Omega)$ and $f(\boldsymbol{\mu}) \in L^2(Q_T)$ for all $\boldsymbol{\mu} \in \mathcal{D}$.

Then, for all $\boldsymbol{\mu} \in \mathcal{D}$, problem (1.4) admits a unique solution $u \in L^2(I_f;X) \cap \mathcal{C}^0(I_f;L^2(\Omega))$. We refer to Section 5.1 in [37] for a more detail description.

Now, we consider a Galerkin approximation of problem (1.4). Let $\{\mathcal{T}_h\}_{h>0}$ be a uniformly regular family of triangulations of Ω (see Definition A.2), then we consider a discrete subspace $X_h = X_h(\Omega) \subset X$ of dimension N_h as an inner approximation. Then, for a given $t \in I_f$, $\boldsymbol{\mu} \in \mathcal{D}$, the semi-discretized equation in space is

$$\begin{cases} \text{Due } u_h^0(\boldsymbol{\mu}) = u_h(0;\boldsymbol{\mu}), \text{ find } u_h(t;\boldsymbol{\mu}) \in X_h \text{ such that} \\ (\partial_t u_h(t;\boldsymbol{\mu}), v_h)_\Omega + A(u_h(t;\boldsymbol{\mu}), v_h;\boldsymbol{\mu}) = (f(t;\boldsymbol{\mu}), v_h)_\Omega, \quad \forall v_h \in X_h. \end{cases}$$

where $u_h(0;\boldsymbol{\mu})$ is an approximation of $u^0(\boldsymbol{\mu})$ in the space X_h.

Along this dissertation, we will use the explicit (or semi-explicit when $A(\cdot,\cdot;\boldsymbol{\mu})$ is nonlinear) Euler scheme for time discretization, although the results presented can be extended to more general numerical schemes.

Let L be a positive integer that defines the number of time steps that we are considering, let $\Delta t = T_f/L$ be the time step and $t_k = k\Delta t$ for $k = 0, 1, \ldots, L$. We consider $u_h^k(\boldsymbol{\mu})$ the approximation of $u_h(t_k;\boldsymbol{\mu})$. Then, the space-time discretization of the PDE (1.4) is

$$\begin{cases} \text{For any } k = 1,\ldots,L \text{ and } \boldsymbol{\mu} \in \mathcal{D}, \text{ assuming known } u_h^{k-1}(\boldsymbol{\mu}) \in X_h, \\ \text{find } u_h^k(\boldsymbol{\mu}) \in X_h \text{ such that, } \forall v_h \in X_h \\ M(u_h^k(\boldsymbol{\mu}), v_h;\boldsymbol{\mu}) + A(u_h^k(\boldsymbol{\mu}), v_h;\boldsymbol{\mu}) = F^k(v_h;\boldsymbol{\mu}) + M(u_h^{k-1}(\boldsymbol{\mu}), v_h;\boldsymbol{\mu}), \end{cases} \tag{1.5}$$

where $M(\cdot,\cdot;\boldsymbol{\mu}) : X \times X \to \mathbb{R}$ is defined by $M(u,v;\boldsymbol{\mu}) = (u,v)_\Omega/\Delta t$ and $F^k(v_h;\boldsymbol{\mu})$ is a suitable approximation of the second member, for example,

$$F^k(v_h;\boldsymbol{\mu}) = \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}\int_\Omega f(t,\mathbf{x};\boldsymbol{\mu})\, v_h \, d\Omega\, dt$$

for all $\boldsymbol{\mu} \in \mathcal{D}$.

1.2 Model-Order Reduction

The resolution of problem (1.5) lead us to solve a linear system with dimension N_h for $k = 1,\ldots,L$. Sometimes, especially if $d = 3$, N_h could be large and the resolution of this problem for a given $\boldsymbol{\mu}$ can take a considerable time and resources, not to mention for every $\boldsymbol{\mu} \in \mathcal{D}$. This is why we are interested in the application of reduced-order modeling (ROM) techniques. These techniques use an approximation of the problem to solve with much lower numerical complexity.

In particular, we are interested in the Reduced Basis (RB) methods which generate a RB problem using solutions of problem (1.5) as follows

$$\begin{cases} \text{For any } k=1,\ldots,L \text{ and } \boldsymbol{\mu}\in\mathcal{D}, \text{ assuming known } u_N^{k-1}(\boldsymbol{\mu})\in X_N, \\ \text{find } u_N^k(\boldsymbol{\mu})\in X_N \text{ such that, } \forall v_N\in X_N \\ M(u_N^k(\boldsymbol{\mu}),v_N;\boldsymbol{\mu})+A(u_N^k(\boldsymbol{\mu}),v_N;\boldsymbol{\mu}) = F^k(v_N;\boldsymbol{\mu})+M(u_N^{k-1}(\boldsymbol{\mu}),v_N;\boldsymbol{\mu}), \end{cases} \tag{1.6}$$

where $u_N^0(\boldsymbol{\mu})$ is the projection of $u_h^0(\boldsymbol{\mu})$ on X_N. The space X_N is built from $u_h^k(\boldsymbol{\mu})\in X_h$ solutions of (1.5) for specifically chosen $\boldsymbol{\mu}\in\mathcal{D}$ and $k=1,\ldots,L.$ According to this, it is easy to think that the space X_N should summarize the information described by the *high fidelity* problem (1.5) and therefore, keep the important one. The selection of k and $\boldsymbol{\mu}$ is the key of this section.

The solution of problem (1.6) by the RB method is divided into two phases:

1. The offline phase in which we compute the RB Space X_N and the RB Matrices.

2. The online phase in which we solve the problem (1.6) for any parameter $\boldsymbol{\mu}\in\mathcal{D}$.

Phase 1 needs the solution of the sampling problem, one of the key problems to obtain accurate ROMs, in particular RB methods. The RB base, which forms the space X_N, is intended to provide certified solutions, in the sense that these provide errors below targeted values.

Reduced Basis Technique

The construction of Reduced Basis spaces is based upon an *a posteriori* error estimator and Greedy Algorithm for optimal sampling.

The Greedy Algorithm is an iterative algorithm that provides the optimal sampling at each stage. In each iteration, we seek for the error between the high fidelity solution and the RB solution $\|u_h^k(\boldsymbol{\mu})-u_N^k(\boldsymbol{\mu})\|_X$, and we select the time and parameter for which the error is larger.

For the RB method, we first need to define $\mathcal{D}_{train}\subset\mathcal{D}$ as a set of parameters to check in the algorithm. The choice of this set is complex, for efficiency should be small but sufficiently large to represent $\mathcal{D}$. The algorithm is described in Algorithm 1. Its application to the steady problem corresponds to letting $k=1$ (assuming that u^1 is the steady solution).

The selection of the new pair $(\boldsymbol{\mu}^N,k^N)$ requires to obtain each $u_h^k(\boldsymbol{\mu})$ for $\boldsymbol{\mu}\in\mathcal{D}_{train}$ and $k=1,\ldots,L$ which may be hard to compute numerically. It is usual to use the *a*

Algorithm 1 Greedy

Set $\varepsilon_{tol} > 0$, $N_{max} \in \mathbb{N}$ and $S_1 = \{(\boldsymbol{\mu}^1, k^1)\}$;
Compute $u_h^{k^1}(\boldsymbol{\mu}^1)$;
$X_1 = \text{span}\{u_h^{k^1}(\boldsymbol{\mu}^1)\}$;
for $N = 2 : N_{\max}$ **do**
    $(\boldsymbol{\mu}^N, k^N) = \arg\max_{\boldsymbol{\mu} \in \mathcal{D}_{train}, k=1,\dots,L} \|u_h^k(\boldsymbol{\mu}) - u_{N-1}^k(\boldsymbol{\mu})\|_X$;
    $\varepsilon_{N-1} = \|u_h^{k^N}(\boldsymbol{\mu}^N) - u_{N-1}^{k^N}(\boldsymbol{\mu}^N)\|_X$;
    if $\varepsilon_{N-1} \leq \varepsilon_{tol}$ **then**
        $N_{max} = N - 1$;
    end if
    Compute $u_h^{k^N}(\boldsymbol{\mu}^N)$;
    $S_N = S_{N-1} \cup \{(\boldsymbol{\mu}^N, k^N)\}$;
    $X_N = X_{N-1} \cup \text{span}\{u_h^{k^N}(\boldsymbol{\mu}^N)\}$;
end for

posteriori error bound $\Delta_N(\boldsymbol{\mu}, k)$ of the error $\|u_h^k(\boldsymbol{\mu}) - u_N^k(\boldsymbol{\mu})\|_X$ which should be simpler to compute. The use of the *a posteriori* error bound $\Delta_N(\boldsymbol{\mu}, k)$ is called "Weak" Greedy Algorithm 2 as it uses the error estimator instead of the error itself.

To avoid redundancies, the RB space should be orthonormal in the X-norm, hence we use the Gram-Schmidt orthonormalization process.

To obtain reliable error estimates, the X-norm should be independent of $\boldsymbol{\mu}$ and k.

Algorithm 2 (Weak) Greedy

Set $\varepsilon_{tol} > 0$, $N_{max} \in \mathbb{N}$ and $S_1 = \{(\boldsymbol{\mu}^1, k^1)\}$;
Compute $u_h^{k^1}(\boldsymbol{\mu}^1)$;
$X_1 = \text{span}\{u_h^{k^1}(\boldsymbol{\mu}^1)\}$;
for $N = 2 : N_{\max}$ **do**
    $(\boldsymbol{\mu}^N, k^N) = \arg\max_{\boldsymbol{\mu} \in \mathcal{D}_{train}, k=1,\dots,L} \Delta_{N-1}(\boldsymbol{\mu}, k)$;
    $\varepsilon_{N-1} = \Delta_{N-1}(\boldsymbol{\mu}^N, k^N)$;
    if $\varepsilon_{N-1} \leq \varepsilon_{tol}$ **then**
        $N_{max} = N - 1$;
    end if
    Compute $u_h^{k^N}(\boldsymbol{\mu}^N)$;
    $S_N = S_{N-1} \cup \{(\boldsymbol{\mu}^N, k^N)\}$;
    $X_N = X_{N-1} \cup \text{span}\{u_h^{k^N}(\boldsymbol{\mu}^N)\}$;
end for

Proper Orthogonal Decomposition (POD)

The Proper Orthogonal Decomposition is a model reduction technique. POD problem consists in finding the orthonormal base that minimizes the error

$$\sum_{\boldsymbol{\mu}\in\mathcal{D}}\sum_{k=1}^{L}||u_h^k(\boldsymbol{\mu})-u_N^k(\boldsymbol{\mu})||_X^2.$$

This minimization problem is equivalent to computing the Singular Value Decomposition (SVD) of a snapshot matrix. Down below, we describe the main procedure:

1. First, we define the set of snapshots $\{u_h^k(\boldsymbol{\mu})\}$ for any $k=1,\dots,L$ and $\boldsymbol{\mu}\in\mathcal{D}_{train}$. Then, we are able to define the matrices $\mathbb{S}_i\in\mathbb{R}^{N_h\times L}$ for $i=1,\dots,n_s$ where $\dim(\mathcal{D}_{train})=n_s$ such that

   $$\mathbb{S}_i=[\underline{u}_h^1(\boldsymbol{\mu}^i),\underline{u}_h^2(\boldsymbol{\mu}^i),\dots,\underline{u}_h^L(\boldsymbol{\mu}^i)],\ \forall i=1,\dots,n_s.$$

   The vectors $\underline{u}_h^k(\boldsymbol{\mu})\in\mathbb{R}^{N_h}$ represent the degrees of freedom of the function $u_h^k(\boldsymbol{\mu})\in X_h$ for $k=1,\dots,L$. Finally, the snapshot matrix $\mathbb{S}\in\mathbb{R}^{N_h\times Ln_s}$ is defined as

   $$\mathbb{S}=[\mathbb{S}_1|\mathbb{S}_2|\dots|\mathbb{S}_{n_s}].$$

2. Now, we define the matrix $\mathbb{X}_h\in\mathbb{R}^{N_h\times N_h}$ as the matrix associated to the inner product $(\cdot,\cdot)_X$ such that

   $$\underline{u}_h^T\mathbb{X}_h\underline{v}_h=(u_h,v_h)_X,\quad \forall u_h,v_h\in X_h.$$

   Then, the matrix $\mathbb{C}=\mathbb{S}^T\mathbb{X}_h\mathbb{S}\in\mathbb{R}^{Ln_s\times Ln_s}$ is the so-called *correlation* matrix related to the X-norm.

3. Finally, we solve the eigen value problem

   $$\mathbb{C}\psi_i=\sigma_i^2\psi_i,\quad i=1,\dots,r$$

   with $\psi_i\in\mathbb{R}^{Ln_s}$ for all $i=1,\dots,r$ and $r=rank(\mathbb{C})$. The resolution of this problem provides r singular values σ_i such that $\sigma_1\geq\sigma_2\geq\dots\geq\sigma_r$. The key here is the choice $N\leq r$ for some criterion over the singular values σ_i for $i=1,\dots,r$.

   If we define ε_{tol} as the tolerance, then we can define a criterion envolving the energy of the modes, that is

   $$I(N)=\frac{\sum_{i=1}^{N}\sigma_i^2}{\sum_{i=1}^{r}\sigma_i^2}\geq 1-\varepsilon_{tol}^2.$$

   or to choose the first N such that $\sigma_i^2\leq\varepsilon_{tol}$ for $i=N+1,\dots,r$.

4. We define the POD-ROM basis such that

$$\underline{\varphi}_i = \frac{1}{\sigma_i}\mathbb{S}\psi_i, \quad \forall i = 1,\ldots,N$$

where we recall that $\underline{\varphi}_i$ represents the degrees of freedom of the basis functions φ_i for $i = 1,\ldots,N$. Then, $X_N = \text{span}\{\varphi_i\}_{i=1}^N$ and that base is already orthonormal respect to the X-norm.

The POD algorithm is described in Algorithm 3.

Algorithm 3 POD with respect to the X-norm

function $\mathbb{V}$ =POD($\mathbb{S}, \mathbb{X}_h, \varepsilon_{tol}$)
  Form $\mathbb{C} = \mathbb{S}^T\mathbb{X}_h\mathbb{S}$;
  Solve the eigen value problem $\mathbb{C}\psi_i = \sigma_i^2\psi_i$, $i = 1,\ldots,r$;
  Choose N in terms of ε_{tol};
  Set $\underline{\varphi}_i = \frac{1}{\sigma_i}\mathbb{S}\psi_i$, $i = 1,\ldots,N$; **return** $\mathbb{V} = [\underline{\varphi}_1,\ldots,\underline{\varphi}_N]$
end function

The principal issue with the POD strategy is the selection of the parameter training set $\mathcal{D}_{train}$. We have no way to estimate the training set or the error *a priori* .

Moreover, the size of $\mathbb{C}$ depends on the choice of $\mathcal{D}_{train}$ and L, therefore, at some point, the resolution of the eigenvalue problem could represent a challenge.

Again, taking $L = 1$, this POD strategy can be used for unsteady problems. We refer to [29, 5, 47] and Chapter 6 in [38] for more details.

POD+Greedy strategy for unsteady problems

The techniques previously introduced can be reduced to steady problems with no hitches. Actually, when we consider steady problems depending on parameters, it is usual to apply any of these two strategies, Greedy or POD.

However, when we talk about unsteady problems, the Greedy algorithm may not converge since the variability of the snapshots, and a POD strategy can be hard to perform because of the correlation matrix size.

This is why we introduce a joint strategy for the unsteady problem that we will use in Chapter 6. The key is to use POD considering the time t as parameter and the Greedy algorithm for the parameter $\boldsymbol{\mu}$. The strategy is shown in Algorithm 4.

We refer to [39, 22] for more information about the POD+Greedy strategy, however we will extend the application of this strategh in Chapter 6.

Algorithm 4 POD+Greedy

Set $\varepsilon_{1,tol}, \varepsilon_{2,tol} > 0$, $N_{max} \in \mathbb{N}$, $\boldsymbol{\mu}^* \in \mathcal{D}_{train}$, $\mathbb{Z} = [\,]$ and $S = \{\,\}$;
while $N < N_{\max}$ **do**
  $S = S \cup \{\mu^*\}$;
  Compute $u_h^k(\mu^*)$ for $k = 1, \ldots, L$;
  Build $\mathbb{S} = [\underline{u}_h^1(\boldsymbol{\mu}^*), \underline{u}_h^2(\boldsymbol{\mu}^*), \ldots, \underline{u}_h^L(\boldsymbol{\mu}^*)]$;
  $[\underline{\xi}_1, \ldots, \underline{\xi}_M]$=POD($\mathbb{S}, \mathbb{X}_h, \varepsilon_{1,tol}$);
  $\mathbb{Z} = [\mathbb{Z}, \underline{\xi}_1, \ldots, \underline{\xi}_M]$;
  $[\underline{\varphi}_1, \ldots, \underline{\varphi}_N]$=POD($\mathbb{Z}, \mathbb{X}_h, \varepsilon_{2,tol}$);
  $X_N = \{\varphi_i\}_{i=1}^N$;
  $\boldsymbol{\mu}^* = \arg\max_{\boldsymbol{\mu} \in \mathcal{D}_{train}} \Delta_N(\boldsymbol{\mu}, L)$;
  $\varepsilon_N = \Delta_N(\boldsymbol{\mu}^*, L)$;
  if $\varepsilon_N \leq \varepsilon_{tol}$ **then**
    $N_{max} = N$;
  end if
end while

Algebraic Construction

Let us recall that X_N is the space generated by the RB functions $\{\varphi_i\}_{i=1}^N$. The reduced solution $u_N^k(\boldsymbol{\mu}) \in X_N$ of (1.6) for any $k = 1, \ldots, L$ can be described as a linear combination of $\{\varphi_i\}_{i=1}^N$,

$$u_N^k = \sum_{i=1}^{N} (\underline{u}_N^k)_i(\boldsymbol{\mu})\varphi_i$$

where $\underline{u}_N^k$ is the solution of the reduced linear system

$$(\mathbb{M}_N + \mathbb{A}_N)\underline{u}_N^k = \mathbf{f}_N^k + \mathbb{M}_N \underline{u}_N^{k-1}, \tag{1.7}$$

being $\mathbb{M}_N, \mathbb{A}_N \in \mathscr{M}^{N\times N}(\mathbb{R})$ and $\mathbf{f}_N^k \in \mathbb{R}^N$ associated to the left and right hand side of (1.6). These matrices depend on the parameter $\boldsymbol{\mu}$ and their construction can take a long computing time depending on N since they are full matrices. To save that time in the online phase, we look for a linear representation of the problem (1.6) with respects to the parameters. This linear representation will allow us to define RB parameter-independent matrices that we store at the offline phase end.

Remark 1.1. *For the Greedy Algorithm 1-2, the construction of the spaces X_N is recursive, this is, in order to build the N-iteration, we need the $N-1$-iteration. The construction of* *the RB matrices could be done in a recursive way too. For example, for the matrix related to the M operator, given $\mathbb{M}_{N-1}$, then,*

$$\mathbb{M}_N = \left[\begin{array}{ccc|c} & & & (\varphi_N,\varphi_1)_\Omega/\Delta t \\ & \mathbb{M}_{N-1} & & \vdots \\ & & & (\varphi_N,\varphi_{N-1})_\Omega/\Delta t \\ \hline (\varphi_1,\varphi_N)_\Omega/\Delta t & \dots & (\varphi_{N-1},\varphi_N)_\Omega/\Delta t & (\varphi_N,\varphi_N)_\Omega/\Delta t \end{array}\right].$$

Physical vs. geometrical parameters

In terms of parameters, we shall identify two kinds, physical or geometric. Dimensionless numbers, as those described in Appendix A.4, are meant to be a physical parameter. In this case, the parameter representation is almost direct. However, it could be non-linear and linearization techniques should be applied. In the next section, we will introduce the EIM method that will help us to deal with this situation.

For the geometric parameter, the dependency is intrinsic to the domain. In this case, we are able to fix a parameter $\boldsymbol{\mu}_r \in \mathcal{D}$ which defines a reference domain $\Omega_r = \Omega(\boldsymbol{\mu}_r)$ and we define problem (1.5) over Ω_r instead of $\Omega(\boldsymbol{\mu})$ thanks to a change of variable. We will see an application in Chapter 2, however, we refer to [41, 42, 38, 14] for more details.

1.3 Empirical Interpolation Method

The aim of the Empirical Interpolation Method (EIM) is to build a linear approximation g_M of non-linear function g respect to the time t and paramter $\boldsymbol{\mu}$ as follows

$$g_M(\mathbf{x};t,\boldsymbol{\mu}) = \sum_{i=1}^{M} \sigma_i(t,\boldsymbol{\mu}) q_i(\mathbf{x}),\ \forall \mathbf{x} \in \Omega, t \in I_f, \boldsymbol{\mu} \in \mathcal{D}$$

for σ_i and q_i for $i = 1,\dots,M$ to be determined.

To build this approximation, let consider the space

$$\mathcal{G} = \{g(\cdot;t,\boldsymbol{\mu}),\ \boldsymbol{\mu} \in \mathcal{D},\ t \in I_f\}$$

as the family of parameter-time-dependent functions. Then, the key of the EIM is to find a reduced approximation space $\mathcal{G}_M = \text{span}\{q_i\}_{i=1}^M \subset \mathcal{G}$ in a first offline phase.

The offline phase could be afforded using the Greedy algorithm. The coefficients $\{\sigma_i\}_{i=1}^M$ are obtained by the solution of

$$\sum_{j=1}^{M} q_j(\mathbf{x}^i)\sigma_j(t,\boldsymbol{\mu}) = g(\mathbf{x}^i;t,\boldsymbol{\mu}),\ i = 1,\dots,M$$

where $T_M = \{\mathbf{x}_i\}_{i=1}^M$ are the interpolation or *magic* points (*cf.* [33]). That is, in matrix form

$$\mathbb{B}\boldsymbol{\sigma}(t,\boldsymbol{\mu}) = \mathbf{g}_M(t,\boldsymbol{\mu}),\ \forall t \in I_f,\ \boldsymbol{\mu} \in \mathcal{D} \tag{1.8}$$

where

$$(\mathbb{B})_{ij} = q_j(\mathbf{x}^i),\ (\boldsymbol{\sigma}(t,\boldsymbol{\mu}))_i = \sigma_i(t,\boldsymbol{\mu}),\ (\mathbf{g}_M)_i = g(\mathbf{x}^i;t,\boldsymbol{\mu}),$$

for $i,j = 1,\ldots,M$. The offline and online phases are described in Algorithm 5.

Algorithm 5 EIM (computable version)

```
function (𝔹,T,G,S)=EIMOFFLINE(g, ε_EIM, M_max, 𝒟_EIM, Δt, L)
    Set S = { }, T = { }, G = { };
    (k^1, μ^1) = arg max_{μ∈𝒟_EIM, k=0,...,L} ||g(·;kΔt, μ)||_∞;
    r_1 = g(·;k^1 Δt; μ^1);
    M = 1;
    while M < M_max do
        x_M = arg sup_{x∈Ω} |r_M(x)|;
        q_M = r_M / r_M(x_M);
        S = S ∪ {(k^M, μ^M)};
        T = T ∪ {x_M};
        G = G ∪ {q_M};
        𝔹_ij = q_j(x_i), 1 ≤ i,j ≤ M;
        (k^{M+1}, μ^{M+1}) = arg max_{μ∈𝒟_EIM, k=0,...,L} ||g(·;kΔt, μ) − g_M(·;kΔt, μ)||_∞;
        r_{M+1} = g(·;k^{M+1}Δt, μ^{M+1}) − g_M(·;k^{M+1}Δt, μ^{M+1}), ε = ||r_{M+1}||_∞;
        M = M + 1;
        if ε ≤ ε_EIM then
            M_max = M;
        end if
    end while
end function
function σ=EIMONLINE(𝔹, t, μ, {x_1, ..., x_M})
    for i = 1 : M do
        g_i = (x_i; t, μ);
    end for
    σ = 𝔹^{-1} g̲;
end function
```

1.4 ROM application to steady Smagorinsky turbulence model

In [11], E. Delgado *et al.* have built a RB Smagorinsky model for the steady case. They introduce *a posteriori* error bound estimator for non-linear problems using the BRR theory (*cf.* [8]). In this section, we will introduce the basics of the theory developed by E. Delgado, which will be the foundation of Chapter 2 and 5.

Let us consider the dimensionless Smagorinsky turbulence model. It is a Large Eddy Simulation (LES) model and it was first introduced by Joseph Smagorinsky in 1963 (see [46]).

LES modelization is based upon the Kolmogorov Theory of statistical equilibrium theory that we will describe in detail in Chapter 6. In essence, we suppose that the Reynolds number is sufficiently high to ensure that energy is transferred from large to small scales, which are responsible for dissipation only. This generates an inertial spectrum in which only inertial forces are important, delimited by two scales which form the inertial range. The LES modelization consists in solving a part of this inertial range, in addition to the large scales of the flow.

In particular, the Smagorinsky model is intrinsically discrete, linked to a space discretization that determines the large scales to solve. It adds an additional term to the averaged Navier-Stokes equations, called the eddy viscosity term, which is related to a grid $\mathcal{T}_h$ that models the energy dissipated by the sub-grid (unresolved) scales.

Then, the Smagorinsky turbulence model in its differential form is formulated as

$$\begin{cases} -\dfrac{1}{\mathrm{Re}}\Delta \mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla\cdot(\nu_t(\mathbf{w})\nabla\mathbf{w}) + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{w} = 0, & \text{in } \Omega, \\ \mathbf{w} = g, & \text{on } \Gamma_D, \\ -p\mathbf{n} + \left(\dfrac{1}{\mathrm{Re}} + \nu_t(\mathbf{w})\right)\dfrac{\partial \mathbf{w}}{\partial \mathbf{n}} = \mathbf{0}, & \text{on } \Gamma_N, \end{cases} \tag{1.9}$$

where Re is the Reynolds number defined in (A.22), $\mathbf{w}$ is the velocity field, p is the pressure per mass density,

$$\nu_t(\mathbf{w}) = \sum_{K\in\mathcal{T}_h} (C_S h_K)^2 |\nabla \mathbf{w}_{|K}|$$

is the eddy viscosity, where $|\cdot|$ denotes the Frobenius norm in $\mathbb{R}^{d\times d}$ defined in (A.1), the constant C_S is the Smagorinsky constant which is estimated to be $C_S \approx 0.18$ (*cf.* [31, 26, 44]) and h_K is the diameter of the element $K\in\mathcal{T}_h$. In Chapter 6 in [25], V. John study the existence and uniqueness of solutions of the Smagorinsky model (1.9) following the results presented by Ladyzhenskaya in [30].

1.4.1 Reduced Basis steady Smagorinsky model

Let us state the basics of the RB approximation to the steady Smagorinsky model, considering the Reynolds number as the parameter μ in a suitable parameter set $\mathcal{D}$.

In order to apply the RB Method, we need a model with homogeneous Dirichlet BCs. To solve this, we decompose the velocity field into two fields, $\mathbf{u}$ the solution of a problem with homogeneous BCs and the lift $\mathbf{u}_D$, which should achieve the non-homogeneous Dirichlet BC $\mathbf{u}_D = g$ on Γ_D. Without lost of generality, we suppose that the lift is divergence free since this simplifies the problem.

Finite Element approximation of Steady Smagorinsky model As in Section 1.1, we shall start from a variational formulation of our problem in a suitable Hilbert space and then build a discretization.

Let us consider the spaces

$$Y = \{\mathbf{v} \in H^1(\Omega) : \mathbf{v}_{|\Gamma_D} = 0\}, \quad Q = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$$

for velocity and pressure, and denote $X = Y \times Q$. Let Y_h and Q_h be Finite Element spaces constructed on the grid $\mathcal{T}_h$ to approximate Y and Q, respectively and $X_h = Y_h \times Q_h$.

Then the variational discretization of problem (1.9) is stated as follows:

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) = (\mathbf{u}_h(\mu), p_h(\mu)) \in X_h \text{ such that} \\ a(\mathbf{u}_h, \mathbf{v}_h; \mu) + a_t(\mathbf{u}_h + \mathbf{u}_D; \mathbf{u}_h + \mathbf{u}_D, \mathbf{v}_h; \mu) \\ +b(\mathbf{v}_h, p_h; \mu) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h; \mu) + c(\mathbf{u}_D, \mathbf{u}_h, \mathbf{v}_h; \mu) \\ +c(\mathbf{u}_h, \mathbf{u}_D, \mathbf{v}_h; \mu) = F(\mathbf{v}_h; \mu), & \forall \mathbf{v}_h \in Y_h, \\ b(\mathbf{u}_h, q_h; \mu) = 0, & \forall q_h \in Q_h. \end{cases} \tag{1.10}$$

where the bilinear forms $a(\cdot,\cdot;\mu)$ and $b(\cdot,\cdot;\mu)$ are defined by

$$a(\mathbf{u}, \mathbf{v}; \mu) = \frac{1}{\mu} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega, \; b(\mathbf{v}, q; \mu) = -\int_\Omega (\nabla \cdot \mathbf{v}) q \, d\Omega;$$

the trilinear form, $c(\cdot,\cdot,\cdot;\mu)$ and the eddy diffusion term, $a_t(\cdot;\cdot,\cdot;\mu)$, are given by

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}; \mu) = \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, d\Omega,$$

$$a_t(\mathbf{u}; \mathbf{v}, \mathbf{w}; \mu) = \int_\Omega \nu_t(\mathbf{u}) \nabla \mathbf{v} : \nabla \mathbf{w} \, d\Omega.$$

Also, the linear form $F(\cdot;\mu)$ is defined by

$$F(\mathbf{v}; \mu) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega - a(\mathbf{u}_D, \mathbf{v}; \mu) - c(\mathbf{u}_D, \mathbf{u}_D, \mathbf{v}; \mu).$$

Lastly, let us define the norms for the velocity and pressure spaces, considering the natural norm for each case. For the velocity space Y and, by extension, for Y_h, we define the inner product as follows

$$(\mathbf{u},\mathbf{v})_T = \int_\Omega \left[\frac{1}{\tilde{\mu}} + \tilde{\nu}_t\right] \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega,\ \forall \mathbf{u},\mathbf{v} \in Y \tag{1.11}$$

where $\tilde{\nu}_t = \nu_t(\mathbf{u}_h(\tilde{\mu}) + \mathbf{u}_D)$ and

$$\tilde{\mu} = \arg\max_{\mu\in\mathcal{D}} \left\{ \sum_{K\in\mathcal{T}_h} (C_S h_K)^2 \min_{\mathbf{x}\in K} |\nabla\mathbf{u}_h(\mu) + \nabla\mathbf{u}_D|(x) \right\}.$$

This inner product induces a "turbulence" norm $\|\cdot\|_T = (\cdot,\cdot)_T^{1/2}$, which is equivalent to the $H_0^1(\Omega)$-norm. We use this norm since it takes into account the turbulence viscosity effects that governs the flow. For the pressure space Q and, by extension, for Q_h, we consider the usual L^2-norm. Finally, for the X and X_h space, we consider the norm

$$\|V\|_X = \sqrt{\|\mathbf{v}\|_T^2 + \|q\|_{L^2(\Omega)}^2}, \quad \forall V = (\mathbf{v},q) \in X. \tag{1.12}$$

Following the notation used in (1.5), we can state the discrete problem

$$\begin{cases} \text{Find } U_h(\mu) = (\mathbf{u}_h(\mu), p_h(\mu)) \in X_h \text{ such that} \\ A(U_h(\mu), V_h;\mu) = F(V_h;\mu),\ \forall V_h = (\mathbf{v}_h, q_h) \in X_h \end{cases} \tag{1.13}$$

where

$$A(U_h,V_h;\mu) = \frac{1}{\mu} A_0(U_h,V_h) + A_1(U_h,V_h) + A_2(U_h,V_h) + A_3(U_h,V_h) \tag{1.14}$$

and

$$\begin{aligned}
A_0(U,V) &= \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega, \\
A_1(U,V) &= \int_\Omega (\nabla\cdot\mathbf{u})q - (\nabla\cdot\mathbf{v})p\, d\Omega + \int_\Omega (\mathbf{u}_D\cdot\nabla)\mathbf{u}\cdot\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{u}_D\cdot\mathbf{v}\, d\Omega, \\
A_2(U,V) &= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, d\Omega, \\
A_3(U,V) &= \int_\Omega \nu_t(\mathbf{u}+\mathbf{u}_D)\nabla(\mathbf{u}+\mathbf{u}_D) : \nabla\mathbf{v}\, d\Omega.
\end{aligned}$$

Note that now $A(\cdot,\cdot;\mu)$ is not bilinear, due to the convection and the eddy diffusion terms. Problem (1.13) is well posed, we will see it by means of the BRR theory of approximation of branches of regular solutions of non-linear problems in Section 1.4.2.

RB Smagorinsky model Now, we are able to define the RB Problem for the Smagorinsky model. We define X_N as the reduced space, then the RB problem is stated as

$$\begin{cases} \text{find } U_N(\mu) = (\mathbf{u}_N(\mu), p_N(\mu)) \in X_N \text{ such that} \\ A(U_N(\mu), V_N; \mu) = F(V_N; \mu), \ \forall V_N = (\mathbf{v}_N, q_N) \in X_N \end{cases} \tag{1.15}$$

The reduced space is $X_N = Y_N \times Q_N$, where Y_N and Q_N are the reduced spaces for velocity and pressure, respectively.

The inf-sup condition is necessary to the well-posedness of problem (1.15). Since Y_N is formed by solutions of (1.13) and these are nearly divergence-free, the pair velocity-pressure (Y_N, Q_N) is not inf-sup stable. In [11], it is proposed the use of an inner pressure *supremizer* operator $T_N^\mu : Q_h \longrightarrow Y_h$ defined as follows

$$(T_N^\mu p_h, \mathbf{v}_h)_T = b(\mathbf{v}_h, p_h; \mu), \ \forall \mathbf{v}_h \in Y_h, \tag{1.16}$$

to enrich the velocity space Y_N in a Greedy Algorithm (*cf.* [47]). This ensures that Y_N, enriched with the supremizer, besides Q_N are inf-sup stable.

The Greedy algorithm to construct the RB approximation of Smagorinsky model is described in Algorithm 6.

Algorithm 6 (Weak) Greedy for steady Smagorinsky model

Set $\varepsilon_{tol} > 0$, $N_{max} \in \mathbb{N}$ and $S_1 = \{\mu^1\}$;
Compute $U_h(\mu^1) = (\mathbf{u}_h(\mu^1), p_h(\mu^1))$;
$Q_1 = \text{span}\{\psi^1 := p_h(\mu^1)\}$;
$Y_1 = \text{span}\{\boldsymbol{\xi}^1 := \mathbf{u}_h(\mu^1), \boldsymbol{\xi}^2 := T_N^\mu \psi^1\}$;
for $N = 2 : N_{\max}$ **do**
  $\mu^N = \arg\max_{\mu \in \mathcal{D}_{train}} \Delta_{N-1}(\mu)$;
  $\varepsilon_{N-1} = \Delta_{N-1}(\boldsymbol{\mu}^N)$;
  if $\varepsilon_{N-1} \leq \varepsilon_{tol}$ **then**
    $N_{max} = N - 1$;
  end if
  Compute $U_h(\mu^N) = (\mathbf{u}_h(\mu^N), p_h(\mu^N))$;
  $S_N = S_{N-1} \cup \{\mu^N\}$;
  $Q_N = Q_{N-1} \cup \text{span}\{\psi^N := p_h(\mu^N)\}$;
  $Y_N = Y_{N-1} \cup \text{span}\{\boldsymbol{\xi}^{2N-1} := \mathbf{u}_h(\mu^N), \boldsymbol{\xi}^{2N} := T_N^\mu \psi^N\}$;
end for

Remark 1.2. *The use of an inner pressure supremizer T_N^μ affects the dimension of the reduced spaces Y_N and Q_N. In our case, using the Greedy Algorithm, the dimension of the velocity space is twice that of the pressure space.*

Finally, we are able to define the reduced parameter independent matrices and tensors that will be stored in the offline phase. Then, we define $\mathbb{A}_N, \mathbb{D}_N \in \mathscr{M}^{2N\times 2N}(\mathbb{R})$, $\mathbb{B}_N \in \mathscr{M}^{N\times 2N}(\mathbb{R})$, $\mathbb{C}_N \in \mathscr{M}^{2N\times 2N\times 2N}(\mathbb{R})$ defined by

$$\begin{array}{lll}
(\mathbb{A}_N)_{ij} & = \int_\Omega \nabla \boldsymbol{\xi}^j : \nabla \boldsymbol{\xi}^i \, d\Omega, & i,j = 1,\dots,2N, \\
(\mathbb{B}_N)_{ij} & = -\int_\Omega (\nabla \cdot \boldsymbol{\xi}^j)\psi^i \, d\Omega, & i = 1,\dots,N, \; j = 1,\dots,2N, \\
(\mathbb{C}_N)_{ijs} & = \int_\Omega (\boldsymbol{\xi}^s \cdot \nabla)\boldsymbol{\xi}^j \cdot \boldsymbol{\xi}^i \, d\Omega, & i,j,s = 1,\dots,2N, \\
(\mathbb{D}_N)_{ij} & = \int_\Omega (\mathbf{u}_D \cdot \nabla)\boldsymbol{\xi}^j \cdot \boldsymbol{\xi}^i + (\boldsymbol{\xi}^j \cdot \nabla)\mathbf{u}_D \cdot \boldsymbol{\xi}^i \, d\Omega, & i,j = 1,\dots,2N.
\end{array}$$

where $Y_N = \text{span}\{\boldsymbol{\xi}^i\}_{i=1}^{2N}$ and $Q_N = \text{span}\{\psi^i\}_{i=1}^{N}$.

Since the Smagorinsky term is nonlinear with respect to the parameter due to the eddy diffusion term, we use the Empirical Interpolation Method defined in Section 1.3 to approximate it. With this method, we get the RB space $\mathcal{G}_M = \text{span}\{q^m\}_{m=1}^M$ and we obtain the following linear decomposition,

$$\nu_t(\mathbf{u}) \approx \sum_{m=1}^{M} \sigma_j(\mu) q^m(\mathbf{x}). \tag{1.17}$$

With the computation of the coefficients $\{\sigma_j(\mu)\}_{j=1}^M$ for a given $\mu \in \mathcal{D}$, we are able to approximate the eddy diffusion term during the online phase. This yields the approximation

$$a_t(\mathbf{u};\mathbf{u},\mathbf{v};\boldsymbol{\mu}) \approx \sum_{m=1}^{M} \sigma_j(\mu) \int_\Omega q^m \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega.$$

Finally, we are able to define the matrix $\mathbb{E}_N^m \in \mathscr{M}^{2N\times 2N}(\mathbb{R})$ for $m = 1,\dots,M$ associated to the eddy diffusion term by

$$(\mathbb{E}_N^m)_{ij} = \int_\Omega q^m \nabla \boldsymbol{\xi}^j : \nabla \boldsymbol{\xi}^i \, d\Omega, \; i,j = 1,\dots,2N.$$

1.4.2 *A posteriori* error bound estimator

In this Section, we summarize Sections 4 and 5 in [11], where the *a posteriori* error bound estimator is derived. It is based on the Brezzi-Rappaz-Raviart (BRR) theory mentioned above (*cf.* [8]).

Thanks to the BRR theory, it is proved that in a neighbourhood of $U_N(\mu)$ solution of (1.15), there exists a unique $U_h(\mu)$ solution of (1.13), with

$$\|U_h(\mu) - U_N(\mu)\|_X \le \Delta_N(\mu), \; \forall \mu \in \mathcal{D}$$

where

$$\Delta_N(\mu) = \frac{\beta_N(\mu)}{2\rho_T(\mu)}\left[1-\sqrt{1-\tau_N(\mu)}\right],\ \tau_N(\mu) = \frac{4\varepsilon_N(\mu)\rho_T(\mu)}{\beta_N(\mu)^2}. \tag{1.18}$$

Here,

- $\beta_N(\mu)$ is the Stability Factor

$$\beta_N(\mu) \equiv \inf_{Z_h\in X_h}\sup_{V_h\in X_h}\frac{\partial_1 A(U_N(\boldsymbol{\mu}),V_h;\mu)(Z_h)}{\|Z_h\|_X\|V_h\|_X}, \tag{1.19}$$

  where the derivative operator of $A(\cdot,\cdot;\mu)$ follows the Definition A.1.

- ρ_T is the Lipschitz continuity of $\partial_1 A(U_N(\boldsymbol{\mu}),V_h;\mu)(Z_h)$ for all V_h, $Z_h \in X_h$ defined by

$$\rho_T = 2C_{4;T} + 4C_{3;2}C_S h_{\max}^{2-d/3}$$

  where $C_{4;T} > 0$ is the Sobolev embedding constant verifying that $\|\cdot\|_{L^4(\Omega)} \leq C_{4;T}\|\cdot\|_T$, $h_{\max} = \max_{K\in\mathcal{T}_h} h_K$ and $C_{3;2} > 0$ is defined in (A.11).

- $\varepsilon_N(\mu)$ is the residual defined by $\varepsilon_N(\mu) = \|\mathscr{R}(U_N(\mu);\mu)\|_{X'}$ is the residual where

$$\langle\mathscr{R}(Z_h;\mu),V_h\rangle = A(Z_h,V_h;\mu) - F(V_h;\mu),\ \forall Z_h,V_h\in X_h.$$

Note that (1.18) only applies when the dual norm of the residual $\varepsilon_N(\mu)$ is small enough to have $\tau(\mu)_N < 1$ for all $\mu \in \mathcal{D}$.

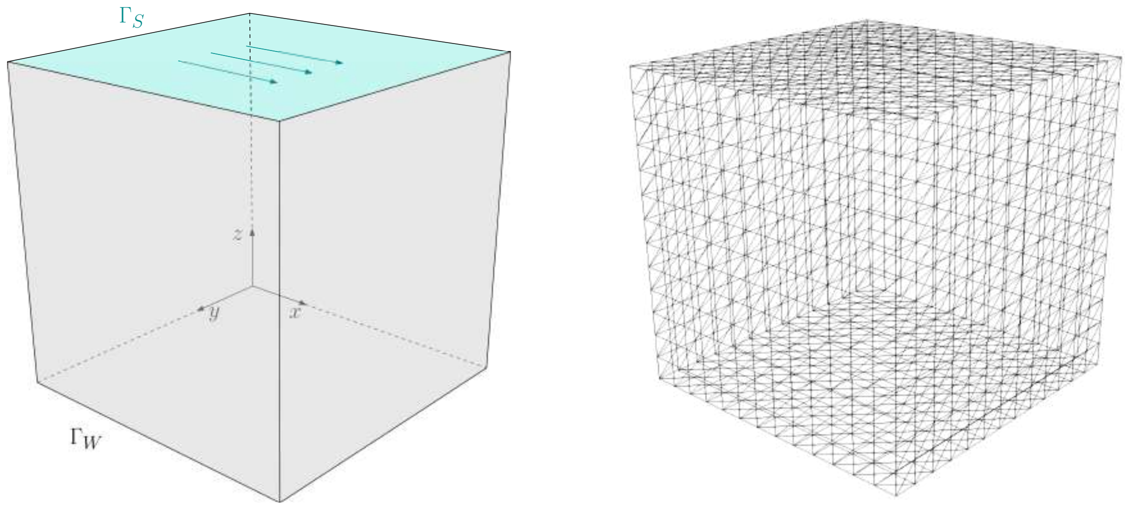
1.5 Application to 3D lid-driven cavity turbulent flow

In this Section, we extend Section 7.2 in [11] to the 3D case, actually to 3D lid-driven cavity flow, using the same software, FreeFem++ v. 4.8 (*cf.* [23]).

To solve the problem presented, we use a FE approximation with the Taylor-Hood finite element, i.e., we consider $\mathbb{P}2-\mathbb{P}1$ for velocity-pressure discretization that are inf-sup stable. Since the pressure is in $L^2(\Omega)$ with a mean of zero, we add a stabilization term for the pressure discretization in the variational formulation, this is, a L^2 penalization.

We consider a lid-driven cavity flow problem in the unit cube as we see in Figure 1.1a. We define a tetrahedral mesh in the cube, with 13 subdivisions for each edge, obtaining a mesh with 13 182 tetrahedral and 2 744 vertices (see Figure 1.1b). Furthermore, the grid element diameter is $h_K = 0.1332$ for each $K \in \mathcal{T}_h$.

The parameter set is an interval such that $\mathcal{D} = [\mathrm{Re}_{min},\mathrm{Re}_{max}]$ with $\mathrm{Re}_{min} = 1\,000$ and $\mathrm{Re}_{max} = 5\,000$. In this range, the flow is steady and laminar, therefore, this test is purely



(a) 3D domain. **(b)** Mesh.

Figure 1.1: Unit cube for the problem.

academic to verify the estimate (1.18). Recall that the eddy viscosity term adds diffusion to our problem and thanks to this, we are able to achieve a steady situation.

As we consider a lid-driven cavity problem, we only need to set Dirichlet BCs. We identify two parts on the boundary, the top Γ_S and the walls Γ_W as it is shown in Figure 1.1a. We impose that $\mathbf{w}_{|\Gamma_W} = \mathbf{0}$ and $\mathbf{w}_{|\Gamma_S} = (g(x,y),0)$ with $g(x,y) = 16x(x-1)y(y-1)$ for all $(x,y) \in \Gamma_S$. As we already mentioned, we need to split the velocity field $\mathbf{w}$ into $\mathbf{u}$ and a lift $\mathbf{u}_D$. For this case, we choose $\mathbf{u}_D$ to be the solution of Stokes problem.

Since the problem (1.10) is nonlinear, we use a semi-implicit linearized evolution approach,

$$\begin{cases} \text{Assuming that } \mathbf{u}_h^0 = \mathbf{0}, \text{ then for } k = 1,\dots,N_{FE}, \\ \text{find } (\mathbf{u}_h^k, p_h^k) = (\mathbf{u}_h^k(\boldsymbol{\mu}), p_h^k(\boldsymbol{\mu})) \in X_h \text{ such that} \\ \dfrac{1}{\Delta t}(\mathbf{u}_h^k, \mathbf{v}_h)_\Omega + a(\mathbf{u}_h^k, \mathbf{v}_h;\mu) + a_t(\mathbf{u}_h^{k-1} + \mathbf{u}_D; \mathbf{u}_h^k + \mathbf{u}_D, \mathbf{v}_h;\mu) \\ +b(\mathbf{v}_h, p_h^k;\mu) + c(\mathbf{u}_h^{k-1}, \mathbf{u}_h^k, \mathbf{v}_h;\mu) + c(\mathbf{u}_D, \mathbf{u}_h^k, \mathbf{v}_h;\mu) \\ +c(\mathbf{u}_h^k, \mathbf{u}_D, \mathbf{v}_h;\mu) = F(\mathbf{v}_h;\mu) + \dfrac{1}{\Delta t}(\mathbf{u}_h^{k-1}, \mathbf{v}_h)_\Omega, \qquad \forall \mathbf{v}_h \in Y_h, \\ b(\mathbf{u}_h^k, q_h;\mu) = 0, \qquad \forall q_h \in Q_h. \end{cases} \tag{1.20}$$

We stop the process when a nearly steady solution has been reached, this is, when for $k = 1,\dots,N_{FE}$ we have that $\|(\mathbf{u}_h^k - \mathbf{u}_h^{k-1})/\Delta t\|_{L^2(\Omega)} < \varepsilon_{FE}$ for $\varepsilon_{FE} = 10^{-11}$ and $\Delta t = 1$. The selection to ε_{FE} affects directly to the residual $\varepsilon_N(\mu)$ for all $\mu \in \mathcal{D}$ in

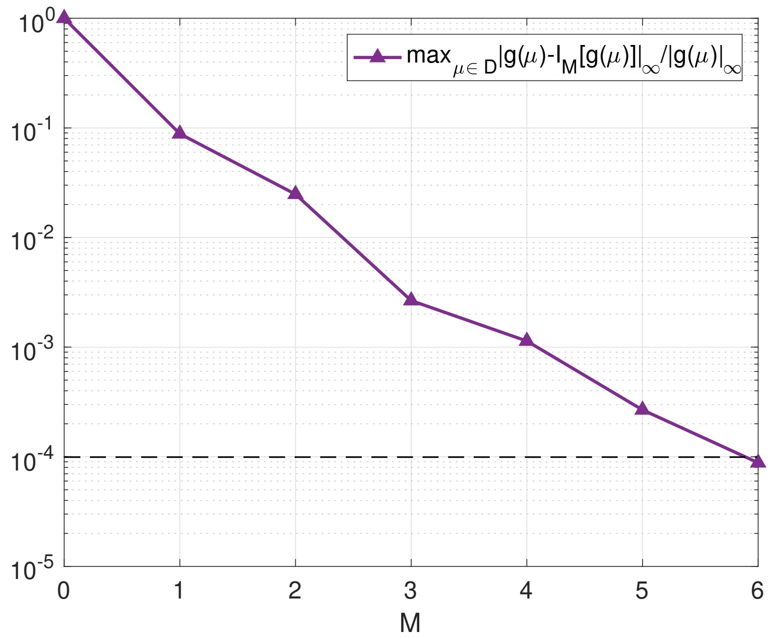


Figure 1.2: Convergence of the EIM algorithm.

the *a posteriori* error estimator computation, since $\varepsilon_h(\mu) = \|\mathscr{R}(U_h(\mu);\mu)\|_{X'} \sim \varepsilon_{FE}$. We obtain $U_h(\mu) = (\mathbf{u}_h(\mu), p_h(\mu)) \in X_h$ the steady solution which we will consider as the *high fidelity* solution.

We compute a reduced approximation of the eddy viscosity by the EIM Algorithm 5. For this purpose, we compute $U_h(\mu)$ the steady solution of (1.20) for $\mu \in \mathcal{D}_{EIM}$ where $\mathcal{D}_{EIM} = \{1\,000, 1\,250, \ldots, 5\,000\}$ with a step of 250.

For the EIM, we obtain the RB space $\mathcal{G} = \text{span}\{q_i\}_{i=1}^M$ along with the magic points $T_M = \{\mathbf{x}_i\}_{i=1}^M$ and $\mathbb{B}$ as we saw in Section 1.3, applied to $\nu_t(\mathbf{u}_h + \mathbf{u}_D)$. We obtain $M = 6$ bases for an error below $\varepsilon_{EIM} = 10^{-4}$, as we see in Figure 1.2. We also compute the error committed for $\mu \in \{1\,000, 1\,025, \ldots, 5\,000\}$ with a step of 25 in Figure 1.3.

Setup for the *a posteriori* error bound estimate For the computation of the *a posteriori* error bound $\Delta_N(\mu)$ defined in (1.18), we need to obtain the Stability Factor $\beta_N(\mu)$ and the Sobolev embedding constant $C_{4;T}$ that appears in the Lipschitz continuity constant ρ_T.

For the Stability Factor $\beta_N(\mu)$, we approximate this quantity by $\beta_h(\mu)$ since $U_N(\mu)$ is assumed to be a good approximation of $U_h(\mu)$. We consider the heuristic strategy introduced in [35]. First, we compute $\beta_h(\mu)$ for $\mu \in \mathcal{D}_{EIM}$ since we already computed

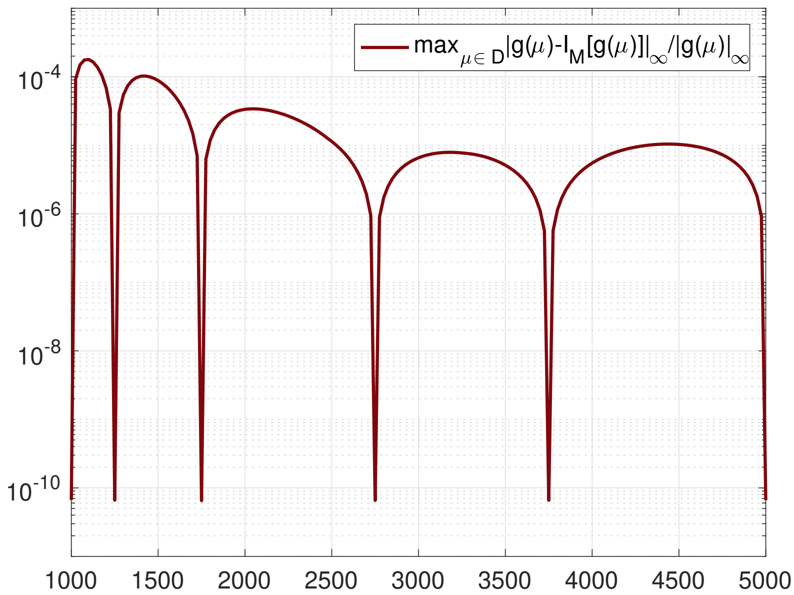


Figure 1.3: Committed EIM Error.

$U_h(\mu)$ for this parameter set. This is based on the computation of the smallest eigenvalue problem

$$\begin{cases} \text{Find } (\alpha, Z_h) = \in \mathbb{R} \times X_h,\ Z_h \neq 0, \text{ such that} \\ \mathbb{F}(\mu)^T \mathbb{X}^{-1} \mathbb{F}(\mu) \underline{Z}_h = \alpha \mathbb{X} \underline{Z}_h, \quad \forall Z_h \in X_h, \end{cases}$$

where the matrices $\mathbb{X}$ and $\mathbb{F}(\mu)$ are the matrices associated to the inner product related to the X-norm (1.12) and the tangent operator $\partial_1 A$, this is,

$$\begin{aligned} \underline{V}_h^T \mathbb{X} \underline{Z}_h &= (V_h, Z_h)_X, && \forall V_h, Z_h \in X_h, \\ \underline{V}_h^T \mathbb{F}(\mu) \underline{Z}_h &= \partial_1 A(U_h(\mu), V_h; \mu)(Z_h), && \forall V_h, Z_h \in X_h,\ \forall \mu \in \mathcal{D}, \end{aligned}$$

resulting that $\beta_h(\mu) = (\alpha_{\min})^{1/2}$.

Then, we apply the Radial Basis Function (RBF) algorithm to obtain an approximation of $\beta_h(\mu)$ for any $\mu \in \mathcal{D}$. In this case, we stop the algorithm when the RBF estimator is below $\varepsilon_\beta = 10^{-4}$. In the RBF, we have tested a total of 51 parameters and we have selected 22. This procedure is more detailed in Section 1.5 in [14], however, we describe the Stability Factor approximation for the unsteady case in Section 5.4.

For the computation of the Sobolev embedding constant $C_{4;T}$, we follow the fixed-point algorithm that can be found in [15] and [34].

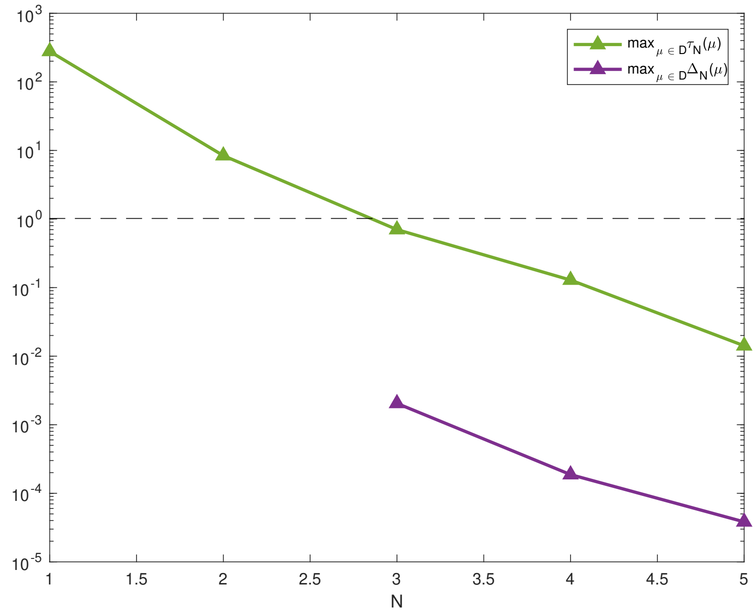


Figure 1.4: Greedy Algorithm convergence.

Greedy Algorithm Finally, we compute the RB space applying the Greedy Algorithm 6. We stop the algorithm when $\Delta_N(\mu) < \varepsilon_{RB}$ with $\varepsilon_{RB} = 10^{-4}$. The training set is $\mathcal{D}_{train} = \{1\,000, 1\,025, 1\,050, \ldots, 5\,000\}$ with $\dim(\mathcal{D}_{train}) = 161$.

In Figure 1.4, we can see the convergence history of the Greedy algorithm. Since for the two first iterations $\tau_N(\mu) \geq 1$, we use this quantity as the estimator as it is proportional to the dual norm of the residual.

In Figure 1.5 we observe the evolution of the estimator in each Greedy iteration. We can see the parameter selection clearly, first, we choose $\mu^1 = 1\,000$ as we fix it as the first parameter, then the algorithm chooses the opposite $\mu^2 = 5\,000$. With 3 basis functions, we obtain $\tau_N(\mu) < 1$ and we are able to compute $\Delta_N(\mu)$. Finally, the algorithm chooses $\mu^3 = 2\,350$, $\mu^4 = 1\,400$ and $\mu^5 = 3\,575$, and we stop the algorithm since we achieve convergence. Each peak in the figure corresponds to a selected parameter, included in the RB base.

Finally, we obtain 5 basis functions, resulting a linear system of dimension 15 in the online phase because of the addition of the inner pressure *supremizer* operator defined in (1.16) to enrich the velocity space.

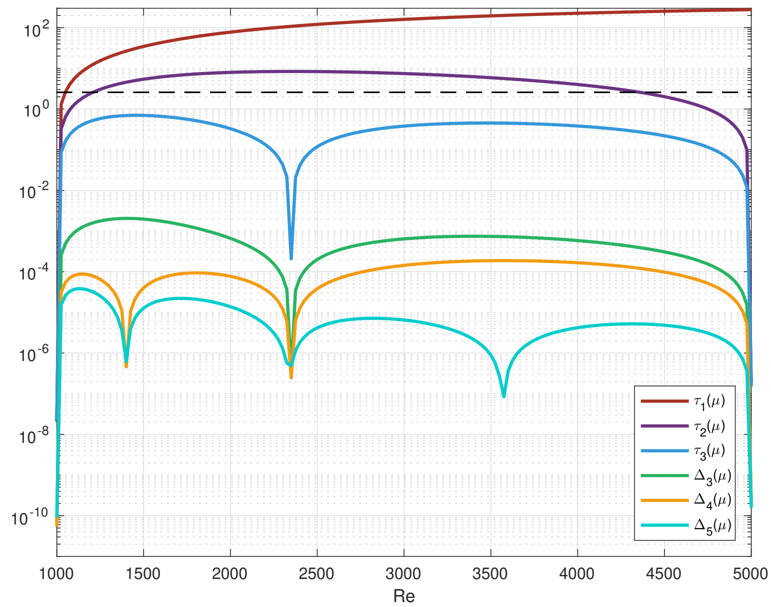


Figure 1.5: Error estimator in each Greedy iteration.

Online phase We randomly choose four Reynolds numbers to compare the computational time and the error between the FE and RB solution. We compute $U_h(\mu)$ and $U_N(\mu)$ on only one processor in a cluster with CPUs AMD EPYC 7542 2.9 GHz.

We summarize results in Table 1.1. The computation of the RB solution took not nearly 2 seconds, while the FE solution took around 112min. The speed-ups are close to 3 000.

Re	2 251	3 003	3 545	4 860
$\Delta_N(\mu)$	$3.06 \cdot 10^{-6}$	$6.47 \cdot 10^{-6}$	$3.25 \cdot 10^{-7}$	$1.89 \cdot 10^{-6}$
$\|\|U_h - U_N\|\|_X$	$4.65 \cdot 10^{-7}$	$8.71 \cdot 10^{-7}$	$4.75 \cdot 10^{-8}$	$2.34 \cdot 10^{-7}$
Efficiency	6.58	7.43	6.84	8.08
T_{FE}	99.69 min	109.14 min	112.81 min	119.42 min
T_{RB}	2.07 s	2.13 s	2.22 s	2.23 s
speedup	2 890	3 071	3 046	3 220

Table 1.1: Errors and speedups.

The efficiency of the *a posteriori* error bound is nearly 10 which is rather large although it is common for non-linear problems.

In figures 1.6-1.7 we show velocity and pressure for the FE and RB problem. Finally, we compute the FE and RB solution for $\mu \in \mathcal{D}_{train}$, and we compute the estimate and the error between the solutions. In Figure 1.8, we validate that the *a posteriori* error bound

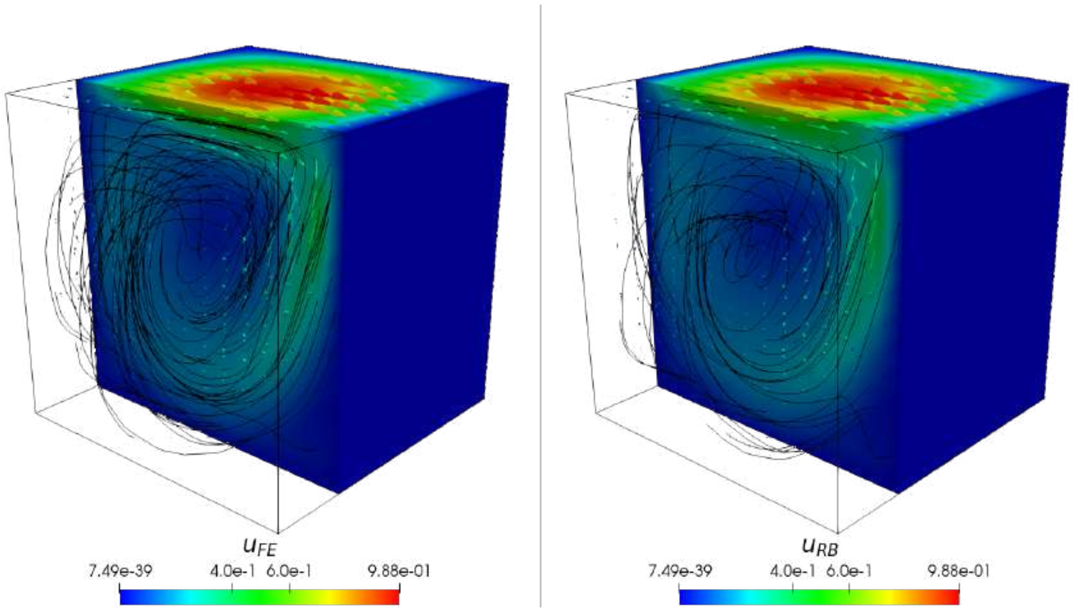


Figure 1.6: Velocity field for Re=3 545.

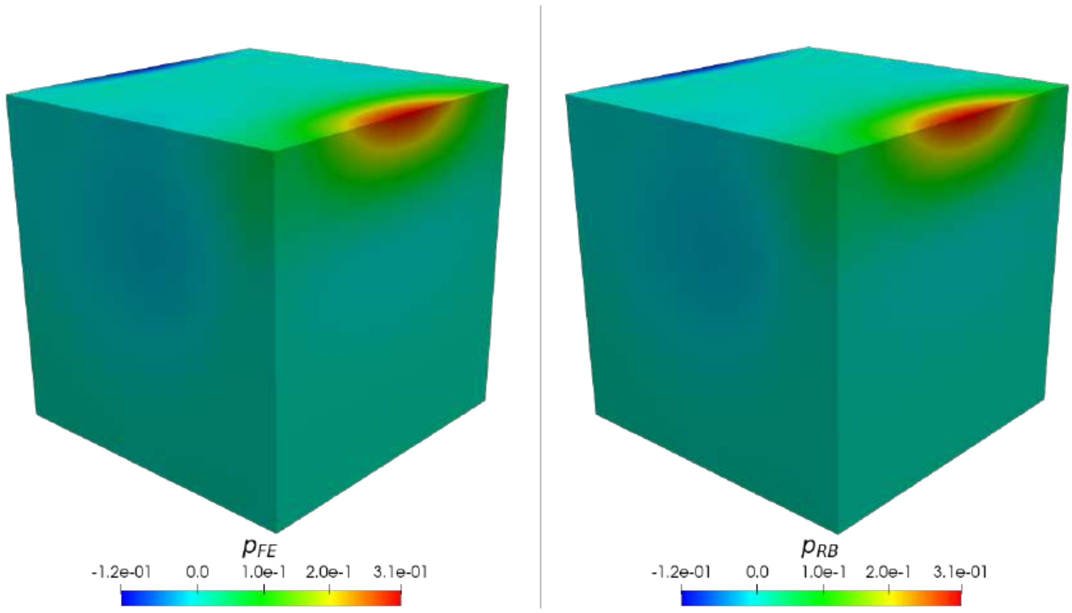


Figure 1.7: Pressure for Re=3 545.

estimator $\Delta_N(\mu)$ is greater than the error between the FE and RB solutions, furthermore, $\max_\mu \|U_h(\mu) - U_N(\mu)\|_X = 5.3031 \cdot 10^{-6}$.

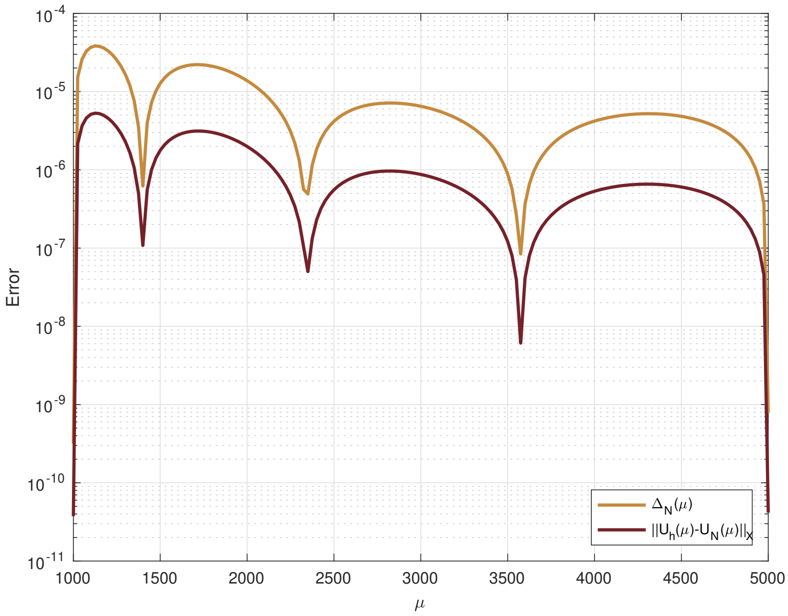


Figure 1.8: Validation of the *a posteriori* error bound estimator.

Part

Multiparameter Reduced Basis steady Smagorinsky model

2

Reduced Basis steady Smagorinsky model with variable geometry

In this chapter, we build a RB model for the air flow within a cloister, as a problem of practical interest of a turbulent flow with multi-parametric geometrical dependence. We consider a steady turbulent flow governed by the 2D Smagorinsky model. The geometrical parameters are the width and the height of the corridors around the cloister.

In Section 2.1 we define the problem establishing the domain, the geometrical parameters, and the procedure to follow for applying the RB Method. In Section 2.2 we define the RB problem explaining how the RB matrices are built. Finally, in Section 2.3 we deduce an *a posteriori* error estimator necessary to build the RB space in the Greedy Algorithm. At the end of the chapter, in Section 2.4, we show the numerical results for a specific case, giving keys for coding and validating the estimator developed in Section 2.3.

2.1 Problem statement

We consider a 2D modeling of the air flow within a cloister. Although this will avoid 3D effects in the aerothermal flow, this still is an adequate approach to determine its optimal dimensions regarding thermal comfort. Indeed, thermal inversions and strong 3D effects will only be produced by strong winds that are assumed to occur only occasionally. We thus consider thermally stratified flows through a 2D modeling.

We consider the 2D steady Smagorinsky model that allows us to solve the largest scales considering a coarse mesh. This model was introduced in Section 1.4. However, we will derive the dimensionless model as we depart from a real situation.

The flow domain is described in Figure 2.1 which can be considered as a vertical section of the real domain (a cloister), where ω and σ are the width and the height of the corridor around the cloister, respectively, and W and H are the total width and height. The parameters for this case are $\boldsymbol{\mu} = (\omega, \sigma)$ while W and H are fixed. We denote by $\Omega(\boldsymbol{\mu}) \in \mathbb{R}^2$ the domain for each $\boldsymbol{\mu} \in \mathcal{D}$ with $\mathcal{D}$ the parameter set.

We take $\mathcal{D} = [\omega_{\min}, \omega_{\max}] \times [\sigma_{\min}, \sigma_{\max}]$ where $\omega_{\min}$, $\omega_{\max}$, $\sigma_{\min}$ and $\sigma_{\max}$ are the extreme values of corridors width and height that make sense for architectural design. In the cases $\sigma = 0$ and $\sigma = H$, there would be no cloister, it would be a question of studying a courtyard, then $[\sigma_{\min}, \sigma_{\max}] \subset (0, H)$. For ω, we consider only up to $W/2$ due to symmetry and we exclude the extremes for the same reason as for the height, then $[\omega_{\min}, \omega_{\max}] \subset (0, W/2)$.

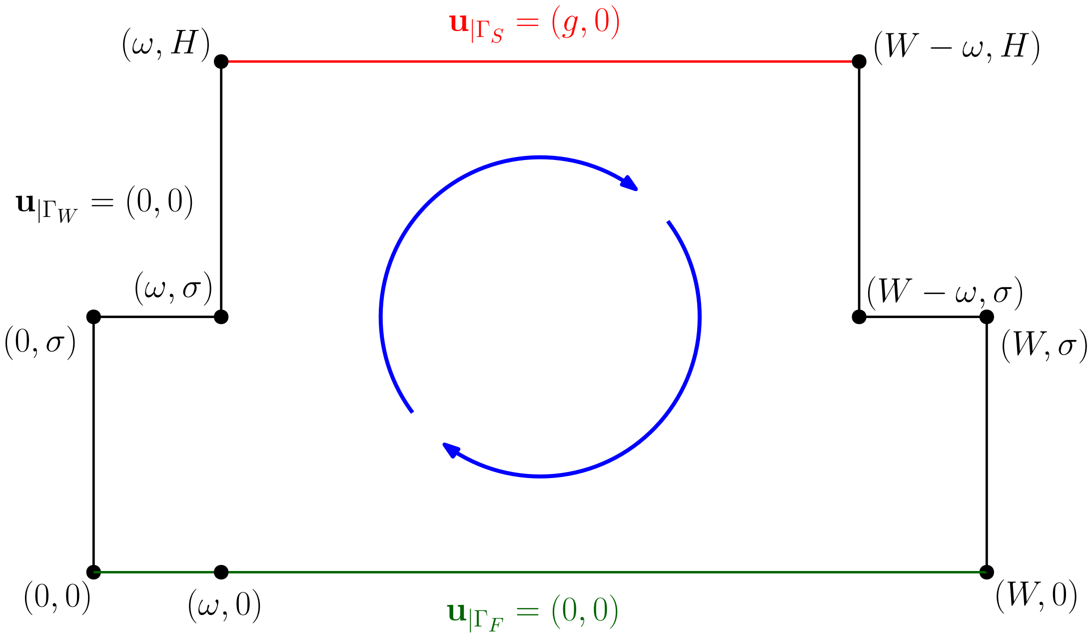


Figure 2.1: Domain and diagram of the problem.

Let $\Gamma(\boldsymbol{\mu}) = \partial\overline{\Omega}(\boldsymbol{\mu})$ be the domain boundary that splits into three parts:

- Open sky boundary: $\Gamma_S(\boldsymbol{\mu})$,
- Floor boundary: $\Gamma_F(\boldsymbol{\mu})$,
- Lateral walls boundary: $\Gamma_W(\boldsymbol{\mu})$.

The flow within the cloister is induced through boundary Γ_S. This setup leads us to a lid-driven cavity turbulent flow problem. We impose homogeneous Dirichlet boundary conditions (BCs) on the velocity on the floor $\Gamma_F(\boldsymbol{\mu})$ and the walls $\Gamma_W(\boldsymbol{\mu})$. On the top we assume an horizontal wind, given by the function $g(x_1) \geq 0$ with $x_1 \in \Gamma_S$.

We set $\{\mathcal{T}_h\}_{h>0}$ as a family of triangulations of $\overline{\Omega}$. In this way, the problem to solve is: Find $\mathbf{w} : \Omega(\boldsymbol{\mu}) \longrightarrow \mathbb{R}^2$ and $p : \Omega(\boldsymbol{\mu}) \longrightarrow \mathbb{R}$ such that

$$\begin{cases} -\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla\cdot(\nu_t(\mathbf{w};\boldsymbol{\mu})\nabla\mathbf{w}) + \nabla p = \mathbf{0}, & \text{in } \Omega(\boldsymbol{\mu}), \\ \nabla\cdot\mathbf{w} = 0, & \text{in } \Omega(\boldsymbol{\mu}), \\ \mathbf{w} = (g(\cdot;\boldsymbol{\mu}),0), & \text{on } \Gamma_S(\boldsymbol{\mu}), \\ \mathbf{w} = \mathbf{0}, & \text{on } \Gamma_W(\boldsymbol{\mu})\cup\Gamma_F(\boldsymbol{\mu}), \end{cases} \tag{2.1}$$

where $\mathbf{w}$ is the velocity field, p is the pressure per mass density, ν is the kinematic viscosity for the air and the turbulent viscosity is given by

$$\nu_t(\mathbf{w};\boldsymbol{\mu}) = \sum_{K\in\mathcal{T}_h} (C_S h_K(\boldsymbol{\mu}))^2 |\nabla\mathbf{w}(\boldsymbol{\mu})_{|K}|,$$

where we recall that h_K is the diameter of the element $K \in \mathcal{T}_h$ and C_S denotes the Smagorinsky constant.

As it is usual, we consider the dimensionless model. In this way, we use a single model for flows within similar domains, and deal with computer quantities close to one to avoid round-off errors. We set U_0 and L_0 as the characteristics velocity and length, respectively, then $\mathbf{w} = U_0\mathbf{w}^*$, $p = \rho U_0^2 p^*$, $\boldsymbol{\mu} = L_0\boldsymbol{\mu}^*$, where $\mathbf{w}^*$ and p^* represent the dimensionless velocity and pressure, $\boldsymbol{\mu}^*$ the dimensionless length parameter, ρ is the mass density of the air (constant in this case), $g = U_0 g^*$, $h_K = L_0 h_K^*$. Henceforth, $(\mathbf{w}^*, p^*)$ satisfy the problem

$$\begin{cases} -\dfrac{1}{\mathrm{Re}}\Delta^*\mathbf{w}^* + (\mathbf{w}^*\cdot\nabla^*)\mathbf{w}^* - \nabla^*\cdot(\nu_t^*(\mathbf{w}^*;\boldsymbol{\mu}^*)\nabla^*\mathbf{w}^*) + \nabla^* p^* = \mathbf{0}, & \text{in } \Omega(\boldsymbol{\mu}^*), \\ \nabla^*\cdot\mathbf{w}^* = 0, & \text{in } \Omega(\boldsymbol{\mu}^*), \\ \mathbf{w}^* = (g^*(\cdot;\boldsymbol{\mu}^*),0), & \text{on } \Gamma_S(\boldsymbol{\mu}^*), \\ \mathbf{w}^* = \mathbf{0}, & \text{on } \Gamma_{W,F}(\boldsymbol{\mu}^*), \end{cases}$$

where $\mathrm{Re} = U_0 L_0/\nu$ is the Reynolds number defined in (A.22), fixed for this problem, and

$$\nu_t^*(\mathbf{w}^*;\boldsymbol{\mu}^*) = \sum_{K\in\mathcal{T}_h} (C_S h_K^*(\boldsymbol{\mu}^*))^2 |\nabla^*\mathbf{w}^*(\boldsymbol{\mu}^*)|. \tag{2.2}$$

The operator ∇^* and Δ^* are defined respect to the dimensionless variable $\mathbf{x}^* = \mathbf{x}/L_0$. From now on, we avoid the *star* notation for simplicity.

In order to apply the RB Method, we need a model with homogeneous Dirichlet BCs. To solve this, we decompose the velocity field as the sum of two fields, $\mathbf{u}$ the solution of a problem with homogeneous Dirichlet BCs and $\mathbf{u}_D$ a function defined in all $\Omega(\boldsymbol{\mu})$ sufficiently regular and that satisfies non-homogeneous Dirichlet BCs, this is,

- $\mathbf{u}_D = (g(\cdot;\boldsymbol{\mu}),0)$ on $\Gamma_S(\boldsymbol{\mu})$,
- $\mathbf{u}_D = \mathbf{0}$ on $\Gamma_W(\boldsymbol{\mu}) \cup \Gamma_F(\boldsymbol{\mu})$.

This function is known as the boundary lifting function or simply a lift of the conditions. Then, the problem leads

For a known $\mathbf{u}_D$, find $(\mathbf{u},p)$ such that

$$\begin{cases} -\dfrac{1}{\mathrm{Re}}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}_D\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_D + \nabla p & \\ -\nabla\cdot(\nu_t(\mathbf{u}+\mathbf{u}_D;\boldsymbol{\mu})\nabla(\mathbf{u}+\mathbf{u}_D)) = \dfrac{1}{\mathrm{Re}}\Delta\mathbf{u}_D - (\mathbf{u}_D\cdot\nabla)\mathbf{u}_D, & \text{in } \Omega(\boldsymbol{\mu}), \\ \nabla\cdot\mathbf{u} = -\nabla\cdot\mathbf{u}_D, & \text{in } \Omega(\boldsymbol{\mu}), \\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma(\boldsymbol{\mu}). \end{cases} \tag{2.3}$$

The total solution $\mathbf{w} = \mathbf{u} + \mathbf{u}_D$ is completely independent of the lift choice. This $\mathbf{u}_D$ can be determined by an analytical expression or by the solution of a simpler problem which preserves the nonhomogeneous BCs, for example, the Stokes problem.

We are interested in looking at weak solutions to establish an approximation, and therefore we start by defining the variational formulation for this problem.

Let us consider the spaces $Y(\boldsymbol{\mu}) = H_0^1(\Omega(\boldsymbol{\mu}))$, $Q(\boldsymbol{\mu}) = L_0^2(\Omega(\boldsymbol{\mu}))$, and $X(\boldsymbol{\mu}) = Y(\boldsymbol{\mu}) \times Q(\boldsymbol{\mu})$. We can define the weak formulation of the problem (2.3), doing the standard procedure:

$$\begin{cases} \text{Find } (\mathbf{u},p) = (\mathbf{u}(\boldsymbol{\mu}),p(\boldsymbol{\mu})) \in X(\boldsymbol{\mu}) \text{ such that} & \\ a(\mathbf{u},\mathbf{v};\boldsymbol{\mu}) + b(\mathbf{v},p;\boldsymbol{\mu}) + c(\mathbf{u},\mathbf{u},\mathbf{v};\boldsymbol{\mu}) & \\ +c(\mathbf{u}_D,\mathbf{u},\mathbf{v};\boldsymbol{\mu}) + c(\mathbf{u},\mathbf{u}_D,\mathbf{v};\boldsymbol{\mu}) & \\ +a_t(\mathbf{u}+\mathbf{u}_D;\mathbf{u}+\mathbf{u}_D,\mathbf{v};\boldsymbol{\mu}) & \\ = -a(\mathbf{u}_D,\mathbf{v};\boldsymbol{\mu}) - c(\mathbf{u}_D,\mathbf{u}_D,\mathbf{v};\boldsymbol{\mu}), & \forall \mathbf{v} \in Y(\boldsymbol{\mu}), \\ b(\mathbf{u},q;\boldsymbol{\mu}) = -b(\mathbf{u}_D,q;\boldsymbol{\mu}), & \forall q \in Q(\boldsymbol{\mu}), \end{cases} \tag{2.4}$$

where the bilinear forms $a(\cdot,\cdot;\boldsymbol{\mu})$ and $b(\cdot,\cdot;\boldsymbol{\mu})$ are given by

$$a(\mathbf{u},\mathbf{v};\boldsymbol{\mu}) = \frac{1}{\mathrm{Re}}\int_{\Omega(\boldsymbol{\mu})} \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega,\ b(\mathbf{v},q;\boldsymbol{\mu}) = -\int_{\Omega(\boldsymbol{\mu})} (\nabla\cdot\mathbf{v})q\, d\Omega;$$

the trilinear form, $c(\cdot,\cdot,\cdot;\boldsymbol{\mu})$ and the eddy viscosity term, $a_t(\cdot;\cdot,\cdot;\boldsymbol{\mu})$, are given by

$$c(\mathbf{u},\mathbf{v},\mathbf{z};\boldsymbol{\mu}) = \int_{\Omega(\boldsymbol{\mu})} (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{z}\,d\Omega,$$

$$a_t(\mathbf{u};\mathbf{v},\mathbf{z};\boldsymbol{\mu}) = \int_{\Omega(\boldsymbol{\mu})} \nu_t(\mathbf{u};\boldsymbol{\mu})\nabla\mathbf{v} : \nabla\mathbf{z}\,d\Omega.$$

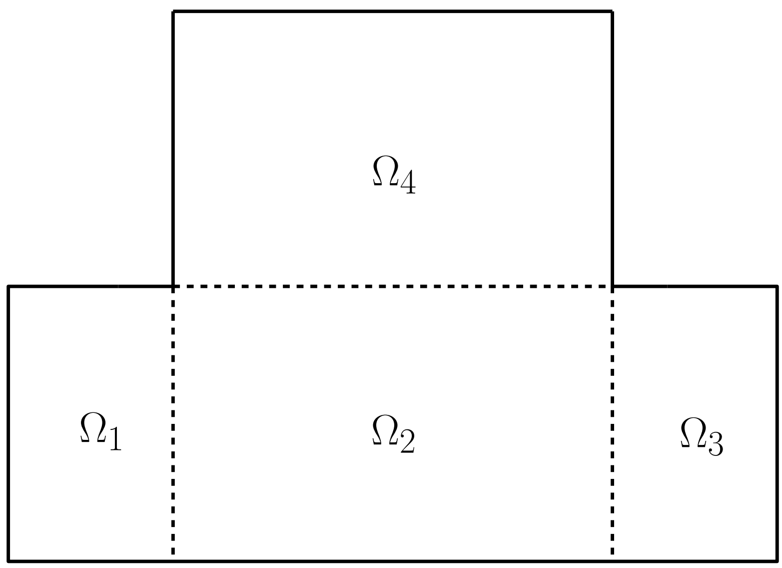


Figure 2.2: Partitioned Geometry.

These forms depend on the geometrical parameters that determine the domain. To apply the RB Method, we need to express these forms as linear expressions with respect to the parameters. With this purpose, we consider a reference domain, on which we build the variational formulation through a change of variables.

We set $\boldsymbol{\mu}_r = (\omega_r, \sigma_r) \in \mathcal{D}$ a couple of reference parameters that define $\Omega_r = \Omega(\boldsymbol{\mu}_r)$ as the reference domain. Then, we consider a smooth mapping $\Phi(\cdot;\boldsymbol{\mu})$ that transforms the reference domain into the original domain for any value of the parameter $\boldsymbol{\mu}$. We have defined Φ as a piecewise linear mapping and because of that we divide the domain in four regions Ω_l for $l = 1,2,3,4$ as is shown in Figure 2.2 and we define a linear transformation for each one of them, this is, Φ_l for $l = 1,2,3,4$ such that $\Phi(\mathbf{x};\boldsymbol{\mu}) =$

$\Phi_l(\mathbf{x};\boldsymbol{\mu})$ if $\mathbf{x} \in \Omega_l$, as follows

$$
\begin{aligned}
\Phi_1(\mathbf{x};\boldsymbol{\mu}) &= \begin{pmatrix} \dfrac{\omega}{\omega_r} & 0 \\ 0 & \dfrac{\sigma}{\sigma_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\
\Phi_2(\mathbf{x};\boldsymbol{\mu}) &= \begin{pmatrix} \dfrac{W-2\omega}{W-2\omega_r} & 0 \\ 0 & \dfrac{\sigma}{\sigma_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \omega - \dfrac{W-2\omega}{W-2\omega_r} \\ 0 \end{pmatrix}, \\
\Phi_3(\mathbf{x};\boldsymbol{\mu}) &= \begin{pmatrix} \dfrac{\omega}{\omega_r} & 0 \\ 0 & \dfrac{\sigma}{\sigma_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} W\dfrac{\omega_r - \omega}{\omega_r} \\ 0 \end{pmatrix}, \\
\Phi_4(\mathbf{x};\boldsymbol{\mu}) &= \begin{pmatrix} \dfrac{W-2\omega}{W-2\omega_r} & 0 \\ 0 & \dfrac{H-\sigma}{H-\sigma_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \omega - \omega_r\dfrac{W-2\omega}{W-2\omega_r} \\ \sigma - \sigma_r\dfrac{H-\sigma}{H-\sigma_r} \end{pmatrix}.
\end{aligned}
\tag{2.5}
$$

Observe that the global mapping is Lipschitz-continuous as

$$\Phi_{1|\Gamma_{12}} = \Phi_{2|\Gamma_{12}},\ \Phi_{2|\Gamma_{23}} = \Phi_{3|\Gamma_{23}} \text{ and } \Phi_{2|\Gamma_{24}} = \Phi_{4|\Gamma_{24}},$$

where $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j$; and $\Phi_l \in \mathcal{C}^\infty(\Omega_l)$ for $l = 1,2,3,4$. The Jacobian determinant for each transformation is

$$|J_{\Phi_1}| = |J_{\Phi_3}| = \frac{\omega\sigma}{\omega_r\sigma_r},\ |J_{\Phi_2}| = \frac{(W-2\omega)\sigma}{(W-2\omega_r)\sigma_r},\ |J_{\Phi_4}| = \frac{(W-2\omega)(H-\sigma)}{(W-2\omega_r)(H-\sigma_r)}, \tag{2.6}$$

Associated with the Jacobian matrix for each transformation, we define three diagonal matrices $\eta^l = \eta^l(\boldsymbol{\mu}), \lambda^l = \lambda^l(\boldsymbol{\mu}), \varphi^l = \varphi^l(\boldsymbol{\mu}) \in \mathscr{M}^{2\times 2}(\mathbb{R})$ for $l = 1,2,3,4$ by

$$\eta^l = J_{\Phi_l}^{-T}|J_{\Phi_l}|,\ \lambda^l = J_{\Phi_l}^{-1}J_{\Phi_l}^{-T}|J_{\Phi_l}|,\ \varphi^l = J_{\Phi_l}^{-1}J_{\Phi_l}^{-T},$$

whose diagonal elements are:

$$\eta^1_{11} = \eta^2_{11} = \eta^3_{11} = \frac{\sigma}{\sigma_r},\ \eta^4_{11} = \frac{H-\sigma}{H-\sigma_r},\ \eta^1_{22} = \eta^3_{22} = \frac{\omega}{\omega_r},\ \eta^2_{22} = \eta^4_{22} = \frac{W-2\omega}{W-2\omega_r}, \tag{2.7}$$

$$\lambda^1_{11} = \lambda^3_{11} = \frac{\omega_r\sigma}{\omega\sigma_r},\ \lambda^2_{11} = \frac{(W-2\omega_r)\sigma}{(W-2\omega)\sigma_r},\ \lambda^4_{11} = \frac{(W-2\omega_r)(H-\sigma)}{(W-2\omega)(H-\sigma_r)}, \tag{2.8}$$

$$\lambda^1_{22} = \lambda^3_{22} = \frac{\omega\sigma_r}{\omega_r\sigma},\ \lambda^2_{22} = \frac{(W-2\omega)\sigma_r}{(W-2\omega_r)\sigma},\ \lambda^4_{22} = \frac{(W-2\omega)(H-\sigma_r)}{(W-2\omega_r)(H-\sigma)}, \tag{2.9}$$

and

$$\varphi_{11}^1 = \varphi_{11}^3 = \frac{\omega_r^2}{\omega^2},\ \varphi_{11}^2 = \varphi_{11}^4 = \frac{(W-2\omega_r)^2}{(W-2\omega)^2}, \tag{2.10}$$

$$\varphi_{22}^1 = \varphi_{22}^2 = \varphi_{22}^3 = \frac{\sigma_r^2}{\sigma^2},\ \varphi_{22}^4 = \frac{(H-\sigma_r)^2}{(H-\sigma)^2}. \tag{2.11}$$

These matrices are well defined since $\boldsymbol{\mu} \in \mathcal{D} \subset (0, W/2) \times (0, H)$.

Now, we can state the weak formulation (2.4) on Ω_r. Let us consider the spaces $Y_r = Y(\boldsymbol{\mu}_r)$, $Q_r = Q(\boldsymbol{\mu}_r)$, and $X_r = Y_r \times Q_r$. Let us define the norms related to the spaces Y_r and Q_r. For the velocity space Y_r, we consider the *turbulence* norm which is the natural norm associated to the problem, defined as

$$(\mathbf{v},\mathbf{z})_T = \int_{\Omega_r} \left[\frac{1}{\mathrm{Re}} + \tilde{\nu}_t\right] \nabla\mathbf{v} : \nabla\mathbf{z}\, d\Omega, \quad \forall \mathbf{v},\mathbf{z} \in Y_r \tag{2.12}$$

where $\tilde{\nu}_t = \nu_t(\mathbf{u}+\mathbf{u}_D;\tilde{\boldsymbol{\mu}})$ and

$$\tilde{\boldsymbol{\mu}} = \arg\min_{\boldsymbol{\mu}\in\mathcal{D}} \sum_{K\in\mathcal{T}_h} (C_S h_K(\boldsymbol{\mu}))^2 \min_{\mathbf{x}\in K} |\nabla(\mathbf{u}(\boldsymbol{\mu}) + \mathbf{u}_D)|(x).$$

The inner product $(\cdot,\cdot)_T$ is parameter independent, it is well defined and it induces a norm $\|\cdot\|_T = (\cdot,\cdot)_T^{1/2}$ in Y_r equivalent to the H_0^1-norm. For the pressure space Q_r, we will use the usual L^2-norm. Finally, we define the X-norm such as

$$\|V\|_X = \sqrt{\|\mathbf{v}\|_T^2 + \|q\|_{L^2(\Omega)}^2}, \quad \forall V = (\mathbf{v},q) \in X_r. \tag{2.13}$$

Then, the weak formulation reads:

$$\left\{\begin{array}{ll} \text{Find } (\mathbf{u},p) = (\mathbf{u}(\boldsymbol{\mu}),p(\boldsymbol{\mu})) \in X_r \text{ such that} & \\ \displaystyle\sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l (a_{kl}(\mathbf{u},\mathbf{v}) + a_{t,kl}(\mathbf{u}+\mathbf{u}_D;\mathbf{u}+\mathbf{u}_D,\mathbf{v};\boldsymbol{\mu})) & \\ +\eta_{kk}^l (b_{kl}(\mathbf{v},p) + c_{kl}(\mathbf{u},\mathbf{u},\mathbf{v}) + c_{kl}(\mathbf{u}_D,\mathbf{u},\mathbf{v}) + c_{kl}(\mathbf{u},\mathbf{u}_D,\mathbf{v})) & \\ \displaystyle= -\sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l a_{kl}(\mathbf{u}_D,\mathbf{v}) + \eta_{kk}^l c_{kl}(\mathbf{u}_D,\mathbf{u}_D,\mathbf{v}), & \forall \mathbf{v} \in Y_r, \\ \displaystyle\sum_{l=1}^{4}\sum_{k=1}^{2} \eta_{kk}^l b_{kl}(\mathbf{u},q) = -\sum_{l=1}^{4}\sum_{k=1}^{2} \eta_{kk}^l b_{kl}(\mathbf{u}_D,q), & \forall q \in Q_r. \end{array}\right. \tag{2.14}$$

Notice that the indices k and l are associated to the derivative direction and the different regions in Ω_r, respectively. Then, for $k = 1,2$ and $l = 1,2,3,4$ the forms associated to the problem (2.14) are defined by

$$a_{kl}(\mathbf{u},\mathbf{v}) = \int_{\Omega_l} \frac{1}{\mathrm{Re}} (\partial_k u_1 \partial_k v_1 + \partial_k u_2 \partial_k v_2)\, d\Omega, \tag{2.15}$$

$$b_{kl}(\mathbf{u},p) = -\int_{\Omega_l} \partial_k u_k p \, d\Omega, \tag{2.16}$$

$$c_{kl}(\mathbf{u},\mathbf{v},\mathbf{z}) = \int_{\Omega_l} (u_k \partial_k v_1 z_1 + u_k \partial_k v_2 z_2) \, d\Omega, \tag{2.17}$$

$$a_{t,kl}(\mathbf{u};\mathbf{v},\mathbf{z};\boldsymbol{\mu}) = \int_{\Omega_l} \nu_t^l(\mathbf{u};\boldsymbol{\mu})(\partial_k v_1 \partial_k z_1 + \partial_k v_2 \partial_k z_2) \, d\Omega, \tag{2.18}$$

where

$$\nu_t^l(\mathbf{u};\boldsymbol{\mu}) = \sum_{K\in\mathcal{T}_{h,l}} (C_S h_K(\boldsymbol{\mu}))^2 \sqrt{\varphi_{11}^l((\partial_1 u_1)^2 + (\partial_1 u_2)^2) + \varphi_{22}^l((\partial_2 u_1)^2 + (\partial_2 u_2)^2)} \tag{2.19}$$

being $\mathcal{T}_{h,l} = \mathcal{T}_h \cap \overline{\Omega}_l$ the mesh corresponding to each region.

Now, we can rewrite this problem in a compact notation as

$$\begin{cases} \text{Find } U(\boldsymbol{\mu}) = (\mathbf{u}(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in X_r \text{ such that} \\ A(U(\boldsymbol{\mu}),V;\boldsymbol{\mu}) = F(V;\boldsymbol{\mu}), \qquad \forall V \in X_r, \end{cases} \tag{2.20}$$

where $A(\cdot,\cdot;\boldsymbol{\mu}) : X_r \times X_r \longrightarrow \mathbb{R}$ is the operator defined by

$$A(U,V;\boldsymbol{\mu}) = A_0(U,V;\boldsymbol{\mu}) + A_1(U,V;\boldsymbol{\mu}) + A_2(U,V;\boldsymbol{\mu}) + A_3(U,V;\boldsymbol{\mu})$$

with

$$\begin{aligned} A_0(U,V;\boldsymbol{\mu}) &= \sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l a_{kl}(\mathbf{u},\mathbf{v}), \\ A_1(U,V;\boldsymbol{\mu}) &= \sum_{l=1}^{4}\sum_{k=1}^{2} \eta_{kk}^l (b_{kl}(\mathbf{v},p) - b_{kl}(\mathbf{u},q) + c_{kl}(\mathbf{u}_D,\mathbf{u},\mathbf{v}) + c_{kl}(\mathbf{u},\mathbf{u}_D,\mathbf{v})), \\ A_2(U,V;\boldsymbol{\mu}) &= \sum_{l=1}^{4}\sum_{k=1}^{2} \eta_{kk}^l c_{kl}(\mathbf{u},\mathbf{u},\mathbf{v}), \\ A_3(U,V;\boldsymbol{\mu}) &= \sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l a_{t,kl}(\mathbf{u}+\mathbf{u}_D;\mathbf{u}+\mathbf{u}_D,\mathbf{v};\boldsymbol{\mu}), \end{aligned} \tag{2.21}$$

while the right hand side is defined as

$$F(V;\boldsymbol{\mu}) = -\sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l a_{kl}(\mathbf{u}_D,\mathbf{v}) + \eta_{kk}^l (c_{kl}(\mathbf{u}_D,\mathbf{u}_D,\mathbf{v}) - b_{kl}(\mathbf{u}_D,q)).$$

Therefore, we are able to state the discrete formulation of (2.20). We consider a finite element discretization Y_h and Q_h inner approximations of Y_r and Q_r, respectively; and define $X_h = Y_h \times Q_h$. By extension, we endow Y_h, Q_h and X_h with the norms for Y_r, Q_r and X_r, respectively. We consider the following discretization of the problem (2.20):

$$\begin{cases} \text{Find } U_h(\boldsymbol{\mu}) = (\mathbf{u}_h(\boldsymbol{\mu}), p_h(\boldsymbol{\mu})) \in X_h \text{ such that} \\ A(U_h(\boldsymbol{\mu}), V_h; \boldsymbol{\mu}) = F(V_h; \boldsymbol{\mu}), \qquad \forall V_h \in X_h, \end{cases} \tag{2.22}$$

In the first part of Section 2.3, we will study the well-posedness of problem (2.22), applying the Brezzi-Rappaz-Raviart (BRR) theory (*cf.* [8]). Thanks to the BRR theory, we are able to deduce an *a posteriori* error bound estimator in the second part of Section 2.3.

2.2 Reduced Basis problem

Now, we are able to define the RB problem from the discrete formulation defined in (2.22).

First, we need to build the RB space for the RB problem as we described in Section 1.4.1. This space must be a subspace of X_h, then, the required basis is built from solutions of the problem (2.22) for a selected number of parameters. These solutions are nearly weakly divergence free, therefore, the RB problem does not meet the inf-sup condition (see Theorem A.8) necessary to obtain a unique reduced solution. We enrich the velocity reduced space by introducing an inner-pressure *supremizer* operator. This procedure was developed by G. Rozza and K. Veroy in [43] for the Stokes equations, and by E. Delgado et al. for the Smagorinsky model in [11] and we already introduced it for the 3D case in Section 1.4.

For any $\boldsymbol{\mu} \in \mathcal{D}$, let $T_N^{\boldsymbol{\mu}} : Q_h \longrightarrow Y_h$ be the inner-pressure *supremizer* defined as,

$$(T_N^{\boldsymbol{\mu}} p_h, \mathbf{v}_h)_T = \sum_{l=1}^{4} \sum_{k=1}^{2} \eta_{kk}^l b_{kl}(\mathbf{v}_h, p_h), \; \forall \mathbf{v}_h \in Y_h. \tag{2.23}$$

Let $X_N = Y_N \times Q_N \subset X_h$ be the RB space of dimension $N \in \mathbb{N}$ defined as

$$Y_N = \text{span}\{\boldsymbol{\xi}^{2i-1} = \mathbf{u}_h(\boldsymbol{\mu}^i), \; \boldsymbol{\xi}^{2i} = T_N^{\boldsymbol{\mu}} p_h(\boldsymbol{\mu}^i), \; i = 1, \dots, N\} \tag{2.24}$$

and

$$Q_N = \text{span}\{\psi^i = p_h(\boldsymbol{\mu}^i), \; i = 1, \dots, N\} \tag{2.25}$$

where we denote by $\{\boldsymbol{\mu}^i\}_{i=1}^N$ the set of selected parameters following the Greedy Algorithm 6 in Chapter 1.

Then, the Reduced Basis Problem is stated as

$$\begin{cases} \text{Find } U_N(\boldsymbol{\mu}) \in X_N \text{ such that} \\ A(U_N(\boldsymbol{\mu}), V_N; \boldsymbol{\mu}) = F(V_N; \boldsymbol{\mu}) \quad \forall V_N \in X_N. \end{cases} \tag{2.26}$$

The solution $U_N(\boldsymbol{\mu}) = (\mathbf{u}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N$ can be expressed as a linear combination of the selected basis, by

$$\mathbf{u}_N(\boldsymbol{\mu}) = \sum_{i=1}^{2N} (\underline{\mathbf{u}}_N)_i(\boldsymbol{\mu})\boldsymbol{\xi}^i, \;\; p_N(\boldsymbol{\mu}) = \sum_{i=1}^{N} (\underline{p}_N)_i(\boldsymbol{\mu})\psi^i$$

where $\underline{\mathbf{u}}_N \in \mathbb{R}^{2N}$ and $\underline{p}_N \in \mathbb{R}^N$ conform the solution of a reduced linear system which is built from parameter-independent matrices and tensors. To develop this system, we build the parameter-independent matrices $\mathbb{A}_N^{kl}, \mathbb{D}_N^{kl} \in \mathscr{M}^{2N\times 2N}(\mathbb{R})$, $\mathbb{B}_N^{kl} \in \mathscr{M}^{N\times 2N}(\mathbb{R})$, $\mathbb{C}_N^{kl} \in \mathscr{M}^{2N\times 2N\times 2N}(\mathbb{R})$ for $k = 1,2$ and $l = 1,2,3,4$ defined by

$$\begin{aligned}
(\mathbb{A}_N^{kl})_{ij} &= a_{kl}(\boldsymbol{\xi}^j, \boldsymbol{\xi}^i), && i,j = 1,\dots,2N,\\
(\mathbb{B}_N^{kl})_{ij} &= b_{kl}(\boldsymbol{\xi}^j, \psi^i), && i = 1,\dots,N,\; j = 1,\dots,2N,\\
(\mathbb{C}_N^{kl})_{ijs} &= c_{kl}(\boldsymbol{\xi}^s, \boldsymbol{\xi}^j, \boldsymbol{\xi}^i), && i,j,s = 1,\dots,2N,\\
(\mathbb{D}_N^{kl})_{ij} &= c_{kl}(\mathbf{u}_D, \boldsymbol{\xi}^j, \boldsymbol{\xi}^i) + c_{kl}(\boldsymbol{\xi}^j, \mathbf{u}_D, \boldsymbol{\xi}^i), && i,j = 1,\dots,2N.
\end{aligned}$$

Since the Smagorinsky term is non-linear with respect to the parameters due to the eddy diffusion term, we use the EIM to approximate it developed in Section 1.3. With this method, we compute an approximation RB space $W_M = \text{span}\{q^m\}_{m=1}^M$ selecting this base by a Greedy algorithm. The eddy viscosity is approximated on this space,

$$\nu_t(\mathbf{u};\boldsymbol{\mu}) \approx \sum_{m=1}^{M} \sigma_j(\boldsymbol{\mu}) q^m(\mathbf{x}). \tag{2.27}$$

With the computation of the coefficients $\{\sigma_j(\boldsymbol{\mu})\}_{j=1}^M$ by solving a linear system for a given $\boldsymbol{\mu} \in \mathcal{D}$, we are able to approximate the eddy diffusion term during the online phase. This procedure has been exposed in Algorithm 5.

Then, for any $k = 1,2$ and $l = 1,2,3,4$,

$$a_{t,kl}(\mathbf{u};\mathbf{u},\mathbf{v};\boldsymbol{\mu}) \approx \sum_{m=1}^{M} \sigma_j(\boldsymbol{\mu}) \int_{\Omega_l} q^m(\partial_k u_1 \partial_k v_1 + \partial_k u_2 \partial_k v_2)\, d\Omega.$$

Finally, we are able to define the matrix $\mathbb{E}_N^{klm} \in \mathscr{M}^{2N\times 2N}(\mathbb{R})$ for $k = 1,2$, $l = 1,2,3,4$ and $m = 1,\dots,M$ associated to the eddy viscosity term by

$$(\mathbb{E}_N^{klm})_{ij} = \int_{\Omega_l} q^m(\partial_k \xi_{j,1} \partial_k \xi_{i,1} + \partial_k \xi_{j,2} \partial_k \xi_{i,2})\, d\Omega,\; i,j = 0,\dots,2N-1.$$

Thanks to these matrices, we are able to build the reduced system for any parameter $\boldsymbol{\mu}$.

2.3 *A posteriori* error estimator

As we saw in Section 1.4 the Greedy algorithm uses the norm of the error between the *high fidelity* solution of (2.22) and the RB solution of (2.26) for each parameter $\boldsymbol{\mu} \in \mathcal{D}_{train}$, which implies the computation of a large number of *high fidelity* solution. This would not be very practical since this would largely increase the offline phase computational cost. To avoid this, we develop *a posteriori* error estimator to bound the exact error.

Since our problem is non-linear, we use the Brezzi-Rappaz-Raviart (BRR) theory (*cf.* [8]), that will help us to build an *a posteriori* error estimator for the Smagorinsky model.

First, we study the well-posedness of problem (2.22), studying the continuity and inf-sup stability of the Fréchet derivative of A, that we denote by $\partial_1 A$.

Secondly, we use the BRR theory to obtain an *a posteriori* error bound estimator. We prove that the tangent operator $\partial_1 A$ is Lipschitz continuous, and we obtain the estimator proving existence and uniqueness of a solution U_h of (2.22) close to a solution U_N of (2.26).

2.3.1 Well-posedness analysis

In this section, we analyse the well-posedness of problem (2.22) studying the directional derivative of the operator A defined in Definition A.1, in the sense that for any solution $U_h(\boldsymbol{\mu})$ of (2.22), there exist $\beta_0 > 0$ and $\gamma_0 \in \mathbb{R}$ such that for all $\boldsymbol{\mu} \in \mathcal{D}$,

$$\begin{aligned} 0 < \beta_0 < \beta_h(\boldsymbol{\mu}) &\equiv \inf_{Z_h \in X_h} \sup_{V_h \in X_h} \frac{\partial_1 A(U_h(\boldsymbol{\mu}), V_h; \boldsymbol{\mu})(Z_h)}{\|Z_h\|_X \|V_h\|_X}, \\ \infty > \gamma_0 < \gamma_h(\boldsymbol{\mu}) &\equiv \sup_{Z_h \in X_h} \sup_{V_h \in X_h} \frac{\partial_1 A(U_h(\boldsymbol{\mu}), V_h; \boldsymbol{\mu})(Z_h)}{\|Z_h\|_X \|V_h\|_X}. \end{aligned} \tag{2.28}$$

For this purpose, we prove that the directional derivative of the operator A is continuous and weakly coercive. In this section, we set $U = (\mathbf{u}, p)$, $V = (\mathbf{v}, q)$ and $Z = (\mathbf{z}, r)$. Moreover, for simplicity, we denote Ω, in stead of the reference domain Ω_r. From the definition of $A(\cdot,\cdot;\boldsymbol{\mu})$ in (2.21) it holds,

$$\partial_1 A_0(U,V;\boldsymbol{\mu})(Z) = A_0(Z,V)$$

$$\partial_1 A_1(U,V;\boldsymbol{\mu})(Z) = A_1(Z,V)$$

$$\partial_1 A_2(U,V;\boldsymbol{\mu})(Z) = \sum_{l=1}^{4} \sum_{k=1}^{2} \eta_{kk}^{l} (c_{kl}(\mathbf{u},\mathbf{z},\mathbf{v}) + c_{kl}(\mathbf{z},\mathbf{u},\mathbf{v}))$$

and

$$\partial_1 A_3(U,V;\boldsymbol{\mu})(Z) = \sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^{l} a_{t,kl}(\mathbf{w};\mathbf{z},\mathbf{v};\boldsymbol{\mu})$$
$$+\sum_{l=1}^{4}\sum_{K\in\mathcal{T}_{h,l}} \int_K (C_S h_K(\boldsymbol{\mu}))^2 \frac{\sum_{k=1}^{2}\lambda_{kk}^{l}(\nabla_k\mathbf{w}\cdot\nabla_k\mathbf{z})}{\sqrt{\sum_{k=1}^{2}\varphi_{kk}^{l}(\nabla_k\mathbf{w}\cdot\nabla_k\mathbf{w})}}\left(\sum_{k=1}^{2}\varphi_{kk}^{l}(\nabla_k\mathbf{w}\cdot\nabla_k\mathbf{v})\right)$$

where, for simplicity, we are denoting by $\mathbf{w} = \mathbf{u}+\mathbf{u}_D$ and $\nabla_k\mathbf{u} = [\partial_k u_1, \partial_k u_2]$ for $k = 1,2$.

Remark 2.1. *For the next results, we introduce the notation*

$$\alpha^* = \max_{l=1,2,3,4}\left(\max_{k=1,2}\alpha_{kk}^{l}\right) \text{ and } \hat{\alpha} = \min_{l=1,2,3,4}\left(\min_{k=1,2}\alpha_{kk}^{l}\right)$$

and similar notations for λ^l, η^l and φ^l for $l = 1,2,3,4$ defined in (2.7)-(2.11). *Moreover,*

$$h^*_{\max} = \max_{\boldsymbol{\mu}\in\mathcal{D}}\max_{K\in\mathcal{T}_h} h_K(\boldsymbol{\mu}).$$

Remark 2.2. *We also introduce the definition of some constants necessary for the next results that are derived from relevant results exposed in Appendix A.2:*

- *The norm related to the inner product* (2.12) *is equivalent to the $H^1_0(\Omega)$-norm, then, there exists a constant $C_{1,2;T} > 0$ such that*

$$\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \le C_{1,2;T}\|\boldsymbol{v}\|_T,\ \forall \boldsymbol{v}\in Y. \tag{2.29}$$

- *We recall the definition of the constant $C_{4;1,2} > 0$ from the application of the Sobolev embedding Theorem A.2 in* (A.9). *Moreover, because of the definition of the previous constant, there exists $C_{4;T} = C_{4;1,2}C_{1,2;T}$. To sum up,*

$$\|\boldsymbol{v}\|_{L^4(\Omega)} \le C_{4;1,2}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)},\ \|\boldsymbol{v}\|_{L^4(\Omega)} \le C_{4;T}\|\boldsymbol{v}\|_T,\ \forall\boldsymbol{v}\in Y. \tag{2.30}$$

- *For all $K\in\mathcal{T}_h$, we apply the Local Inverse Inequality Theorem A.3 as in* (A.11), *introducing the constant $C_{3;2} > 0$, this is,*

$$\|\nabla\boldsymbol{v}_h\|_{L^3(K)} \le C_{3;2}h_K^{-d/6}\|\nabla\boldsymbol{v}_h\|_{L^2(K)}, \quad \forall K\in\mathcal{T}_h, \forall\boldsymbol{v}_h\in Y_h. \tag{2.31}$$

We start by studying the continuity of the directional derivative $\partial_1 A(U_h(\boldsymbol{\mu}),\cdot;\boldsymbol{\mu})(\cdot)$ for a given $U_h(\boldsymbol{\mu})\in X_h$.

Proposition 2.1. *For any $\boldsymbol{\mu} \in \mathcal{D}$ and $U_h = U_h(\boldsymbol{\mu}) \in X_h$, it holds*

$$|\partial_1 A(U_h, V_h; \boldsymbol{\mu})(Z_h)| \leq \gamma_h^* \|Z_h\|_X \|V_h\|_X, \quad \forall Z_h, V_h \in X_h,$$

where $\gamma_h^* = \max_{\boldsymbol{\mu} \in \mathcal{D}} \gamma_h(\boldsymbol{\mu})$ and

$$\begin{aligned}\gamma_h(\boldsymbol{\mu}) = \lambda^* \frac{C_{1,2;T}^2}{Re} + \eta^* C_{1,2;T} + \eta^* C_{4;T} C_{1,2;T} (\|\boldsymbol{u}_h\|_{L^4(\Omega)} + \|\boldsymbol{u}_D\|_{L^4(\Omega)}) \\ + \eta^* C_{4;T}^2 (\|\nabla \boldsymbol{u}_h\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_D\|_{L^2(\Omega)}) \\ + \lambda^* \left(1 + \frac{\varphi^*}{\sqrt{\hat{\varphi}}}\right) C_S^2 (h_{\max}^*)^{2-d/3} C_{1,2;T}^2 C_{3;2}^2 \|\nabla \boldsymbol{u}_h + \nabla \boldsymbol{u}_D\|_{L^3(\Omega)}.\end{aligned}$$

where the dependency on $\boldsymbol{\mu}$ is produced by $\boldsymbol{u}_h$.

Proof. Since

$$|\partial_1 A(U_h, V_h; \boldsymbol{\mu})(Z_h)| \leq \sum_{i=0}^{3} |\partial_1 A_i(U_h, V_h; \boldsymbol{\mu})(Z_h)|$$

we study each term separately.

- First,

$$\begin{aligned}|\partial_1 A_0(U_h, V_h; \boldsymbol{\mu})(Z_h)| = |A_0(Z_h, V_h)| = \left| \sum_{l=1}^{4} \sum_{k=1}^{2} \lambda_{kk}^l a_{kl}(\mathbf{z}_h, \mathbf{v}_h) \right| \\ \leq \lambda^* \left| \frac{1}{\text{Re}} \int_\Omega \nabla \mathbf{z}_h : \nabla \mathbf{v}_h \, d\Omega \right|.\end{aligned}$$

  Then, we use the Cauchy-Schwarz Inequality A.1, and (2.29)

$$|\partial_1 A_0(U_h, V_h; \boldsymbol{\mu})(Z_h)| \leq \lambda^* \frac{1}{\text{Re}} \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|\nabla \mathbf{v}_h\|_{L^2(\Omega)} \leq \lambda^* \frac{C_{1,2;T}^2}{\text{Re}} \|\mathbf{z}_h\|_T \|\mathbf{v}_h\|_T$$

  Now, from the definition of the X-norm (2.13),

$$|\partial_1 A_0(U_h, V_h; \boldsymbol{\mu})(Z_h)| \leq \lambda^* \frac{C_{1,2;T}^2}{\text{Re}} \|Z_h\|_X \|V_h\|_X.$$

- Second,

$$\begin{aligned}&|\partial_1 A_1(U_h, V_h; \boldsymbol{\mu})(Z_h)| = |A_1(Z_h, V_h)| \\ &= \left| \sum_{l=1}^{4} \sum_{k=1}^{2} \eta_{kk}^l (b_{kl}(\mathbf{v}_h, r_h) - b_{kl}(\mathbf{z}_h, q_h) + c_{kl}(\mathbf{u}_D, \mathbf{z}_h, \mathbf{v}_h) + c_{kl}(\mathbf{z}_h, \mathbf{u}_D, \mathbf{v}_h)) \right| \\ &\leq \eta^* \left| \int_\Omega (\nabla \cdot \mathbf{v}_h) r_h - (\nabla \cdot \mathbf{z}_h) q_h + (\mathbf{u}_D \cdot \nabla) \mathbf{z}_h \cdot \mathbf{v}_h + (\mathbf{z}_h \cdot \nabla) \mathbf{u}_D \cdot \mathbf{v}_h \, d\Omega \right| \\ &\leq \eta^* (\|\nabla \mathbf{v}_h\|_{L^2(\Omega)} \|r_h\|_{L^2(\Omega)} + \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}) \\ &+ \eta^* (\|\mathbf{u}_D\|_{L^4(\Omega)} \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)} + \|\mathbf{z}_h\|_{L^4(\Omega)} \|\nabla \mathbf{u}_D\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)})\end{aligned}$$

Then, we use (2.29) and (2.30) and the definition of the X-norm (2.13),

$$
\begin{aligned}
|\partial_1 A_1(U_h,V_h;\boldsymbol{\mu})(Z_h)| &\leq \eta^* C_{1,2;T}(\|\mathbf{v}_h\|_T \|r_h\|_{L^2(\Omega)} + \|\mathbf{z}_h\|_T \|q_h\|_{L^2(\Omega)}) \\
&\quad + \eta^* C_{4;T}(C_{1,2;T}\|\mathbf{u}_D\|_{L^4(\Omega)} + C_{4;T}\|\nabla \mathbf{u}_D\|_{L^2(\Omega)})\|\mathbf{v}_h\|_T \|\mathbf{z}_h\|_T \\
&\leq \eta^*(C_{1,2;T} + C_{4;T}C)\|V_h\|_X \|Z_h\|_X,
\end{aligned}
$$

where $C = (C_{1,2;T}\|\mathbf{u}_D\|_{L^4(\Omega)} + C_{4;T}\|\nabla \mathbf{u}_D\|_{L^2(\Omega)})$.

- The following term,

$$
\begin{aligned}
|\partial_1 A_2(U_h,V_h;\boldsymbol{\mu})(Z_h)| &= \left| \sum_{l=1}^{4}\sum_{k=1}^{2} \eta_{kk}^l (c_{kl}(\mathbf{u}_h,\mathbf{z}_h,\mathbf{v}_h) + c_{kl}(\mathbf{z}_h,\mathbf{u}_h\mathbf{v}_h)) \right| \\
&\leq \eta^* \left| \int_\Omega (\mathbf{u}_h\cdot\nabla)\mathbf{z}_h\cdot\mathbf{v}_h + (\mathbf{z}_h\cdot\nabla)\mathbf{u}_h\cdot\mathbf{v}_h \, d\Omega \right| \\
&\leq \eta^* \Big(\|\mathbf{u}_h\|_{L^4(\Omega)} \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)} \\
&\quad + \|\mathbf{z}_h\|_{L^4(\Omega)} \|\nabla \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)}) \Big) \\
&\leq \eta^* \left(C_{1,2;T}C_{4;T}\|\mathbf{u}_h\|_{L^4(\Omega)} + C_{4;T}^2 \|\nabla \mathbf{u}_h\|_{L^2(\Omega)} \right) \|\mathbf{v}_h\|_T \|\mathbf{z}_h\|_T \\
&\leq \eta^* \left(C_{1,2;T}C_{4;T}\|\mathbf{u}_h\|_{L^4(\Omega)} + C_{4;T}^2 \|\nabla \mathbf{u}_h\|_{L^2(\Omega)} \right) \|V_h\|_X \|Z_h\|_X.
\end{aligned}
$$

- Finally,

$$
\begin{aligned}
|\partial_1 A_3(U_h,V_h,\boldsymbol{\mu})(Z_h)| &\leq \left| \sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l a_{t,kl}(\mathbf{w}_h;\mathbf{z}_h,\mathbf{v}_h;\boldsymbol{\mu}) \right| \\
&\quad + \left| \sum_{l=1}^{4} \sum_{K\in\mathcal{T}_{h,l}} \int_K (C_S h_K)^2 \frac{\sum_{k=1}^2 \lambda_{kk}^l (\nabla_k \mathbf{w}_h\cdot\nabla_k \mathbf{z}_h)}{\sqrt{\sum_{k=1}^2 \varphi_{kk}^l |\nabla_k \mathbf{w}_h|}} \left(\sum_{k=1}^{2} \varphi_{kk}^l (\nabla_k \mathbf{w}_h\cdot\nabla_k \mathbf{v}_h) \right) \right| \\
&\leq \lambda^* \sum_{K\in\mathcal{T}_h} \left| \int_K (C_S h_K(\boldsymbol{\mu}))^2 |\nabla \mathbf{w}_h| \nabla \mathbf{z}_h : \nabla \mathbf{v}_h \right| \\
&\quad + \frac{\lambda^*\varphi^*}{\sqrt{\hat{\varphi}}} \sum_{K\in\mathcal{T}_h} \left| \int_K (C_S h_K(\boldsymbol{\mu}))^2 \frac{\nabla \mathbf{w}_h : \nabla \mathbf{z}_h}{|\nabla \mathbf{w}_h|} (\nabla \mathbf{w}_h : \nabla \mathbf{v}_h) \right| \\
&\leq \lambda^* \left(1 + \frac{\varphi^*}{\sqrt{\hat{\varphi}}} \right) \sum_{K\in\mathcal{T}_h} \int_K (C_S h_K(\boldsymbol{\mu}))^2 |\nabla \mathbf{w}_h| |\nabla \mathbf{z}_h| |\nabla \mathbf{v}_h|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lambda^* \left(1+\frac{\varphi^*}{\sqrt{\hat{\varphi}}}\right) \sum_{K\in\mathcal{T}_h} (C_S h_K(\boldsymbol{\mu}))^2 \|\nabla \mathbf{w}_h\|_{L^3(K)} \|\nabla \mathbf{z}_h\|_{L^3(K)} \|\nabla \mathbf{v}_h\|_{L^3(K)} \\
&\leq \lambda^* \left(1+\frac{\varphi^*}{\sqrt{\hat{\varphi}}}\right) C_{3;2}^2 \sum_{K\in\mathcal{T}_h} C_S^2 h_K(\boldsymbol{\mu})^{2-d/3} \|\nabla \mathbf{w}_h\|_{L^3(K)} \|\nabla \mathbf{z}_h\|_{L^2(K)} \|\nabla \mathbf{v}_h\|_{L^2(K)} \\
&\leq \lambda^* \left(1+\frac{\varphi^*}{\sqrt{\hat{\varphi}}}\right) C_S^2 (h^*_{\max})^{2-d/3} C_{3;2}^2 \|\nabla \mathbf{w}_h\|_{L^3(\Omega)} \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|\nabla \mathbf{v}_h\|_{L^2(\Omega)} \\
&\leq \lambda^* \left(1+\frac{\varphi^*}{\sqrt{\hat{\varphi}}}\right) C_S^2 (h^*_{\max})^{2-d/3} C_{1,2;T}^2 C_{3;2}^2 \|\nabla \mathbf{w}_h\|_{L^3(\Omega)} \|\mathbf{z}_h\|_T \|\mathbf{v}_h\|_T \\
&\leq \lambda^* \left(1+\frac{\varphi^*}{\sqrt{\hat{\varphi}}}\right) C_S^2 (h^*_{\max})^{2-d/3} C_{1,2;T}^2 C_{3;2}^2 \|\nabla \mathbf{w}_h\|_{L^3(\Omega)} \|V_h\|_X \|V_h\|_X.
\end{aligned}
$$

where we have used (2.31) and that $\mathbf{w}_h = \mathbf{u}_h + \mathbf{u}_D$.

Consequently, the continuity constant is

$$
\begin{aligned}
\gamma_h(\boldsymbol{\mu}) = \lambda^* \frac{C_{1,2;T}^2}{\mathrm{Re}} &+ \eta^*(C_{1,2;T} + C_{4;T} C) \\
&+ \eta^* \left(C_{1,2;T} C_{4;T} \|\mathbf{u}_h\|_{L^4(\Omega)} + C_{4;T}^2 \|\nabla \mathbf{u}_h\|_{L^2(\Omega)} \right) \\
&\quad + \lambda^* \left(1+\frac{\varphi^*}{\sqrt{\hat{\varphi}}}\right) C_S^2 (h^*_{\max})^{2-d/3} C_{1,2;T}^2 C_{3;2}^2 \|\nabla \mathbf{u}_h + \nabla \mathbf{u}_D\|_{L^3(\Omega)}.
\end{aligned}
$$

where the dependency on $\boldsymbol{\mu}$ is produced by $\mathbf{u}_h$. □

Now, we study the coercivity of the derivative operator.

Proposition 2.2. *For any $\boldsymbol{\mu} \in \mathcal{D}$ and $U_h = U_h(\boldsymbol{\mu}) \in X_h$, let us suppose that $\|\nabla \boldsymbol{u}_D\|_{L^2(\Omega)} < 1/C$ and*

$$
\|\nabla \boldsymbol{u}_h\|_{L^2(\Omega)} \leq \frac{1}{C} - \|\nabla \boldsymbol{u}_D\|_{L^2(\Omega)},
$$

where the positive constant C is defined by

$$
C = \frac{\hat{\lambda} \min\{1, \sqrt{\hat{\varphi}}\}}{2\eta^* C_{1,2;T}^2 C_{4;1,2}^2}
$$

with $C_{1,2;T}$ and $C_{4;1,2}$ defined in (2.29)-(2.30)*. Then it holds,*

$$
\partial_1 A(U_h, Z_h; \boldsymbol{\mu})(Z_h) \geq \tilde{\beta}_h \|z_h\|_T^2, \quad \forall Z_h \in X_h,
$$

where

$$\tilde{\beta}_h = \hat{\lambda}\min\{1,\sqrt{\hat{\varphi}}\} - 2\eta^* C_{1,2;T}^2 C_{4;1,2}^2(\|\nabla \boldsymbol{u}_h\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_D\|_{L^2(\Omega)}) > 0.$$

Proof. By definition,

$$\partial_1 A(U_h, Z_h; \boldsymbol{\mu})(Z_h) = \sum_{i=0}^{3} \partial_1 A_i(U_h, Z_h; \boldsymbol{\mu})(Z_h)$$

and we study each term separately. We recall that forms A_i for $i = 0,1,2,3$ are defined in (2.21). Along the proof, we will use that $\mathbf{w}_h = \mathbf{u}_h + \mathbf{u}_D$ for simplicity.

- The derivative of the diffusion term is bounded as

$$\partial_1 A_0(U_h, Z_h; \boldsymbol{\mu})(Z_h) = \sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l a_{kl}(\mathbf{z}_h, \mathbf{z}_h) \geq \hat{\lambda}\frac{1}{\mathrm{Re}}\|\nabla \mathbf{z}_h\|_{L^2(\Omega)}^2.$$

  The eddy viscosity term is treated by

$$\begin{aligned}
\partial_1 A_3(U_h, Z_h; \boldsymbol{\mu})(Z_h) &= \sum_{l=1}^{4}\sum_{k=1}^{2} \lambda_{kk}^l a_{t,kl}(\mathbf{w}_h; \mathbf{z}_h, \mathbf{z}_h; \boldsymbol{\mu}) \\
&+ \sum_{l=1}^{4}\sum_{K\in\mathcal{T}_{h,l}} \int_K (C_S h_K)^2 \frac{\sum_{k=1}^{2} \lambda_{kk}^l \nabla_k \mathbf{w}_h \cdot \nabla_k \mathbf{z}_h}{\sqrt{\sum_{k=1}^{2} \varphi_{kk}^l |\nabla_k \mathbf{w}_h|^2}} \left(\sum_{k=1}^{2} \varphi_{kk}^l \nabla_k \mathbf{w}_h \cdot \nabla_k \mathbf{z}_h\right) \\
&\geq \hat{\lambda}\sqrt{\hat{\varphi}} \sum_{K\in\mathcal{T}_h} \int_K (C_S h_K)^2 |\nabla \mathbf{w}_h||\nabla \mathbf{z}_h|^2 + \frac{\hat{\lambda}\hat{\varphi}}{\sqrt{\varphi^*}} \sum_{K\in\mathcal{T}_h} \int_K (C_S h_K)^2 \frac{|\nabla \mathbf{w}_h : \nabla \mathbf{z}_h|^2}{|\nabla \mathbf{w}_h|}.
\end{aligned}$$

  Using that this last addend is positive and adding these two terms, we obtain

$$\begin{aligned}
&\partial_1 A_0(U_h, Z_h; \boldsymbol{\mu})(Z_h) + \partial_1 A_3(U_h, Z_h; \boldsymbol{\mu})(Z_h) \\
&\qquad \geq \hat{\lambda}\frac{1}{\mathrm{Re}}\int_\Omega |\nabla \mathbf{z}_h|^2\, d\Omega + \hat{\lambda}\sqrt{\hat{\varphi}}\int_\Omega (C_S h_K)^2 |\nabla \mathbf{w}_h||\nabla \mathbf{z}_h|^2\, d\Omega \\
&\qquad + \frac{\hat{\lambda}\hat{\varphi}}{\sqrt{\varphi^*}} \sum_{K\in\mathcal{T}_h} \int_K (C_S h_K)^2 \frac{|\nabla \mathbf{w}_h : \nabla \mathbf{z}_h|^2}{|\nabla \mathbf{w}_h|} \\
&\qquad \geq \hat{\lambda}\min\{1,\sqrt{\hat{\varphi}}\}\|\mathbf{z}_h\|_T^2.
\end{aligned} \tag{2.32}$$

- On the other hand, we bound from above the rest of the terms, such as

$$
\begin{aligned}
&|\partial_1 A_1(U_h, Z_h; \boldsymbol{\mu})(Z_h) + \partial_1 A_2(U_h, Z_h; \boldsymbol{\mu})(Z_h)| \\
&\quad = \left| \sum_{l=1}^{4} \sum_{k=1}^{2} \eta_{kk}^{l} (c_{kl}(\mathbf{w}_h, \mathbf{z}_h, \mathbf{z}_h) + c_{kl}(\mathbf{z}_h, \mathbf{w}_h, \mathbf{z}_h)) \right| \\
&\quad \leq \eta^* \left| \int_\Omega (\mathbf{w}_h \cdot \nabla)\mathbf{z}_h \cdot \mathbf{z}_h + (\mathbf{z}_h \cdot \nabla)\mathbf{w}_h \cdot \mathbf{z}_h \, d\Omega \right| \\
&\quad \leq \eta^* \int_\Omega |\mathbf{w}_h||\nabla \mathbf{z}_h||\mathbf{z}_h| + |\mathbf{z}_h|^2 |\nabla \mathbf{w}_h| \, d\Omega \\
&\quad \leq \eta^* \left(\|\mathbf{w}_h\|_{L^4(\Omega)} \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|\mathbf{z}_h\|_{L^4(\Omega)} + \|\mathbf{z}_h\|^2_{L^4(\Omega)} \|\nabla \mathbf{w}_h\|_{L^2(\Omega)} \right) \\
&\quad \leq 2\eta^* C^2_{4;1,2} \|\nabla \mathbf{w}_h\|_{L^2(\Omega)} \|\nabla \mathbf{z}_h\|^2_{L^2(\Omega)} \\
&\quad \leq 2\eta^* C^2_{1,2;T} C^2_{4;1,2} \|\nabla \mathbf{w}_h\|_{L^2(\Omega)} \|\mathbf{z}_h\|^2_T
\end{aligned}
$$

  where $C_{4;1,2}$ is defined in (2.30) and thanks to the Triangle Inequality, we obtain that

$$
\begin{aligned}
&|\partial_1 A_1(U_h, Z_h; \boldsymbol{\mu})(Z_h) + \partial_1 A_2(U_h, Z_h; \boldsymbol{\mu})(Z_h)| \\
&\qquad\qquad \leq 2\eta^* C^2_{1,2;T} C^2_{4;1,2} (\|\nabla \mathbf{u}_h\|_{L^2(\Omega)} + \|\nabla \mathbf{u}_D\|_{L^2(\Omega)}) \|\mathbf{z}_h\|^2_T.
\end{aligned}
$$

  Then,

$$
\begin{aligned}
&\partial_1 A_1(U_h, Z_h; \boldsymbol{\mu})(Z_h) + \partial_1 A_2(U_h, Z_h; \boldsymbol{\mu})(Z_h) \\
&\qquad \geq -2\eta^* C^2_{1,2;T} C^2_{4;1,2} (\|\nabla \mathbf{u}_h\|_{L^2(\Omega)} + \|\nabla \mathbf{u}_D\|_{L^2(\Omega)}) \|\mathbf{z}_h\|^2_T. \quad (2.33)
\end{aligned}
$$

Finally, regrouping (2.32) and (2.33), we obtain

$$
\partial_1 A(U_h, Z_h; \boldsymbol{\mu})(Z_h) \geq \tilde{\beta}_h \|\mathbf{z}_h\|^2_T,
$$

where

$$
\tilde{\beta}_h = \hat{\lambda} \min\{1, \sqrt{\hat{\varphi}}\} - 2\eta^* C^2_{1,2;T} C^2_{4;1,2} (\|\nabla \mathbf{u}_h\|_{L^2(\Omega)} + \|\nabla \mathbf{u}_D\|_{L^2(\Omega)})
$$

Since $\tilde{\beta}_h$ should be positive, we define $C > 0$

$$
C = \frac{\hat{\lambda} \min\{1, \sqrt{\hat{\varphi}}\}}{2\eta^* C^2_{1,2;T} C^2_{4;1,2}}
$$

then, if $\|\nabla \mathbf{u}_D\|_{L^2(\Omega)} < 1/C$ and $\|\nabla \mathbf{u}_h\|_{L^2(\Omega)} \leq 1/C - \|\nabla \mathbf{u}_D\|_{L^2(\Omega)}$, we are able to guarantee that the constant is positive and therefore, the operator is coercive. □

Remark 2.3. *If we choose Y_h and Q_h in order to verify the inf-sup condition in Theorem A.8 (for example, considering Taylor-Hood finite element spaces), we are able to ensure that the operator $\partial_1 A$ verifies* (2.28). *Moreover, if the pair of spaces (Y_N, Q_N), are inf-sup stables and the data are small enough, the problem* (2.26) *is also well-posed.*

According to BRR theory, we can conclude that problem (2.22) admits a solution $U_h \in X_h$ for small data which is locally unique.

2.3.2 The building of an *a posteriori* error estimator

As a previous result, we need to prove that the tangent operator $\partial_1 A$ is Lipschitz.

Lemma 2.1. *For all $\boldsymbol{\mu} \in \mathcal{D}$ and $U_h^1, U_h^2, Z_h, V_h \in X_h$, it holds*

$$|\partial_1 A(U_h^1, V_h; \boldsymbol{\mu})(Z_h) - \partial_1 A(U_h^2, V_h; \boldsymbol{\mu})(Z_h)| \leq \rho_T \|U_h^1 - U_h^2\|_X \|Z_h\|_X \|V_h\|_X \qquad (2.34)$$

for a positive constant

$$\rho_T = 2\eta^* C_{1,2;T} C_{4;T}^2 + \left(\lambda^* \sqrt{\varphi^*} + 3\lambda^* \frac{\varphi^*}{\sqrt{\hat{\varphi}}} \right) C_S^2 (h_{\max}^*)^{2-d/2} C_{1,2;T}^3 C_{3;2}^3,$$

where constant $C_{1,2;T}$, $C_{4;T}$ and $C_{3;2}$ are defined in (2.29)-(2.31).

Proof. By definition

$$\begin{aligned}\partial_1 A(U_h^1, V_h; \boldsymbol{\mu})(Z_h) - \partial_1 A(U_h^2, V_h; \boldsymbol{\mu})(Z_h) &= \partial_1 A_2(U_h^1, V_h; \boldsymbol{\mu})(Z_h) \\ - \partial_1 A_2(U_h^2, V_h; \boldsymbol{\mu})(Z_h) + \partial_1 A_3(U_h^1, V_h; \boldsymbol{\mu})(Z_h) &- \partial_1 A_3(U_h^2, V_h; \boldsymbol{\mu})(Z_h), \qquad (2.35)\end{aligned}$$

remaining the terms associated with the nonlinear part of the operator A. First, we study the first couple of terms above

$$\begin{aligned}&\partial_1 A_2(U_h^1, V_h; \boldsymbol{\mu})(Z_h) - \partial_1 A_2(U_h^2, V_h; \boldsymbol{\mu})(Z_h) \\ &= \sum_{l=1}^{4} \sum_{k=1}^{2} \eta_{kk}^l (c_{kl}(\mathbf{u}_h^1, \mathbf{z}_h, \mathbf{v}_h) - c_{kl}(\mathbf{u}_h^2, \mathbf{z}_h, \mathbf{v}_h) + c_{kl}(\mathbf{z}_h, \mathbf{u}_h^1, \mathbf{v}_h) - c_{kl}(\mathbf{z}_h, \mathbf{u}_h^2, \mathbf{v}_h)) \\ &= \sum_{l=1}^{4} \sum_{k=1}^{2} \eta_{kk}^l (c_{kl}(\mathbf{u}_h^1 - \mathbf{u}_h^2, \mathbf{z}_h, \mathbf{v}_h) + c_{kl}(\mathbf{z}_h, \mathbf{u}_h^1 - \mathbf{u}_h^2, \mathbf{v}_h)) \\ &\leq \eta^* \left(\int_\Omega \left((\mathbf{u}_h^1 - \mathbf{u}_h^2) \cdot \nabla \right) \mathbf{z}_h \cdot \mathbf{v}_h \, d\Omega + \int_\Omega (\mathbf{z}_h \cdot \nabla)(\mathbf{u}_h^1 - \mathbf{u}_h^2) \cdot \mathbf{v}_h \, d\Omega \right).\end{aligned}$$

Then,

$$\left| \int_\Omega \left((\mathbf{u}_h^1 - \mathbf{u}_h^2) \cdot \nabla \right) \mathbf{z}_h \cdot \mathbf{v}_h \, d\Omega \right| \leq \int_\Omega |\mathbf{u}_h^1 - \mathbf{u}_h^2| |\nabla \mathbf{z}_h| |\mathbf{v}_h| \, d\Omega$$
$$\leq \|\mathbf{u}_h^1 - \mathbf{u}_h^2\|_{L^4(\Omega)} \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)} \leq C_{1,2;T} C_{4;T}^2 \|\mathbf{u}_h^1 - \mathbf{u}_h^2\|_T \|\mathbf{z}_h\|_T \|\mathbf{v}_h\|_T$$
$$\leq C_{1,2;T} C_{4;T}^2 \|U_h^1 - U_h^2\|_X \|Z_h\|_X \|V_h\|_X,$$

and analogously,

$$\left| \int_\Omega (\mathbf{z}_h \cdot \nabla)(\mathbf{u}_h^1 - \mathbf{u}_h^2) \cdot \mathbf{v}_h \, d\Omega \right| \leq \|\mathbf{z}_h\|_{L^4(\Omega)} \|\nabla (\mathbf{u}_h^1 - \mathbf{u}_h^2)\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)}$$
$$\leq C_{1,2;T} C_{4;T}^2 \|U_h^1 - U_h^2\|_X \|Z_h\|_X \|V_h\|_X,$$

and finally,

$$|\partial_1 A_2(U_h^1, V_h; \boldsymbol{\mu})(Z_h) - \partial_1 A_2(U_h^2, V_h; \boldsymbol{\mu})(Z_h)|$$
$$\leq 2\eta^* C_{1,2;T} C_{4;T}^2 \|U_h^1 - U_h^2\|_X \|Z_h\|_X \|V_h\|_X, \quad (2.36)$$

where the constants $C_{1,2;T}$ and $C_{4;T}$ were defined in (2.29) and (2.30), respectively.

Now we study the second couple of terms in (2.35),

$$\partial_1 A_3(U_h^1, V_h; \boldsymbol{\mu})(Z_h) - \partial_1 A_3(U_h^2, V_h; \boldsymbol{\mu})(Z_h)$$
$$= \sum_{l=1}^{4} \sum_{k=1}^{2} \lambda_{kk}^l (a_{t,kl}(\mathbf{w}_h^1; \mathbf{z}_h, \mathbf{v}_h; \boldsymbol{\mu}) - a_{t,kl}(\mathbf{w}_h^2; \mathbf{z}_h, \mathbf{v}_h; \boldsymbol{\mu}))$$
$$+ \sum_{l=1}^{4} \sum_{K \in \mathcal{T}_{h,l}} \int_K (C_S h_K)^2 \left[\frac{\sum_{k=1}^2 \lambda_{kk}^l (\nabla_k \mathbf{w}_h^1 \cdot \nabla_k \mathbf{z}_h)}{\sqrt{\sum_{k=1}^2 \varphi_{kk}^l (\nabla_k \mathbf{w}_h^1 \cdot \nabla_k \mathbf{w}_h^1)}} \left(\sum_{k=1}^{2} \varphi_{kk}^l (\nabla_k \mathbf{w}_h^1 \cdot \nabla_k \mathbf{v}_h) \right) \right. \quad (2.37)$$
$$\left. - \frac{\sum_{k=1}^2 \lambda_{kk}^l (\nabla_k \mathbf{w}_h^2 \cdot \nabla_k \mathbf{z}_h)}{\sqrt{\sum_{k=1}^2 \varphi_{kk}^l (\nabla_k \mathbf{w}_h^2 \cdot \nabla_k \mathbf{w}_h^2)}} \left(\sum_{k=1}^{2} \varphi_{kk}^l (\nabla_k \mathbf{w}_h^2 \cdot \nabla_k \mathbf{v}_h) \right) \right].$$

where we recall that $\mathbf{w}_h^i = \mathbf{u}_h^i + \mathbf{u}_D$ for $i = 1, 2$ and the form $a_{t,kl}(\cdot; \cdot, \cdot; \boldsymbol{\mu})$ was defined in (2.18). It holds

$$\left| \sum_{l=1}^{4} \sum_{k=1}^{2} \lambda_{kk}^l (a_{t,kl}(\mathbf{w}_h^1; \mathbf{z}_h, \mathbf{v}_h; \boldsymbol{\mu}) - a_{t,kl}(\mathbf{w}_h^2; \mathbf{z}_h, \mathbf{v}_h; \boldsymbol{\mu})) \right|$$
$$\leq \lambda^* \left| \int_\Omega (\nu_t(\mathbf{w}_h^1; \boldsymbol{\mu}) - \nu_t(\mathbf{w}_h^2; \boldsymbol{\mu}))(\nabla \mathbf{z}_h : \nabla \mathbf{v}_h) \, d\Omega \right|$$
$$\leq \lambda^* \sqrt{\varphi^*} \left| \sum_{K \in \mathcal{T}_h} \int_K (C_S h_K)^2 \left(|\nabla \mathbf{w}_h^1| - |\nabla \mathbf{w}_h^2| \right) (\nabla \mathbf{z}_h : \nabla \mathbf{v}_h) \right|$$

$$
\begin{aligned}
&\leq \lambda^* \sqrt{\varphi^*} \sum_{K \in \mathcal{T}_h} \int_K (C_S h_K)^2 |\nabla(\mathbf{u}_h^1 - \mathbf{u}_h^2)| |\nabla \mathbf{z}_h| |\nabla \mathbf{v}_h| \\
&\leq \lambda^* \sqrt{\varphi^*} \sum_{K \in \mathcal{T}_h} (C_S h_K)^2 \|\nabla(\mathbf{u}_h^1 - \mathbf{u}_h^2)\|_{L^3(K)} \|\nabla \mathbf{z}_h\|_{L^3(K)} \|\nabla \mathbf{v}_h\|_{L^3(K)} \\
&\leq \lambda^* \sqrt{\varphi^*} C_{3;2}^3 \sum_{K \in \mathcal{T}_h} (C_S h_K)^{2-d/2} \|\nabla(\mathbf{u}_h^1 - \mathbf{u}_h^2)\|_{L^2(K)} \|\nabla \mathbf{z}_h\|_{L^2(K)} \|\nabla \mathbf{v}_h\|_{L^2(K)} \\
&\leq \lambda^* \sqrt{\varphi^*} C_S^2 (h_{\max}^*)^{2-d/2} C_{3;2}^3 \|\nabla(\mathbf{u}_h^1 - \mathbf{u}_h^2)\|_{L^2(\Omega)} \|\nabla \mathbf{z}_h\|_{L^2(\Omega)} \|\nabla \mathbf{v}_h\|_{L^2(\Omega)} \\
&\leq \lambda^* \sqrt{\varphi^*} C_S^2 (h_{\max}^*)^{2-d/2} C_{1,2;T}^3 C_{3;2}^3 \|U_h^1 - U_h^2\|_X \|Z_h\|_X \|V_h\|_X.
\end{aligned}
$$

where we have applied the Local Inverse Inequality (2.31).

For the second term, using the same results and procedure that in the first term, we obtain:

$$
\begin{aligned}
&\left| \sum_{l=1}^{4} \sum_{K \in \mathcal{T}_{h,l}} \int_K (C_S h_K)^2 \left[\frac{\sum_{k=1}^2 \lambda_{kk}^l (\nabla_k \mathbf{w}_h^1 \cdot \nabla_k \mathbf{z}_h)}{\sqrt{\sum_{k=1}^2 \varphi_{kk}^l (\nabla_k \mathbf{w}_h^1 \cdot \nabla_k \mathbf{w}_h^1)}} \left(\sum_{k=1}^2 \varphi_{kk}^l (\nabla_k \mathbf{w}_h^1 \cdot \nabla_k \mathbf{v}_h) \right) \right. \right. \\
&\qquad \left. \left. - \frac{\sum_{k=1}^2 \lambda_{kk}^l (\nabla_k \mathbf{w}_h^2 \cdot \nabla_k \mathbf{z}_h)}{\sqrt{\sum_{k=1}^2 \varphi_{kk}^l (\nabla_k \mathbf{w}_h^2 \cdot \nabla_k \mathbf{w}_h^2)}} \left(\sum_{k=1}^2 \varphi_{kk}^l (\nabla_k \mathbf{w}_h^2 \cdot \nabla_k \mathbf{v}_h) \right) \right] \right| \\
&\leq \lambda^* \frac{\varphi^*}{\sqrt{\hat{\varphi}}} \left| \sum_{K \in \mathcal{T}_h} \int_K (C_S h_K)^2 \left(\frac{\nabla \mathbf{w}_h^1 : \nabla \mathbf{z}_h}{|\nabla \mathbf{w}_h^1|} (\nabla \mathbf{w}_h^1 : \nabla \mathbf{v}_h) - \frac{\nabla \mathbf{w}_h^2 : \nabla \mathbf{z}_h}{|\nabla \mathbf{w}_h^2|} (\nabla \mathbf{w}_h^2 : \nabla \mathbf{v}_h) \right) \right| \\
&= \lambda^* \frac{\varphi^*}{\sqrt{\hat{\varphi}}} \left| \sum_{K \in \mathcal{T}_h} \int_K (C_S h_K)^2 \left(\frac{\nabla \mathbf{w}_h^1 : \nabla \mathbf{z}_h}{|\nabla \mathbf{w}_h^1|} (\nabla(\mathbf{w}_h^1 - \mathbf{w}_h^2) : \nabla \mathbf{v}_h) + \frac{\nabla(\mathbf{w}_h^1 - \mathbf{w}_h^2) : \nabla \mathbf{z}_h}{|\nabla \mathbf{w}_h^2|} (\nabla \mathbf{w}_h^2 : \nabla \mathbf{v}_h) \right) \right. \\
&\qquad \left. + \sum_{K \in \mathcal{T}_h} \int_K (C_S h_K)^2 \frac{(|\nabla \mathbf{w}_h^2| - |\nabla \mathbf{w}_h^1|)}{|\nabla \mathbf{w}_h^1| |\nabla \mathbf{w}_h^2|} (\nabla \mathbf{w}_h^1 : \nabla \mathbf{z}_h)(\nabla \mathbf{w}_h^2 : \nabla \mathbf{v}_h) \right| \\
&\leq 3\lambda^* \frac{\varphi^*}{\sqrt{\hat{\varphi}}} \sum_{K \in \mathcal{T}_h} \int_K (C_S h_K)^2 |\nabla(\mathbf{u}_h^1 - \mathbf{u}_h^2)| |\nabla \mathbf{z}_h| |\nabla \mathbf{v}_h| \\
&\leq 3\lambda^* \frac{\varphi^*}{\sqrt{\hat{\varphi}}} \sum_{K \in \mathcal{T}_h} (C_S h_K)^2 \|\mathbf{u}_h^1 - \mathbf{u}_h^2\|_{L^3(K)} \|\mathbf{z}_h\|_{L^3(K)} \|\mathbf{v}_h\|_{L^3(K)} \\
&\leq 3\lambda^* \frac{\varphi^*}{\sqrt{\hat{\varphi}}} C_S^2 (h_{\max}^*)^{2-d/2} C_{1,2;T}^3 C_{3;2}^3 \|U_h^1 - U_h^2\|_X \|Z_h\|_X \|V_h\|_X.
\end{aligned}
$$

Using these bounds in (2.37) and summing with (2.36), we conclude that there exists

$$
\rho_T = 2\eta^* C_{1,2;T} C_{4;T}^2 + \left(\lambda^* \sqrt{\varphi^*} + 3\lambda^* \frac{\varphi^*}{\sqrt{\hat{\varphi}}} \right) C_S^2 (h_{\max}^*)^{2-d/2} C_{1,2;T}^3 C_{3;2}^3,
$$

such that (2.34) is satisfied. □

Remark 2.4. *Note that the Lipschitz constant ρ_T only depends on possible values of $\boldsymbol{\mu}$, this is, the selection of $\mathcal{D}$ and the characteristic mesh parameters.*

It also depends on $(h^*_{\max})^{2-d/2}$ which tends to zero since $2-d/2>0$ for any $d<4$.

As we explained, in order to apply the RB Method, we develop an *a posteriori* error estimator. For its construction, we first introduce some definitions.

Definition 2.1. *The Stability Factor $\beta_N(\boldsymbol{\mu})$ and the continuity factor $\gamma_N(\boldsymbol{\mu})$, associated with $\partial_1 A(U_N(\boldsymbol{\mu}),\cdot;\boldsymbol{\mu})$ are defined by*

$$\beta_N(\boldsymbol{\mu}) \equiv \inf_{Z_h\in X_h}\sup_{V_h\in X_h} \frac{\partial_1 A(U_N(\boldsymbol{\mu}),V_h;\boldsymbol{\mu})(Z_h)}{\|Z_h\|_X\|V_h\|_X}, \tag{2.38}$$

$$\gamma_N(\boldsymbol{\mu}) \equiv \sup_{Z_h\in X_h}\sup_{V_h\in X_h} \frac{\partial_1 A(U_N(\boldsymbol{\mu}),V_h;\boldsymbol{\mu})(Z_h)}{\|Z_h\|_X\|V_h\|_X}, \tag{2.39}$$

for a given $U_N(\boldsymbol{\mu})\in X_N$, solution of (2.26).

Note that from Proposition 2.2, $\beta_N(\boldsymbol{\mu})>0$ whenever the data is small enough and the pair (Y_h,Q_h) are inf-sup stable. This condition implies that the tangent operator $\partial_1 A(U_N(\boldsymbol{\mu}),\cdot;\boldsymbol{\mu})$ is an isomorphism of X_h into its self for all $\boldsymbol{\mu}\in\mathcal{D}$.

Definition 2.2. *For a given $U_N(\boldsymbol{\mu})\in X_N$ solution of* (2.26)*, the supremizer operator $T_N^{\boldsymbol{\mu}}: X_h \longrightarrow X_h$ is defined by*

$$T_N^{\boldsymbol{\mu}} Z_h = \arg\sup_{V_h\in X_h} \frac{\partial_1 A(U_N(\boldsymbol{\mu}),V_h;\boldsymbol{\mu})(Z_h)}{\|V_h\|_X}. \tag{2.40}$$

This *supremizer* is unlike the inner-pressure *supremizer* presented in (2.23). The pressure *supremizer* is related to the pressure and it is defined to enrich the velocity space, while this last *supremizer* is related to the tangent operator $\partial_1 A$.

Remark 2.5. *From definitions 2.1 and 2.2, we deduce the relation between the factors and the supremizer operator:*

$$\beta_N(\boldsymbol{\mu}) \equiv \inf_{Z_h\in X_h} \frac{\|T_N^{\boldsymbol{\mu}} Z_h\|_X}{\|Z_h\|_X}, \quad \gamma_N(\boldsymbol{\mu}) \equiv \sup_{Z_h\in X_h} \frac{\|T_N^{\boldsymbol{\mu}} Z_h\|_X}{\|Z_h\|_X}.$$

In particular, we have

$$\beta_N(\boldsymbol{\mu}) \le \frac{\partial_1 A(U_N(\boldsymbol{\mu}),T_N^{\boldsymbol{\mu}} Z_h;\boldsymbol{\mu})(Z_h)}{\|T_N^{\boldsymbol{\mu}} Z_h\|_X\|Z_h\|_X},\ \forall Z_h\in X_h, \tag{2.41}$$

for a given $U_N\in X_N$.

The construction of an *a posteriori* error estimator goes through the proof of the existence and uniqueness of the solution of problem (2.22).

Theorem 2.1 (Uniqueness). *Let $\boldsymbol{\mu} \in \mathcal{D}$, and assume that $\beta_N(\boldsymbol{\mu}) > 0$. If problem* (2.22) *admits a solution $U_h(\boldsymbol{\mu})$ such that*

$$U_h(\boldsymbol{\mu}) \in B_X\left(U_N(\boldsymbol{\mu}), \frac{\beta_N(\boldsymbol{\mu})}{\rho_T}\right)$$

then, this solution is unique.

For the proof of this theorem and the next one, we define the following operators:

- The residual $\mathscr{R}(\cdot;\boldsymbol{\mu}) : X_h \longrightarrow X_h'$ by

$$\langle \mathscr{R}(Z_h;\boldsymbol{\mu}), V_h\rangle = A(Z_h,V_h;\boldsymbol{\mu}) - F(V_h;\boldsymbol{\mu}),\ \forall Z_h,V_h \in X_h. \tag{2.42}$$

- The derivative of the operator $A(\cdot,\cdot;\boldsymbol{\mu})$, $\mathcal{DA}(U_h(\boldsymbol{\mu});\boldsymbol{\mu}) : X_h \longrightarrow X_h'$, by

$$\langle \mathcal{DA}(U_h(\boldsymbol{\mu});\boldsymbol{\mu})Z_h, V_h\rangle = \partial_1 A(U_h,V_h;\boldsymbol{\mu})(Z_h),\ \forall Z_h,V_h \in X_h. \tag{2.43}$$

- The operator $H : X_h \longrightarrow X_h$ by

$$H(Z_h;\boldsymbol{\mu}) = Z_h - \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})^{-1}\mathscr{R}(Z_h;\boldsymbol{\mu}),\ \forall Z_h \in X_h. \tag{2.44}$$

Proof. The strategy of this proof is to show that $H(\cdot;\boldsymbol{\mu})$ is a contraction in the sense of Definition A.3. If there exists a fixed point U_h, this point is a solution of problem (2.22) since

$$H(U_h(\boldsymbol{\mu});\boldsymbol{\mu}) = U_h(\boldsymbol{\mu}) - \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})^{-1}\mathscr{R}(U_h(\boldsymbol{\mu});\boldsymbol{\mu}) = U_h(\boldsymbol{\mu}) \\ \Longleftrightarrow \mathscr{R}(U_h(\boldsymbol{\mu});\boldsymbol{\mu}) = 0.$$

Let us study if the operator H is a contraction, proving in this way that it has a unique fixed point. By definition, we have

$$H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu}) = (Z_h^1 - Z_h^2) - \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})^{-1}(\mathscr{R}(Z_h^1;\boldsymbol{\mu}) - \mathscr{R}(Z_h^2;\boldsymbol{\mu})). \tag{2.45}$$

Using the residual definition (2.42) and the mean value theorem, it holds

$$\mathscr{R}(Z_h^1;\boldsymbol{\mu}) - \mathscr{R}(Z_h^2;\boldsymbol{\mu}) = \mathcal{DA}(\xi;\boldsymbol{\mu})(Z_h^1 - Z_h^2), \tag{2.46}$$

where $\xi = sZ_h^1 + (1-s)Z_h^2$, for some $s \in (0,1)$.

To prove this, we define $T : [0,1] \longrightarrow \mathbb{R}$, by

$$T(s) = \langle \mathscr{R}(sZ_h^1 + (1-s)Z_h^2;\boldsymbol{\mu}), V_h\rangle,\ \forall V_h \in X_h.$$

The function T is derivable for all $s \in (0,1)$ and its derivative is

$$T'(s) = \langle \mathcal{DA}(sZ_h^1 + (1-s)Z_h^2;\boldsymbol{\mu})(Z_h^1 - Z_h^2), V_h\rangle.$$

If we apply the mean value theorem to T we have (2.46).

Now, multiplying (2.45) by $\mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})$ and applying (2.46), we can write

$$\mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})) = [\mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}) - \mathcal{DA}(\xi;\boldsymbol{\mu})](Z_h^1 - Z_h^2).$$

Then, thanks to the Lipschitz condition (2.34), we can write

$$\begin{aligned}\langle \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})), V_h\rangle \\ \leq \rho_T \|U_N(\boldsymbol{\mu}) - \xi\|_X \|Z_h^1 - Z_h^2\|_X \|V_h\|_X \end{aligned} \quad (2.47)$$

Taking $Z_h = H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})$ in (2.41), then,

$$\begin{aligned}\beta_N(\boldsymbol{\mu})\|H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})\|_X \|T_N^{\boldsymbol{\mu}}(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu}))\|_X \\ \leq \langle \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})), T_N^{\boldsymbol{\mu}}(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu}))\rangle,\end{aligned}$$

and applying (2.47) to $V_h = T_N^{\boldsymbol{\mu}}(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu}))$ we have that,

$$\begin{aligned}\beta_N(\boldsymbol{\mu})\|H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})\|_X \|T_N^{\boldsymbol{\mu}}(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu}))\|_X \\ \leq \rho_T \|U_N(\boldsymbol{\mu}) - \xi\|_X \|Z_h^1 - Z_h^2\|_X \|T_N^{\boldsymbol{\mu}}(H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu}))\|_X\end{aligned}$$

from which we obtain that

$$\|H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})\|_X \leq \frac{\rho_T}{\beta_N(\boldsymbol{\mu})} \|U_N(\boldsymbol{\mu}) - \xi\|_X \|Z_h^1 - Z_h^2\|_X.$$

If Z_h^1 and Z_h^2 are in $B_X(U_N(\boldsymbol{\mu}), \alpha)$ then $\|U_N(\boldsymbol{\mu}) - \xi\|_X \leq \alpha$ since $\xi = sZ_h^1 + (1-s)Z_h^2$ for some $s \in (0,1)$ and

$$\|H(Z_h^1;\boldsymbol{\mu}) - H(Z_h^2;\boldsymbol{\mu})\|_X \leq \frac{\rho_T}{\beta_N(\boldsymbol{\mu})} \alpha \|Z_h^1 - Z_h^2\|_X$$

Then, $H(\cdot;\boldsymbol{\mu})$ is a contraction if $\alpha < \dfrac{\beta_N(\boldsymbol{\mu})}{\rho_T}$.

In conclusion, if there exists a fixed point U_h in $B_X\left(U_N(\boldsymbol{\mu}), \dfrac{\beta_N(\boldsymbol{\mu})}{\rho_T}\right)$, this point is unique. □

Now we state an *a posteriori* error bound estimator that will be deduced in the Existence Theorem 2.2. First,

$$\tau_N(\boldsymbol{\mu}) = \frac{4\varepsilon_N(\boldsymbol{\mu})\rho_T}{\beta_N(\boldsymbol{\mu})^2} \tag{2.48}$$

where the Stability Factor $\beta_N(\boldsymbol{\mu})$ is defined in (2.38), the Lipschitz constant ρ_T is defined in (2.34) and

$$\varepsilon_N(\boldsymbol{\mu}) = \|\mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})\|_{X'} = \sup_{V_h\in X_h} \frac{\langle \mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}), V_h\rangle}{\|V_h\|_X} \tag{2.49}$$

that is, $\varepsilon_N(\boldsymbol{\mu})$ is the dual norm of the residual.

Then, we define

$$\Delta_N(\boldsymbol{\mu}) = \frac{\beta_N(\boldsymbol{\mu})}{2\rho_T}\left[1-\sqrt{1-\tau_N(\boldsymbol{\mu})}\right], \ \forall \boldsymbol{\mu}\in\mathcal{D}. \tag{2.50}$$

Thanks to the next result, we will see that $\Delta_N(\boldsymbol{\mu})$ is an *a posteriori* error estimator.

Theorem 2.2 (Existence)**.** *Assume that* $\beta_N(\boldsymbol{\mu}) > 0$ *and* $\tau_N(\boldsymbol{\mu}) \leq 1$ *for all* $\boldsymbol{\mu}\in\mathcal{D}$. *Then there exists a unique solution* $U_h(\boldsymbol{\mu})$ *of* (2.22) *such that the error with respect to* $U_N(\boldsymbol{\mu})$ *solution of* (2.26)*, is bounded by the a posteriori error bound estimator, i.e.,*

$$\|U_h(\boldsymbol{\mu})-U_N(\boldsymbol{\mu})\|_X \leq \Delta_N(\boldsymbol{\mu}), \tag{2.51}$$

with effectivity,

$$\Delta_N(\boldsymbol{\mu}) \leq \left[\frac{2\gamma_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})} + \tau_N(\boldsymbol{\mu})\right] \|U_h(\boldsymbol{\mu})-U_N(\boldsymbol{\mu})\|_X. \tag{2.52}$$

Proof. The strategy of this proof is to show the existence of $U_h(\boldsymbol{\mu})\in X_h$ close to $U_N(\boldsymbol{\mu})\in X_N$ solution of (2.22) and (2.26). To do this, it is enough to prove that we are under the hypothesis of the Schauder Fixed-Point Theorem A.7, that is that $H(\cdot;\boldsymbol{\mu})$ is a contraction in a certain compact set K of X_h. Then, the fixed point exists and if this compact verifies that

$$K \subset B_X\left(U_N(\boldsymbol{\mu}), \frac{\beta_N(\boldsymbol{\mu})}{\rho_T}\right),$$

then, we guarantee also the uniqueness of this fixed point thanks to Theorem 2.1.

We consider the definitions (2.42)-(2.44). Then,

$$\begin{aligned} H(Z_h;\boldsymbol{\mu}) - U_N(\boldsymbol{\mu}) &= Z_h - U_N(\boldsymbol{\mu}) - \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})^{-1}\mathscr{R}(Z_h;\boldsymbol{\mu}) \\ &= Z_h - U_N(\boldsymbol{\mu}) - \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})^{-1}[\mathscr{R}(Z_h;\boldsymbol{\mu}) - \mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})] \\ &\quad - \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})^{-1}\mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}). \end{aligned}$$

Applying $\mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})$, we obtain

$$\begin{aligned}\langle \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu})),V_h\rangle &= \langle \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(Z_h-U_N(\boldsymbol{\mu})),V_h\rangle \\ &-\langle \mathscr{R}(Z_h;\boldsymbol{\mu})-\mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}),V_h\rangle-\langle \mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}),V_h\rangle,\ \forall V_h\in X_h\end{aligned}$$

Following the same idea as in Theorem 2.1, we can find $\xi(\boldsymbol{\mu})=sZ_h+(1-s)U_N(\boldsymbol{\mu})$ with $s\in(0,1)$ such that

$$\mathscr{R}(Z_h;\boldsymbol{\mu})-\mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})=\mathcal{DA}(\xi(\boldsymbol{\mu});\boldsymbol{\mu})(Z_h-U_N(\boldsymbol{\mu})).$$

In this way, and thanks to Lemma 2.1 and the definition of $\varepsilon_N(\boldsymbol{\mu})$ in (2.49), we obtain that:

$$\begin{aligned}&\langle \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu})),V_h\rangle\\ &\quad=\langle \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(Z_h-U_N(\boldsymbol{\mu})),V_h\rangle\\ &\quad-\langle \mathcal{DA}(\xi(\boldsymbol{\mu});\boldsymbol{\mu})(Z_h-U_N(\boldsymbol{\mu})),V_h\rangle-\langle \mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}),V_h\rangle\\ &\quad=\langle(\mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})-\mathcal{DA}(\xi(\boldsymbol{\mu});\boldsymbol{\mu}))(Z_h-U_N(\boldsymbol{\mu})),V_h\rangle\\ &\quad-\langle \mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}),V_h\rangle\\ &\quad\le \rho_T\|U_N(\boldsymbol{\mu})-\xi(\boldsymbol{\mu})\|_X\|Z_h-U_N(\boldsymbol{\mu})\|_X\|V_h\|_X+\varepsilon_N(\boldsymbol{\mu})\|V_h\|_X\\ &\quad\le\left(\rho_T\|Z_h-U_N(\boldsymbol{\mu})\|_X^2+\varepsilon_N(\boldsymbol{\mu})\right)\|V_h\|_X.\end{aligned}$$

Then, using (2.41)

$$\begin{aligned}&\beta_N(\boldsymbol{\mu})\|H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu})\|_X\|T_N^{\boldsymbol{\mu}}(H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu}))\|_X\\ &\quad\le\langle \mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})(H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu})),T_N^{\boldsymbol{\mu}}(H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu}))\rangle\\ &\quad\quad\le\left(\rho_T\|Z_h-U_N(\boldsymbol{\mu})\|_X^2+\varepsilon_N(\boldsymbol{\mu})\right)\|T_N^{\boldsymbol{\mu}}(H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu}))\|_X,\end{aligned}$$

simplifying,

$$\|H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu})\|_X\le\frac{\rho_T}{\beta_N(\boldsymbol{\mu})}\|Z_h-U_N(\boldsymbol{\mu})\|_X^2+\frac{\varepsilon_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})}.$$

and since $Z_h\in B_X(U_N(\boldsymbol{\mu}),\alpha)$ we obtain

$$\|H(Z_h;\boldsymbol{\mu})-U_N(\boldsymbol{\mu})\|_X\le\frac{\rho_T}{\beta_N(\boldsymbol{\mu})}\alpha^2+\frac{\varepsilon_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})}.$$

To ensure that H maps $B_X(U_N(\boldsymbol{\mu}),\alpha)$ into a part of itself, we are seeking the values of α such that

$$\frac{\rho_T}{\beta_N(\boldsymbol{\mu})}\alpha^2+\frac{\varepsilon_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})}\leq\alpha.$$

This holds if α is between the two roots of the second order equation

$$\rho_T\alpha^2-\beta_N(\boldsymbol{\mu})\alpha+\varepsilon_N(\boldsymbol{\mu})=0$$

which are,

$$\begin{aligned}\alpha_\pm&=\frac{\beta_N(\boldsymbol{\mu})\pm\sqrt{\beta_N(\boldsymbol{\mu})^2-4\rho_T\varepsilon_N(\boldsymbol{\mu})}}{2\rho_T}\\&=\frac{\beta_N(\boldsymbol{\mu})}{2\rho_T}\left[1\pm\sqrt{1-\frac{4\rho_T\varepsilon_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})^2}}\right]\\&=\frac{\beta_N(\boldsymbol{\mu})}{2\rho_T}\left[1\pm\sqrt{1-\tau_N(\boldsymbol{\mu})}\right].\end{aligned}$$

Observe that as $\tau_N(\boldsymbol{\mu})\leq 1$, then $\alpha_-\leq\alpha_+\leq\dfrac{\beta_N(\boldsymbol{\mu})}{\rho_T}$.

Then, $H(\cdot;\boldsymbol{\mu})$ is compact on $B_X(U_N(\boldsymbol{\mu}),\alpha)$ if $\alpha\in[\alpha_-,\alpha_+]$, then, there exists a unique solution $U_h(\boldsymbol{\mu})$ (2.22) in the ball $B_X(U_N(\boldsymbol{\mu}),\alpha)$. To obtain the estimator, we take $\alpha=\alpha_-=\Delta_N(\boldsymbol{\mu})$.

To prove the efficiency, let us define the error $E_h(\boldsymbol{\mu})=U_h(\boldsymbol{\mu})-U_N(\boldsymbol{\mu})$. From the definition of the residual and applying the mean value theorem, for some $s\in(0,1)$ we have that

$$\begin{aligned}\langle\mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}),V_h\rangle&=A(U_N(\boldsymbol{\mu}),V_h;\boldsymbol{\mu})-F(V_h;\boldsymbol{\mu})\\&=A(U_N(\boldsymbol{\mu}),V_h;\boldsymbol{\mu})-A(U_h(\boldsymbol{\mu}),V_h;\boldsymbol{\mu})\\&=\partial_1A(sU_h(\boldsymbol{\mu})+(1-s)U_N(\boldsymbol{\mu}),V_h;\boldsymbol{\mu})(E_h(\boldsymbol{\mu}))\\&=\langle\mathcal{DA}(sU_h(\boldsymbol{\mu})+(1-s)U_N(\boldsymbol{\mu});\boldsymbol{\mu})E_h(\boldsymbol{\mu}),V_h\rangle\\&=\langle(\mathcal{DA}(sU_h(\boldsymbol{\mu})+(1-s)U_N(\boldsymbol{\mu});\boldsymbol{\mu})-\mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}))E_h(\boldsymbol{\mu}),V_h\rangle\\&\quad+\langle\mathcal{DA}(U_N(\boldsymbol{\mu});\boldsymbol{\mu})E_h(\boldsymbol{\mu}),V_h\rangle.\end{aligned}$$

Thanks to the Lemma 2.1, and taking into account the definition of $\gamma_N(\boldsymbol{\mu})$ and $\beta_N(\boldsymbol{\mu})$ in

(2.38) and (2.39) respectively, we obtain,

$$\langle \mathscr{R}(U_N(\boldsymbol{\mu});\boldsymbol{\mu}), V_h\rangle \leq \rho_T \|s(U_N(\boldsymbol{\mu}) - U_h(\boldsymbol{\mu}))\|_X \|E_h(\boldsymbol{\mu})\|_X \|V_h\|_X + \gamma_N(\boldsymbol{\mu})\|E_h(\boldsymbol{\mu})\|_X \|V_h\|_X.$$

Taking supreme in $V_h \in X_h$ and using the definition of (2.49)

$$\varepsilon_N(\boldsymbol{\mu}) \leq \rho_T \|E_h(\boldsymbol{\mu})\|_X^2 + \gamma_N(\boldsymbol{\mu})\|E_h(\boldsymbol{\mu})\|_X.$$

Since $0 \leq \tau_N(\boldsymbol{\mu}) \leq 1$ and $1 - \sqrt{1 - \tau_N(\boldsymbol{\mu})} \leq \tau_N(\boldsymbol{\mu})$, we have that

$$\frac{2\rho_T}{\beta_N(\boldsymbol{\mu})}\Delta_N(\boldsymbol{\mu}) \leq \tau_N(\boldsymbol{\mu}),$$

and then

$$\Delta_N(\boldsymbol{\mu}) \leq \frac{2\varepsilon_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})}.$$

It follows that

$$\Delta_N(\boldsymbol{\mu}) \leq \frac{2\rho_T}{\beta_N(\boldsymbol{\mu})}\|E_h(\boldsymbol{\mu})\|_X^2 + \frac{2\gamma_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})}\|E_h(\boldsymbol{\mu})\|_X.$$

Thanks to (2.51), we know that $\|E_h(\boldsymbol{\mu})\|_X \leq \Delta_N(\boldsymbol{\mu})$, then

$$\frac{2\rho_T}{\beta_N(\boldsymbol{\mu})}\|E_h(\boldsymbol{\mu})\|_X \leq \tau_N(\boldsymbol{\mu}),$$

it follows that

$$\Delta_N(\boldsymbol{\mu}) \leq \left[\frac{2\gamma_N(\boldsymbol{\mu})}{\beta_N(\boldsymbol{\mu})} + \tau_N(\boldsymbol{\mu})\right]\|E_h(\boldsymbol{\mu})\|_X.$$

□

Finally, we have proved that $\Delta_N(\boldsymbol{\mu})$ is an *a posteriori* error estimator with an efficiency described in (2.52). The estimator is composed of different elements such that the inf-sup and Lipschitz constants and finally the norm of the residual. In the next section, we apply the RB method to the Smagorinsky model and we will see how we can compute these factors in practice.

2.4 Numerical results

In this section, we present a numerical test coded in FreeFem++ v. 4.8 (*cf.* [23]). To solve the problems presented along this chapter, we use a FE approximation with the Taylor-Hood finite element, i.e., we consider $\mathbb{P}2-\mathbb{P}1$ for velocity-pressure that are inf-sup stable. Additionally, it is necessary to impose that the pressure mean is zero and we can not ensure this using $\mathbb{P}1$ as the finite element space for the pressure. For this reason, we add a stabilization term in the variational formulation, this is a L^2 penalization, as the pressure is defined up to an additive constant.

2.4.1 Problem statement

To state the flow domain, we consider that the typically total size of the courtyard is 28m wide (W) and 6m high (H). Then, we are able to state the set of parameters fixing the maximum and the minimum width and height of the corridor around the courtyard,

$$\omega_{\min} = 2\text{m},\ \omega_{\max} = 4\text{m},\ \sigma_{\min} = 2.5\text{m},\ \sigma_{\max} = 3\text{m},$$

and we set

$$\omega_r = \frac{\omega_{\min} + \omega_{\max}}{2} = 3\text{m},\ \sigma_r = \frac{\sigma_{\min} + \sigma_{\max}}{2} = 2.75\text{m}$$

as the reference parameters and the characteristic length as $L_0 = W - 2 * \omega_r = 22$m.

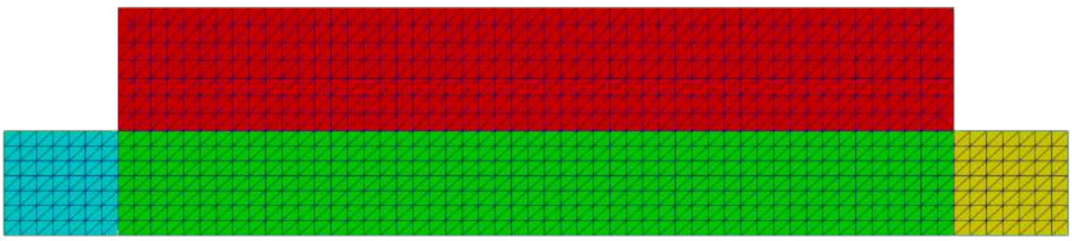


Figure 2.1: Final mesh.

In Figure 2.1, we can see the final dimensionless reference geometry, split into four regions and the mesh with 1 624 triangles and 892 vertices. The diameter of each triangle h_K is constant in each region and it can be set by (ω, σ).

The characteristic velocity U_0 is determined by the maximum velocity that we have at the top of the courtyard. We set a constant horizontal wind of $2.13 \cdot 10^{-3}$m/s, and due to the kinematic viscosity for the air is $\nu = 1.51 \cdot 10^{-5}\text{m}^2/\text{s}$, we obtain a Reynolds number such as $\mathrm{Re} = 3\,100$. This is a small wind velocity, but for larger values the flow becomes unsteady and the steady model cannot be applied.

Second, we set the lifting of the BCs for the problem. In this case, we choose to solve the following Stokes equations taking into account the non-homogeneous BCs

$$\begin{cases} -\dfrac{1}{\mathrm{Re}}\Delta \mathbf{u}_D + \nabla p_D = \mathbf{0}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u}_D = 0, & \text{in } \Omega, \\ \mathbf{u}_D = (g_D, 0), & \text{on } \Gamma_S, \\ \mathbf{u}_D = \mathbf{0}, & \text{on } \Gamma_W \cup \Gamma_F, \end{cases} \tag{2.53}$$

where g_D is a dimensionless regularization of the Dirichlet boundary condition given by

$$g_D(x) = \begin{cases} \sin\left(\pi \dfrac{x-\omega_r}{2a}\right), & \text{if} \quad x \le a + \omega_r, \\ 1, & \text{if} \quad a + \omega_r < x < W - (\omega_r + a), \\ \sin\left(\pi \dfrac{W-(\omega_r + x)}{2a}\right), & \text{if} \quad W - (\omega_r + a) \le x, \end{cases} \tag{2.54}$$

with $a = 0.1$.

Finally, we are ready to solve the problem (2.22). This problem is non-linear, hence we use a semi-implicit evolution approach, similarly to (1.20) and we finish the process when a steady solution has been reached, this is when

$$\sum_{l=1}^{4} \left\| \sqrt{|J_{\Phi_l}|} \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|_{L^2(\Omega_l)}^2 < \varepsilon_{FE}^2$$

with $\varepsilon_{FE} = 10^{-11}$ and $\Delta t = 5 \cdot 10^{-2}$.

2.4.2 Empirical Interpolation Method (EIM)

Since the parameter dependency of the turbulent eddy viscosity defined in (2.19) is non-linear, we need a tensorized approximation to handle this term in the RB formulation. For this approximation, we use the EIM Algorithm 5. Since it is a problem with two geometrical parameters, the number of solutions needed to perform the EIM could be high.

We stop the construction of the EIM base when we reach an error below $\varepsilon_{EIM} = 10^{-4}$. In Figure 2.2 we see the convergence of the algorithm that finishes with a total of 77 basis functions.

In Figure 2.3 we show the normalized error between the turbulent eddy viscosity and the interpolation for 625 parameters. The maximum error is $1.302 \cdot 10^{-3}$ which means that we have obtained a good interpolation for the eddy viscosity term.

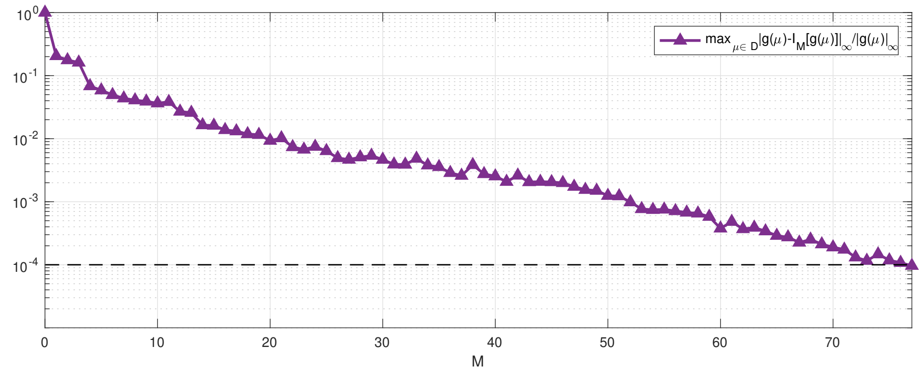


Figure 2.2: Convergence of the EIM algorithm.

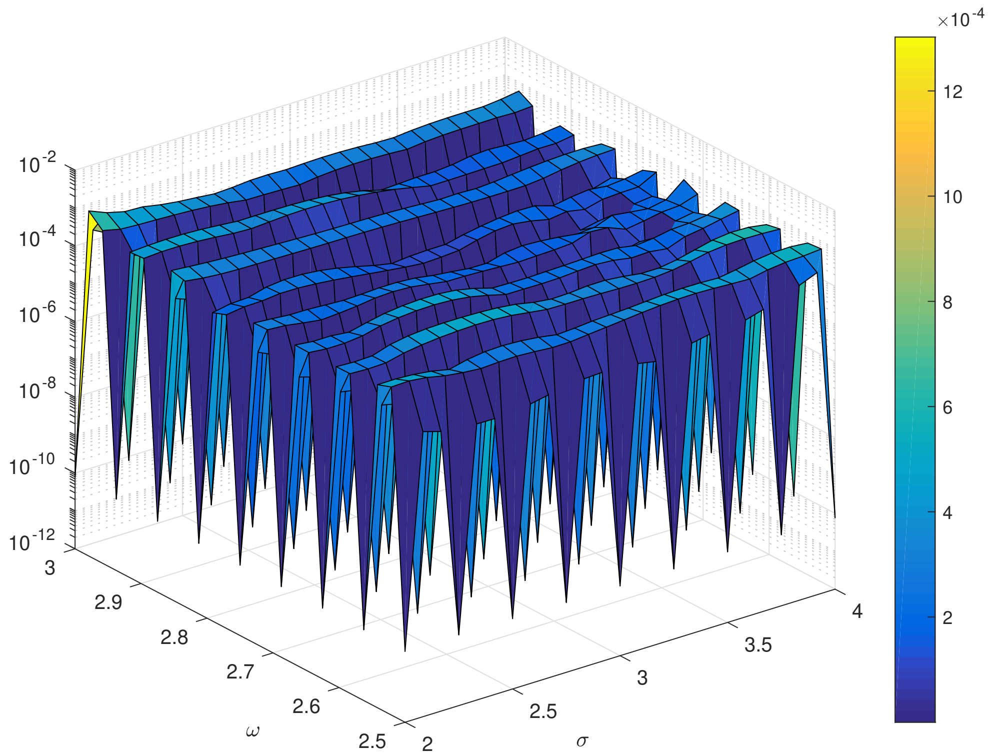


Figure 2.3: Normalized error in L^∞-norm of the turbulent eddy viscosity.

2.4.3 Setup for an *a posteriori* error bound estimate

This section is intended to detail the computation of the different items that take part in the construction of the *a posteriori* error bound estimate defined in (2.50).

- For the computation of the Stability Factor (2.38) we substitute this quantity for

$\beta_h(\boldsymbol{\mu})$ since $U_N(\boldsymbol{\mu})$ is intended to be a good approximation of $U_h(\boldsymbol{\mu})$ drawn from the solutions that we already computed for the EIM.

To compute the inf-sup stability factor, we use the procedure exposed in Section 1.5 in [14] using the Radial Basis Function (RBF) algorithm to approximate the value of the inf-sup stability factor for the new parameters. First, we compute $\beta_h(\boldsymbol{\mu})$ for the 81 solutions getting the lowest eigenvalue in problem

$$\begin{cases} \text{Find } (\alpha, Z_h) \in \mathbb{R} \times X_h,\ Z_h \neq 0, \text{ such that} \\ \mathbb{F}(\mu)^T \mathbb{X}^{-1} \mathbb{F}(\mu) \underline{Z}_h = \alpha \mathbb{X} \underline{Z}_h, \quad \forall Z_h \in X_h, \end{cases}$$

where the matrices $\mathbb{X}$ and $\mathbb{F}(\mu)$ are the matrices associated to the inner product related to the X-norm (2.13) and the tangent operator $\partial_1 A$, this is,

$$\begin{aligned} &\underline{V}_h^T \mathbb{X} \underline{Z}_h = (V_h, Z_h)_X, && \forall V_h, Z_h \in X_h, \\ &\underline{V}_h^T \mathbb{F}(\mu) \underline{Z}_h = \partial_1 A(U_h(\mu), V_h; \mu)(Z_h), && \forall V_h, Z_h \in X_h,\ \forall \mu \in \mathcal{D}, \end{aligned}$$

resulting that $\beta_h(\boldsymbol{\mu}) = (\alpha_{\min})^{1/2}$.

Secondly, we apply the RBF algorithm to obtain an approximation of $\beta_h(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathcal{D}$. In this case, we stop the algorithm when the estimator is below $\varepsilon_\beta = 10^{-4}$. In the RBF, we tested 729 solutions and at the end we have selected 82 parameters.

- The residual (2.49) is computed in each iteration solving a variational problem applying the Riesz representation Theorem A.5.

- To compute the Lipschitz constant ρ_T defined in (2.34) we need to compute η^*, λ^*, φ^*, $\hat{\varphi}$ and $h^*_{\max}$ which are easy to obtain since these parameters only depend on $\mathcal{D}$ and $C_{4;T}$ defined in (2.30). This last constant is approximated by a fixed-point algorithm described in [15, 34].

For the first iterations, $\tau_N(\boldsymbol{\mu})$ is not lower than one since the residual is still large. While $\tau_N(\boldsymbol{\mu}) > 1$, we use as *a posteriori* error bound estimator the proper $\tau_N(\boldsymbol{\mu})$.

2.4.4 Offline phase

Finally, we compute the RB space applying the Greedy algorithm for the selection of $\dim(\mathcal{D}_{train}) = 625$. We stop the algorithm when $\Delta_N(\boldsymbol{\mu}) < \varepsilon_{RB}$ with $\varepsilon_{RB} = 5 \cdot 10^{-4}$.

In each iteration, we use the Gram-Schmidt algorithm to orthonormalize the reduced basis functions to improve the condition number of the reduced problem.

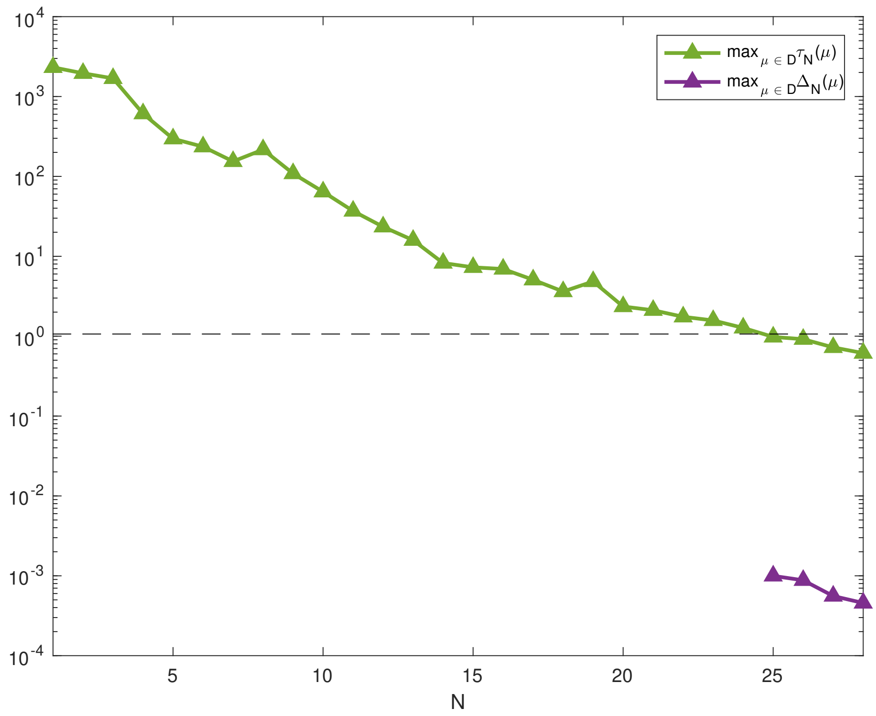


Figure 2.4: Greedy Algorithm convergence.

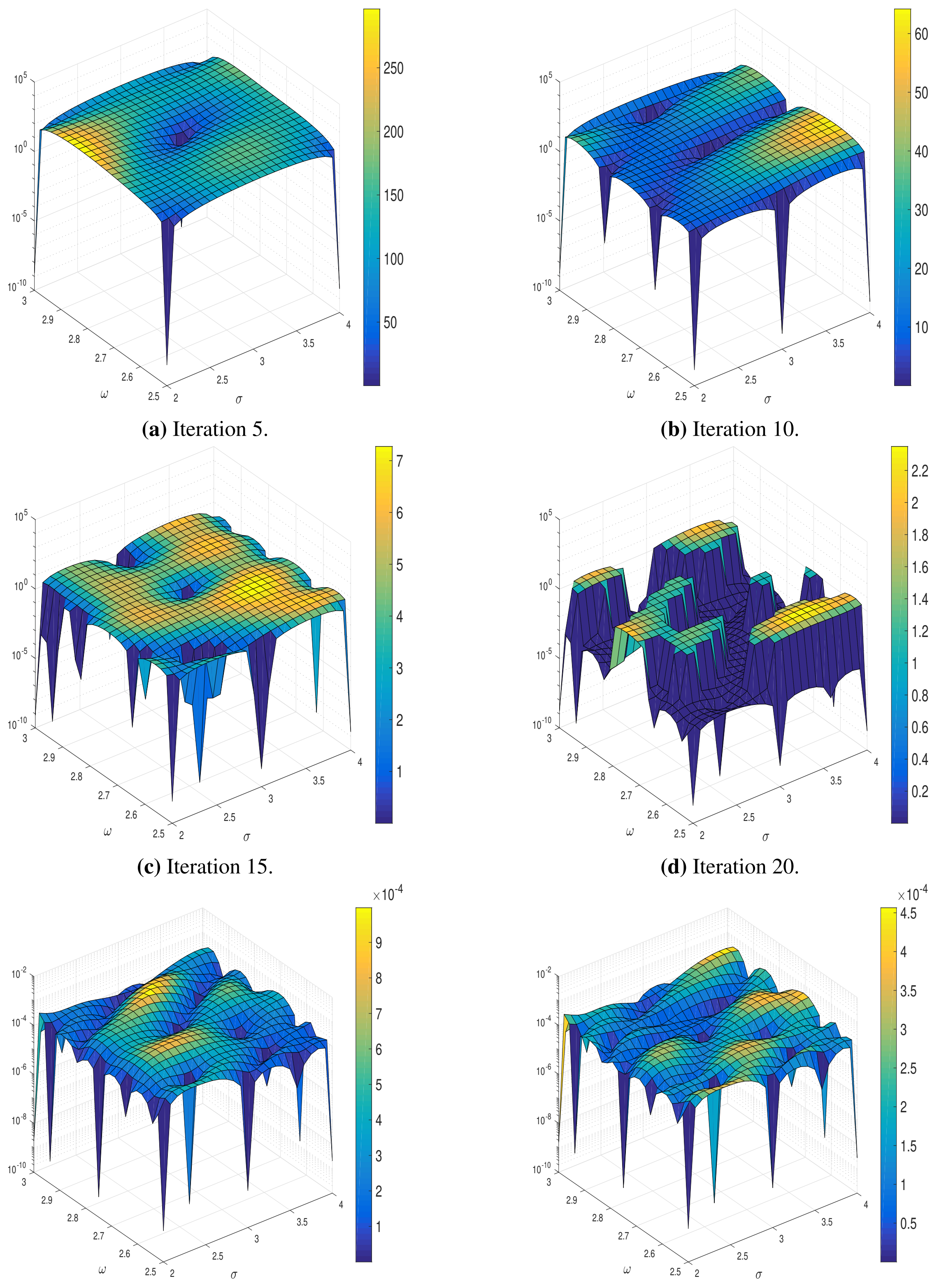
Finally, in Figure 2.4 we show the Greedy Algorithm convergence. We observe that $\tau_N(\boldsymbol{\mu})$ is lower than one when we achieve 25 basis functions, then we are able to compute $\Delta_N(\boldsymbol{\mu})$ finally stopping the algorithm with 28 basis function.

It is interesting to take a look at the parameter selection. In Figure 2.5 we show the *a posteriori* error bound estimator for six different iterations of the Greedy Algorithm. As we have explained before, in figures 2.5a-2.5d we show $\tau_N(\boldsymbol{\mu})$ while in figures 2.5e and 2.5f we show $\Delta_N(\boldsymbol{\mu})$. We observe the progressive decrease of errors.

The Greedy algorithm starts with $\boldsymbol{\mu}^0 = (\omega_{\text{min}}, \sigma_{\text{min}})$ as the first parameter, and in the following iterations it selects $\boldsymbol{\mu}^1 = (\omega_{\text{max}}, \sigma_{\text{max}})$, $\boldsymbol{\mu}^2 = (\omega_{\text{max}}, \sigma_{\text{min}})$ and $\boldsymbol{\mu}^3 = (\omega_{\text{min}}, \sigma_{\text{max}})$ finishing with $\boldsymbol{\mu}^4 = (2.771, 3)$ that is close to $(\omega_{\text{mean}}, \sigma_{\text{mean}})$. This situation is expected since the algorithm is catching the main information in these iterations. This is described in Figure 2.5a.

In Figure 2.5b we still see a homogeneous distribution of the parameter selection and that the total error is decreasing along figures 2.5c and 2.5d.

Finally, in Figure 2.5e we use the *a posteriori* error bound estimator and we see that in three more iterations, the maximum is lower than ε_{RB} and we finish the algorithm (Figure 2.5f).



(a) Iteration 5.

(b) Iteration 10.

(c) Iteration 15.

(d) Iteration 20.

(e) Iteration 25.

(f) Iteration 28.

Figure 2.5: $\Delta_N(\boldsymbol{\mu})$ through the Greedy Algorithm.

Data	Case 1	Case 2	Case 3
Δ_N	$1.21 \cdot 10^{-4}$	$1.07 \cdot 10^{-4}$	$1.89 \cdot 10^{-4}$
$\|\|U_h - U_N\|\|_X$	$5.93 \cdot 10^{-6}$	$3.73 \cdot 10^{-6}$	$7.28 \cdot 10^{-6}$
Efficiency	20.4	28.69	25.96
speedup	142	167	148

Table 2.1: Errors and speedups.

We have tried to keep going with the algorithm looking for decreasing the error to $\varepsilon_{RB} = 10^{-4}$, however, in a few more iterations, the algorithm has reached a point when it selects parameters that already have been selected. This makes the reduced system be singular, thus stopping the algorithm. This can be due to a low precision of the solution of the linear problems that appear in the full order model.

2.4.5 Online phase

In this section, we validate an *a posteriori* error bound estimator $\Delta_N(\boldsymbol{\mu})$ and we compare the computational time between the Reduced Basis and the Finite Element solutions.

To do this, we choose three random pairs of parameters:

- Case 1: $\omega = 2.891$m, $\sigma = 2.734$m,
- Case 2: $\omega = 2.649$m, $\sigma = 2.65$m,
- Case 3: $\omega = 2.469$m, $\sigma = 2.923$m,

and we compute the $U_h(\boldsymbol{\mu})$ and $U_N(\boldsymbol{\mu})$ for each one studying the error between them. This computation has been made on a Mac Book Pro 2017 with a 2.3 GHz Intel Core i5 processor.

The computational time for the Finite Element (FE) solutions $U_h(\boldsymbol{\mu})$ is between 3 and 3.60min while for the Reduced Basis solutions $U_N(\boldsymbol{\mu})$ is between 1.3 and 1.48s, which is a considerable improvement. In the Table 2.1 we show the value of the estimator $\Delta_N(\boldsymbol{\mu})$, the error between both solutions in the X-norm, the estimator efficiency and the computing time relation between the FE and the RB solutions. The speed-ups factors are close to 150.

Finally, in Figures 2.6-2.7 we show a comparison between the estimate $\Delta_N(\boldsymbol{\mu})$ and the real error $\|U_h - U_N\|_X$. As we can see, both have the same shape and the real error is lower than the estimate, which validates the estimate and the computation made in this section.

Additionally, in figures 2.8 and 2.9 we show the velocity field and the pressure, respectively, for the three cases. The domain represented in these figures is $\Omega(\boldsymbol{\mu})$ for each case.

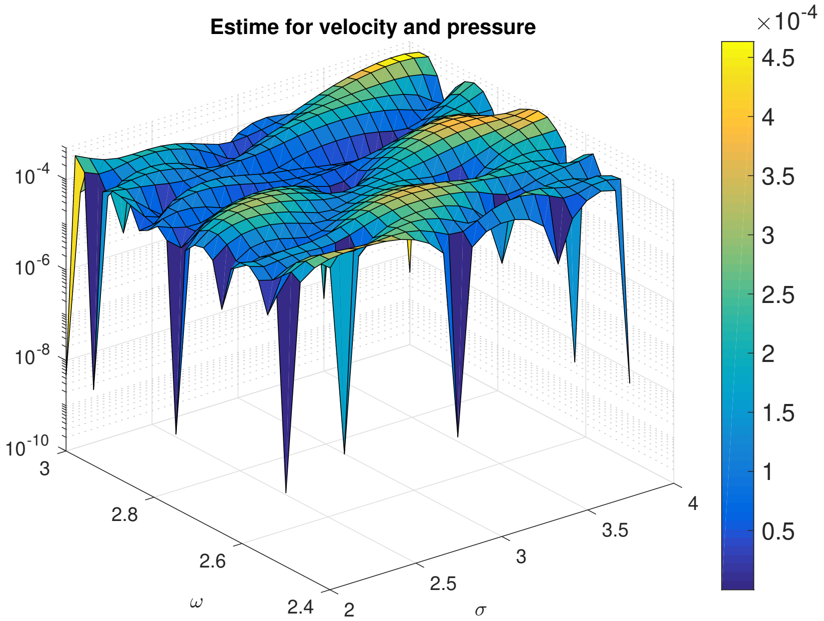


Figure 2.6: Final estimate $\Delta_N(\boldsymbol{\mu})$.

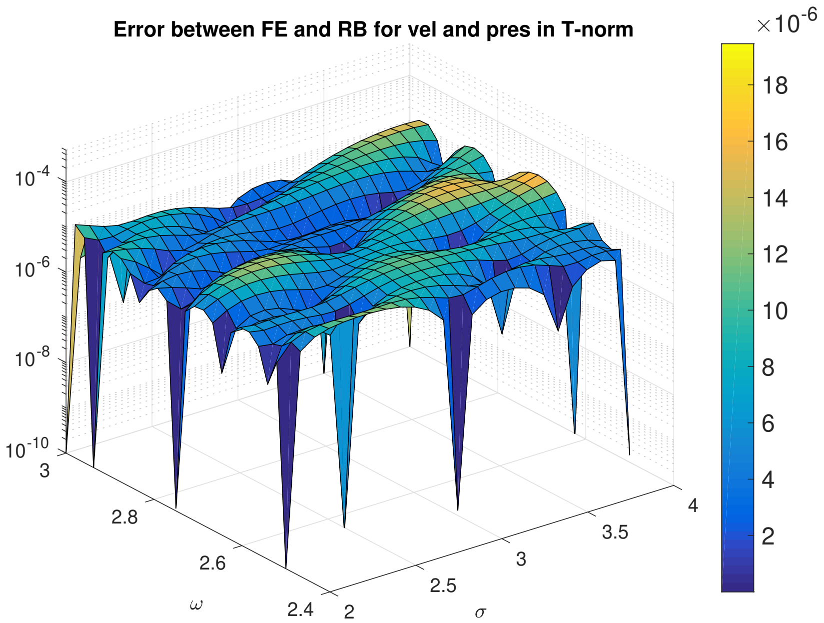


Figure 2.7: Final error $\|U_h - U_N\|_X$.

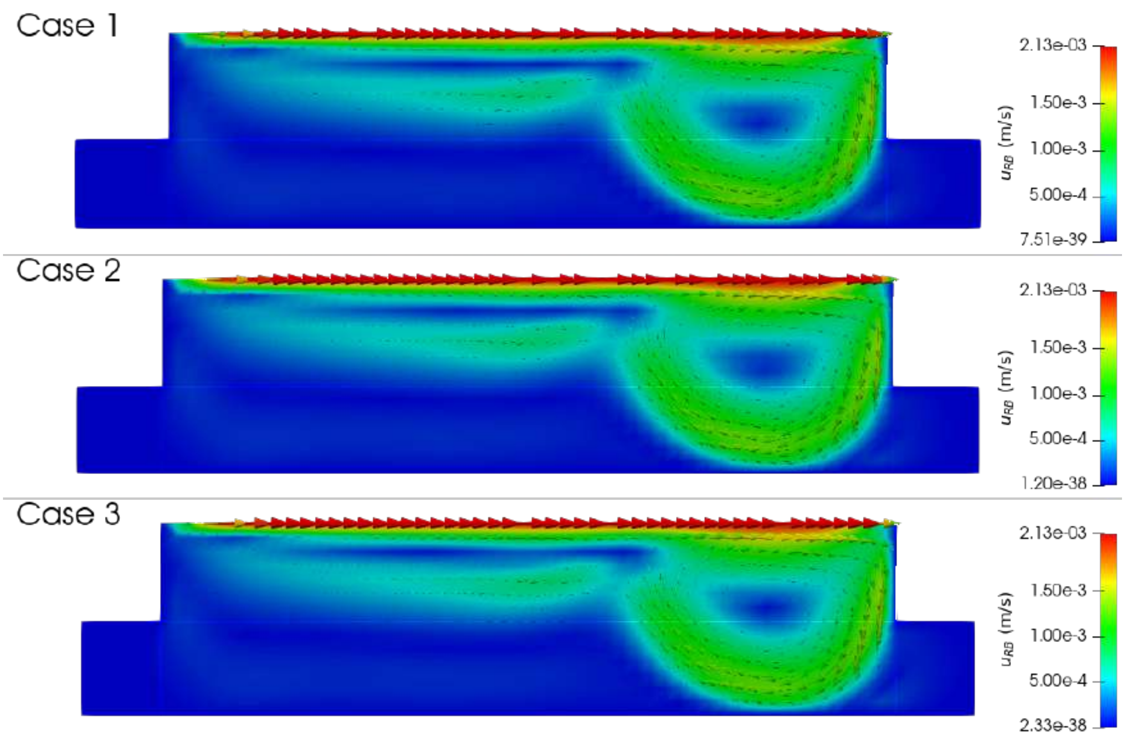


Figure 2.8: Velocity field for each case.

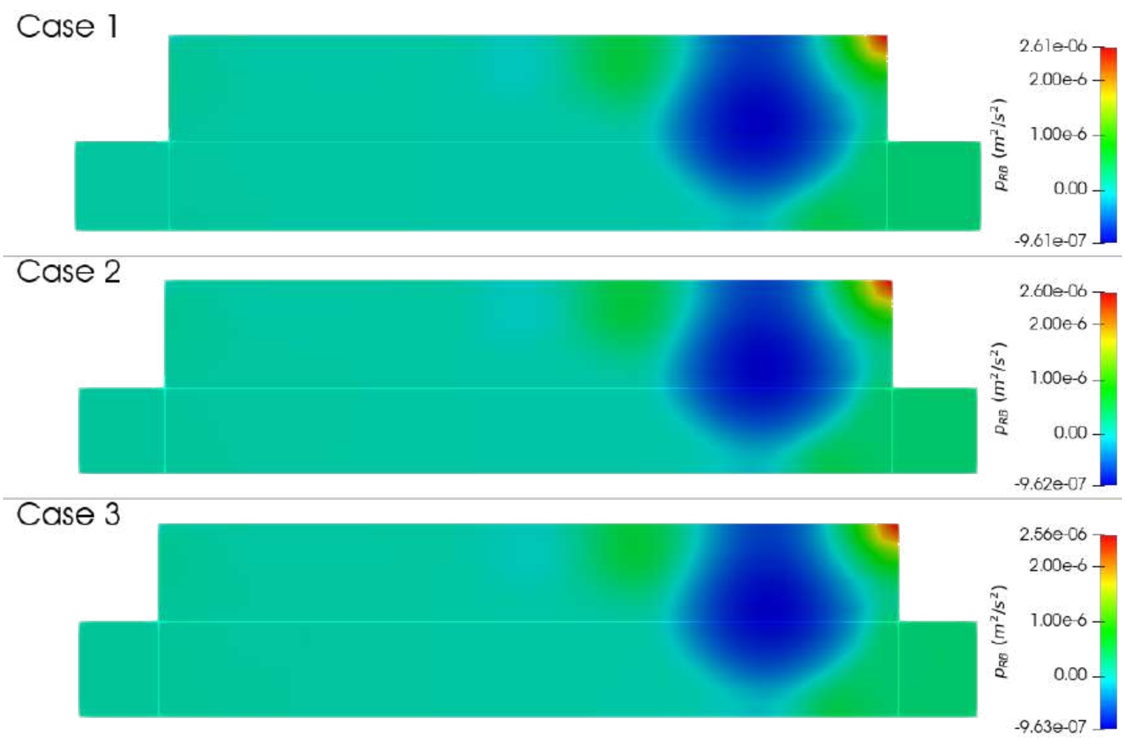


Figure 2.9: Pressure for each case.

3

Reduced Basis steady Smagorinsky model for forced convection flow

In this chapter, we afford the thermal comfort optimization of a cloister by means of Reduced Basis modeling. We first build the RB model and then apply it to the thermal comfort optimization of the cloister. We consider forced convection, in which the momentum conservation equations are decoupled from the temperature equation.

We introduce the new problem in Section 3.1, bringing in the temperature to the Smagorinsky model with the Boussinesq approximation, defining the related RB problem, and explaining the connection with the results presented in Chapter 2. Next, we build an *a posteriori* error bound estimator for the temperature in Section 3.2, following an classical technique using the Lax-Milgram Lemma A.1.

Finally, we show some numerical results in Section 3.3 for a realistic case validating the estimate. In Section 3.4, we also solve an optimization problem which goal is to choose the pair of parameters that give us the best cloister configuration minimizing the minimizing the difference between the temperature and an ideal comfort temperature near the ground.

3.1 Problem statement

In this section, we introduce an aero-thermal model for the cloister. We modify the Smagorinsky model (2.1) including the Boussinesq approximation for temperature.

We consider forced convection in which buoyancy forces are not considered in the momentum conservation equation. This is an acceptable approximation for moderate temperature gradients between the ground and the outer air. We refer to Chapter 14 in [49] for more information. In principle, a steady thermal flow governed by the Smagorinsky turbulence model would obey to the following equations:

$$
\begin{cases}
-\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla\cdot(\nu_t(\mathbf{w};\boldsymbol{\mu})\nabla\mathbf{w}) + \nabla p = -\mathbf{g}\beta(\theta-\theta_0), & \text{in } \Omega(\boldsymbol{\mu}),\\
\nabla\cdot\mathbf{w} = 0, & \text{in } \Omega(\boldsymbol{\mu}),\\
(\mathbf{w}\cdot\nabla)\theta - \nabla\cdot((\kappa+\kappa_t(\mathbf{w}))\nabla\theta) = Q, & \text{in } \Omega(\boldsymbol{\mu}),\\
\mathbf{w} = (g(\cdot;\boldsymbol{\mu}),0), & \text{on } \Gamma_S(\boldsymbol{\mu}),\\
\mathbf{w} = \mathbf{0}, & \text{on } \Gamma_W(\boldsymbol{\mu})\cup\Gamma_F(\boldsymbol{\mu}),\\
\theta = \theta_S, & \text{on } \Gamma_S(\boldsymbol{\mu}),\\
\theta = \theta_F, & \text{on } \Gamma_F(\boldsymbol{\mu}),\\
-\mathbf{n}\cdot((\kappa+\kappa_t(\mathbf{w};\boldsymbol{\mu}))\nabla\theta) = \dfrac{\alpha}{\rho c_p}(\theta-\theta_0), & \text{on } \Gamma_W(\boldsymbol{\mu}).
\end{cases}
\tag{3.1}
$$

An additional term appears in the momentum conservation equation $(3.1)_1$, $-\mathbf{g}\beta(\theta-\theta_0)$, that models the buoyancy force, where β is the coefficient of expansion of the fluid, $\mathbf{g}$ is the gravitational acceleration, and θ_0 represent a reference temperature.

In the energy conservation equation $(3.1)_3$, the first term represents the transport of heat due to the motion of the fluid and it is known as the advection term, while the second one is the diffusion term where we introduce the eddy diffusivity $\kappa_t(\cdot;\boldsymbol{\mu})$, which usually in turbulence modeling is assumed to be proportional to $\nu_t(\cdot;\boldsymbol{\mu})$. At the right-hand side we find the heat generation term Q. The coefficient κ is known as the thermal diffusivity and is defined as $\kappa = k/\rho c_p$, where k is thermal conductivity, ρ is mass density and c_p is the specific heat capacity (see Table 1).

Finally, we impose Dirichlet boundary conditions at the top of the domain ($\Gamma_S(\boldsymbol{\mu})$) and on the floor ($\Gamma_F(\boldsymbol{\mu})$). Moreover, we impose the heat flow on the walls ($\Gamma_W(\boldsymbol{\mu})$), where α is the heat transfer or film coefficient. This is stated in the last equations $(3.1)_6$-$(3.1)_8$.

For simplicity, we do not consider any distributed heating for this problem, then $Q=0$. For a perfect gas, $\beta = 1/\theta$, then the buoyancy term is written as $-\mathbf{g}(\theta-\theta_0)/\theta$. If the absolute variation of the temperature from the ground to the outer air is sufficiently small, then the buoyancy effects can be neglected, and we obtain a variation of the temperature depending on the velocity, not the other way round. This is the situation that we consider, called "forced convection".

To sum up, we assume forced convection, which gives us the following model

$$
(1)\begin{cases}
-\nu\Delta\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{w}-\nabla\cdot(\nu_t(\mathbf{w};\boldsymbol{\mu})\nabla\mathbf{w})+\nabla p=\mathbf{0}, & \text{in } \Omega(\boldsymbol{\mu}),\\
\nabla\cdot\mathbf{w}=0, & \text{in } \Omega(\boldsymbol{\mu}),\\
\mathbf{w}=(g(\cdot;\boldsymbol{\mu}),0), & \text{on } \Gamma_S(\boldsymbol{\mu}),\\
\mathbf{w}=\mathbf{0}, & \text{on } \Gamma_W(\boldsymbol{\mu})\cup\Gamma_F(\boldsymbol{\mu}),
\end{cases} \tag{3.2}
$$

$$
(2)\begin{cases}
(\mathbf{w}\cdot\nabla)\theta-\nabla\cdot((\kappa+\kappa_t(\mathbf{w};\boldsymbol{\mu}))\nabla\theta)=0, & \text{in } \Omega(\boldsymbol{\mu}),\\
\theta=\theta_S, & \text{on } \Gamma_S(\boldsymbol{\mu}),\\
\theta=\theta_F, & \text{on } \Gamma_F(\boldsymbol{\mu}),\\
-\mathbf{n}\cdot((\kappa+\kappa_t(\mathbf{w};\boldsymbol{\mu}))\nabla\theta)=\dfrac{\alpha}{\rho c_p}(\theta-\theta_0), & \text{on } \Gamma_W(\boldsymbol{\mu}).
\end{cases} \tag{3.3}
$$

where we first compute the velocity $\mathbf{w}$ and the pressure p in (3.2) and in a second step, we compute the temperature θ in (3.3).

As the velocity does not depend on the temperature, we just need to build a RB problem for the temperature.

We start by obtaining the dimensionless problem following the same procedure as in Section 2.1. We consider $\theta_0=\theta_C$ where θ_C is an ideal comfort temperature and we set the dimensionless temperature $\theta^*=\theta/\theta_C$. We obtain the dimensionless equations

$$
\begin{cases}
(\mathbf{w}^*\cdot\nabla^*)\theta^*-\nabla^*\cdot\left(\left(\dfrac{1}{\mathrm{Pe}}+\kappa_t^*(\mathbf{w}^*;\boldsymbol{\mu}^*)\right)\nabla^*\theta^*\right)=0, & \text{in } \Omega(\boldsymbol{\mu}^*),\\
\theta^*=\dfrac{\theta_S}{\theta_C}, & \text{on } \Gamma_S(\boldsymbol{\mu}^*),\\
\theta^*=\dfrac{\theta_F}{\theta_C}, & \text{on } \Gamma_F(\boldsymbol{\mu}^*),\\
-\mathbf{n}\cdot\left(\left(\dfrac{1}{\mathrm{Pe}}+\kappa_t^*(\mathbf{w}^*;\boldsymbol{\mu}^*)\right)\nabla^*\theta^*\right)=\dfrac{\mathrm{Nu}}{\mathrm{Pe}}(\theta^*-1), & \text{on } \Gamma_W(\boldsymbol{\mu}^*),
\end{cases} \tag{3.4}
$$

where $\mathbf{w}=U_0\mathbf{w}^*$,

$$\kappa_t^*(\mathbf{w}^*;\boldsymbol{\mu}^*)=\frac{1}{\mathrm{Pr}}\nu_t^*(\mathbf{w}^*;\boldsymbol{\mu}^*),$$

where we recall that $\nu_t^*(\mathbf{w}^*;\boldsymbol{\mu}^*)$ was defined in (2.2), Pr, Pe and Nu are the Prandlt, Péclet and Nusselt numbers defined in Appendix (A.21), (A.23) and (A.24) respectively, such that

$$\mathrm{Pr}=\frac{c_p\mu}{k},\quad \mathrm{Pe}=\mathrm{Re}\,\mathrm{Pr},\quad \mathrm{Nu}=\frac{\alpha L}{k},$$

where μ is the dynamical viscosity. Again, from now on, we avoid the *star* notation for simplicity.

Once more, we need homogeneous boundary conditions on Γ_S and Γ_F for the temperature to apply the RB method. Following the same procedure as for the velocity, we consider a lift θ_D for the temperature, thus the problem to solve will be

$$\begin{cases} (\mathbf{w}\cdot\nabla)\hat{\theta} - \nabla\cdot\left(\left(\frac{1}{\mathrm{Pe}} + \kappa_t(\mathbf{w};\boldsymbol{\mu})\right)\nabla(\hat{\theta}+\theta_D)\right) = -(\mathbf{w}\cdot\nabla)\theta_D, & \text{in } \Omega(\boldsymbol{\mu}), \\ \hat{\theta} = 0, & \text{on } \Gamma_{S,F}(\boldsymbol{\mu}), \\ -\mathbf{n}\cdot\left(\left(\frac{1}{\mathrm{Pe}} + \kappa_t(\mathbf{w};\boldsymbol{\mu})\right)\nabla(\hat{\theta}+\theta_D)\right) = \frac{\mathrm{Nu}}{\mathrm{Pe}}(\hat{\theta}+\theta_D-1), & \text{on } \Gamma_W(\boldsymbol{\mu}). \end{cases} \tag{3.5}$$

In this case, θ_D only should satisfy

- $\theta_D = \theta_S/\theta_C$ on $\Gamma_S(\boldsymbol{\mu})$,
- $\theta_D = \theta_F/\theta_C$ on $\Gamma_F(\boldsymbol{\mu})$.

It is easy to define an analytic expression for θ_D, for example, we can take

$$\theta_D(\mathbf{x}) = \frac{\theta_S-\theta_F}{\theta_C H}x_2 + \frac{\theta_F}{\theta_C}. \tag{3.6}$$

Again, $\theta = \hat{\theta} + \theta_D$ is unique and does not depend on the lift choice.

We obtain the dimensionless equations. Let us consider the space

$$\Theta(\boldsymbol{\mu}) = \{\theta \in H^1(\Omega(\boldsymbol{\mu})) \mid \theta_{|\Gamma_S\cup\Gamma_F} = 0\}$$

endowed with the $H_0^1(\Omega(\boldsymbol{\mu}))$-norm. Then, problem (3.5) admits the variational formulation

$$\begin{cases} \text{For a given } \mathbf{u} = \mathbf{u}(\boldsymbol{\mu}) \in Y(\boldsymbol{\mu}),\ \mathbf{w} = \mathbf{u} + \mathbf{u}_D, \\ \text{find } \theta^{\mathbf{u}} = \theta^{\mathbf{u}}(\boldsymbol{\mu}) \in \Theta(\boldsymbol{\mu}) \text{ such that} \\ c^\theta(\mathbf{w},\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) + a^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) + a_t^\theta(\mathbf{w},\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) \\ +d^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) = -c^\theta(\mathbf{w},\theta_D,\theta^{\mathbf{v}};\boldsymbol{\mu}) - a^\theta(\theta_D,\theta^{\mathbf{v}};\boldsymbol{\mu}) \\ -a_t^\theta(\mathbf{w},\theta_D,\theta^{\mathbf{v}};\boldsymbol{\mu}) - d^\theta(\theta_D-1,\theta^{\mathbf{v}};\boldsymbol{\mu}), \quad \forall\theta^{\mathbf{v}} \in \Theta(\boldsymbol{\mu}) \end{cases} \tag{3.7}$$

where its solution $\theta^{\mathbf{u}}$ is the temperature associated to the velocity $\mathbf{u}$ solution of (2.4). The bilinear forms $a^\theta(\cdot,\cdot;\boldsymbol{\mu})$ and $d^\theta(\cdot,\cdot;\boldsymbol{\mu})$ are defined by

$$a^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) = \frac{1}{\mathrm{Pe}}\int_{\Omega(\boldsymbol{\mu})}\nabla\theta^{\mathbf{u}}\cdot\nabla\theta^{\mathbf{v}}d\Omega,\ d^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) = \frac{\mathrm{Nu}}{\mathrm{Pe}}\int_{\Gamma_W(\boldsymbol{\mu})}\theta^{\mathbf{u}}\theta^{\mathbf{v}}\,d\Gamma,$$

and the trilinear forms, $c^\theta(\cdot,\cdot,\cdot;\boldsymbol{\mu})$ and $a_t^\theta(\cdot,\cdot,\cdot;\boldsymbol{\mu})$ are given by

$$c^\theta(\mathbf{w},\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) = \int_{\Omega(\boldsymbol{\mu})} (\mathbf{w}\cdot\nabla\theta^{\mathbf{u}})\theta^{\mathbf{v}} d\Omega,$$

$$a_t^\theta(\mathbf{w},\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) = \int_{\Omega(\boldsymbol{\mu})} \kappa_t(\mathbf{w};\boldsymbol{\mu})\nabla\theta^{\mathbf{u}}\cdot\nabla\theta^{\mathbf{v}} d\Omega.$$

Again, these forms need to be expressed as integrals on the reference domain. We follow the same procedure as in Chapter 2 using the transformation (2.5) and defining the matrices from equations (2.6)-(2.11).

Let define $\Omega_r = \Omega(\boldsymbol{\mu}_r)$ and by extension $\Theta_r = \Theta(\boldsymbol{\mu}_r)$, then problem (3.7) reads:

$$\begin{cases} \text{For a given } \mathbf{u} = \mathbf{u}(\boldsymbol{\mu}) \in Y_r,\ \mathbf{w} = \mathbf{u} + \mathbf{u}_D, \\ \text{find } \theta^{\mathbf{u}} = \theta^{\mathbf{u}}(\boldsymbol{\mu}) \in \Theta_r \text{ such that for all } \theta^{\mathbf{v}} \in \Theta_r \\ \displaystyle\sum_{l=1}^{4}\sum_{k=1}^{2}[\eta_{kk}^l c_{kl}^\theta(\mathbf{w},\theta^{\mathbf{u}},\theta^{\mathbf{v}}) + \gamma_{kk}^l(a_{kl}^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}}) + a_{t,kl}^\theta(\mathbf{w},\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}))] \\ \displaystyle +\sum_{l=1}^{4}|J_{\Phi_l}|d_l^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}}) = -\sum_{l=1}^{4}\sum_{k=1}^{2}[\eta_{kk}^l c_{kl}^\theta(\mathbf{w},\theta_D,\theta^{\mathbf{v}}) \\ \displaystyle +\gamma_{kk}^l(a_{kl}^\theta(\theta_D,\theta^{\mathbf{v}}) + a_{t,kl}^\theta(\mathbf{w},\theta_D,\theta^{\mathbf{v}};\boldsymbol{\mu}))] - \sum_{l=1}^{4}|J_{\Phi_l}|d_l^\theta(\theta_D - 1,\theta^{\mathbf{v}}) \end{cases} \tag{3.8}$$

where for $k = 1,2$ and $l = 1,2,3,4$:

$$a_{kl}^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}}) = \frac{1}{\mathrm{Pe}}\int_{\Omega_l} \partial_k\theta^{\mathbf{u}}\partial_k\theta^{\mathbf{v}}\, d\Omega,\ d_l^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}}) = \frac{\mathrm{Nu}}{\mathrm{Pe}}\int_{\Gamma_{W,l}} \theta^{\mathbf{u}}\theta^{\mathbf{v}}\, d\Omega,$$

$$c_{kl}^\theta(\mathbf{u},\theta^{\mathbf{u}},\theta^{\mathbf{v}}) = \int_{\Omega_l} u_k\partial_k\theta^{\mathbf{u}}\theta^{\mathbf{v}}\, d\Omega,\ a_{t,kl}^\theta(\mathbf{u},\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}) = \int_{\Omega_l} \kappa_t(\mathbf{u};\boldsymbol{\mu})\partial_k\theta^{\mathbf{u}}\partial_k\theta^{\mathbf{v}}\, d\Omega,$$

where $\kappa_t^l(\mathbf{u};\boldsymbol{\mu}) = \nu_t^l(\mathbf{u};\boldsymbol{\mu})/Pr$ and we recall that $\nu_t^l(\cdot;\boldsymbol{\mu})$ has been defined in (2.19) and $\Gamma_{W,l} = \Gamma_W \cap \bar{\Omega}_l$. Observe that $d_2^\theta(\theta^{\mathbf{u}},\theta^{\mathbf{v}}) = 0$ for all $\theta^{\mathbf{u}},\ \theta^{\mathbf{v}} \in \Omega_r$ since $\Gamma_{W,2} = \emptyset$.

We assume that the velocity and the pressure are computed by the RB method introduced in Chapter 2. Therefore, from now on, we set $\mathbf{w}_N = \mathbf{u}_N + \mathbf{u}_D$ where $\mathbf{u}_N = \mathbf{u}_N(\boldsymbol{\mu}) \in Y_N$ is the solution of (2.3).

Now, we are able to state the discrete formulation for problem (3.8). Let Θ_h be a finite dimensional space, inner approximation of Θ_r. By extension, we endow Θ_h with the $H_0^1(\Omega)$-norm. Then, we consider the following discretization in its compact form:

$$\begin{cases} \text{For a given } \mathbf{u}_N = \mathbf{u}_N(\boldsymbol{\mu}) \in Y_N,\ \mathbf{w}_N = \mathbf{u}_N + \mathbf{u}_D, \\ \text{find } \theta_h^{\mathbf{u}_N}(\boldsymbol{\mu}) \in \Theta_h \text{ such that} \\ A^\theta(\mathbf{w}_N,\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu}),\theta_h^{\mathbf{v}};\boldsymbol{\mu}) = F^\theta(\mathbf{w}_N,\theta_h^{\mathbf{v}};\boldsymbol{\mu}), \quad \forall\theta_h^{\mathbf{v}} \in \Theta_h \end{cases} \tag{3.9}$$

where the operators $A^\theta(\mathbf{w}_N,\cdot,\cdot;\boldsymbol{\mu}):\Theta_h\times\Theta_h\longrightarrow\mathbb{R}$ and $F^\theta(\mathbf{w}_N,\cdot;\boldsymbol{\mu}):\Theta_h\longrightarrow\mathbb{R}$ are defined by

$$A^\theta(\mathbf{w}_N,\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu})=\sum_{l=1}^{4}\sum_{k=1}^{2}[\eta_{kk}^{l}c_{kl}^{\theta}(\mathbf{w}_N,\theta^{\mathbf{u}},\theta^{\mathbf{v}})$$
$$+\gamma_{kk}^{l}(a_{kl}^{\theta}(\theta^{\mathbf{u}},\theta^{\mathbf{v}})+a_{t,kl}^{\theta}(\mathbf{w}_N,\theta^{\mathbf{u}},\theta^{\mathbf{v}};\boldsymbol{\mu}))]+\sum_{l=1}^{4}|J_{\Phi_l}|d_l^{\theta}(\theta^{\mathbf{u}},\theta^{\mathbf{v}})$$

and

$$F^\theta(\mathbf{w}_N,\theta^{\mathbf{v}};\boldsymbol{\mu})=-\sum_{l=1}^{4}\sum_{k=1}^{2}[\eta_{kk}^{l}c_{kl}^{\theta}(\mathbf{w}_N,\theta_D,\theta^{\mathbf{v}})$$
$$+\gamma_{kk}^{l}(a_{kl}^{\theta}(\theta_D,\theta^{\mathbf{v}})+a_{t,kl}^{\theta}(\mathbf{w}_N,\theta_D,\theta^{\mathbf{v}};\boldsymbol{\mu}))]-\sum_{l=1}^{4}|J_{\Phi_l}|d_l^{\theta}(\theta_D-1,\theta^{\mathbf{v}}).$$

We state the RB problem establishing RB spaces and matrices. Let $\Theta_N\subset\Theta_h$ be the RB space with $N\in\mathbb{N}$ the number of basis functions

$$\Theta_N=\operatorname{span}\{\vartheta^i=\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu}^i),\ i=1,\dots,N\},$$

where $\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu}^i)$ is the solution of (3.9) for a suitable parameter $\boldsymbol{\mu}^i$ for $i=1,\dots,N$. We state the RB problem by

$$\begin{cases}\text{For a given } \mathbf{u}_N=\mathbf{u}_N(\boldsymbol{\mu})\in Y_N,\ \mathbf{w}_N=\mathbf{u}_N+\mathbf{u}_D,\\ \text{find } \theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})\in\Theta_N \text{ such that}\\ A^\theta(\mathbf{w}_N,\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu}),\theta_N^{\mathbf{v}};\boldsymbol{\mu})=F^\theta(\mathbf{w}_N,\theta_N^{\mathbf{v}};\boldsymbol{\mu}),\quad\forall\theta_N^{\mathbf{v}}\in\Theta_N.\end{cases}\tag{3.10}$$

Note that as the solution $\theta_N(\boldsymbol{\mu})$ can be expressed as a linear combination of the selected basis functions

$$\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})=\sum_{i=1}^{N}\underline{\theta}_N^i(\boldsymbol{\mu})\vartheta^i,$$

and equation (3.10) is linear, then the coefficients $\{\underline{\theta}_N^i\}_{i=1}^N$ are obtained from the solution of the reduced linear system.

3.2 *A posteriori* error estimator

In this section, we study the well-posed of problem (3.9), which guaranties the existence and uniqueness of solution applying the Lax-Milgram Lemma A.1. Then, supporting us on this result, we are able to obtain an *a posteriori* error bound estimator. We follow the procedure explained in Section 3.6 in [38].

First, we study continuity and coercivity of the operator $A^{\theta}(\mathbf{w}_N,\cdot,\cdot;\boldsymbol{\mu})$, using the notation introduced in Remark 2.1. For simplicity of notation, we denote the reference domain by Ω instead of Ω_r. This will not source of confusion as there are no other domains to deal with. Hereunder, we define some constants necessary for the results.

Remark 3.1. *For the next results, we introduce some constants that are derived from the application of relevant results exposed in Appendix A.2:*

- *We recall the definition of the constant $C_{4;1,2} > 0$ from the application of the Sobolev Embedding Theorem A.2 in* (A.9)*, this is,*

$$\|\theta_h^{\boldsymbol{v}}\|_{L^4(\Omega)} \leq C_{4;1,2}\|\theta_h^{\boldsymbol{v}}\|_{H_0^1(\Omega)}, \quad \forall \theta_h^{\boldsymbol{v}} \in \Theta_h. \tag{3.11}$$

- *For all $K \in \mathcal{T}_h$, we apply the Local Inverse Inequality Theorem A.3 as in* (A.11) *introducing the constant $C_{3;2} > 0$ such that*

$$\|\nabla\theta_h^{\boldsymbol{v}}\|_{L^3(K)} \leq C_{3;2} h_K^{-d/6}\|\nabla\theta_h^{\boldsymbol{v}}\|_{L^2(K)}, \quad \forall K \in \mathcal{T}_h, \forall \theta_h^{\boldsymbol{v}} \in \Theta_h. \tag{3.12}$$

- *We recall the constant $C_{\Gamma_W} > 0$ as a result of applying the Trace Theorem A.6 on Γ_W and the equivalence between the $H^1(\Omega)$ and $H_0^1(\Omega)$ norms, this is,*

$$\|\theta_h^{\boldsymbol{v}}\|_{L^2(\Gamma_W)} \leq C_{\Gamma_W}\|\theta_h^{\boldsymbol{v}}\|_{H_0^1(\Omega)}, \quad \forall \theta_h^{\boldsymbol{v}} \in \Theta_h. \tag{3.13}$$

We recall that $\mathbf{w}_N = \mathbf{u}_N + \mathbf{u}_D$ where $\mathbf{u}_N$ is the solution of (2.3).

Theorem 3.1 (Continuity)**.** *For any $\boldsymbol{\mu} \in \mathcal{D}$ and $\boldsymbol{u}_N = \boldsymbol{u}_N(\boldsymbol{\mu}) \in Y_N$, it holds*

$$|A^{\theta}(\boldsymbol{w}_N, \theta_h^{\boldsymbol{u}_N}, \theta_h^{\boldsymbol{v}}; \boldsymbol{\mu})| \leq \gamma_h(\boldsymbol{\mu})\|\theta_h^{\boldsymbol{u}_N}\|_{H_0^1(\Omega)}\|\theta_h^{\boldsymbol{v}}\|_{H_0^1(\Omega)}, \quad \forall \theta_h^{\boldsymbol{u}_N}, \theta_h^{\boldsymbol{v}} \in \Theta_h,$$

where

$$\gamma_h(\boldsymbol{\mu}) = \eta^* C_{4;1,2}^2\|\nabla\boldsymbol{w}_N\|_{L^2(\Omega)} + \gamma^*\left(\frac{1}{Pe} + \sqrt{\varphi^*}C_{3;2}^2\frac{C_S^2(h_{\max}^*)^{2-d/3}}{\mathrm{Pr}}\|\nabla\boldsymbol{w}_N\|_{L^3(\Omega)}\right) + C_{\Gamma_W}^2\max_{l=1,3,4}|J_{\Phi_l}|\frac{Nu}{Pe}$$

where the constants $C_{4;1,2}$, $C_{3;2}$ and C_{Γ_W} are defined in (3.11)-(3.13).

Proof. First, we know that

$$|A^\theta(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu})| \le \sum_{l=1}^{4}\sum_{k=1}^{2}\eta_{kk}^l|c_{kl}^\theta(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}})| + \sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^l|a_{kl}^\theta(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}})|$$
$$+\sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^l|a_{t,kl}^\theta(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu})| + \sum_{l=1}^{4}|J_{\Phi_l}||d_l^\theta(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}})|.$$

We separately study each term:

- At first,

$$\begin{aligned}\sum_{l=1}^{4}\sum_{k=1}^{2}\eta_{kk}^l|c_{kl}^\theta(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}})| &\le \eta^*\int_\Omega |\mathbf{w}_N||\nabla\theta_h^{\mathbf{u}_N}||\theta_h^{\mathbf{v}}|d\Omega\\ &\le \eta^*\|\mathbf{w}_N\|_{L^4(\Omega)}\|\nabla\theta_h^{\mathbf{u}_N}\|_{L^2(\Omega)}\|\theta_h^{\mathbf{v}}\|_{L^4(\Omega)}\\ &\le \eta^* C_{4;1,2}^2\|\nabla\mathbf{w}_N\|_{L^2(\Omega)}\|\theta_h^{\mathbf{u}_N}\|_{H_0^1(\Omega)}\|\theta_h^{\mathbf{v}}\|_{H_0^1(\Omega)},\end{aligned}$$

  where $C_{4;1,2}$ is defined in (3.11).

- Next,

$$\sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^l|a_{kl}^\theta(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}})| \le \frac{\gamma^*}{\mathrm{Pe}}\int_\Omega|\nabla\theta_h^{\mathbf{u}_N}||\nabla\theta_h^{\mathbf{v}}|d\Omega \le \frac{\gamma^*}{\mathrm{Pe}}\|\theta_h^{\mathbf{u}_N}\|_{H_0^1(\Omega)}\|\theta_h^{\mathbf{v}}\|_{H_0^1(\Omega)}.$$

- Moreover,

$$\begin{aligned}&\sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^l|a_{t,kl}^\theta(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu})|\\ &\quad\le \gamma^*\sqrt{\varphi^*}\sum_{K\in\mathcal{T}_h}\int_K\frac{(C_Sh_K)^2}{\mathrm{Pr}}|\nabla\mathbf{w}_N||\nabla\theta_h^{\mathbf{u}_N}||\nabla\theta_h^{\mathbf{v}}|dK\\ &\quad\le \gamma^*\sqrt{\varphi^*}\sum_{K\in\mathcal{T}_h}\frac{(C_Sh_K)^2}{\mathrm{Pr}}\|\nabla\mathbf{w}_N\|_{L^3(K)}\|\nabla\theta_h^{\mathbf{u}_N}\|_{L^3(K)}\|\nabla\theta_h^{\mathbf{v}}\|_{L^3(K)}\\ &\quad\le \gamma^*\sqrt{\varphi^*}\sum_{K\in\mathcal{T}_h}C_{3;2}^2\frac{C_S^2h_K^{2-d/3}}{\mathrm{Pr}}\|\nabla\mathbf{w}_N\|_{L^3(K)}\|\nabla\theta_h^{\mathbf{u}_N}\|_{L^2(K)}\|\nabla\theta_h^{\mathbf{v}}\|_{L^2(K)}\\ &\quad\le \gamma^*\sqrt{\varphi^*}C_{3;2}^2\frac{C_S^2(h_{\max}^*)^{2-d/3}}{\mathrm{Pr}}\|\nabla\mathbf{w}_N\|_{L^3(\Omega)}\|\nabla\theta_h^{\mathbf{u}_N}\|_{L^2(\Omega)}\|\nabla\theta_h^{\mathbf{v}}\|_{L^2(\Omega)}\end{aligned}$$

  where we have used the Local Inverse Inequality (3.12).

- Finally,

$$\sum_{l=1}^{4}|J_{\Phi_l}||d_l^{\theta}(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}})| \le \max_{l=1,3,4}|J_{\Phi_l}|\frac{\text{Nu}}{\text{Pe}}\int_{\Gamma_W}\theta_h^{\mathbf{u}_N}\theta_h^{\mathbf{v}}\,d\Gamma$$
$$\le \max_{l=1,3,4}|J_{\Phi_l}|\frac{\text{Nu}}{\text{Pe}}\|\theta_h^{\mathbf{u}_N}\|_{L^2(\Gamma_W)}\|\theta_h^{\mathbf{v}}\|_{L^2(\Gamma_W)}$$
$$\le C_{\Gamma_W}^2\max_{l=1,3,4}|J_{\Phi_l}|\frac{\text{Nu}}{\text{Pe}}\|\theta_h^{\mathbf{u}_N}\|_{H_0^1(\Omega)}\|\theta_h^{\mathbf{v}}\|_{H_0^1(\Omega)}$$

where C_{Γ_W} was defined in (3.13). Note that in the max operator, the l-index is taking the values $1,3,4$ since $l=2$ refers to Ω_2 and $\Omega_2\cap\Gamma=\emptyset$.

From these bounds, the continuity constant is defined as

$$\gamma_h(\boldsymbol{\mu}) = \eta^* C_{4;1,2}^2\|\nabla\mathbf{w}_N\|_{L^2(\Omega)} + \gamma^*\left(\frac{1}{\text{Pe}} + \sqrt{\varphi^*}C_{3;2}^2\frac{C_S^2(h_{\max}^*)^{2-d/3}}{\text{Pr}}\|\nabla\mathbf{w}_N\|_{L^3(\Omega)}\right) + C_{\Gamma_W}^2\max_{l=1,3,4}|J_{\Phi_l}|\frac{\text{Nu}}{\text{Pe}}$$

and the dependency on $\boldsymbol{\mu}$ takes places through $\mathbf{w}_N = \mathbf{u}_N + \mathbf{u}_D$. □

Theorem 3.2 (Coercivity). *For any $\boldsymbol{\mu}\in\mathcal{D}$ and $\boldsymbol{u}_N=\boldsymbol{u}_N(\boldsymbol{\mu})\in Y_N$, let us suppose that $\|\boldsymbol{w}_N-\boldsymbol{w}\|_{H_0^1(\Omega(\boldsymbol{\mu}))}$ is sufficiently small. Then it holds,*

$$A^{\theta}(\boldsymbol{w}_N,\theta_h^{\boldsymbol{u}_N},\theta_h^{\boldsymbol{u}_N};\boldsymbol{\mu}) \ge \beta(\boldsymbol{\mu})\|\theta_h^{\boldsymbol{u}_N}\|_{H_0^1(\Omega)}^2,\quad \forall\theta_h^{\boldsymbol{u}_N}\in\Theta_h,$$

where

$$\beta(\boldsymbol{\mu}) = \hat{\gamma}\left(\frac{1}{Pe} + \min_{\boldsymbol{x}\in\Omega}\frac{\nu_t(\boldsymbol{w}_N;\boldsymbol{\mu})}{Pr}\right) - \frac{\varepsilon_N(\boldsymbol{\mu})}{2}$$

and

$$\varepsilon_N(\boldsymbol{\mu}) = C_{4;1,2}^2\max_{l=1,2,3,4}|J_{\Phi_l}|^{1/2}\|\boldsymbol{w}_N-\boldsymbol{w}\|_{H_0^1(\Omega(\boldsymbol{\mu}))}.$$

Proof. Let us recall that

$$A^{\theta}(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N};\boldsymbol{\mu}) = \sum_{l=1}^{4}\sum_{k=1}^{2}\eta_{kk}^l c_{kl}^{\theta}(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N}) + \sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^l a_{kl}^{\theta}(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N}) + \sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^l a_{t,kl}^{\theta}(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N};\boldsymbol{\mu}) + \sum_{l=1}^{4}|J_{\Phi_l}|d_l^{\theta}(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N}).$$

It holds

$$\sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^l a_{kl}^{\theta}(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N}) \ge \frac{\hat{\gamma}}{\text{Pe}}\|\nabla\theta_h^{\mathbf{u}_N}\|_{L^2(\Omega)}^2$$

and

$$\sum_{l=1}^{4}\sum_{k=1}^{2}\gamma_{kk}^{l}a_{t,kl}^{\theta}(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N};\boldsymbol{\mu})\geq\hat{\gamma}\sum_{K\in\mathcal{T}_h}\int_K\frac{\nu_t(\mathbf{w}_N;\boldsymbol{\mu})}{\Pr}|\nabla\theta_h^{\mathbf{u}_N}|dK$$
$$\geq\frac{\hat{\gamma}}{\Pr}\min_{\mathbf{x}\in\Omega}\nu_t(\mathbf{w}_N;\boldsymbol{\mu})\|\nabla\theta_h^{\mathbf{u}_N}\|_{L^2(\Omega)}^2$$

We bound the boundary term by

$$\sum_{l=1}^{4}|J_{\Phi_l}|d_l^{\theta}(\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N})\geq\min_{l=1,3,4}|J_{\Phi_l}|\int_{\Gamma_W}|\theta_h^{\mathbf{u}_N}|^2\,d\Gamma=\min_{l=1,3,4}|J_{\Phi_l}|\|\theta_h^{\mathbf{u}_N}\|_{L^2(\Gamma_W)}^2\geq 0.$$

Finally, we upper bound the last term and we prove that it is sufficiently small not to affect the coercivity. We have

$$\sum_{l=1}^{4}\sum_{k=1}^{2}\eta_{kk}^{l}c_{kl}^{\theta}(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N})=\int_{\Omega(\boldsymbol{\mu})}(\mathbf{w}_N\cdot\nabla)(\theta_h^{\mathbf{u}_N})^2d\Omega$$
$$=-\frac{1}{2}\int_{\Omega(\boldsymbol{\mu})}\nabla\cdot\mathbf{w}_N(\theta_h^{\mathbf{u}_N})^2d\Omega \quad (3.14)$$

Since $\mathbf{w}_N=\mathbf{u}_N+\mathbf{u}_D$ and $\mathbf{u}_N$ is the solution of problem (2.26), it verifies that for all $q_N\in Q_N$,

$$\int_{\Omega(\boldsymbol{\mu})}\nabla\cdot\mathbf{w}_N q_N=0.$$

Therefore, $\mathbf{w}_N$ is weakly divergence free and we can not ensure that (3.14) is null, since we can not guarantee that $(\theta_h^{\mathbf{u}_N})^2$ belongs to Q_N. However, $\mathbf{w}$ solution of (2.1) is divergence free, then

$$\left|\int_{\Omega(\boldsymbol{\mu})}\nabla\cdot\mathbf{w}_N(\theta_N^{\mathbf{u}_N})^2\,d\Omega\right|=\left|\int_{\Omega(\boldsymbol{\mu})}\nabla\cdot(\mathbf{w}_N-\mathbf{w})(\theta_N^{\mathbf{u}_N})^2\,d\Omega\right|$$
$$\leq\|\nabla(\mathbf{w}_N-\mathbf{w})\|_{L^2(\Omega(\boldsymbol{\mu}))}\|(\theta_N^{\mathbf{u}_N})^2\|_{L^2(\Omega(\boldsymbol{\mu}))}$$
$$=\|\mathbf{w}_N-\mathbf{w}\|_{H_0^1(\Omega(\boldsymbol{\mu}))}\|\theta_N^{\mathbf{u}_N}\|_{L^4(\Omega(\boldsymbol{\mu}))}^2$$

Hence,

$$\|\theta_N^{\mathbf{u}_N}\|_{L^4(\Omega(\boldsymbol{\mu}))}^2=\left(\int_{\Omega(\boldsymbol{\mu})}(\theta_N^{\mathbf{u}_N})^4\right)^{1/2}=\left(\sum_{l=1}^{4}\int_{\Omega_l}|J_{\Phi_l}|(\theta_N^{\mathbf{u}_N})^4\right)^{1/2}$$
$$\leq\max_{l=1,2,3,4}|J_{\Phi_l}|^{1/2}\left(\int_{\Omega}(\theta_N^{\mathbf{u}_N})^4\right)^{1/2}$$
$$=\max_{l=1,2,3,4}|J_{\Phi_l}|^{1/2}\|\theta_N^{\mathbf{u}_N}\|_{L^4(\Omega)}^2$$
$$\leq C_{4;1,2}^2\max_{l=1,2,3,4}|J_{\Phi_l}|^{1/2}\|\theta_N^{\mathbf{u}_N}\|_{H_0^1(\Omega)}^2$$

where the constant $C_{4;1,2}$ is defined in (3.11). Then,

$$\sum_{l=1}^{4}\sum_{k=1}^{2}\eta_{kk}^{l}c_{kl}^{\theta}(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{u}_N}) > -\frac{\varepsilon_N(\boldsymbol{\mu})}{2}\|\theta_N^{\mathbf{u}_N}\|_{H_0^1(\Omega)}^2$$

where

$$\varepsilon_N(\boldsymbol{\mu}) = C_{4;1,2}^2 \max_{l=1,2,3,4}|J_{\Phi_l}|^{1/2}\|\mathbf{w}_N-\mathbf{w}\|_{H_0^1(\Omega(\boldsymbol{\mu}))}.$$

Finally, the coercivity constant is

$$\beta(\boldsymbol{\mu}) = \hat{\gamma}\left(\frac{1}{Pe}+\min_{\mathbf{x}\in\Omega}\frac{\nu_t(\mathbf{w}_N;\boldsymbol{\mu})}{Pr}\right)-\frac{\varepsilon_N(\boldsymbol{\mu})}{2},$$

which is positive for small enough $\|\mathbf{w}_N-\mathbf{w}\|_{H_0^1(\Omega(\boldsymbol{\mu}))}$. □

Remark 3.2. *If N is sufficiently large, then we can obtain that $\|\boldsymbol{w}_N-\boldsymbol{w}\|_{H_0^1(\Omega(\boldsymbol{\mu}))}$ to be sufficiently small to guarantee that $\beta(\boldsymbol{\mu})$ be positive and in this way, to ensure the coercivity of the form A^θ.*

Once we obtain the continuity and coercivity factors, we are able to infer an *a posteriori* error estimator.

Lemma 3.1. *Let $\boldsymbol{\mu}\in\mathcal{D}$. We denote $e_h^{\boldsymbol{u}_N}(\boldsymbol{\mu}) = \theta_h^{\boldsymbol{u}_N}(\boldsymbol{\mu})-\theta_N^{\boldsymbol{u}_N}(\boldsymbol{\mu})\in\Theta_h$ the error between the high fidelity solution of problem* (3.9) *and the reduced solution of* (3.10)*. Then,*

$$\frac{\|\mathscr{R}(\boldsymbol{w}_N;\theta_N^{\boldsymbol{u}_N}(\boldsymbol{\mu}))\|_{\Theta_h'}}{\gamma_h(\boldsymbol{\mu})}\le\|e_h^{\boldsymbol{u}_N}(\boldsymbol{\mu})\|_{H_0^1(\Omega)}\le\frac{\|\mathscr{R}(\boldsymbol{w}_N;\theta_N^{\boldsymbol{u}_N}(\boldsymbol{\mu}))\|_{\Theta_h'}}{\beta(\boldsymbol{\mu})},\tag{3.15}$$

where the residual operator $\mathscr{R}(\boldsymbol{w}_N;\theta_N^{\boldsymbol{u}_N}(\boldsymbol{\mu}))\in\Theta_h'$ is defined by

$$\langle\mathscr{R}(\boldsymbol{w}_N,\theta_N^{\boldsymbol{u}_N}(\boldsymbol{\mu}));\theta_h^{\boldsymbol{v}}\rangle = F^\theta(\boldsymbol{w}_N,\theta_h^{\boldsymbol{v}};\boldsymbol{\mu})-A^\theta(\boldsymbol{w}_N,\theta_N^{\boldsymbol{u}_N}(\boldsymbol{\mu}),\theta_h^{\boldsymbol{v}};\boldsymbol{\mu}),\quad\forall\theta_h^{\boldsymbol{v}}\in\Theta_h.$$

Proof. We represent $\mathscr{R}(\mathbf{w}_N;\cdot)$ in Θ_h by means of the L^2-inner product by $r(\mathbf{w}_N)\in\Theta_h$:

$$(r(\mathbf{w}_N),\theta_h^{\mathbf{v}})_\Omega = \langle\mathscr{R}(\mathbf{w}_N;\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})),\theta_h^{\mathbf{v}}\rangle,\quad\forall\theta_h^{\mathbf{v}}\in\Theta_h.\tag{3.16}$$

Now taking into account that the problem defined by $e_h^{\mathbf{u}_N}$ can be expressed by

$$\begin{aligned}A^\theta(\mathbf{w}_N,e_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu}) &= A^\theta(\mathbf{w}_N,\theta_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu})-A^\theta(\mathbf{w}_N,\theta_N^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu})\\ &= F^\theta(\mathbf{w}_N,\theta_h^{\mathbf{v}};\boldsymbol{\mu})-A^\theta(\mathbf{w}_N,\theta_N^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu})\\ &= \langle\mathscr{R}(\mathbf{w}_N;\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})),\theta_h^{\mathbf{v}}\rangle.\end{aligned}\tag{3.17}$$

and thanks to the continuity of the operator $A^\theta(\mathbf{w}_N,\cdot,\cdot;\boldsymbol{\mu})$ in Theorem 3.1, it holds

$$|(r(\mathbf{w}_N),\theta_h^{\mathbf{v}})_\Omega| = |A^\theta(\mathbf{w}_N,e_h^{\mathbf{u}_N},\theta_h^{\mathbf{v}};\boldsymbol{\mu})| \le \gamma_h(\boldsymbol{\mu})\|e_h^{\mathbf{u}_N}\|_{H_0^1(\Omega)}\|\theta_h^{\mathbf{v}}\|_{H_0^1(\Omega)}.$$

Due to the definition of dual norm, we obtain that

$$\|\mathscr{R}(\mathbf{w}_N;\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu}))\|_{\Theta_h'} \le \gamma_h(\boldsymbol{\mu})\|e_h^{\mathbf{u}_N}\|_{H_0^1(\Omega)}.$$

On the other hand, we apply the Lax-Milgram Lemma A.1 to the problem (3.17),

$$\|e_h^{\mathbf{u}_N}\|_{H_0^1(\Omega)} \le \frac{\|\mathscr{R}(\mathbf{w}_N;\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu}))\|_{\Theta_h'}}{\beta(\boldsymbol{\mu})}.$$

□

Remark 3.3. *In view of the preceding lemma, we state an a posteriori error estimator as follows*

$$\Delta_{\theta,N}(\boldsymbol{\mu}) = \frac{\|\mathscr{R}(\boldsymbol{w}_N;\theta_N^{\boldsymbol{u}_N}(\boldsymbol{\mu}))\|_{\Theta_h'}}{\beta(\boldsymbol{\mu})}. \tag{3.18}$$

Then the estimation (3.15) *can be rewritten as*

$$\frac{\Delta_{\theta,N}(\boldsymbol{\mu})\beta(\boldsymbol{\mu})}{\gamma_h(\boldsymbol{\mu})} \le \|e_h^{\boldsymbol{u}_N}(\boldsymbol{\mu})\|_{H_0^1(\Omega)} \le \Delta_{\theta,N}(\boldsymbol{\mu}).$$

The efficiency is the quotient between the upper and lower bounds of the error, and in this case is $\gamma_h(\boldsymbol{\mu})/\beta(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathcal{D}$.

3.3 Numerical results

In this section, we apply the RB method to problem (3.9) computing the RB spaces using the estimate (3.18).

We consider the velocity field $\mathbf{u}_N \in Y_N$ solution of (2.26), for this reason we keep the same domain configuration as in Section 2.4.1. Again, we recall that $\mathbf{w}_N = \mathbf{u}_N + \mathbf{u}_D$.

3.3.1 Problem statement

We set the floor temperature equal to the comfort one that we assume to be 295.15K (22°C). We assume that the temperature of the outer air is 303.15K (30°C). Since the characteristic temperature was chosen as $\theta_C = \theta_F$, the dimensionless temperature on the floor should be 1, and 1.4 on the top.

We set the Prandlt number associated to the air equal to 0.71 and the Nusselt number associated to the transfer of temperature at the walls equal to 2 (*cf.* [4]). The Péclet number appearing in the temperature equation is defined by $\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr}$. Considering that $\mathrm{Re} = 3\,100$, then $\mathrm{Pe} = 2\,201$. Finally, we set the parameter set $\mathcal{D}$ as in Section 2.4.1, that is $\boldsymbol{\mu} = (\omega,\sigma) \in \mathcal{D} = [2\mathrm{m},4\mathrm{m}] \times [2.5\mathrm{m},3\mathrm{m}]$.

3.3.2 Offline phase

We again use the EIM applied to approximate the eddy diffusion, taking into account the approximation of the eddy viscosity, already computed in Chapter 2.

In this case, the computation of an *a posteriori* estimator is simpler than the one developed in Chapter 2. Besides, we need to compute an approximation for the coercivity constant for each $\boldsymbol{\mu} \in \mathcal{D}$ and the residual associated to the problem (3.9).

To compute the coercivity constant, we apply the same procedure as for the inf-sup constant in Chapter 2. We compute $\beta_h(\boldsymbol{\mu})$ for 81 pairs of parameters with the procedure described in Section 3.7 in [38], solving an eigenvalue problem, and then, we apply the RBF algorithm explained in Section 1.5 in [14]. We stop the algorithm when the estimator is below $\varepsilon_\beta = 10^{-4}$ testing 729 parameter pairs and selecting 82 parameters.

For the residual, we use a representation $r(\mathbf{u}_N)$ in Θ_h as in (3.16), solving a FE problem.

Finally, we compute the offline phase stopping the algorithm when $\Delta_{\theta,N}(\boldsymbol{\mu}) < \varepsilon_{\theta,RB}$ with $\varepsilon_{\theta,RB} = 5 \cdot 10^{-3}$. Again, we orthonormalize the RB functions using the Gram-Schmidt algorithm with $H_0^1(\Omega)$-norm to the temperature space Θ_h. Along this phase, we have tested a total of 625 parameters. In Figure 3.1 we can see the Greedy Algorithm convergence. It stops when 17 basis functions are selected.

3.3.3 Online Phase

In this section, we compare the computational time for the full discrete model and the RB model, and we validate the *a posteriori* error bound estimator $\Delta_{\theta,N}(\boldsymbol{\mu})$.

We select the three random cases, already chosen in Section 2.4.5:

- Case 1: $\omega = 2.891$m, $\sigma = 2.734$m,
- Case 2: $\omega = 2.649$m, $\sigma = 2.65$m,
- Case 3: $\omega = 2.469$m, $\sigma = 2.923$m.

First, we compute the high fidelity problem (computing $\theta_h^{\mathbf{u}_h}(\boldsymbol{\mu})$ from $\mathbf{u}_h(\boldsymbol{\mu})$ solution of (2.22)) and the full reduced problem (computing $\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})$ from $\mathbf{u}_N(\boldsymbol{\mu})$ solution of (2.26)). We measure the error between both solutions and the time that takes to compute them and we compare them to obtain the speedup.

In the Table 3.1 we can observe the committed error in $H_0^1(\Omega)$-norm and the speedup. As we can see, the errors are admissible and we reduce the computational time from

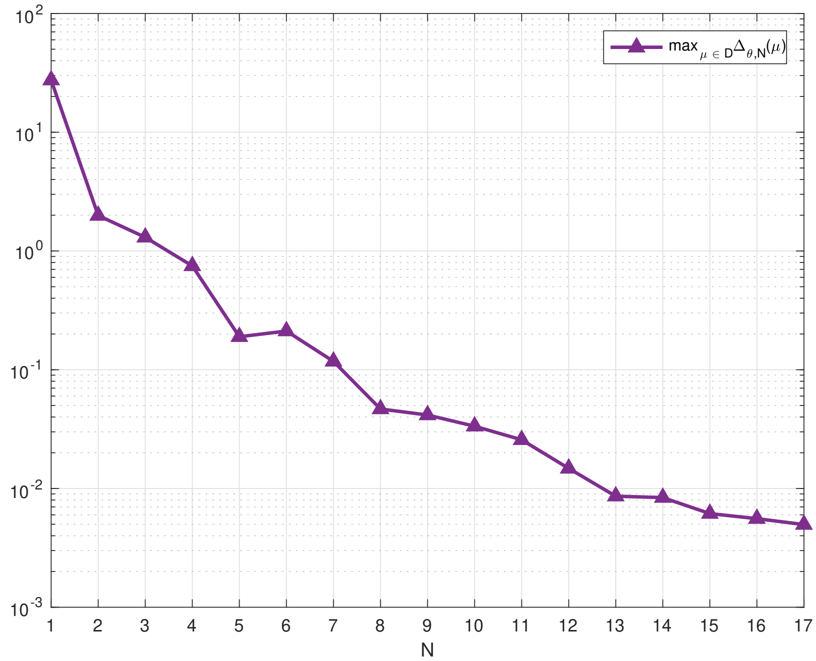


Figure 3.1: Greedy Algorithm convergence.

Data	Case 1	Case 2	Case 3
$\Vert\theta_h^{\mathbf{u}_h} - \theta_N^{\mathbf{u}_N}\Vert_{H_0^1(\Omega)}$	$5.10403 \cdot 10^{-4}$	$6.24142 \cdot 10^{-4}$	$1.09348 \cdot 10^{-3}$
speedup	134	146	156

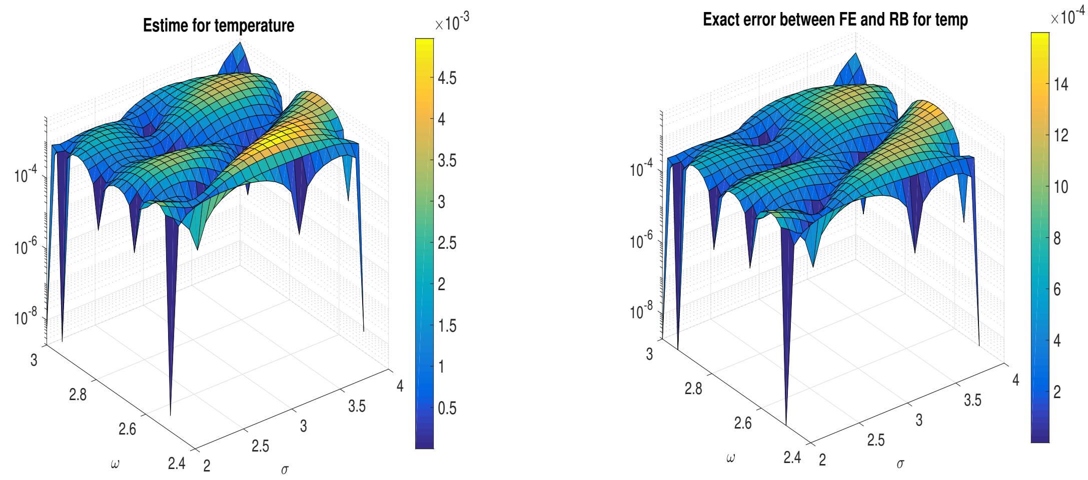
Table 3.1: Errors and speedups.

3.5min to 1.33s. This computation has been done on only one processor of a cluster with CPUs AMD EPYC 7542 2.9 GHz.

In order to validate the *a posteriori* error bound estimator $\Delta_{\theta,N}(\boldsymbol{\mu})$, we consider the finite element solution $\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu})$ computed from the velocity $\mathbf{u}_N(\boldsymbol{\mu})$ and we compare the error between $\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu})$ and $\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})$ in the $H_0^1(\Omega)$-norm and the estimator $\Delta_{\theta,N}(\boldsymbol{\mu})$ for 625 pair of parameters.

In Figure 3.2 we show two graphics in 3D with the same limits in the axes. In Figure 3.2a the vertical scale goes from $5.03 \cdot 10^{-9}$ to $5 \cdot 10^{-3}$ and in Figure 3.2b it goes from $1.61 \cdot 10^{-9}$ to $1.6 \cdot 10^{-3}$. Both graphics have the same shape and the estimate is above the error between the finite element and the RB solution. With this, we can validate our estimate for the temperature.

Additionally, we compute the estimate and the error for the three previous cases of parameters. The full high fidelity model computes the temperature associated to $\mathbf{u}_h(\boldsymbol{\mu})$,



(a) Final $\Delta_{\theta,N}(\boldsymbol{\mu})$. **(b)** $\|\theta_h^{\mathbf{u}_N} - \theta_N^{\mathbf{u}_N}\|_{H_0^1(\Omega)}$.

Figure 3.2: Validation of the estimate.

that is, $\theta_h^{\mathbf{u}_h}(\boldsymbol{\mu})$, however, we are using the temperature associated to $\mathbf{u}_N(\boldsymbol{\mu})$, that is, $\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu})$ as the high fidelity solution. We show the results in Table 3.2 and we also see that $\|\theta_h^{\mathbf{u}_h} - \theta_N^{\mathbf{u}_N}\|_{H_0^1(\Omega)}$ is close to $\|\theta_h^{\mathbf{u}_N} - \theta_N^{\mathbf{u}_N}\|_{H_0^1(\Omega)}$ at least for these three cases.

Data	Case 1	Case 2	Case 3
$\|\theta_h^{\mathbf{u}_N} - \theta_N^{\mathbf{u}_N}\|_{H_0^1(\Omega)}$	$5.10131 \cdot 10^{-4}$	$6.30339 \cdot 10^{-4}$	$1.11298 \cdot 10^{-3}$
$\Delta_{\theta,N}$	$1.76917 \cdot 10^{-3}$	$2.78895 \cdot 10^{-3}$	$3.81012 \cdot 10^{-3}$
Efficiency	3.47	4.42	3.42

Table 3.2: Validation for the three cases of the estimate.

Finally, in Figure 3.3 we show the temperature in Kelvin (K) for the three cases. It is interesting to compare them with the velocity field shown in Figure 3.4. We observe that the highest temperate values are in the biggest eddy, while the temperature keeps colder in the corridors as expected from buoyancy effects, even if these have not been taken into account to compute the velocity.

3.4 Application: geometrical optimization of a cloister

In this section, we optimize the cloister geometry with the goal of maximizing the comfort measured as the difference between an ideal comfort temperature and the computed temperature, in a region where the visitors of the cloister may find themselves. To do

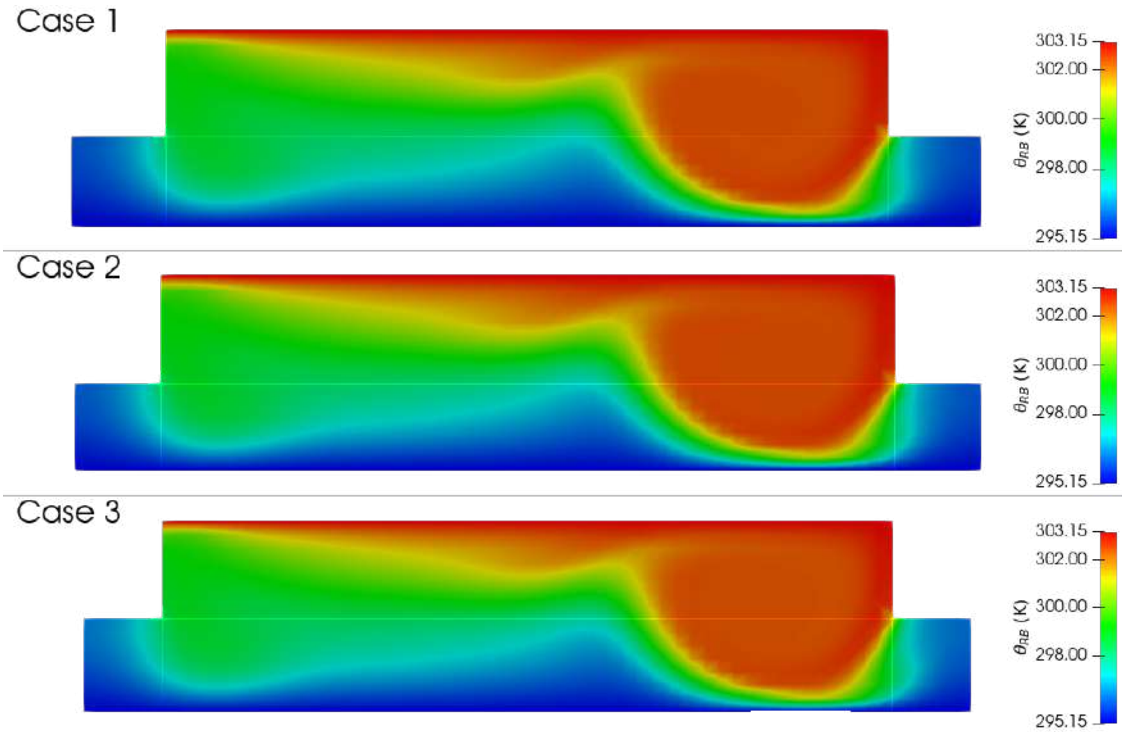


Figure 3.3: Temperature for the three test cases.

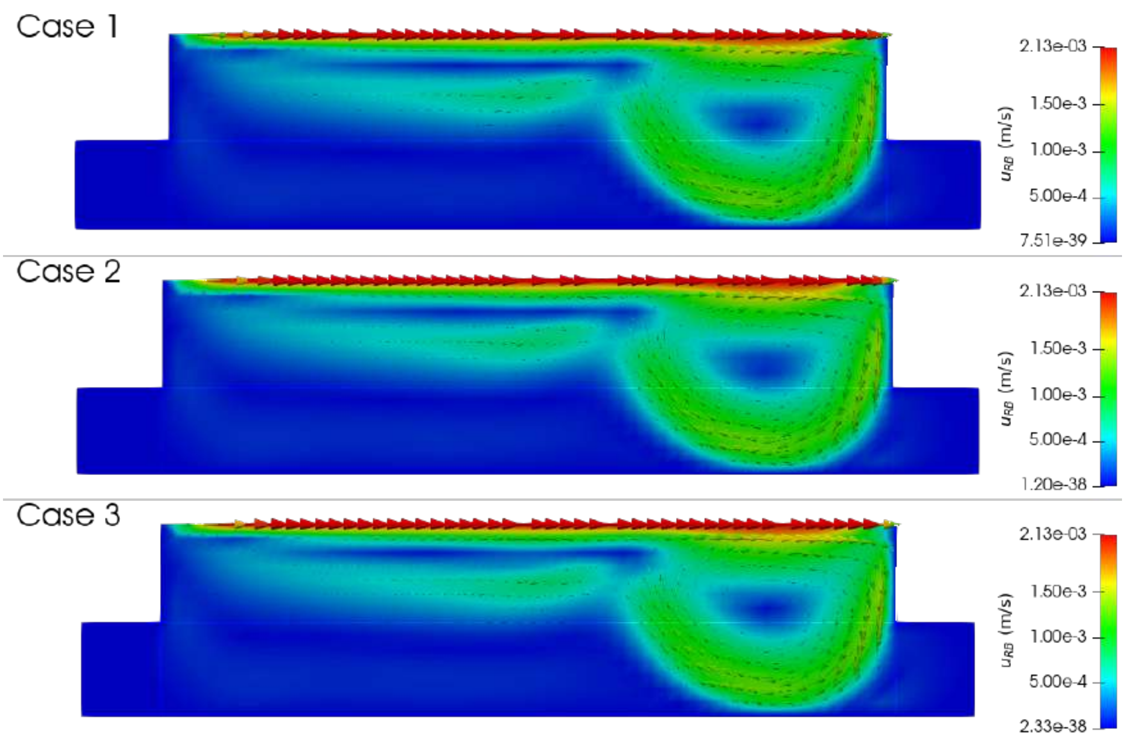


Figure 3.4: Velocity for the three test cases.

this, we define the "thermal comfort" functional $\mathscr{J} : \mathcal{D} \longrightarrow \mathbb{R}^+$ by

$$\mathscr{J}(\boldsymbol{\mu}) = \frac{1}{\mathscr{A}(\Omega_{\mathbb{I}}(\boldsymbol{\mu}))} \int_{\Omega_{\mathbb{I}}(\boldsymbol{\mu})} |\theta(\boldsymbol{\mu}) - \theta_C|^2 d\Omega \tag{3.19}$$

where $\mathscr{A}(\Omega)$ represents the area of Ω, $\Omega_{\mathbb{I}}(\boldsymbol{\mu}) = \bigcup_{l\in\mathbb{I}} \Omega_l \subset \Omega(\boldsymbol{\mu})$, $\theta(\boldsymbol{\mu})$ is the solution of (3.3) and $\mathbb{I}$ represents the set of indices of the domains where the visitors of the cloister may find them selves. The purpose is to minimize this functional for $\boldsymbol{\mu} \in \mathcal{D}$ and choosing $\Omega_{\mathbb{I}}$ depending on our interest:

- we choose $\mathbb{I} = \{1,2,3\}$ if we are interested on the ground floor,
- we choose $\mathbb{I} = \{1,3\}$ if we are interested on the corridors.

We have to solve the constrained minimization problem

$$\boldsymbol{\mu}_{opt} = \arg \min_{\boldsymbol{\mu}\in\mathcal{D}_{ad}} \mathscr{J}_N(\boldsymbol{\mu})$$

where $\mathcal{D}_{ad}$ is the set of admissible parameters from the architectural point of view.

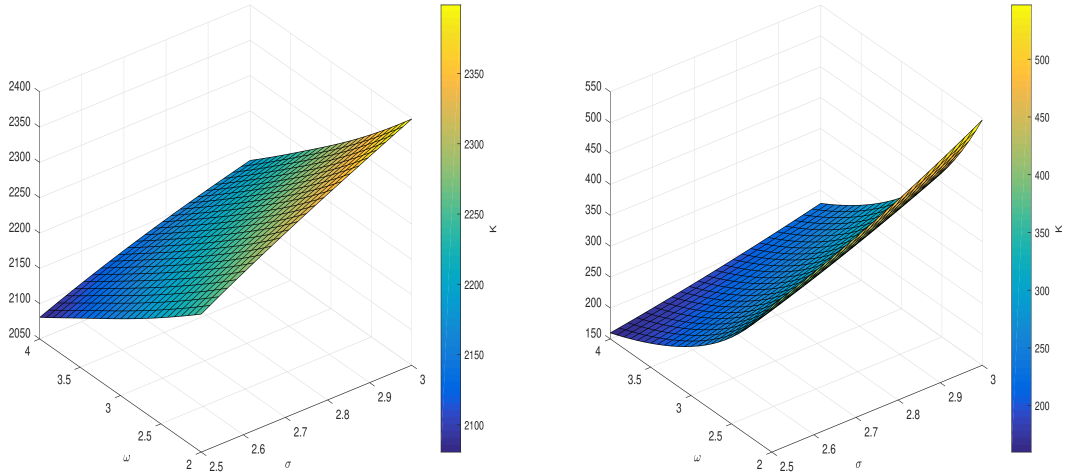
Instead of applying optimization methods to solve the optimization problem, we compute the functional for a training set $\mathcal{D}_{train} \subset \mathcal{D}_{ad}$ and we directly look for the minimum. This is possible thanks to the RB solution of the temperature equation, whose computational cost is very low. With this method, we obtain additional information about the functional based on the parameters and in particular, the minimum.

Since we assume that $\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu})$ is a good approximation of $\theta(\boldsymbol{\mu})$ solution of (3.3) and we build $\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})$ from $\theta_h^{\mathbf{u}_N}(\boldsymbol{\mu})$, we can say that we are not loosing the important information on the temperature behavior using the RB method for (3.19) with $\theta = \theta_N^{\mathbf{u}_N}$, denoted by $\mathscr{J}_N(\boldsymbol{\mu})$.

Again, we select a total of 625 pairs of parameters and we compute $\mathscr{J}_N(\boldsymbol{\mu})$, this takes less than 16 minutes of computing time. We remember that $\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})$ solution of (3.10) is a dimensionless variable defined on Ω_r with $\theta_N^{\mathbf{u}_N} = 0$ on $\Gamma_S \cup \Gamma_F$, then we need to set the Dirichlet boundary condition θ_D to $\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})$. In practice, we need to define a dimensionless functional $\mathscr{J}_N^*(\boldsymbol{\mu})$ and for simplicity, we define it on $\Omega_{\mathbb{I}}(\boldsymbol{\mu}_r)$.

On one side, we know that the physical dimensions of the functional in (3.19) is θ_0^2, then we define $\mathscr{J}_N^* = \mathscr{J}_N/\theta_C^2$. On the other side, we use the mapping defined in (2.5) to define the functional on $\Omega_{\mathbb{I}}(\boldsymbol{\mu}_r)$. Finally, we obtain that

$$\mathscr{J}_N^*(\boldsymbol{\mu}) = \frac{1}{\mathscr{A}^*(\mathbb{I};\boldsymbol{\mu})} \sum_{i\in\mathbb{I}} \int_{\Omega_i} |J_{\Phi_i}||\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu}) + \theta_D - 1|^2 d\Omega, \tag{3.20}$$



(a) $\mathscr{J}_N(\boldsymbol{\mu})$ for $\mathbb{I} = \{1,2,3\}$. **(b)** $\mathscr{J}_N(\boldsymbol{\mu})$ for $\mathbb{I} = \{1,3\}$.

Figure 3.5: Computation of the functional for $\theta_N^{\mathbf{u}_N}(\boldsymbol{\mu})$.

where

$$\mathscr{A}^*(\mathbb{I};\boldsymbol{\mu}) = \sum_{i\in\mathbb{I}} \int_{\Omega_i} |J_{\Phi_i}| d\Omega = \begin{cases} \sigma W & \text{if } \mathbb{I} = \{1,2,3\} \\ 2\sigma\omega & \text{if } \mathbb{I} = \{1,3\} \end{cases}. \tag{3.21}$$

In Figure 3.5 we show the functional $\mathscr{J}_N(\boldsymbol{\mu})$ for both cases. In these graphics, we can see that the functional minimum it is achieved for a corridor width of 4m and a height of 2.5m, which correspond to the maximum width and the minimum height of the corridor. This makes sense since, with this configuration, the air barely flows through the corridors, it stays in the interior of the courtyard and does not carry the hot air that comes from the top of the corridors.

In the hypothetical case that the corridors were higher, the horizontal flow that comes into the corridors would be increased, and so would be the temperature. Similarly, if the corridors were narrower, their area would decrease and so would the amount of air in them, on the contrary, it would not affect the flow that comes into the corridors.

A Mac Book Pro 2017 with a 2.3 GHz Intel Core i5 processor takes 7.79min for case $\mathbb{I} = \{1,2,3\}$ and 8.04min for case $\mathbb{I} = \{1,3\}$ using two processors.

Open problems

To consider more realistic situations, we set some open problems related to the modeling carried on this part:

- ☐ Extension to free convection, where buoyancy effects are included in the momentum conservation equation. An estimator involving velocity, pressure, and temperature must be developed.

- ☐ Cloister modeling in 3D. It may be necessary to change the mapping Φ developed in (2.5) and the corresponding *a posteriori* error estimate.

- ☐ Inclusion of radiation effects Q to the problem using the previous modeling in 3D. This is necessary since radiation depends on the position of the sun and this movement is in 3D.

- ☐ Inclusion of unsteady and seasonal effects.

- ☐ Inclusion of tempering elements in the cloister, such as fountains, vegetation, etc.

Part II

Unsteady Reduced Smagorinsky model

4

Numerical approximation of transient Smagorinsky model

We are interested in the development of an *a posteriori* error bound estimator using the Brezzi-Rappaz-Raviart (BRR) theory (*cf.* [8]) for the transient Smagorinsky model. To do this, it is necessary to define the problem on Hilbert spaces, therefore, we study *a priori* estimates of the time-discrete Smagorinsky model and the conditions to formulate the problem on these spaces.

We introduce the problem to study in Section 4.1 and we develop the *a priori* estimates for velocity and pressure in Sections 4.2 and 4.3.

4.1 Problem statement

Let be Ω a bounded polyhedral domain in $\mathbb{R}^d$, with $d = 2,3$ with the boundary domain $\Gamma = \partial\Omega$. Let $T_f > 0$ be a chosen final finite time, then, we define the interval $I_f = (0, T_f)$ and $Q_T = I_f \times \Omega$.

The $\{\mathcal{T}_h\}_{h>0}$ be a uniformly regular family of triangulations of $\overline{\Omega}$ and h the maximum diameter among every element $K \in \mathcal{T}_h$.

Now, we are under the conditions to define the following unsteady Smagorinsky Model with homogeneous Dirichlet boundary conditions:

Find $(\mathbf{u}, p)$ such that

$$\begin{cases} \partial_t \mathbf{u} - \nabla \cdot \left(\left(\frac{1}{\mu} + \nu_t(\mathbf{u}) \right) \nabla \mathbf{u} \right) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q_T, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } Q_T, \\ \mathbf{u} = \mathbf{0} & \text{on } I_f \times \Gamma, \\ \mathbf{u}(0, \cdot) = \mathbf{0} & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $\mu = \text{Re}$ the Reynolds number is considered the parameter for this problem, $\mathbf{u}$ is the velocity field and p is the pressure per mass density. Moreover, $\mathbf{f} \in L^2(Q_T)$ is the source of the problem and the eddy diffusion term is given by

$$\nu_t(\mathbf{u}) = \sum_{K \in \mathcal{T}_h} (C_S h_K)^2 |\nabla \mathbf{u}_{|K}| \chi_K \tag{4.2}$$

where C_S is the Smagorinsky constant. The parameter μ belongs to a suitable parameter set $\mathcal{D} \subset \mathbb{R}$.

Remark 4.1. *For simplicity, we select an homogeneous boundary and initial conditions for the problem* (4.1). *However the results exposed in this chapter can be extended to general boundary and initial conditions, using a lift function as in Part I.*

To establish a weak formulation of the problem (4.1) we consider the following Banach space

$$W_0^{1,3}(\Omega) = \{\mathbf{v} \in W^{1,3}(\Omega) : \mathbf{v}_{|\Gamma} = \mathbf{0}\}$$

equipped with its natural norm. We consider the following weak formulation of (4.1):

Find $\mathbf{u} \in H^1(0, T_f; L^2(\Omega)) \cap L^3(0, T_f; W_0^{1,3}(\Omega))$ such that $\mathbf{u}(0, \cdot) = \mathbf{0}$ and $p \in L^2(0, T_f; L^{3/2}(\Omega))$ solution of the problem

$$\begin{cases} \int_{Q_T} \partial_t \mathbf{u} \cdot \mathbf{v} \, dQ_T + \int_{Q_T} \left(\frac{1}{\mu} + \nu_t(\mathbf{u}) \right) \nabla \mathbf{u} : \nabla \mathbf{v} \, dQ_T - \int_{Q_T} p \nabla \cdot \mathbf{v} \, dQ_T \\ + \int_{Q_T} q \nabla \cdot \mathbf{u} \, dQ_T + \frac{1}{2} \int_{Q_T} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} - (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{u} \, dQ_T \\ = \int_{Q_T} \mathbf{f} \cdot \mathbf{v} \, dQ_T \end{cases} \tag{4.3}$$

for all $\mathbf{v} \in H^1(0, T_f; L^2(\Omega)) \cap L^3(0, T_f; W_0^{1,3}(\Omega))$ and $q \in L^2(0, T_f; L^{3/2}(\Omega))$.

The existence and uniqueness of this problem has been deeply studied. Due to the lack of regularity for the velocity in the equation (4.1), it is usual to define the velocity on divergence free spaces to recover the pressure using the De Rham Theorem (*cf.* Theorem 3.1 in [12], Theorem B. 73 in [18]). One proof for divergence free spaces can be found in Chapter 6 in [25] by J. Volker, who follow the proof presented by Ladyzhenskaya in [30]. For the pressure, the space $L^{3/2}(\Omega)$ is a Banach space, therefore it is not possible to apply the BRR theory with this space.

However, in practice, we use a discrete model as the *high fidelity* solution, therefore we would like to study this model instead of (4.3).

We follow the Space-Time Discretization presented by T. Chacón and R. Lewandowski in Section 10.3 in [12], choosing $\varepsilon = \theta = 1$ obtaining the full implicit Euler scheme and we follow the proofs for the *a priori* estimates developed in Chapter 6 in [25] by J. Volker.

First, we set the discretization in time using a semi-implicit Euler scheme. Let L be a positive integer that defines the number of time steps that we are considering, let be the time step $\Delta t = T_f/L$ and consider the discrete times of solution $t_k = k\Delta t$, $k = 0,1,\ldots,L$. Let $\mathbf{u}^k$ be the approximation of $\mathbf{u}(t_k,\cdot)$ for any $k = 0,1,\ldots,L$.

Then, for the space discretization, we define two discrete spaces $Y_h \subset W^{1,3}(\Omega)$ and $Q_h \subset L^2(\Omega)$ that are inner approximations, where the subscript $h > 0$ denotes the mesh diameter. The norms associated to these spaces will be described by the *a priori* estimates along the chapter.

From the initial condition in (4.1), we set $\mathbf{u}_h^0 = \mathbf{0}$. This setting gives us the following model:

For $k = 1,\ldots,L$, and assuming known $\mathbf{u}_h^{k-1} \in Y_h$, find $(\mathbf{u}_h^k, p_h^k) \in Y_h \times Q_h$ solution of

$$\begin{cases} \dfrac{1}{\Delta t} m(\mathbf{u}_h^k,\mathbf{v}_h^k) - \dfrac{1}{\Delta t} m(\mathbf{u}_h^{k-1},\mathbf{v}_h^k) + \dfrac{1}{\mu} a(\mathbf{u}_h^k,\mathbf{v}_h^k) & \\ +a_t(\mathbf{u}_h^k;\mathbf{u}_h^k,\mathbf{v}_h^k) + c(\mathbf{u}_h^k,\mathbf{u}_h^k,\mathbf{v}_h^k) + b(\mathbf{v}_h^k,p_h^k) = \langle \mathbf{f}^k,\mathbf{v}_h^k\rangle_\Omega, & \forall \mathbf{v}_h^k \in Y_h, \\ b(\mathbf{u}_h^k,q_h^k) = 0, & \forall q^k \in Q_h, \end{cases} \tag{4.4}$$

where $\mathbf{f}^k$ is the average value of $\mathbf{f}$ in $[t_{k-1},t_k]$, the bilinear forms $m(\cdot,\cdot)$, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$m(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}\, d\Omega, \quad a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega, \tag{4.5}$$

$$b(\mathbf{v},q) = -\int_\Omega \nabla\cdot\mathbf{v}, q\, d\Omega; \tag{4.6}$$

the trilinear form $c(\cdot,\cdot,\cdot)$ is given by

$$c(\mathbf{u},\mathbf{z},\mathbf{v}) = \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{z}\cdot\mathbf{v} - (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{z}\, d\Omega; \tag{4.7}$$

and the nonlinear Smagorinsky term, $a_t(\cdot;\cdot,\cdot)$ is defined by

$$a_t(\mathbf{u};\mathbf{z},\mathbf{v}) = \int_\Omega \nu_t(\mathbf{u})\nabla\mathbf{z} : \nabla\mathbf{v}\, d\Omega. \tag{4.8}$$

To simplify the notation, we shall consider the following discrete functions:

- $\tilde{\mathbf{v}}_h : (0,T_f) \longrightarrow Y_h$ is the piecewise constant function defined by

$$\tilde{\mathbf{v}}_h(t) := \mathbf{v}_h^k \text{ if } t \in (t_{k-1},t_k).$$

- $\tilde{q}_h : (0,T_f) \longrightarrow Q_h$ is the piecewise constant in time function defined by

$$\tilde{q}_h(t) := q_h^k \text{ if } t \in (t_{k-1},t_k).$$

- $\partial_t^\star \tilde{\mathbf{v}}_h : (0,T_f) \longrightarrow Y_h$ is the discrete derivative of $\tilde{\mathbf{v}}_h$ defined by

$$\partial_t^\star \tilde{\mathbf{v}}_h(t) := \frac{\mathbf{v}_h^k - \mathbf{v}_h^{k-1}}{\Delta t} \text{ if } t \in [t_{k-1},t_k].$$

- $\tilde{\mathbf{f}} : (0,T_f) \longrightarrow L^2(\Omega)$ is the piecewise constant in time function defined by $\tilde{\mathbf{f}}(t) = \mathbf{f}^k$ if $t \in (t_{k-1},t_k)$.

We can define $\tilde{Y}_h$ as the space formed by the piecewise constant functions that belongs to Y_h and $\tilde{Q}_h$ formed by the piecewise constant functions that belongs to Q_h.

Furthermore, we can define an equivalent problem to (4.4) as follows:

Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \tilde{Y}_h \times \tilde{Q}_h$ solution of

$$\begin{cases} \tilde{m}(\partial_t^* \tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h) + \dfrac{1}{\mu}\tilde{a}(\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h) + \tilde{a}_t(\tilde{\mathbf{u}}_h; \tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h) & \\ +\tilde{c}(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h) + \tilde{b}(\tilde{\mathbf{v}}_h, \tilde{p}_h) = \langle \tilde{\mathbf{f}}, \tilde{\mathbf{v}}_h \rangle_{Q_T}, & \forall \tilde{\mathbf{v}}_h \in \tilde{Y}_h, \\ \tilde{b}(\tilde{\mathbf{u}}_h, \tilde{q}_h) = 0, & \forall \tilde{q}_h \in \tilde{Q}_h, \end{cases} \tag{4.9}$$

where the forms $\tilde{m}(\cdot,\cdot)$, $\tilde{a}(\cdot,\cdot)$, $\tilde{b}(\cdot,\cdot)$, $\tilde{c}(\cdot,\cdot,\cdot)$ and $\tilde{a}_t(\cdot;\cdot,\cdot)$ are the same as the forms defined in (4.5) - (4.8) by changing the integration domain by the time-space domain $Q_T = I_f \times \Omega$, instead of Ω.

Note that for a given Banach Space X and any $n \in \mathbb{N}$, the continuous norms applied to the piecewise functions $\tilde{\mathbf{v}} : (0,T_f) \longrightarrow X$ such that $\tilde{\mathbf{v}}(t) := \mathbf{v}^k$, for all $t \in (t_{k-1},t_k)$ and $k = 1,\ldots,L$ are related to its discrete norms, this is,

$$\|\tilde{\mathbf{v}}\|_{L^n(X)} = \left(\int_0^{T_f} \|\tilde{\mathbf{v}}(s)\|_X^n ds\right)^{1/n} = \left(\sum_{k=1}^L \int_{t_{k-1}}^{t_k} \|\tilde{\mathbf{v}}(s)\|_X^n ds\right)^{1/n} = \left(\sum_{k=1}^L \Delta t \|\mathbf{v}^k\|_X^n\right)^{1/n},$$

and

$$\|\tilde{\mathbf{v}}\|_{L^\infty(X)} = \max_{t\in(0,T_f)} \|\tilde{\mathbf{v}}(t)\|_X = \max_{k=1,\ldots,L} \|\mathbf{v}^k\|_X.$$

Remark 4.2. *We shall denote by C a constant that may change from one to other occurrence, but which always is independent of h.*

Remark 4.3. *Note that, since $\boldsymbol{f}^k = \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} \boldsymbol{f}(s)\,ds$ for $k = 1,\dots,L$, then $\|\tilde{\boldsymbol{f}}\|_{L^n(X)} \le \|\boldsymbol{f}\|_{L^n(X)}$ for any Banach space X. This result is well know, however, we follow with the proof for more completeness. This is, for any $n \ge 1$*

$$\|\tilde{\boldsymbol{f}}\|^n_{L^n(X)} = \int_0^{T_f} \|\tilde{\boldsymbol{f}}(s)\|^n_X \,ds = \sum_{k=1}^{L}\int_{t_{k-1}}^{t_k} \|\tilde{\boldsymbol{f}}(s)\|^n_X \,ds = \sum_{k=1}^{L} \Delta t \|\boldsymbol{f}^k\|^n_X$$
$$= \sum_{k=1}^{L} \Delta t^{1-n} \left\| \int_{t_{k-1}}^{t_k} \boldsymbol{f}(s)\,ds \right\|^n_X \le \sum_{k=1}^{L} \Delta t^{1-n} \left(\int_{t_{k-1}}^{t_k} \|\boldsymbol{f}(s)\|_X \,ds \right)^n.$$

Using the Hölder's Inequality A.1 on the interval (t_{k-1}, t_k),

$$\|\tilde{\boldsymbol{f}}\|^n_{L^n(X)} \le \sum_{k=1}^{L} \Delta t^{1-n} \left(\int_{t_{k-1}}^{t_k} 1\,ds \right)^{n-1} \left(\int_{t_{k-1}}^{t_k} \|\boldsymbol{f}(s)\|^n_X \,ds \right) = \sum_{k=1}^{L} \int_{t_{k-1}}^{t_k} \|\boldsymbol{f}(s)\|^n_X \,ds = \|\boldsymbol{f}\|^n_{L^n(X)}.$$

4.2 *A priori* error estimates for the velocity

In this section, we will derive some *a priori* error estimates for the velocity in terms of norms in time and space.

Lemma 4.1. *Assume that $\boldsymbol{f} \in L^1(0, T_f; L^2(\Omega))$ and $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \tilde{Y}_h \times \tilde{Q}_h$ is a solution of* (4.9). *Then this solution satisfies*

$$\|\tilde{\boldsymbol{u}}_h\|_{L^\infty(L^2)} \le C\|\boldsymbol{f}\|_{L^1(L^2)}. \tag{4.10}$$

Proof. For any $k = 1,\dots,L$, $(\mathbf{u}_h^k, p_h^k) \in Y_h \times Q_h$ is a solution of (4.4). We take $\mathbf{v}_h^k = \mathbf{u}_h^k$ and we obtain from the first equation in (4.4),

$$\frac{1}{\Delta t} m(\mathbf{u}_h^k, \mathbf{u}_h^k) - \frac{1}{\Delta t} m(\mathbf{u}_h^{k-1}, \mathbf{u}_h^k) + \frac{1}{\mu} a(\mathbf{u}_h^k, \mathbf{u}_h^k)$$
$$+ a_t(\mathbf{u}_h^k; \mathbf{u}_h^k, \mathbf{u}_h^k) + b(\mathbf{u}_h^k, p_h^k) + c(\mathbf{u}_h^k, \mathbf{u}_h^k, \mathbf{u}_h^k) = \langle \mathbf{f}^k, \mathbf{u}_h^k \rangle_\Omega$$

We note that

$$a(\mathbf{u}_h^k, \mathbf{u}_h^k) = \|\nabla \mathbf{u}_h^k\|^2_{L^2(\Omega)},\ c(\mathbf{u}_h^k, \mathbf{u}_h^k, \mathbf{u}_h^k) = 0,$$

and that $b(\mathbf{u}_h^k, p_h^k) = 0$ since $\mathbf{u}_h^k$ verifies the second equation in (4.4).

Furthermore, using the property (A.3) and multiplying by $2\Delta t$ we obtain:

$$\|\mathbf{u}_h^k\|_{L^2(\Omega)}^2 - \|\mathbf{u}_h^{k-1}\|_{L^2(\Omega)}^2 + \|\mathbf{u}_h^k - \mathbf{u}_h^{k-1}\|_{L^2(\Omega)}^2 + 2\frac{1}{\mu}\Delta t\|\nabla \mathbf{u}_h^k\|_{L^2(\Omega)}^2 + 2\Delta t a_t(\mathbf{u}_h^k;\mathbf{u}_h^k,\mathbf{u}_h^k) = 2\Delta t\langle \mathbf{f}^k,\mathbf{u}_h^k\rangle_\Omega \quad (4.11)$$

Then, as $a_t(\mathbf{u}_h^k;\mathbf{u}_h^k,\mathbf{u}_h^k) \geq 0$, we have

$$\|\mathbf{u}_h^k\|_{L^2(\Omega)}^2 - \|\mathbf{u}_h^{k-1}\|_{L^2(\Omega)}^2 \leq 2\Delta t\langle \mathbf{f}^k,\mathbf{u}_h^k\rangle_\Omega$$

Summing for $k = 1,\dots,m$ for some $m \geq 1$, we obtain:

$$\|\mathbf{u}_h^m\|_{L^2(\Omega)}^2 \leq 2\sum_{k=1}^m \Delta t\langle \mathbf{f}^k,\mathbf{u}_h^k\rangle_\Omega \leq 2\sum_{k=1}^m \Delta t\|\mathbf{f}^k\|_{L^2(\Omega)}\|\mathbf{u}_h^k\|_{L^2(\Omega)} \leq 2\sum_{k=1}^m \Delta t\|\mathbf{f}^k\|_{L^2(\Omega)}\|\tilde{\mathbf{u}}_h\|_{L^\infty(L^2)}$$

Taking the maximum in $m = 1,\dots,L$:

$$\|\tilde{\mathbf{u}}_h\|_{L^\infty(L^2)} \leq 2\sum_{k=1}^L \Delta t\|\mathbf{f}^k\|_{L^2(\Omega)} = 2\|\tilde{\mathbf{f}}\|_{L^1(L^2)} \leq 2\|\mathbf{f}\|_{L^1(L^2)}$$

and therefore we obtain the result for $C = 2$. □

To take into account the eddy diffusion effects in our estimates, we consider the following norm. Let $X \subseteq H_0^1(\Omega)$ be a Banach Space, we define the turbulence inner product $(\cdot,\cdot)_T$ in this space as

$$(\mathbf{v},\mathbf{z})_T = \int_\Omega \left[\frac{1}{\bar{\mu}} + \nu_t^\star\right] \nabla \mathbf{v} : \nabla \mathbf{z}\, d\Omega,\ \forall \mathbf{v},\mathbf{z} \in X \quad (4.12)$$

where $\nu_t^\star = \nu_t(\mathbf{u}(\bar{\mu}))$, with

$$\bar{\mu} = \arg\min_{\mu\in\mathcal{D}} \left\{ \sum_{K\in\mathcal{T}_h} (C_S h_K)^2 \min_{x\in K} |\nabla \mathbf{u}(\mathbf{x},\mu)_{|K}|\chi_K \right\},$$

where $\mathbf{u}(\mu)$ is the minimum on time of the solution $\mathbf{u}(\cdot,\mu)$ of (4.3) for $\mu \in \mathcal{D}$.

Remark 4.4. *The norm associated to the inner product defined in* (4.12) *is equivalent to the $H_0^1(\Omega)$-norm.*

We shall denote by $\|\cdot\|_{L_T^p(H_0^1)}$ the norm in $L^p(0,T_f;H_0^1(\Omega))$ defined by

$$\|\mathbf{v}\|_{L_T^p(H_0^1)} = \left(\int_0^{T_f} \|\mathbf{v}(t)\|_T^p dt\right)^{1/p},\ \text{for } 1 \leq p < +\infty, \quad (4.13)$$

$$\|\mathbf{v}\|_{L_T^\infty(H_0^1)} = \max_{t\in(0,T_f)} \|\mathbf{v}(t)\|_T, \quad (4.14)$$

where the T-norm is induced by the inner product (4.12).

Lemma 4.2. *Under the same assumptions of Lemma 4.1, it holds*

$$\|\tilde{\boldsymbol{u}}_h\|^2_{L^\infty(L^2)} + \frac{1}{\mu}\|\nabla\tilde{\boldsymbol{u}}_h\|^2_{L^2(Q_T)} + (C_S h_{\min})^2\|\nabla\tilde{\boldsymbol{u}}_h\|^3_{L^3(Q_T)} \le C\|\boldsymbol{f}\|^2_{L^1(L^2)}. \tag{4.15}$$

where $h_{\min}$ is the minimum size of all elements in the mesh $\mathcal{T}_h$.

Proof. From (4.11), we sum for $k = 1, \dots, m$:

$$\|\mathbf{u}_h^m\|^2_{L^2(\Omega)} + \sum_{k=1}^m 2\Delta t\left(\mu\|\nabla\mathbf{u}_h^k\|^2_{L^2(\Omega)} + a_t(\mathbf{u}_h^k;\mathbf{u}_h^k,\mathbf{u}_h^k)\right) \le \sum_{k=1}^m 2\Delta t\langle\mathbf{f}^k,\mathbf{u}_h^k\rangle_\Omega$$

We observe that

$$a_t(\mathbf{u}_h^k;\mathbf{u}_h^k,\mathbf{u}_h) = \sum_{K\in\mathcal{T}_h}(C_S h_K)^2\int_K|\nabla\mathbf{u}_h^k|^3\,dK \ge (C_S h_{\min})^2\|\nabla\mathbf{u}_h^k\|^3_{L^3(\Omega)}.$$

In particular,

$$\|\mathbf{u}_h^m\|^2_{L^2(\Omega)} + \sum_{k=1}^m \frac{\Delta t}{\mu}\|\nabla\mathbf{u}_h^k\|^2_{L^2(\Omega)} + \sum_{k=1}^m \Delta t(C_S h_{\min})^2\|\nabla\mathbf{u}_h^k\|^3_{L^3(\Omega)} \le \sum_{k=1}^m 2\Delta t\langle\mathbf{f}^k,\mathbf{u}_h^k\rangle_\Omega.$$

Moreover, thanks to (4.10)

$$\sum_{k=1}^m \Delta t\langle\mathbf{f}^k,\mathbf{u}_h^k\rangle_\Omega \le \sum_{k=1}^m \Delta t\|\mathbf{f}^k\|_{L^2(\Omega)}\|\mathbf{u}_h^k\|_{L^2(\Omega)}$$
$$\le 2\sum_{k=1}^m \Delta t\|\mathbf{f}^k\|_{L^2(\Omega)}\|\tilde{\mathbf{f}}\|_{L^1(L^2)} = 2\|\tilde{\mathbf{f}}\|^2_{L^1(L^2)} \le 2\|\mathbf{f}\|^2_{L^1(L^2)}.$$

Taking the maximum in $m = 1, \dots, L$, we obtain the result. □

Corollary 4.1. *Under the same assumptions as in Lemma 4.2, we also obtain*

$$\|\tilde{\boldsymbol{u}}_h\|^2_{L^\infty(L^2)} + \|\tilde{\boldsymbol{u}}_h\|^2_{L^2_T(H^1_0)} \le C\|\boldsymbol{f}\|^2_{L^1(L^2)}. \tag{4.16}$$

Now we prove some estimates for $\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}$ with $k = 1, \dots, L$.

Corollary 4.2. *For any $k = 1, \dots, L$*

$$\|\nabla\boldsymbol{u}_h^k\|_{L^3(\Omega)} \le C_{3;2}h_{\min}^{-d/6}\Delta t^{-1/2}\sqrt{2\mu}\|\boldsymbol{f}\|_{L^1(L^2)}. \tag{4.17}$$

Proof. Let us consider the Inverse Inequality Theorem A.4

$$\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)} \le C_{3;2}h_{\min}^{-d/6}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)}$$
$$\le C_{3;2}h_{\min}^{-d/6}\Delta t^{-1/2}\left(\sum_{k=1}^m \Delta t\|\nabla\mathbf{u}_h^k\|^2_{L^2(\Omega)}\right)^{1/2} = C_{3;2}h_{\min}^{-d/6}\Delta t^{-1/2}\|\nabla\tilde{\mathbf{u}}_h\|_{L^2(Q_T)}.$$

Continuing this estimate with (4.15) we get the result. □

Corollary 4.3. *For any $k = 1, \dots, L$*

$$\|\nabla \boldsymbol{u}_h^k\|_{L^3(\Omega)} \leq \Delta t^{-1/3}(C_S h_{\min})^{-2/3}\sqrt[3]{2}\|\boldsymbol{f}\|_{L^1(L^2)}^{2/3}. \tag{4.18}$$

Proof. It is straightforward that

$$\|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)} \leq \Delta t^{-1/3}\|\nabla \tilde{\mathbf{u}}_h\|_{L^3(Q_T)}$$

Then using the estimate (4.15) we get the result. □

Corollary 4.4. *For any $k = 1, \dots, L$*

$$\|\nabla \boldsymbol{u}_h^k\|_{L^3(\Omega)} \leq 2C_{1,3;2}h_{\min}^{-1-d/6}\|\boldsymbol{f}\|_{L^1(L^2)}. \tag{4.19}$$

Proof. We use the Inverse Inequality Theorem A.4

$$\|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)} \leq C_{1,3;2}h_{\min}^{-1-d/6}\|\mathbf{u}_h^k\|_{L^2(\Omega)} \leq C_{1,3;2}h_{\min}^{-1-d/6}\|\tilde{\mathbf{u}}_h\|_{L^\infty(L^2)}$$

Then using estimate (4.10) we get the result. □

Remark 4.5. *Applying the Sobolev embedding Theorem A.2, we obtain that $W^{1,3}(\Omega) \subset L^6(\Omega)$ for $d = 2, 3$. This is, there exists a constant $C_{6;1,3} > 0$ such that*

$$\|\boldsymbol{u}\|_{L^6(\Omega)} \leq C_{6;1,3}\|\nabla \boldsymbol{u}\|_{L^3(\Omega)}. \tag{4.20}$$

This inequality will be used along the proof of the next lemma.

Lemma 4.3. *Under the assumptions of Lemma 4.1 and assuming that $\boldsymbol{f} \in L^2(Q_T)$, it holds*

$$\|\partial_t^* \tilde{\boldsymbol{u}}_h\|_{L^2(Q_T)}^2 + \frac{1}{\mu}\|\nabla \tilde{\boldsymbol{u}}_h\|_{L^\infty(L^2)}^2 + (C_S h_{\min})^2\|\nabla \tilde{\boldsymbol{u}}_h\|_{L^\infty(L^3)}^3$$
$$\leq C_1 \exp\left(C_2 h_{\min}^{-8/3}\|\boldsymbol{f}\|_{L^1(L^2)}^{2/3}\right) h_{\min}^{-8}\|\boldsymbol{f}\|_{L^2(Q_T)}^8 + \|\boldsymbol{f}\|_{L^2(Q_T)}^2 \tag{4.21}$$

where $C_1, C_2 > 0$ depends on Q_T and with Δt sufficiently small.

Proof. We take the momentum equation (4.4) and we chose as test function $\mathbf{v}_h^k = \mathbf{u}_h^k - \mathbf{u}_h^{k-1}$:

$$\Delta t \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|_{L^2(\Omega)}^2 + a(\mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1}) + a_t(\mathbf{u}_h^k; \mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1})$$
$$+ c(\mathbf{u}_h^k, \mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1}) = \langle \mathbf{f}^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1}\rangle_\Omega.$$

Then,

$$\Delta t \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{1}{\mu} \|\nabla \mathbf{u}_h^k\|_{L^2(\Omega)}^2 + a_t(\mathbf{u}_h^k; \mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1})$$
$$= -c(\mathbf{u}_h^k, \mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1}) + a(\mathbf{u}_h^k, \mathbf{u}_h^{k-1}) + \langle \mathbf{f}^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1} \rangle_\Omega.$$

We sum for $k = 1, \ldots, m$:

$$\sum_{k=1}^m \Delta t \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{1}{\mu} \sum_{k=1}^m \|\nabla \mathbf{u}_h^k\|_{L^2(\Omega)}^2 + \sum_{k=1}^m a_t(\mathbf{u}_h^k; \mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1})$$
$$= \frac{1}{\mu} \sum_{k=1}^m a(\mathbf{u}_h^k, \mathbf{u}_h^{k-1}) - \sum_{k=1}^m c(\mathbf{u}_h^k, \mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1}) + \sum_{k=1}^m \langle \mathbf{f}^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1} \rangle_\Omega$$

The main idea is to compensate the right side with the left side. We will treat each term individually.

- First, we use the property (A.4) to obtain

$$\begin{aligned} \frac{1}{\mu} \sum_{k=1}^m a(\mathbf{u}_h^k, \mathbf{u}_h^{k-1}) &= \sum_{k=1}^m \frac{1}{\mu} (\nabla \mathbf{u}_h^k, \nabla \mathbf{u}_h^{k-1})_\Omega \\ &\leq \frac{1}{\mu} \sum_{k=1}^m \left[\frac{1}{2} \|\nabla \mathbf{u}_h^k\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \mathbf{u}_h^{k-1}\|_{L^2(\Omega)}^2 \right] \\ &= \frac{1}{\mu} \sum_{k=1}^{m-1} \|\nabla \mathbf{u}_h^k\|_{L^2(\Omega)}^2 + \frac{1}{2\mu} \|\nabla \mathbf{u}_h^m\|_{L^2(\Omega)}^2. \end{aligned}$$

- We integrate by parts for the antisymmetric term obtaining

$$\sum_{k=1}^m c(\mathbf{u}_h^k, \mathbf{u}_h^k, \mathbf{u}_h^k - \mathbf{u}_h^{k-1}) =$$
$$= \frac{1}{2} \sum_{k=1}^m \Delta t \left((\mathbf{u}_h^k \cdot \nabla) \mathbf{u}_h^k, \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right)_\Omega - \frac{1}{2} \sum_{k=1}^m \Delta t \left((\mathbf{u}_h^k \cdot \nabla) \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}, \mathbf{u}_h^k \right)_\Omega$$
$$= \sum_{k=1}^m \Delta t \left((\mathbf{u}_h^k \cdot \nabla) \mathbf{u}_h^k, \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right)_\Omega + \frac{1}{2} \sum_{k=1}^m \Delta t \left(\nabla \cdot \mathbf{u}_h^k, \mathbf{u}_h^k \cdot \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right)_\Omega$$

We will study each term separately. Using the property (A.5) with $c = 1/2$ we obtain that

$$\sum_{k=1}^{m} \Delta t \left((\mathbf{u}_h^k \cdot \nabla)\mathbf{u}_h^k, \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right)_\Omega \leq \sum_{k=1}^{m} 2\Delta t \|(\mathbf{u}_h^k \cdot \nabla)\mathbf{u}_h^k\|^2_{L^2(\Omega)} + \frac{1}{8}\sum_{k=1}^{m} \Delta t \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|^2_{L^2(\Omega)}.$$

It holds

$$\begin{aligned}\|(\mathbf{u}_h^k \cdot \nabla)\mathbf{u}_h^k\|^2_{L^2(\Omega)} = \int_\Omega |(\mathbf{u}_h^k \cdot \nabla)\mathbf{u}_h^k| \, d\Omega^2 &\leq 3 \int_\Omega (\mathbf{u}_h^k \cdot \mathbf{u}_h^k)(\nabla \mathbf{u}_h^k : \nabla \mathbf{u}_h^k) \, d\Omega \\ &\leq 3 \|\mathbf{u}_h^k\|^2_{L^6(\Omega)} \|\nabla \mathbf{u}_h^k\|^2_{L^3(\Omega)} \\ &\leq 3C^2_{6;1,3} \|\nabla \mathbf{u}_h^k\|^4_{L^3(\Omega)}.\end{aligned}$$

Henceforth,

$$\sum_{k=1}^{m} \Delta t \left((\mathbf{u}_h^k \cdot \nabla)\mathbf{u}_h^k, \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right)_\Omega \leq 6C^2_{6;1,3} \sum_{k=1}^{m} \Delta t \|\nabla \mathbf{u}_h^k\|^4_{L^3(\Omega)} + \frac{1}{8}\sum_{k=1}^{m} \Delta t \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|^2_{L^2(\Omega)}.$$

On the other hand,

$$\frac{1}{2}\sum_{k=1}^{m} \Delta t \left(\nabla \cdot \mathbf{u}_h^k, \mathbf{u}_h^k \cdot \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right)_\Omega \leq \frac{1}{2}\sum_{k=1}^{m} \Delta t \|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)} \|\mathbf{u}_h^k\|_{L^6(\Omega)} \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|_{L^2(\Omega)}.$$

Using again the same Sobolev embedding and the property (A.5) with the constant equal to 1, we obtain that

$$\frac{1}{2}\sum_{k=1}^{m} \Delta t \left(\nabla \cdot \mathbf{u}_h^k, \mathbf{u}_h^k \cdot \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right)_\Omega \leq \frac{C_{6;1,3}}{2} \sum_{k=1}^{m} \Delta t \|\nabla \mathbf{u}_h^k\|^4_{L^3(\Omega)} + \frac{1}{8}\sum_{k=1}^{m} \Delta t \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|^2_{L^2(\Omega)}.$$

Summing up,

$$\sum_{k=1}^{m} c(\mathbf{u}_h^k,\mathbf{u}_h^k,\mathbf{u}_h^k-\mathbf{u}_h^{k-1}) \le (12C_{6;1,3}^2+C_{6;1,3})\frac{1}{2}\sum_{k=1}^{m}\Delta t\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}^4 + \frac{1}{4}\sum_{k=1}^{m}\Delta t\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}^2.$$

- Using again the property (A.5) with $c=1$,

$$\sum_{k=1}^{m}\langle\mathbf{f}^k,\mathbf{u}_h^k-\mathbf{u}_h^{k-1}\rangle_\Omega = \sum_{k=1}^{m}\Delta t\left\langle\mathbf{f}^k,\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\rangle_\Omega \le \sum_{k=1}^{m}\Delta t\|\mathbf{f}^k\|_{L^2(\Omega)}^2+\frac{1}{4}\sum_{k=1}^{m}\Delta t\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}^2$$

- Using now the property (A.6),

$$\begin{aligned}\sum_{k=1}^{m} a_t(\mathbf{u}_h^k,\mathbf{u}_h^k,\mathbf{u}_h^k-\mathbf{u}_h^{k-1}) &= \sum_{k=1}^{m}\sum_{K\in\mathcal{T}_h}(C_Sh_K)^2\left(|\nabla\mathbf{u}_h^k|\nabla\mathbf{u}_h^k,\nabla\mathbf{u}_h^k-\nabla\mathbf{u}_h^{k-1}\right)_K\\ &\ge\frac{1}{3}\sum_{K\in\mathcal{T}_h}(C_Sh_K)^2\left[\sum_{k=1}^{m}\|\nabla\mathbf{u}_h^k\|_{L^3(K)}^3-\|\nabla\mathbf{u}_h^{k-1}\|_{L^3(K)}^3\right]\\ &=\frac{1}{3}\sum_{K\in\mathcal{T}_h}(C_Sh_K)^2\|\nabla\mathbf{u}_h^m\|_{L^3(K)}^3\ge\frac{1}{3}(C_Sh_{\min})^2\|\nabla\mathbf{u}_h^m\|_{L^3(\Omega)}^3.\end{aligned}$$

Summing up,

$$\begin{aligned}&\sum_{k=1}^{m}\Delta t\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}^2+\frac{1}{\mu}\sum_{k=1}^{m}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)}^2+\frac{1}{3}(C_Sh_{\min})^2\|\nabla\mathbf{u}_h^m\|_{L^3(\Omega)}^3\\ &\le\frac{1}{\mu}\sum_{k=1}^{m-1}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)}^2+\frac{1}{2\mu}\|\nabla\mathbf{u}_h^m\|_{L^2(\Omega)}^2+\frac{C^*}{2}\sum_{k=1}^{m}\Delta t\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}^4\\ &\quad+\frac{1}{2}\sum_{k=1}^{m}\Delta t\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}^2+\sum_{k=1}^{m}\Delta t\|\mathbf{f}^k\|_{L^2(\Omega)}^2\end{aligned}$$

where $C^* = 12C_{6;1,3}^2 + C_{6;1,3}$, and we obtain that

$$\sum_{k=1}^{m} \Delta t \left\| \frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{1}{\mu} \|\nabla \mathbf{u}_h^m\|_{L^2(\Omega)}^2 + \frac{2}{3}(C_S h_{\min})^2 \|\nabla \mathbf{u}_h^m\|_{L^3(\Omega)}^3 \leq C^* \sum_{k=1}^{m} \Delta t \|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)}^4 + 2\sum_{k=1}^{m} \Delta t \|\mathbf{f}^k\|_{L^2(\Omega)}^2 \quad (4.22)$$

Let us prove a bound for $\nabla \tilde{\mathbf{u}}_h \in L^4(0, T_f; L^3(\Omega))$. From the previous expression,

$$\|\nabla \mathbf{u}_h^m\|_{L^3(\Omega)}^3 \leq \frac{3}{2(C_S h_{\min})^2} \left(C^* \sum_{k=1}^{m} \Delta t \|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)}^4 + 2\sum_{k=1}^{m} \Delta t \|\mathbf{f}^k\|_{L^2(\Omega)}^2 \right)$$

Then, we apply the discrete Gronwall Lemma A.3 using $k = \Delta t$, $B = 0$ and

$$a_m = \|\nabla \mathbf{u}_h^m\|_{L^3(\Omega)}^3, \; b_m = 0, \; c_m = 2C' h_{\min}^{-2} \|\mathbf{f}^m\|_{L^2(\Omega)}^2,$$
$$\gamma_m = C' C^* h_{\min}^{-2} \|\nabla \mathbf{u}_h^m\|_{L^3(\Omega)}.$$

where $C' = \dfrac{3}{2C_S^2}$.

In order to apply this lemma, we need to guarantee that $\Delta t \gamma_m \leq \rho$ with $\rho < 1$ for all $m = 1, \ldots, L$. From Corollary 4.4 we obtain that

$$\Delta t \|\nabla \mathbf{u}_h^m\|_{L^3(\Omega)} \leq 2C_{1,3;2} \Delta t h_{\min}^{-1-d/6} \|\mathbf{f}\|_{L^1(L^2)}$$

Then, multiplying by $C' C^* h_{\min}^{-2}$,

$$\Delta t \gamma_m \leq 2C_{1,3;2} C' C^* \Delta t h_{\min}^{-3-d/6} \|\mathbf{f}\|_{L^1(L^2)} \leq \rho$$

if and only if

$$\Delta t \leq \frac{h_{\min}^{3+d/6}}{2C_{1,3;2} C' C^* \|\mathbf{f}\|_{L^1(L^2)}} \cdot \rho$$

for some $\rho \in (0,1)$. We select Corollary 4.4 since it seems to be the least restrictive compared to Corollary 4.2 and 4.3.

Now, applying the discrete Gronwall Lemma A.3,

$$\begin{aligned}
\|\nabla \mathbf{u}_h^m\|_{L^3(\Omega)} &\leq \exp\left(\sum_{k=1}^{m} \frac{\Delta t \gamma_k}{1 - \Delta t \gamma_k} \right) \cdot \left(2C' h_{\min}^{-2} \sum_{k=1}^{m} \Delta t \|\mathbf{f}^k\|_{L^2(\Omega)}^2 \right) \\
&\leq \exp\left(\frac{1}{1-\rho} \sum_{k=1}^{m} \Delta t \gamma_k \right) \cdot \left(2C' h_{\min}^{-2} \sum_{k=1}^{m} \Delta t \|\mathbf{f}^k\|_{L^2(\Omega)}^2 \right) \\
&= \exp\left(\frac{C' C^* h_{\min}^{-2}}{1-\rho} \sum_{k=1}^{m} \Delta t \|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)} \right) \cdot \left(2C' h_{\min}^{-2} \sum_{k=1}^{m} \Delta t \|\mathbf{f}^k\|_{L^2(\Omega)}^2 \right)
\end{aligned}$$

And taking the supreme in $m = 1, \dots, L$,

$$\|\nabla \tilde{\mathbf{u}}_h\|_{L^\infty(L^3)} \leq \exp\left(\frac{C'C^* h_{\min}^{-2}}{1-\rho}\|\nabla \tilde{\mathbf{u}}_h\|_{L^1(L^3)}\right)\left(2C' h_{\min}^{-2}\|\mathbf{f}\|^2_{L^2(Q_T)}\right)$$

The exponential term is bounded since $\nabla \tilde{\mathbf{u}}_h \in L^3(0,T_f;L^3(\Omega))$ from (4.15) and $L^3(0,T_f) \hookrightarrow L^1(0,T_f)$, then, there exists $C'_{T_f} > 0$ such that

$$\|\nabla \tilde{\mathbf{u}}_h\|_{L^1(L^3)} \leq C'_{T_f}\|\nabla \tilde{\mathbf{u}}_h\|_{L^3(Q_T)} \leq \frac{\sqrt[3]{2}C'_{T_f}}{(C_S h_{\min})^{2/3}}\|\mathbf{f}\|^{2/3}_{L^1(L^2)}$$

with this, we have proven that $\tilde{\mathbf{u}}_h \in L^\infty(0,T_f,W^{1,3}(\Omega))$.

Again, since $L^\infty(0,T_f) \hookrightarrow L^4(0,T_f)$, there exists $C^\star_{T_f} > 0$ such that

$$\|\nabla \tilde{\mathbf{u}}_h\|_{L^4(L^3)} \leq C^\star_{T_f}\|\nabla \tilde{\mathbf{u}}_h\|_{L^\infty(L^3)}.$$

Then, taking the supreme in $m = 1, \dots, L$ in (4.22) we obtain

$$\begin{aligned}
&\|\partial_t^* \tilde{\mathbf{u}}_h\|^2_{L^2(Q_T)} + \frac{1}{\mu}\|\nabla \tilde{\mathbf{u}}_h\|^2_{L^\infty(L^2)} + (C_S h_{\min})^2 \frac{2}{3}\|\nabla \tilde{\mathbf{u}}_h\|^3_{L^\infty(L^3)} \\
&\quad \leq C^*\|\nabla \tilde{\mathbf{u}}_h\|^4_{L^4(L^3)} + 2\|\mathbf{f}\|^2_{L^2(Q_T)} \\
&\quad \leq C^*(C^*_{T_f})^4\|\nabla \tilde{\mathbf{u}}_h\|^4_{L^\infty(L^3)} + \|\mathbf{f}\|^2_{L^2(Q_T)} \\
&\quad \leq C^*(C^*_{T_f})^4 \exp\left(\frac{4C'C^* h_{\min}^{-2}}{1-\rho}\|\nabla \tilde{\mathbf{u}}_h\|_{L^1(L^3)}\right)\left(2C' h_{\min}^{-2}\|\mathbf{f}\|^2_{L^2(Q_T)}\right)^4 + \|\mathbf{f}\|^2_{L^2(Q_T)} \\
&\quad \leq 16C^*(C^\star_{T_f}C')^4 \exp\left(\frac{4\sqrt[3]{2}C'C^*C'_{T_f}h_{\min}^{-8/3}}{C_S^{2/3}(1-\rho)}\|\mathbf{f}\|^{2/3}_{L^1(L^2)}\right) h_{\min}^{-8}\|\mathbf{f}\|^8_{L^2(Q_T)} + \|\mathbf{f}\|^2_{L^2(Q_T)}
\end{aligned}$$

and we obtain the result. □

Corollary 4.5. *Under the same assumptions as in Lemma 4.3, we also obtain*

$$\|\partial_t^* \tilde{\boldsymbol{u}}_h\|^2_{L^2(Q_T)} + \|\tilde{\boldsymbol{u}}_h\|^2_{L^\infty_T(H^1_0)} \leq C_1 \exp\left(C_2 h_{\min}^{-8/3}\|\boldsymbol{f}\|^{2/3}_{L^1(L^2)}\right) h_{\min}^{-8}\|\boldsymbol{f}\|^8_{L^2(Q_T)} + \|\boldsymbol{f}\|^2_{L^2(Q_T)} \quad (4.23)$$

where $C_1, C_2 > 0$ depends on Q_T and Δt sufficiently small.

Remark 4.6. *We are working with the Smagorinsky model, therefore we have set a mesh $\mathcal{T}_h$ with $h > 0$. The Lemma 4.3 shows an estimate with negative power on $h_{\min}$ which should not be a issue since $h_{\min} \neq 0$.*

4.3 *A priori* error estimates for the pressure

In this section, we develop the usual *a priori* estimate for the pressure in $L^2(\Omega)(0,T_f;L^{3/2}(\Omega))$ and using an Inverse Inequality, we obtain an *a priori* estimate in $L^2(Q_T)$.

Lemma 4.4. *Assume that* $\boldsymbol{f}\in L^2(0,T_f;H^{-1}(\Omega))$ *and* $(\tilde{\boldsymbol{u}}_h,\tilde{p}_h)\in\tilde{Y}_h\times\tilde{Q}_h$ *is a solution of* (4.9). *Then it holds*

$$\|\tilde{p}_h\|^2_{L^2(L^{3/2})}\leq C\Bigg(\|\partial_t^*\tilde{\boldsymbol{u}}_h\|^2_{L^2(Q_T)}+\frac{1}{\mu^2}\|\nabla\tilde{\boldsymbol{u}}_h\|^2_{L^2(Q_T)}+((C_S h_{\max})^4+1)\|\nabla\tilde{\boldsymbol{u}}_h\|^4_{L^4(L^3)}$$
$$+\|\tilde{\boldsymbol{u}}_h\|^4_{L^4(L^3)}+\|\boldsymbol{f}\|^2_{L^2(H^{-1}(\Omega))}\Bigg),\quad (4.24)$$

where $C>0$ *is a constant that depends on* Ω*.*

Proof. From the momentum equation of (4.4) for $\mathbf{v}_h^k=\mathbf{v}_h\in Y_h$, we have that

$$(p_h^k,\nabla\cdot\mathbf{v}_h)_\Omega=\left(\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t},\mathbf{v}_h\right)_\Omega+\frac{1}{\mu}(\nabla\mathbf{u}_h^k,\nabla\mathbf{v}_h)_\Omega+(\nu_t(\mathbf{u}_h^k)\nabla\mathbf{u}_h^k,\nabla\mathbf{v}_h)_\Omega$$
$$+\frac{1}{2}\left[((\mathbf{u}_h^k\cdot\nabla)\mathbf{u}_h^k,\mathbf{v}_h)_\Omega-((\mathbf{u}_h^k\cdot\nabla)\mathbf{v}_h,\mathbf{u}_h^k)_\Omega\right]-\langle\mathbf{f}^k,\mathbf{v}_h\rangle_\Omega\quad (4.25)$$

then

$$(p_h^k,\nabla\cdot\mathbf{v}_h)_\Omega\leq\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)}+\frac{1}{\mu}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}$$
$$+(C_S h_{\max})^2\|\nabla\mathbf{u}_h^k\|^2_{L^3(\Omega)}\|\nabla\mathbf{v}_h\|_{L^3(\Omega)}+\frac{1}{2}\|\mathbf{u}_h^k\|_{L^3(\Omega)}\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}\|\mathbf{v}_h\|_{L^3(\Omega)}$$
$$+\frac{1}{2}\|\mathbf{u}_h^k\|_{L^3(\Omega)}\|\nabla\mathbf{v}_h\|_{L^3(\Omega)}\|\mathbf{u}_h^k\|_{L^3(\Omega)}+\|\mathbf{f}^k\|_{H^{-1}(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}\quad (4.26)$$

If we use the Inf-Sup Condition Theorem A.8 for $p_1=3/2$ and $p_2=3$, there exits a constant $\alpha>0$ such that

$$\alpha\|p_h^k\|_{L^{3/2}(\Omega)}\leq\sup_{\mathbf{v}_h\in W^{1,3}(\Omega)}\frac{(\nabla\cdot\mathbf{v}_h,p_h^k)_\Omega}{\|\mathbf{v}_h\|_{W^{1,3}(\Omega)}}.\quad (4.27)$$

Now we try to get a bound of (4.26) depending on $\|\mathbf{v}_h\|_{W^{1,3}(\Omega)}$ so we can use (4.27). Note that

$$W^{1,3}(\Omega)\hookrightarrow L^3(\Omega)\hookrightarrow L^2(\Omega),$$

with continuous injections. Then for any $\mathbf{v} \in W^{1,3}(\Omega)$, there exists $C_{2;3} > 0$ such that

$$\|\mathbf{v}\|_{L^2(\Omega)} \leq C_{2;3}\|\mathbf{v}\|_{L^3(\Omega)} \leq C_{2;3}\|\mathbf{v}\|_{W^{1,3}(\Omega)},$$

and

$$\|\nabla\mathbf{v}\|_{L^2(\Omega)} \leq C_{2;3}\|\nabla\mathbf{v}\|_{L^3(\Omega)} \leq C_{2;3}\|\mathbf{v}\|_{W^{1,3}(\Omega)},$$

then

$$\begin{aligned}(p_h^k, \nabla\cdot\mathbf{v}_h)_\Omega \leq\ & C_{2;3}\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{W^{1,3}(\Omega)} + \frac{C_{2;3}}{\mu}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{W^{1,3}(\Omega)} \\ &+ C_{2;3}(C_S h_{\max})^2\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}^2\|\mathbf{v}_h\|_{W^{1,3}(\Omega)} + \frac{1}{2}\|\mathbf{u}_h^k\|_{L^3(\Omega)}\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}\|\mathbf{v}_h\|_{W^{1,3}(\Omega)} \\ &+ \frac{1}{2}\|\mathbf{u}_h^k\|_{L^3(\Omega)}\|\mathbf{v}_h\|_{W^{1,3}(\Omega)}\|\mathbf{u}_h^k\|_{L^3(\Omega)} + C_{2;3}\|\mathbf{f}^k\|_{H^{-1}(\Omega)}\|\mathbf{v}_h\|_{W^{1,3}(\Omega)}\end{aligned} \tag{4.28}$$

Finally, using (4.27) and (4.28)

$$\begin{aligned}\alpha\|p_h^k\|_{L^{3/2}(\Omega)} \leq\ & C_{2;3}\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)} + \frac{C_{2;3}}{\mu}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)} + C_{2;3}(C_S h_{\max})^2\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}^2 \\ &+ \frac{1}{2}\|\mathbf{u}_h^k\|_{L^3(\Omega)}\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)} + \frac{1}{2}\|\mathbf{u}_h^k\|_{L^3(\Omega)}^2 + C_{2;3}\|\mathbf{f}^k\|_{H^{-1}(\Omega)}\end{aligned}$$

and applying the Poincaré Inequality A.2 with constant $C_{3;1,3} > 0$, we obtain that

$$\begin{aligned}\|p_h^k\|_{L^{3/2}(\Omega)} \leq C\Bigg(&\left\|\frac{\mathbf{u}_h^k-\mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)} + \frac{1}{\mu}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)} + (C_S h_{\max})^2\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}^2 \\ &+\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}^2 + \|\mathbf{u}_h^k\|_{L^3(\Omega)}^2 + \|\mathbf{f}^k\|_{H^{-1}(\Omega)}\Bigg),\end{aligned}$$

where $C = \max\{C_{2;3}, C_{3;1,3}/2\}/\alpha$. Taking the square, multiplying by Δt and summing for $k = 1,\ldots,L$, we obtain

$$\begin{aligned}\|\tilde{p}_h\|_{L^2(L^{3/2})}^2 \leq C\Bigg(&\|\partial_t^*\tilde{\mathbf{u}}_h\|_{L^2(Q_T)}^2 + \frac{1}{\mu^2}\|\nabla\tilde{\mathbf{u}}_h\|_{L^2(Q_T)}^2 + (C_S h_{\max})^4\|\nabla\tilde{\mathbf{u}}_h\|_{L^4(L^3)}^4 \\ &+\|\tilde{\mathbf{u}}_h\|_{L^4(L^3)}^4 + \|\nabla\tilde{\mathbf{u}}_h\|_{L^4(L^3)}^4 + \|\tilde{\mathbf{f}}\|_{L^2(H^{-1}(\Omega))}^2\Bigg),\end{aligned}$$

and we obtain the result. The norms related to $\tilde{\mathbf{u}}_h$ in the second hand are bounded as we saw in Section 4.2. □

Now, applying the Local Inverse Inequality Theorem A.3 we are able to obtain a better estimate for the pressure.

Corollary 4.6. *Assuming the same assumptions as in Lemma 4.4, it holds*

$$\|\tilde{p}_h\|^2_{L^2(Q_T)} \leq C\Bigg(\|\partial_t^* \tilde{\boldsymbol{u}}_h\|^2_{L^2(Q_T)} + \frac{1}{\mu^2}\|\nabla\tilde{\boldsymbol{u}}_h\|^2_{L^2(Q_T)} + (C_S^4 h_{\max}^{4-d/3}+1)\|\nabla\tilde{\boldsymbol{u}}_h\|^4_{L^4(L^3)} + \|\tilde{\boldsymbol{f}}\|^2_{L^2(H^{-1})}\Bigg) \quad (4.29)$$

Proof. We follow the same idea as in the proof of Lemma 4.4. We use the Local Inverse Inequality Theorem A.3 in (A.11) for the eddy viscosity term, this is, there exists a constant $C_{3;2} > 0$ such that

$$\begin{aligned}(\nu_t(\mathbf{u}_h^k)\nabla\mathbf{u}_h^k, \nabla\mathbf{v}_h)_\Omega &\leq \sum_{K\in\mathcal{T}_h}(C_S h_K)^2\|\nabla\mathbf{u}_h^k\|^2_{L^3(K)}\|\nabla\mathbf{v}_h\|_{L^3(K)} \\ &\leq C_{3;2}\sum_{K\in\mathcal{T}_h} C_S^2 h_K^{2-d/6}\|\nabla\mathbf{u}_h^k\|^2_{L^3(K)}\|\nabla\mathbf{v}_h\|_{L^2(K)} \\ &\leq C_{3;2}C_S^2 h_{\max}^{2-d/6}\|\nabla\mathbf{u}_h^k\|^2_{L^3(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}\end{aligned}$$

From this and (4.25), we obtain that

$$\begin{aligned}(p_h^k, \nabla\cdot\mathbf{v}_h)_\Omega \leq{}& \left\|\frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)} + \frac{1}{\mu}\|\nabla\mathbf{u}_h^k\|_{L^2(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)} \\ &+ C_{3;2}C_S^2 h_{\max}^{2-d/6}\|\nabla\mathbf{u}_h^k\|^2_{L^3(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)} + \frac{1}{2}\|\mathbf{u}_h^k\|_{L^2(\Omega)}\|\nabla\mathbf{u}_h^k\|_{L^3(\Omega)}\|\mathbf{v}_h\|_{L^6(\Omega)} \\ &+ \frac{1}{2}\|\mathbf{u}_h^k\|_{L^3(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}\|\mathbf{u}_h^k\|_{L^6(\Omega)} + \|\mathbf{f}_h^k\|_{H^{-1}(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}\end{aligned}$$

Now we use the Inf-Sup Condition Theorem A.8 for $p_1 = 2$ and $p_2 = 2$, then there exists a constant $\alpha' > 0$ such that

$$\alpha'\|p_h^k\|_{L^2(\Omega)} \leq \sup_{\mathbf{v}_h\in H^1(\Omega)}\frac{(\nabla\cdot\mathbf{v}_h, p_h^k)_\Omega}{\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}}.$$

Then we study each term separately:

- First, using Poincaré Inequality A.2 with constant $C_{2;1,2} > 0$

$$\left\|\frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}\|\mathbf{v}_h\|_{L^2(\Omega)} \leq C_{2;1,2}\left\|\frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}.$$

- For the next term, we use the injection $L^3(\Omega) \hookrightarrow L^2(\Omega)$ with the constant $C_{2;3} > 0$ and applying the Poincaré Inequality A.2 with constant $C_{3;1,3} > 0$, we obtain that

$$\|\mathbf{u}_h^k\|_{L^2(\Omega)} \leq C_{2;3}\|\mathbf{u}_h^k\|_{L^3(\Omega)} \leq C_{2;3}C_{3;1,3}\|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)}.$$

  Furthermore, we know from the Sobolev embedding Theorem A.2 that $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ for $d = 2,3$ with constant $C_{6;1,2} > 0$, this is,

$$\|\mathbf{v}_h\|_{L^6(\Omega)} \leq C_{6;1,2}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)}.$$

  To sum up:

$$\|\mathbf{u}_h^k\|_{L^2(\Omega)}\|\nabla \mathbf{u}_h^k\|_{L^3(\Omega)}\|\mathbf{v}_h\|_{L^6(\Omega)} \leq C_{2;3}C_{3;1,3}C_{6;1,2}\|\nabla \mathbf{u}_h^k\|^2_{L^3(\Omega)}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)}.$$

- Thanks to the constant $C_{6;1,3} > 0$ defined in (4.20),

$$\|\mathbf{u}_h^k\|_{L^3(\Omega)}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)}\|\mathbf{u}_h^k\|_{L^6(\Omega)} \leq C_{3;1,3}C_{6;1,3}\|\nabla \mathbf{u}_h^k\|^2_{L^3(\Omega)}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)}.$$

Bringing back to the beginning of the proof, we obtain

$$\begin{aligned}(p_h^k \nabla\cdot \mathbf{v}_h)_\Omega \leq{}& C_{2;1,2}\left\|\frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)} + \frac{1}{\mu}\|\nabla \mathbf{u}_h^k\|_{L^2(\Omega)}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)} \\ &+ (C_{3;2}C_S^2 h_{\max}^{2-d/6} + \frac{1}{2}C_{3;1,3}(C_{2;3}C_{6;1,2} + C_{6;1,3}))\|\nabla \mathbf{u}_h^k\|^2_{L^3(\Omega)}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)} \\ &+ \|\mathbf{f}_h^k\|_{H^{-1}(\Omega)}\|\nabla \mathbf{v}_h\|_{L^2(\Omega)}.\end{aligned} \tag{4.30}$$

By Inf-Sup Condition Theorem A.8 for $p_1 = 2$ and $p_2 = 2$ there exists a constant $\alpha' > 0$ such that

$$\alpha'\|p_h^k\|_{L^2(\Omega)} \leq \sup_{\mathbf{v}_h \in H^1(\Omega)} \frac{(\nabla\cdot \mathbf{v}_h, p_h^k)_\Omega}{\|\nabla \mathbf{v}_h\|_{L^2(\Omega)}}$$

and using Inequality (4.30), we obtain

$$\begin{aligned}\alpha'\|p_h^k\|_{L^2(\Omega)} \leq{}& C_{2;1,2}\left\|\frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)} + \frac{1}{\mu}\|\nabla \mathbf{u}_h^k\|_{L^2(\Omega)} \\ &+ (C_{3;2}C_S^2 h_{\max}^{2-d/6} + \frac{1}{2}C_{3;1,3}(C_{2;3}C_{6;1,2} + C_{6;1,3}))\|\nabla \mathbf{u}_h^k\|^2_{L^3(\Omega)} + \|\mathbf{f}_h^k\|_{H^{-1}(\Omega)}.\end{aligned}$$

Taking the square, multiplying by Δt and summing for $k = 1,\dots,L$, we obtain the result.

□

With this last result, we obtain an estimate for the pressure in a Hilbert space and we also have proved that under a certain constrain over Δt, the velocity filed is in $L^\infty(0,T_f;W^{1,3}(\Omega))$.

These results are the key for the next chapter. As we said, we need Hilbert spaces to develop an *a posteriori* error bound estimator and the velocity regularity obtained will help us to prove it.

5

A posteriori error estimation for unsteady Smagorinsky model

The aim of this chapter is to build a space-time error estimator based on the BRR theory for the unsteady Smagorinsky model. We shall extend the theory described in Section 1.4 and Chapter 2 considering the Reynolds number as the parameter.

In Section 5.1 we establish the general notation to apply the RB method. In Section 5.2, we study the well-posedness of the problem over some suitable norms. Then, in Section 5.3, we obtain an *a posteriori* error bound estimator for the unsteady case. In Section 5.4, we clarify the approximation of the inf-sup Stability Factor for the unsteady case and we overview the difficulties in its computation.

The notation used along this chapter has been introduced in Section 4.1. In order to avoid unnecessary repetition, we refer to the reader to Section 4.1 for notation description.

5.1 Problem statement

We consider the unsteady dimensionless Smagorinsky model introduced in (4.9), that is:

Find $(\tilde{\mathbf{w}}_h, \tilde{p}_h) \in \tilde{Y}_h \times \tilde{Q}_h$ solution of

$$\begin{cases} (\partial_t^* \tilde{\mathbf{w}}_h, \tilde{\mathbf{v}}_h)_{Q_T} + \dfrac{1}{\mathrm{Re}}(\nabla \tilde{\mathbf{w}}_h, \nabla \tilde{\mathbf{v}}_h)_{Q_T} + (\nu_t(\tilde{\mathbf{w}}_h)\nabla \tilde{\mathbf{w}}_h, \nabla \tilde{\mathbf{v}}_h)_{Q_T} & \\ +\dfrac{1}{2}[((\tilde{\mathbf{w}}_h \cdot \nabla)\tilde{\mathbf{w}}_h, \tilde{\mathbf{v}}_h)_{Q_T} - ((\tilde{\mathbf{w}}_h \cdot \nabla)\tilde{\mathbf{v}}_h, \tilde{\mathbf{w}}_h)_{Q_T}] & \\ -(\nabla \cdot \tilde{\mathbf{v}}_h, \tilde{p}_h)_{Q_T} = \langle \tilde{\mathbf{f}}, \tilde{\mathbf{v}}_h \rangle_{Q_T}, & \forall \tilde{\mathbf{v}}_h \in \tilde{Y}_h, \\ (\nabla \cdot \tilde{\mathbf{w}}_h, \tilde{q}_h)_{Q_T} = 0, & \forall \tilde{q}_h \in \tilde{Q}_h. \end{cases}$$

where the eddy viscosity term is given by

$$\nu_t(\tilde{\mathbf{w}}) = \sum_{K \in \mathcal{T}_h} (C_S h_K)^2 |\nabla \tilde{\mathbf{w}}_{|K}| \chi_K.$$

To simplify notation, we introduce the space $\tilde{X}_h = \tilde{Y}_h \times \tilde{Y}_h \times \tilde{Q}_h$, this is, the space formed by piecewise constant functions $\tilde{V}_h$ related to the velocity field $\tilde{\mathbf{v}}_h \in \tilde{Y}_h$, its discrete time derivative $\partial_t^* \tilde{\mathbf{v}}_h \in \tilde{Y}_h$ and pressure $\tilde{q}_h \in \tilde{Q}_h$.

In order to consider nonhomogeneous Dirichlet BCs for the application of the RB method, we set up a divergence free lift $\mathbf{u}_D$ that will take part of the formulation, then, $\tilde{\mathbf{w}}_h = \tilde{\mathbf{u}}_h + \mathbf{u}_D$ where $\tilde{\mathbf{u}}_h$ verifies homogeneous Dirichlet BCs. We assume that $\mathbf{u}_D \in L^\infty(0, T_f; H_0^1(\Omega))$ and for simplicity, it must verify that $\mathbf{u}_D(0,\cdot)$ is equal to the initial condition, then, $\tilde{\mathbf{w}}_h(0,\cdot) = 0$. We consider $\mu = \mathrm{Re}$ the parameter of the problem and $\mathcal{D}$ the parameter set. Then the problem to solve in its compact form is defined as follows

$$\begin{cases} \text{Given } \mu \in \mathcal{D} \text{ find } \tilde{U}_h(\mu) = (\tilde{\mathbf{u}}_h(\mu), \tilde{p}_h(\mu)) \in \tilde{X}_h \text{ such that} \\ \tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu) = F(\tilde{V}_h; \mu), \quad \forall \tilde{V}_h \in \tilde{X}_h, \end{cases} \tag{5.1}$$

where A is defined as

$$\tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu) = \tilde{A}_0(\tilde{U}_h(\mu), \tilde{V}_h; \mu) + \tilde{A}_1(\tilde{U}_h(\mu), \tilde{V}_h; \mu) + \tilde{A}_2(\tilde{U}_h(\mu), \tilde{V}_h; \mu) \tag{5.2}$$

with

$$\begin{aligned}
\tilde{A}_0(\tilde{U}, \tilde{V}; \mu) &= \int_{Q_T} \partial_t^* \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} \, dQ_T \\
\tilde{A}_1(\tilde{U}, \tilde{V}; \mu) &= \frac{1}{\mu} \int_{Q_T} \nabla \tilde{\mathbf{u}} : \nabla \tilde{\mathbf{v}} \, dQ_T + \frac{1}{2} \int_{Q_T} [(\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} - (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{v}} \cdot \tilde{\mathbf{u}}] \, dQ_T \\
&\quad + \frac{1}{2} \int_{Q_T} [(\mathbf{u}_D \cdot \nabla) \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} - (\mathbf{u}_D \cdot \nabla) \tilde{\mathbf{v}} \cdot \tilde{\mathbf{u}}] \, dQ_T \\
&\quad + \frac{1}{2} \int_{Q_T} [(\tilde{\mathbf{u}} \cdot \nabla) \mathbf{u}_D \cdot \tilde{\mathbf{v}} - (\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{v}} \cdot \mathbf{u}_D] \, dQ_T \\
&\quad + \int_{Q_T} \nu_t(\tilde{\mathbf{u}} + \mathbf{u}_D) \nabla (\tilde{\mathbf{u}} + \mathbf{u}_D) : \nabla \tilde{\mathbf{v}} \, dQ_T \\
\tilde{A}_2(\tilde{U}, \tilde{V}; \mu) &= \int_{Q_T} (\nabla \cdot \tilde{\mathbf{u}}) \tilde{q} \, dQ_T - \int_{Q_T} (\nabla \cdot \tilde{\mathbf{v}}) \tilde{p} \, dQ_T \\
F(\tilde{V}; \mu) &= \int_{Q_T} \mathbf{f} \cdot \tilde{\mathbf{v}} \, dQ_T - \frac{1}{\mu} \int_{Q_T} \nabla \mathbf{u}_D : \nabla \tilde{\mathbf{v}} \, dQ_T \\
&\quad - \frac{1}{2} \int_{Q_T} (\mathbf{u}_D \cdot \nabla) \mathbf{u}_D \cdot \tilde{\mathbf{v}} \, dQ_T + \frac{1}{2} \int_{Q_T} (\mathbf{u}_D \cdot \nabla) \tilde{\mathbf{v}} \cdot \mathbf{u}_D \, dQ_T.
\end{aligned} \tag{5.3}$$

where $\tilde{U} = (\tilde{\mathbf{u}}, \partial_t^* \tilde{\mathbf{u}}, \tilde{p})$ and $\tilde{V} = (\tilde{\mathbf{v}}, \partial_t^* \tilde{\mathbf{v}}, \tilde{q})$. Note that $\tilde{A}_0$ is the operator related to the derivative on time, $\tilde{A}_1$ involves convection and viscous effects and $\tilde{A}_2$ involves the pressure.

From now on, we consider the following norm on $\tilde{X}_h$:

$$\|\tilde{V}_h\|_{\tilde{X}} = \sqrt{\|\tilde{\mathbf{v}}_h\|^2_{L^2_T(H^1_0)} + \|\tilde{q}_h\|^2_{L^2(Q_T)}}, \tag{5.4}$$

where we recall that the $L^p_T(H^1_0(\Omega))$-norm was introduced in (4.13)-(4.14) for $p \in [1, +\infty]$.

With this setting, we can state the Reduced Basis problem as

$$\begin{cases} \text{Given } \mu \in \mathcal{D}, \text{ find } \tilde{U}_N(\mu) \in \tilde{X}_N \text{ such that} \\ \tilde{A}(\tilde{U}_N(\mu), \tilde{V}_N; \mu) = F(\tilde{V}_N; \mu), \quad \forall \tilde{V}_N \in \tilde{X}_N \end{cases} \tag{5.5}$$

where $\tilde{X}_N = \tilde{Y}_N \times \tilde{Y}_N \times \tilde{Q}_N$, with $\tilde{Y}_N \subset \tilde{Y}_h$ and $\tilde{Q}_N \subset \tilde{Q}_h$ are the reduced basis spaces.

Remark 5.1. *The results of existence and stability proved in Chapter 4 still hold for problem* (5.1) *for smooth enough lift* $\mathbf{u}_D$.

5.2 Well-posedness analysis

To study the well-posedness of the model, we use the Brezzi-Rappaz-Raviart (BRR) theory (see, e.g. [8]). This theory provides a technique to obtain an *a posteriori* error estimator for nonlinear parametric problems and it has already applied to the steady case. We study the directional derivative defined in Definition A.1 of operator $\tilde{A}$ to apply this theory. Let $\tilde{Z} = (\tilde{\mathbf{z}}, \partial_t^* \tilde{\mathbf{z}}, \tilde{r})$, it holds,

$$\begin{aligned}
\partial_1 \tilde{A}_0(\tilde{U}, \tilde{V}; \mu)(\tilde{Z}) =& \int_{Q_T} \partial_t^* \tilde{\mathbf{z}} \cdot \tilde{\mathbf{v}} \, dQ_T, \\
\partial_1 \tilde{A}_1(\tilde{U}, \tilde{V}; \mu)(\tilde{Z}) =& \frac{1}{\mu} \int_{Q_T} \nabla \tilde{\mathbf{z}} : \nabla \tilde{\mathbf{v}} \, dQ_T \\
&+ \frac{1}{2} \int_{Q_T} [(\tilde{\mathbf{w}} \cdot \nabla) \tilde{\mathbf{z}} \cdot \tilde{\mathbf{v}} - (\tilde{\mathbf{w}} \cdot \nabla) \tilde{\mathbf{v}} \cdot \tilde{\mathbf{z}}] \, dQ_T \\
&+ \frac{1}{2} \int_{Q_T} [(\tilde{\mathbf{z}} \cdot \nabla) \tilde{\mathbf{w}} \cdot \tilde{\mathbf{v}} - (\tilde{\mathbf{z}} \cdot \nabla) \tilde{\mathbf{v}} \cdot \tilde{\mathbf{w}}] \, dQ_T \\
&+ \int_{Q_T} \nu_t(\tilde{\mathbf{w}}) \nabla \tilde{\mathbf{z}} : \nabla \tilde{\mathbf{v}} \, dQ_T \\
&+ \sum_{K \in \mathcal{T}_h} \int_0^{T_f} \int_K (C_S h_K)^2 \frac{\nabla(\tilde{\mathbf{w}}) : \nabla \tilde{\mathbf{z}}}{|\nabla(\tilde{\mathbf{w}})|} \nabla(\tilde{\mathbf{w}}) : \nabla \tilde{\mathbf{v}} \, dK dt, \\
\partial_1 \tilde{A}_2(\tilde{U}, \tilde{V}; \mu)(\tilde{Z}) =& \int_{Q_T} (\nabla \cdot \tilde{\mathbf{z}}) \tilde{q} \, dQ_T - \int_{Q_T} (\nabla \cdot \tilde{\mathbf{v}}) \tilde{r} \, dQ_T
\end{aligned} \tag{5.6}$$

where $\tilde{\mathbf{w}} = \tilde{\mathbf{u}} + \mathbf{u}_D$.

Remark 5.2. *Before to start, we define some constants derived from relevant results presented in Appendix A.2:*

- *The norm related to the inner product* (2.12) *is equivalent to the* $H_0^1(\Omega)$*-norm, then, there exists a constant* $C_{1,2;T} > 0$ *such that*

$$\|\nabla\tilde{\boldsymbol{v}}_h\|_{L^2(\Omega)} \le C_{1,2;T}\|\tilde{\boldsymbol{v}}_h\|_T, \;\forall\tilde{\boldsymbol{v}}_h \in \tilde{Y}_h, \forall t \in I_f. \tag{5.7}$$

- *Applying the Sobolev embedding Theorem A.2 we obtain that* $H_0^1(\Omega) \hookrightarrow L^4(\Omega) \hookrightarrow L^2(\Omega)$ *with constants* $C_{4;1,2}, C_{2;1,2} > 0$*. Thanks to the previous point, there exist constants* $C_{4;T} = C_{4;1,2}C_{1,2;T}$ *and* $C_{2;T} = C_{2;1,2}C_{1,2;T}$ *such that*

$$\|\tilde{\boldsymbol{v}}_h\|_{L^4(\Omega)} \le C_{4;T}\|v\|_T, \;\|\tilde{\boldsymbol{v}}_h\|_{L^2(\Omega)} \le C_{2;T}\|v\|_T, \;\forall\tilde{\boldsymbol{v}}_h \in \tilde{Y}_h, \;\forall t \in I_f. \tag{5.8}$$

- *For all* $K \in \mathcal{T}_h$*, we apply the Local Inverse Inequality Theorem A.3 as in* (A.11)*, introducing the constant* $C_{3;2} > 0$*, this is,*

$$\|\nabla\tilde{\boldsymbol{v}}_h\|_{L^3(K)} \le C_{3;2}h_K^{-d/6}\|\nabla\tilde{\boldsymbol{v}}_h\|_{L^2(K)}, \quad \forall K \in \mathcal{T}_h, \forall\tilde{\boldsymbol{v}}_h \in \tilde{Y}_h, \;\forall t \in I_f. \tag{5.9}$$

We start proving the continuity of operator $\partial_1\tilde{A}$.

Proposition 5.1. *For any* $\mu \in \mathcal{D}$ *and* $\tilde{U}_h(\mu) \in \tilde{X}_h$*, it holds*

$$|\partial_1\tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)| \le \gamma_h\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}}, \;\forall\tilde{Z}_h, \tilde{V}_h \in \tilde{X}_h, \tag{5.10}$$

where $\gamma_h = \max_{\mu\in\mathcal{D}}\gamma_h(\mu)$ *and*

$$\gamma_h(\mu) = 2\frac{C_{2;T}^2}{\Delta t} + 2C_{1,2;T} + \frac{C_{1,2;T}^2}{\mu} + 2C_{1,2;T}C_{4;T}^2\|\tilde{\boldsymbol{w}}_h\|_{L_T^\infty(H_0^1)} + 2(C_{1,2;T}C_{3;2}C_S)^2 h_{\max}^{2-d/3}\|\nabla\tilde{\boldsymbol{w}}_h\|_{L^\infty(L^3)}.$$

with $\tilde{\boldsymbol{w}}_h = \tilde{\boldsymbol{u}}_h + \boldsymbol{u}_D$*, and the constants* $C_{1,2;T}$*,* $C_{2;T}$*,* $C_{4;T}$ *and* $C_{3;2}$ *are defined in* (5.7)-(5.9).

Proof. We start from

$$|\partial_1\tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)| \le |\partial_1\tilde{A}_0(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)| + |\partial_1\tilde{A}_1(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)| + |\partial_1\tilde{A}_2(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)|$$

We study each term separately using the constants defined in (5.7)-(5.9).

The first term involves the derivative in time,

$$
\begin{aligned}
|\partial_1\tilde{A}_0(\tilde{U}_h(\mu),\tilde{V}_h)(\tilde{Z}_h)| &= \left|\sum_{k=1}^{L}\Delta t\int_\Omega \frac{\mathbf{z}_h^k-\mathbf{z}_h^{k-1}}{\Delta t}\cdot\mathbf{v}_h^k\,d\Omega\right| \\
&\leq \sum_{k=1}^{L}\Delta t\left\|\frac{\mathbf{z}_h^k-\mathbf{z}_h^{k-1}}{\Delta t}\right\|_{L^2(\Omega)}\|\mathbf{v}_h^k\|_{L^2(\Omega)} \leq \frac{C_{2;T}^2}{\Delta t}\sum_{k=1}^{L}\Delta t\|\mathbf{z}_h^k-\mathbf{z}_h^{k-1}\|_T\|\mathbf{v}_h^k\|_T \qquad (5.11)\\
&\leq 2\frac{C_{2;T}^2}{\Delta t}\|\tilde{\mathbf{z}}_h\|_{L_T^2(H_0^1)}\|\tilde{\mathbf{v}}_h\|_{L_T^2(H_0^1)} \leq 2\frac{C_{2;T}^2}{\Delta t}\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}}
\end{aligned}
$$

Next, we study $|\partial_1\tilde{A}_1(\tilde{U}_h(\mu),\tilde{V}_h;\mu)(\tilde{Z}_h)|$ term by term,

$$
\begin{aligned}
\left|\frac{1}{\mu}\int_{Q_T}\nabla\tilde{\mathbf{z}}_h:\nabla\tilde{\mathbf{v}}_h\,dQ_T\right| &\leq \frac{1}{\mu}\sum_{k=1}^{L}\Delta t\left|\int_\Omega\nabla\mathbf{z}_h^k:\nabla\mathbf{v}_h^k\,d\Omega\right| \\
&\leq \frac{1}{\mu}\sum_{k=1}^{L}\Delta t\|\nabla\mathbf{z}_h^k\|_{L^2(\Omega)}\|\nabla\mathbf{v}_h^k\|_{L^2(\Omega)} \leq \frac{C_{1,2;T}^2}{\mu}\sum_{k=1}^{L}\Delta t\|\mathbf{z}_h^k\|_T\|\mathbf{v}_h^k\|_T
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\left|\frac{1}{2}\int_{Q_T}[(\tilde{\mathbf{w}}_h\cdot\nabla)\tilde{\mathbf{z}}_h\cdot\tilde{\mathbf{v}}_h-(\tilde{\mathbf{w}}_h\cdot\nabla)\tilde{\mathbf{v}}_h\cdot\tilde{\mathbf{z}}_h]\,dQ_T\right.\\
&\qquad\left.+\frac{1}{2}\int_{Q_T}[(\tilde{\mathbf{z}}_h\cdot\nabla)\tilde{\mathbf{w}}_h\cdot\tilde{\mathbf{v}}_h-(\tilde{\mathbf{z}}_h\cdot\nabla)\tilde{\mathbf{v}}_h\cdot\tilde{\mathbf{w}}_h]\,dQ_T\right|\\
&\leq\frac{1}{2}\sum_{k=1}^{L}\Delta t\int_\Omega\left|(\mathbf{w}_h^k\cdot\nabla)\mathbf{z}_h^k\cdot\mathbf{v}_h^k-(\mathbf{w}_h^k\cdot\nabla)\mathbf{v}_h^k\cdot\mathbf{z}_h^k\right|d\Omega\\
&\quad+\frac{1}{2}\sum_{k=1}^{L}\Delta t\int_\Omega\left|(\mathbf{z}_h^k\cdot\nabla)\mathbf{w}_h^k\cdot\mathbf{v}_h^k-(\mathbf{z}_h^k\cdot\nabla)\mathbf{v}_h^k\cdot\mathbf{w}_h^k\right|d\Omega\\
&\leq\frac{1}{2}\sum_{k=1}^{L}\Delta t\Big[\|\mathbf{w}_h^k\|_{L^4(\Omega)}\|\nabla\mathbf{z}_h^k\|_{L^2(\Omega)}\|\mathbf{v}_h^k\|_{L^4(\Omega)}+\|\mathbf{w}_h^k\|_{L^4(\Omega)}\|\nabla\mathbf{v}_h^k\|_{L^2(\Omega)}\|\mathbf{z}_h^k\|_{L^4(\Omega)}\\
&\quad+\|\mathbf{z}_h^k\|_{L^4(\Omega)}\|\nabla\mathbf{w}_h^k\|_{L^2(\Omega)}\|\mathbf{v}_h^k\|_{L^4(\Omega)}+\|\mathbf{z}_h^k\|_{L^4(\Omega)}\|\nabla\mathbf{v}_h^k\|_{L^2(\Omega)}\|\mathbf{w}_h^k\|_{L^4(\Omega)}\Big]\\
&\leq 2C_{1,2;T}C_{4;T}^2\sum_{k=1}^{L}\Delta t\|\mathbf{w}_h^k\|_T\|\mathbf{z}_h^k\|_T\|\mathbf{v}_h^k\|_T.
\end{aligned}
$$

where we use (5.7) and (5.8) and the Hölder's Inequality A.1.

Finally, the viscosity term is bounded by

$$\left| \int_{Q_T} \nu_t(\tilde{\mathbf{w}}_h) \nabla \tilde{\mathbf{z}}_h : \nabla \tilde{\mathbf{v}}_h \, dQ_T + \sum_{K \in \mathcal{T}_h} \int_0^{T_f} \Delta t \int_K (C_S h_K)^2 \frac{\nabla \tilde{\mathbf{w}}_h : \nabla \tilde{\mathbf{z}}_h}{|\nabla \tilde{\mathbf{w}}_h|} \nabla \tilde{\mathbf{w}}_h : \nabla \tilde{\mathbf{v}}_h \, dK dt \right|$$
$$\leq 2 \sum_{K \in \mathcal{T}_h} \sum_{k=1}^{L} \Delta t \int_K (C_S h_K)^2 \|\nabla \mathbf{w}_h^k\|_{L^3(K)} \|\nabla \mathbf{z}_h^k\|_{L^3(K)} \|\nabla \mathbf{v}_h^k\|_{L^3(K)} \, dK$$
$$\leq 2 C_{3;2}^2 \sum_{K \in \mathcal{T}_h} \sum_{k=1}^{L} \Delta t \int_K C_S^2 h_K^{2-d/3} \|\nabla \mathbf{w}_h^k\|_{L^3(K)} \|\nabla \mathbf{z}_h^k\|_{L^2(K)} \|\nabla \mathbf{v}_h^k\|_{L^2(K)} \, dK$$
$$\leq 2 (C_{3;2} C_S)^2 h_{\max}^{2-d/3} \sum_{k=1}^{L} \Delta t \left[\|\nabla \mathbf{w}_h^k\|_{L^3(\Omega)} \|\nabla \mathbf{z}_h^k\|_{L^2(\Omega)} \|\nabla \mathbf{v}_h^k\|_{L^2(\Omega)} \right]$$
$$\leq 2 (C_{1,2;T} C_{3;2} C_S)^2 h_{\max}^{2-d/3} \sum_{k=1}^{L} \Delta t \left[\|\nabla \mathbf{w}_h^k\|_{L^3(\Omega)} \|\mathbf{z}_h^k\|_T \|\mathbf{v}_h^k\|_T \right]$$

where $\tilde{\mathbf{w}}_h = \tilde{\mathbf{u}}_h + \mathbf{u}_D$ and the constants $C_{1,2;T}$ is defined in (5.7) and $C_{3;2}$ comes from the application of the Local Inverse Inequality (5.9).

To sum up,

$$|\partial_1 \tilde{A}_1(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)|$$
$$\leq \sum_{k=1}^{L} \Delta t \left(\frac{C_{1,2;T}^2}{\mu} + 2 C_{1,2;T} C_{4;T}^2 \|\mathbf{w}_h^k\|_T \right.$$
$$\left. + 2 (C_{1,2;T} C_{3;2} C_S)^2 h_{\max}^{2-d/3} \|\nabla \mathbf{w}_h^k\|_{L^3(\Omega)} \right) \|\mathbf{z}_h^k\|_T \|\mathbf{v}_h^k\|_T$$
$$\leq \left(\frac{C_{1,2;T}^2}{\mu} + 2 C_{1,2;T} C_{4;T}^2 \|\tilde{\mathbf{w}}_h\|_{L_T^\infty(H_0^1)} \right. \tag{5.12}$$
$$\left. + 2 (C_{1,2;T} C_{3;2} C_S)^2 h_{\max}^{2-d/3} \|\nabla \tilde{\mathbf{w}}_h\|_{L^\infty(L^3)} \right) \|\tilde{\mathbf{z}}_h\|_{L_T^2(H_0^1)} \|\tilde{\mathbf{v}}_h\|_{L_T^2(H_0^1)}$$
$$\leq \left(\frac{C_{1,2;T}^2}{\mu} + 2 C_{1,2;T} C_{4;T}^2 \|\tilde{\mathbf{w}}_h\|_{L_T^\infty(H_0^1)} \right.$$
$$\left. + 2 (C_{1,2;T} C_{3;2} C_S)^2 h_{\max}^{2-d/3} \|\nabla \tilde{\mathbf{w}}_h\|_{L^\infty(L^3)} \right) \|\tilde{Z}_h\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}}$$

Finally,

$$|\partial_1 \tilde{A}_2(\tilde{U}_h(\mu), \tilde{V}_h)(\tilde{Z}_h)| = \left| \int_{Q_T} (\nabla \cdot \tilde{\mathbf{z}}_h) \tilde{q}_h \, dQ_T - \int_{Q_T} (\nabla \cdot \tilde{\mathbf{v}}_h) \tilde{r}_h \, dQ_T \right|$$
$$\leq \int_0^{T_f} \|\nabla \tilde{\mathbf{z}}_h\|_{L^2(\Omega)} \|\tilde{q}_h\|_{L^2(\Omega)} \, dt + \int_0^{T_f} \|\nabla \tilde{\mathbf{v}}_h\|_{L^2(\Omega)} \|\tilde{r}_h\|_{L^2(\Omega)} \, dt$$
$$\leq C_{1,2;T} \|\tilde{\mathbf{z}}_h\|_{L_T^2(H_0^1)} \|\tilde{q}_h\|_{L^2(Q_T)} + C_{1,2;T} \|\tilde{\mathbf{v}}_h\|_{L_T^2(H_0^1)} \|\tilde{r}_h\|_{L^2(Q_T)}$$
$$\leq 2 C_{1,2;T} \|\tilde{Z}_h\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}}$$

We get the result with

$$\gamma_h(\mu) = 2\frac{C_{2;T}^2}{\Delta t} + 2C_{1,2;T} + \frac{C_{1,2;T}^2}{\mu} + 2C_{1,2;T}C_{4;T}^2\|\tilde{\mathbf{w}}_h\|_{L_T^\infty(H_0^1)} + 2(C_{1,2;T}C_{3;2}C_S)^2 h_{\max}^{2-d/3}\|\nabla\tilde{\mathbf{w}}_h\|_{L^\infty(L^3)}.$$

and we set $\gamma_h = \max_{\mu\in\mathcal{D}} \gamma_h(\mu)$. □

The continuity constant γ_h grows as Δt^{-1} because of the derivative in time. This dependency is admissible for moderate time steps. If we include the time derivative in the $\tilde{X}$-norm, the constant continuity does not depend on the time step. However, the dependency on Δt^{-1} appears when proving the inf-sup condition.

Proposition 5.2. *For any $\mu \in \mathcal{D}$ and $\tilde{U}_h(\mu) \in \tilde{X}_h$, let us suppose that $\|\boldsymbol{u}_D\|_{L_T^\infty(H_0^1)} < \dfrac{1}{C_{1,2;T}C_{4;T}^2}$ and*

$$\|\tilde{\boldsymbol{u}}_h\|_{L_T^\infty(H_0^1)} < \frac{1}{C_{1,2;T}C_{4;T}^2} - \|\boldsymbol{u}_D\|_{L_T^\infty(H_0^1)}.$$

where the constants $C_{1,2;T}$ and $C_{4;T}$ are defined in (5.7) *and* (5.8) *Then, for all $\mu \in \mathcal{D}$ it holds,*

$$\beta_h(\mu) = \inf_{\tilde{Z}_h\in\tilde{X}_h}\sup_{\tilde{V}_h\in\tilde{X}_h} \frac{\partial_1\tilde{A}(\tilde{U}_h(\mu),\tilde{V}_h;\mu)(\tilde{Z}_h)}{\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}}}, \tag{5.13}$$

where

$$\beta_h(\mu) = \left(C_t + (1+\gamma_h^2)\left(\frac{1+\gamma_h^2}{\tilde{\beta}_h^2(\mu)} + \frac{2\sqrt{CC_t}}{\tilde{\beta}_h(\mu)}\right)\right)^{-1/2}$$

with

$$\tilde{\beta}_h(\mu) = 1 - C_{1,2;T}C_{4;T}^2\|\tilde{\boldsymbol{w}}_h\|_{L_T^\infty(H_0^1)}, \quad C_t = \frac{T_f}{\Delta t}$$

and γ_h is the continuity constant deduced in Proposition 5.1 that grows as Δt^{-1}.

Proof. We base this proof on Theorem 1 in [10]. The key is to prove that for a given $\mu \in \mathcal{D}$, and $\tilde{Z}_h \in \tilde{X}_h$,

$$\|\tilde{Z}_h\|_{\tilde{X}}\beta_h(\mu) \le \tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h) = \sup_{\tilde{V}_h\in\tilde{X}_h}\frac{\partial_1\tilde{A}(\tilde{U}_h(\mu),\tilde{V}_h;\mu)(\tilde{Z}_h)}{\|\tilde{V}_h\|_{\tilde{X}}}.$$

First, we start deducing an estimate for the velocity in its related norm. We select $\tilde{V}_h = \tilde{Z}_h \in X_h$, it holds

$$\partial_1 \tilde{A}(\tilde{U}_h(\mu),\tilde{Z}_h;\mu)(\tilde{Z}_h) = \partial_1 \tilde{A}_0(\tilde{U}_h(\mu),\tilde{Z}_h)(\tilde{Z}_h) + \partial_1 \tilde{A}_1(\tilde{U}_h(\mu),\tilde{Z}_h;\mu)(\tilde{Z}_h) + \partial_1 \tilde{A}_2(\tilde{U}_h(\mu),\tilde{Z}_h)(\tilde{Z}_h).$$

The time derivative term is positive since

$$\begin{aligned}
\partial_1 \tilde{A}_0(\tilde{U}_h(\mu),\tilde{Z}_h)(\tilde{Z}_h) &= \int_{Q_T} \partial_t^* \tilde{\mathbf{z}}_h \cdot \tilde{\mathbf{z}}_h \, dQ_T = \sum_{k=1}^{L} \Delta t \left(\frac{\mathbf{z}_h^k - \mathbf{z}_h^{k-1}}{\Delta t}, \mathbf{z}_h^k \right) \\
&= \frac{1}{2} \sum_{k=1}^{L} \left(\|\mathbf{z}_h^k\|^2_{L^2(\Omega)} - \|\mathbf{z}_h^{k-1}\|^2_{L^2(\Omega)} + \|\mathbf{z}_h^k - \mathbf{z}_h^{k-1}\|^2_{L^2(\Omega)} \right) \\
&= \frac{1}{2} \|\mathbf{z}_h^L\|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{k=1}^{L} \|\mathbf{z}_h^k - \mathbf{z}_h^{k-1}\|^2_{L^2(\Omega)};
\end{aligned}$$

For the second term, on one side, we bound the terms related to viscosity,

$$\begin{aligned}
&\int_{Q_T} \left(\frac{1}{\mu} + \nu_t(\tilde{\mathbf{w}}_h) \right) |\nabla \tilde{\mathbf{z}}_h|^2 \, dQ_T + \sum_{K \in \mathcal{T}_h} \int_0^{T_f} \int_K (C_S h_K)^2 \frac{|\nabla \tilde{\mathbf{w}}_h : \nabla \tilde{\mathbf{z}}_h|^2}{|\nabla \tilde{\mathbf{w}}_h|} \, d\Omega dt \\
&\qquad \geq \int_{Q_T} \left(\frac{1}{\mu} + \nu_t(\tilde{\mathbf{w}}_h) \right) |\nabla \tilde{\mathbf{z}}_h|^2 \, dQ_T \geq \|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)};
\end{aligned}$$

where we recall that $\tilde{\mathbf{w}}_h = \tilde{\mathbf{u}}_h + \mathbf{u}_D$. On another hand,

$$\begin{aligned}
&\frac{1}{2} \int_{Q_T} [(\tilde{\mathbf{z}}_h \cdot \nabla) \tilde{\mathbf{w}}_h \cdot \tilde{\mathbf{z}}_h - (\tilde{\mathbf{z}}_h \cdot \nabla) \tilde{\mathbf{z}}_h \cdot \tilde{\mathbf{w}}_h] \, dQ_T \\
&\quad = \frac{1}{2} \sum_{k=1}^{L} \Delta t \int_\Omega [(\mathbf{z}_h^k \cdot \nabla) \mathbf{w}_h^k \cdot \mathbf{z}_h^k - (\mathbf{z}_h^k \cdot \nabla) \mathbf{z}_h^k \cdot \mathbf{w}_h^k] \, d\Omega \\
&\quad \leq \frac{1}{2} \sum_{k=1}^{L} \Delta t [\|\mathbf{z}_h^k\|_{L^4(\Omega)} \|\nabla \mathbf{w}_h^k\|_{L^2(\Omega)} \|\mathbf{z}_h^k\|_{L^4(\Omega)} + \|\mathbf{z}_h^k\|_{L^4(\Omega)} \|\nabla \mathbf{z}_h^k\|_{L^2(\Omega)} \|\mathbf{w}_h^k\|_{L^4(\Omega)}] \\
&\quad \leq C_{1,2;T} C_{4;T}^2 \sum_{k=1}^{L} \Delta t \|\mathbf{w}_h^k\|_T \|\mathbf{z}_h^k\|_T^2 \leq C_{1,2;T} C_{4;T}^2 \|\tilde{\mathbf{w}}_h\|_{L^\infty_T(H^1_0)} \|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)}.
\end{aligned}$$

Lastly, $\partial_1 \tilde{A}_2(\tilde{U}_h(\mu),\tilde{Z}_h)(\tilde{Z}_h) = 0$, then,

$$\partial_1 \tilde{A}(\tilde{U}_h(\mu),\tilde{Z}_h;\mu)(\tilde{Z}_h) \geq (1 - C_{1,2;T} C_{4;T}^2 \|\tilde{\mathbf{w}}_h\|_{L^\infty_T(H^1_0)}) \|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)}. \tag{5.14}$$

We need that $1 - C_{1,2;T} C_{4;T}^2 \|\tilde{\mathbf{w}}_h\|_{L_T^\infty(H_0^1)} > 0$ and that is equivalent to

$$\frac{1}{C_{1,2;T} C_{4;T}^2} > \|\tilde{\mathbf{u}}_h\|_{L_T^\infty(H_0^1)} + \|\mathbf{u}_D\|_{L_T^\infty(H_0^1)}$$

Then, if we suppose that $\|\mathbf{u}_D\|_T < \dfrac{1}{C_{1,2;T} C_{4;T}^2}$, we need that

$$\|\tilde{\mathbf{u}}_h\|_{L_T^\infty(H_0^1)} < \frac{1}{C_{1,2;T} C_{4;T}^2} - \|\mathbf{u}_D\|_{L_T^\infty(H_0^1)}.$$

Note that inequality (5.14) is possible thanks to Corollary 4.5 for small data. Finally, there exists $\tilde{\beta}_h(\mu) = 1 - C_{1,2;T} C_{4;T}^2 \|\tilde{\mathbf{w}}_h\|_{L_T^\infty(H_0^1)} > 0$ such that

$$\|\tilde{\mathbf{z}}_h\|_{L_T^2(H_0^1)}^2 \tilde{\beta}_h(\mu) \leq \tilde{S}(\tilde{U}_h(\mu), \tilde{Z}_h) \|\tilde{Z}_h\|_{\tilde{X}}, \quad \forall \tilde{Z}_h \in \tilde{X}_h.$$

Next, we deduce the pressure estimate. Setting U_h^k, V_h^k, $Z_h^k \in \tilde{X}_h$ with $X_h = Y_h \times Y_h \times Q_h$ defined by

$$U_h^k = (\mathbf{u}_h^k, \mathbf{u}_h^{k-1}, p_h^k),\ V_h^k = (\mathbf{v}_h^k, \mathbf{v}_h^{k-1}, q_h^k),\ Z_h^k = (\mathbf{z}_h^k, \mathbf{z}_h^{k-1}, r_h^k),$$

from definition (5.6), we can express

$$\partial_1 \tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h) = \sum_{k=1}^{L} \Delta t\, \partial_1 A(U_h^k(\mu), V_h^k; \mu)(Z_h^k) \tag{5.15}$$

where

$$\begin{aligned}\partial_1 A(U_h^k(\mu), V_h^k; \mu)(Z_h^k) = \partial_1 A_0(U_h^k(\mu), V_h^k; \mu)(Z_h^k) + \partial_1 A_1(U_h^k(\mu), V_h^k; \mu)(Z_h^k) \\ + \partial_1 A_2(U_h^k(\mu), V_h^k; \mu)(Z_h^k).\end{aligned}$$

with the forms $\partial_1 A_1$ and $\partial_1 A_2$ as the same as $\partial_1 \tilde{A}_1$ and $\partial_1 \tilde{A}_2$ by changing the integration domain to Ω instead of Q_T and

$$\partial_1 A_0(U_h^k(\mu), V_h^k; \mu)(Z_h^k) = \int_\Omega \frac{\mathbf{z}_h^k - \mathbf{z}_h^{k-1}}{\Delta t} \cdot \mathbf{v}_h^k \, d\Omega.$$

Choosing the test function $V_h^k = (\mathbf{v}_h^k, \mathbf{v}_h^{k-1}, 0)$, for all $k = 1, \ldots, L$, we have

$$\begin{aligned}\partial_1 A(U_h^k(\mu), V_h^k; \mu)(Z_h^k) = \partial_1 A_0(U_h^k(\mu), V_h^k; \mu)(Z_h^k) \\ + \partial_1 A_1(U_h^k(\mu), V_h^k; \mu)(Z_h^k) - \int_\Omega (\nabla \cdot \mathbf{v}_h^k) r_h^k \, d\Omega\end{aligned}$$

from where

$$\int_{\Omega}(\nabla\cdot\mathbf{v}_h^k)r_h^k\,d\Omega = -\partial_1 A(U_h^k(\mu),V_h^k;\mu)(Z_h^k)$$
$$+\partial_1 A_0(U_h^k(\mu),V_h^k;\mu)(Z_h^k)+\partial_1 A_1(U_h^k(\mu),V_h^k;\mu)(Z_h^k)$$

Now, we define a supreme S_k related to each time step, for a given $\mu\in\mathcal{D}$:

$$S_k(U_h^k(\mu),Z_h^k) = \sup_{V_h^k\in X_h}\frac{\partial_1 A(U_h^k(\mu),V_h^k;\mu)(Z_h^k)}{\|V_h^k\|_X}, \quad \forall Z_h^k\in X_h, \tag{5.16}$$

with

$$\|V_h^k\|_X = \sqrt{\|\mathbf{v}_h^k\|_T^2+\|q_h^k\|_{L^2(\Omega)}^2}, \quad \forall V_h^k\in X_h.$$

Because of the V_h^k selection, we obtain that

$$\partial_1 A(U_h^k(\mu),V_h^k;\mu)(Z_h^k)\le S_k(U_h^k(\mu),Z_h^k)\|\mathbf{v}_h^k\|_T,\ \forall Z_h^k\in X_h$$

Analogously to (5.11) and (5.12), it can be obtained:

$$\partial_1 A_0(U_h^k(\mu),V_h^k;\mu)(Z_h^k)\le\frac{C_{2;T}^2}{\Delta t}\|\mathbf{z}_h^k-\mathbf{z}_h^{k-1}\|_T\|\mathbf{v}_h^k\|_T,$$
$$\partial_2 A_0(U_h^k(\mu),V_h^k;\mu)(Z_h^k)\le\gamma_h^k(\mu)\|\mathbf{z}_h^k\|_T\|\mathbf{v}_h^k\|_T,$$

where

$$\gamma_h^k(\mu)=\frac{C_{1,2;T}^2}{\mu}+2C_{1,2;T}C_{4;T}^2\|\mathbf{w}_h^k\|_T+2(C_{1,2;T}C_{3;2}C_S)^2h_{\max}^{2-d/3}\|\nabla\mathbf{w}_h^k\|_{L^3(\Omega)}.$$

with $\mathbf{w}_h^k=\mathbf{u}_h^k+\mathbf{u}_D$. Then,

$$\int_{\Omega}(\nabla\cdot\mathbf{v}_h^k)r_h^k\,d\Omega\le S_k(U_h^k(\mu),Z_h^k)\|\mathbf{v}_h^k\|_T+\frac{C_{2;T}^2}{\Delta t}\|\mathbf{z}_h^k-\mathbf{z}_h^{k-1}\|_T\|\mathbf{v}_h^k\|_T+\gamma_h^k(\mu)\|\mathbf{z}_h^k\|_T\|\mathbf{v}_h^k\|_T$$

Dividing by $\|\mathbf{v}_h^k\|_T$, we are able to apply the Inf-sup condition Theorem A.8 since the T-norm is equivalent to the $H_0^1(\Omega)$-norm. There exists $\alpha>0$ such that

$$\alpha\|r_h^k\|_{L^2(\Omega)}\le S_k(U_h^k(\mu),Z_h^k)+\frac{C_{2;T}^2}{\Delta t}\|\mathbf{z}_h^k-\mathbf{z}_h^{k-1}\|_T+\gamma_h^*\|\mathbf{z}_h^k\|_T.$$

where $\gamma_h^*=\max\limits_{\mu\in\mathcal{D},k=1,\dots,L}\gamma_h^k(\mu)$. Then,

$$\alpha^2\|r_h^k\|_{L^2(\Omega)}^2\le 3\left(S_k^2(U_h^k(\mu),Z_h^k)+\frac{C_{2;T}^4}{\Delta t^2}\|\mathbf{z}_h^k-\mathbf{z}_h^{k-1}\|_T^2+(\gamma_h^*)^2\|\mathbf{z}_h^k\|_T^2\right),$$

and multiplying by Δt and summing on k,

$$\alpha^2\|\tilde{r}_h\|^2_{L^2(Q_T)} \leq 3\sum_{k=1}^{L}\Delta t\left[S_k^2(U_h^k(\mu),Z_h^k)+3\frac{C_{2;T}^4}{\Delta t^2}\|\mathbf{z}_h^k-\mathbf{z}_h^{k-1}\|_T^2+3(\gamma_h^*)^2\|\mathbf{z}_h^k\|_T^2\right],$$

We need an upper bound of this inequality in terms of the supreme $\tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h)$ and $\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H_0^1)}$. On one side,

$$\begin{aligned}\alpha^2\|\tilde{r}_h\|^2_{L^2(Q_T)} \leq& 3\sum_{k=1}^{L}\Delta t\left[S_k^2(U_h^k(\mu),Z_h^k)\right]+3\left(\frac{2C_{2;T}^4}{\Delta t^2}+(\gamma_h^*)^2\right)\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H_0^1)}\\ \leq& 3\sum_{k=1}^{L}\Delta t\left[S_k^2(U_h^k(\mu),Z_h^k)\right]+6\gamma_h^2\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H_0^1)},\end{aligned}$$

where γ_h is the continuity constant deduced in Proposition 5.1. On the other side, let us define the functions $\tilde{V}_h^k \in \tilde{X}_h$ for all $k=1,\dots,L$ by

$$\tilde{V}_h^k = \begin{cases} V_h^k & \text{if } t\in(t_{k-1},t_k),\\ 0 & \text{otherwise,}\end{cases}$$

Then, using the definition of $\tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)$,

$$\tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)=\sup_{\tilde{V}_h\in\tilde{X}_h}\frac{\partial_1\tilde{A}(\tilde{U}_h(\mu),\tilde{V}_h;\mu)(\tilde{Z}_h)}{\|\tilde{V}_h\|_{\tilde{X}}}\geq \sup_{\tilde{V}_h^k\in\tilde{X}_h}\frac{\partial_1\tilde{A}(\tilde{U}_h(\mu),\tilde{V}_h^k;\mu)(\tilde{Z}_h)}{\|\tilde{V}_h^k\|_{\tilde{X}}},$$

for all $k=1,\dots,L$. Note that $\tilde{V}_h^k$ belongs to $\tilde{X}_h$ and

$$\|\tilde{V}_h^k\|^2_{\tilde{X}}=\sum_{n=1}^{L}\Delta t\left(\|\mathbf{v}_h^n\|_T^2+\|q_h^n\|^2_{L^2(\Omega)}\right)=\Delta t\|\mathbf{v}_h^k\|_T^2+\|q_h^k\|^2_{L^2(\Omega)}=\Delta t\|V_h^k\|_X^2$$

with $V_h^k\in X_h$. Moreover, note that from (5.15),

$$\partial_1\tilde{A}(\tilde{U}_h(\mu),\tilde{V}_h^k;\mu)(\tilde{Z}_h)=\Delta t\,\partial_1 A(U_h^k(\mu),V_h^k;\mu)(Z_h^k)$$

then for all $k=1,\dots,L$,

$$\tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)\geq\sup_{V_h^k\in X_h}\frac{\Delta t\,\partial_1 A(U_h^k(\mu),V_h^k;\mu)(Z_h^k)}{\sqrt{\Delta t}\|V_h^k\|_X}=\sqrt{\Delta t}S_k(U_h^k(\mu),Z_h^k).$$

in particular,

$$\sum_{k=1}^{L}\Delta t S_k^2(U_h^k(\mu),Z_h^k)\leq\sum_{k=1}^{L}\tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)=L\tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)=\frac{T_f}{\Delta t}\tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)$$

and we obtain the estimate for the pressure,

$$\|\tilde{r}_h\|^2_{L^2(Q_T)} \le C\left(C_t \tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h) + \gamma_h^2 \|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)}\right); \tag{5.17}$$

where $C = 6/\alpha^2$, $C_t = T_f/\Delta t$. Recovering the velocity estimate, we obtain that

$$\begin{aligned}
\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)}\tilde{\beta}_h(\mu) \le & \tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)\|\tilde{Z}_h\|_{\tilde{X}} \le \tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)\left(\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)} + \|\tilde{r}_h\|^2_{L^2(Q_T)}\right)^{1/2} \\
\le & \tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)\left((1+C\gamma_h^2)\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)} + CC_t\tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h)\right)^{1/2} \\
\le & (1+C\gamma_h^2)^{1/2}\tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)\|\tilde{\mathbf{z}}_h\|_{L^2_T(H^1_0)} + \sqrt{CC_t}\tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h)
\end{aligned}$$

and using the Young Inequality,

$$\begin{aligned}
\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)}\tilde{\beta}_h(\mu) \le & \frac{(1+C\gamma_h^2)^{1/2}}{2\varepsilon}\tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h) \\
& + \frac{\varepsilon(1+C\gamma_h^2)^{1/2}}{2}\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)} + \sqrt{CC_t}\tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h)
\end{aligned}$$

Choosing $\varepsilon = \tilde{\beta}_h(\mu)/(1+C\gamma_h^2)^{1/2}$, we obtain the velocity estimate

$$\|\tilde{\mathbf{z}}_h\|^2_{L^2_T(H^1_0)} \le \left(\frac{1+\gamma_h^2}{\tilde{\beta}_h^2(\mu)} + \frac{2\sqrt{CC_t}}{\tilde{\beta}_h(\mu)}\right)\tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h),$$

and the pressure, combining with (5.17)

$$\|\tilde{r}_h\|^2_{L^2(Q_T)} \le C\left(C_t + C_t\left(\frac{1+\gamma_h^2}{\tilde{\beta}_h^2(\mu)} + \frac{2\sqrt{CC_t}}{\tilde{\beta}_h(\mu)}\right)\right)\tilde{S}^2(\tilde{U}_h(\mu),\tilde{Z}_h).$$

Finally,

$$\|\tilde{Z}_h\|_{\tilde{X}}\beta_h(\mu) \le \tilde{S}(\tilde{U}_h(\mu),\tilde{Z}_h)$$

with

$$\beta_h(\mu) = \left(C_t + (1+\gamma_h^2)\left(\frac{1+\gamma_h^2}{\tilde{\beta}_h^2(\mu)} + \frac{2\sqrt{CC_t}}{\tilde{\beta}_h(\mu)}\right)\right)^{-1/2}.$$

□

Remark 5.3. *In this case, the Stability Factor $\beta_h(\mu)$ grows as γ_h^{-2}, the continuity constant, this is, Δt^2. Again, if we include the time derivative in the $\tilde{X}$-norm, we are not able to deduce an estimate for the derivative in time respect to the supreme $\tilde{S}(\tilde{U}_h(\mu),\cdot)$ as for velocity and pressure, therefore, the inf-sup* (5.13) *is not ensure. As we will see, the Stability Factor takes part in an a posteriori error estimator, for this reason, the previous proof is important.*

Remark 5.4. *If Δt is fixed, we are able to ensure the existence and uniqueness of problem* (5.1)*, thanks to the BRR theory.*

5.3 *A posteriori* error estimator

In this section we construct an *a posteriori* error estimator for the Greedy algorithm and as we said in the previous section, we will use the BRR theory for its construction.

We begin by proving that the derivative of the operator $\tilde{A}(\cdot,\cdot;\mu)$ is locally Lipschitz continuous.

Lemma 5.1. *For all $\mu \in \mathcal{D}$ and $\tilde{U}_h^1, \tilde{U}_h^2, \tilde{Z}_h, \tilde{V}_h \in \tilde{X}_h$, it holds*

$$|\partial_1 \tilde{A}(\tilde{U}_h^1, \tilde{V}_h;\mu)(\tilde{Z}_h) - \partial_1 \tilde{A}(\tilde{U}_h^2, \tilde{V}_h;\mu)(\tilde{Z}_h)| \leq \rho_T \|\tilde{U}_h^1 - \tilde{U}_h^2\|_{\tilde{X}} \|\tilde{Z}_h\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}}, \tag{5.18}$$

for a positive constant

$$\rho_T = \Delta t^{-1/2}(2C_{1,2;T}C_{4;T}^2 + 4(C_{1,2;T}C_{3;2})^3 C_S^2 h_{\max}^{2-d/2}),$$

and the constants $C_{1,2;T}, C_{4;T}$ and $C_{3;2}$ are defined in (5.7)-(5.9).

Proof. As $\tilde{\mathbf{w}}_h^1 - \tilde{\mathbf{w}}_h^2 = \tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2$, we have that

$$\begin{aligned}
&|\partial_1 \tilde{A}(\tilde{U}_h^1, \tilde{V}_h;\mu)(\tilde{Z}_h) - \partial_1 \tilde{A}(\tilde{U}_h^2, \tilde{V}_h;\mu)(\tilde{Z}_h)| \\
&\quad \leq \frac{1}{2}\left|\int_{Q_T} ((\tilde{\mathbf{w}}_h^1 - \tilde{\mathbf{w}}_h^2)\cdot\nabla)\tilde{\mathbf{z}}_h \cdot \tilde{\mathbf{v}}_h \, dQ_T\right| + \frac{1}{2}\left|\int_{Q_T} ((\tilde{\mathbf{w}}_h^1 - \tilde{\mathbf{w}}_h^2)\cdot\nabla)\tilde{\mathbf{v}}_h \cdot \tilde{\mathbf{z}}_h \, dQ_T\right| \\
&\quad + \frac{1}{2}\left|\int_{Q_T} (\tilde{\mathbf{z}}_h\cdot\nabla)(\tilde{\mathbf{w}}_h^1 - \tilde{\mathbf{w}}_h^2)\cdot \tilde{\mathbf{v}}_h \, dQ_T\right| + \frac{1}{2}\left|\int_{Q_T} (\tilde{\mathbf{z}}_h\cdot\nabla)\tilde{\mathbf{v}}_h\cdot(\tilde{\mathbf{w}}_h^1 - \tilde{\mathbf{w}}_h^2) \, dQ_T\right| \\
&\quad + \left|\int_{Q_T} (\nu_t(\tilde{\mathbf{w}}_h^1) - \nu_t(\tilde{\mathbf{w}}_h^2))\nabla\tilde{\mathbf{z}}_h : \nabla\tilde{\mathbf{v}}_h dQ_T\right| \\
&\quad + \left|\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2 (|\nabla\tilde{\mathbf{w}}_h^1| - |\nabla\tilde{\mathbf{w}}_h^2|)\nabla\tilde{\mathbf{z}}_h : \nabla\tilde{\mathbf{v}}_h \, dK dt\right| \\
&\quad + \left|\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2 \frac{\nabla\tilde{\mathbf{w}}_h^1 : \nabla\tilde{\mathbf{z}}_h}{|\nabla\tilde{\mathbf{w}}_h^1|}(\nabla\tilde{\mathbf{w}}_h^1 : \nabla\tilde{\mathbf{v}}_h) \, dK dt\right. \\
&\quad \left. - \sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2 \frac{\nabla\tilde{\mathbf{w}}_h^2 : \nabla\tilde{\mathbf{z}}_h}{|\nabla\tilde{\mathbf{w}}_h^2|}(\nabla\tilde{\mathbf{w}}_h^2 : \nabla\tilde{\mathbf{v}}_h) \, dK dt\right|.
\end{aligned}$$

Now, we bound each term separately, using the definition of the constants $C_{1,2;T}$ and $C_{4;T}$ in (5.7)-(5.8), and the inverse inequality

$$\|\tilde{\mathbf{v}}_h\|_{L_T^\infty(H_0^1)} \leq \frac{1}{\Delta t}\sum_{k=1}^{L}\Delta t\|\mathbf{v}_h^k\|_T \leq \frac{1}{\Delta t}\left(\sum_{k=1}^{L}\Delta t^2\|\mathbf{v}_h^k\|_T^2\right)^{1/2} \leq \Delta t^{-1/2}\|\tilde{\mathbf{v}}_h\|_{L_T^2(H_0^1)}$$

in time for all $\tilde{\mathbf{v}}_h \in \tilde{Y}_h$. Then,

$$\begin{aligned}
\frac{1}{2}\left|\int_{Q_T}((\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2)\cdot\nabla)\ \tilde{\mathbf{z}}_h\cdot\tilde{\mathbf{v}}_h\,dQ_T\right| &+ \frac{1}{2}\left|\int_{Q_T}((\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2)\cdot\nabla)\tilde{\mathbf{v}}_h\cdot\tilde{\mathbf{z}}_h\,dQ_T\right| \\
&\leq \frac{1}{2}\int_0^{T_f}\|\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2\|_{L^4(\Omega)}\|\nabla\tilde{\mathbf{z}}_h\|_{L^2(\Omega)}\|\tilde{\mathbf{v}}_h\|_{L^4(\Omega)}\,dt \\
&+\frac{1}{2}\int_0^{T_f}\|\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2\|_{L^4(\Omega)}\|\nabla\tilde{\mathbf{v}}_h\|_{L^2(\Omega)}\|\tilde{\mathbf{z}}_h\|_{L^4(\Omega)}\,dt \\
&\leq C_{1,2;T}C_{4;T}^2\int_0^{T_f}\|\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2\|_T\|\tilde{\mathbf{z}}_h\|_T\|\tilde{\mathbf{v}}_h\|_T\,dt \\
&\leq C_{1,2;T}C_{4;T}^2\|\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2\|_{L^2_T(H_0^1)}\|\tilde{\mathbf{z}}_h\|_{L^2_T(H_0^1)}\|\tilde{\mathbf{v}}_h\|_{L^\infty_T(H_0^1)} \\
&\leq C_{1,2;T}C_{4;T}^2\Delta t^{-1/2}\|\tilde{U}_h^1-\tilde{U}_h^2\|_{\tilde{X}}\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}};
\end{aligned}$$

and in the same way,

$$\begin{aligned}
\frac{1}{2}\left|\int_{Q_T}(\tilde{\mathbf{z}}_h\cdot\nabla)(\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2)\cdot\tilde{\mathbf{v}}_h\,dQ_T\right| &+\frac{1}{2}\left|\int_{Q_T}(\tilde{\mathbf{z}}_h\cdot\nabla)\tilde{\mathbf{v}}_h\cdot(\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2)\,dQ_T\right| \\
&\leq C_{1,2;T}C_{4;T}^2\Delta t^{-1/2}\|\tilde{U}_h^1-\tilde{U}_h^2\|_{\tilde{X}}\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K(C_Sh_K)^2(|\nabla\tilde{\mathbf{w}}_h^1|-|\nabla\tilde{\mathbf{w}}_h^2|)\nabla\tilde{\mathbf{z}}_h:\nabla\tilde{\mathbf{v}}_h\,dKdt\right| \\
&\quad\leq \sum_{K\in\mathcal{T}_h}\int_0^{T_f}(C_Sh_K)^2\|\nabla\tilde{\mathbf{u}}_h^1-\nabla\tilde{\mathbf{u}}_h^2\|_{L^3(K)}\|\nabla\tilde{\mathbf{z}}_h\|_{L^3(K)}\|\nabla\tilde{\mathbf{v}}_h\|_{L^3(K)}\,dt \\
&\quad\leq C_{3;2}^3\sum_{K\in\mathcal{T}_h}\int_0^{T_f}C_S^2h_K^{2-d/2}\|\nabla\tilde{\mathbf{u}}_h^1-\nabla\tilde{\mathbf{u}}_h^2\|_{L^2(K)}\|\nabla\tilde{\mathbf{z}}_h\|_{L^2(K)}\|\nabla\tilde{\mathbf{v}}_h\|_{L^2(K)}\,dt \\
&\quad\leq C_{3;2}^3C_S^2h_{\max}^{2-d/2}\int_0^{T_f}\|\nabla\tilde{\mathbf{u}}_h^1-\nabla\tilde{\mathbf{u}}_h^2\|_{L^2(\Omega)}\left(\sum_{K\in\mathcal{T}_h}\|\nabla\tilde{\mathbf{z}}_h\|_{L^2(K)}\|\nabla\tilde{\mathbf{v}}_h\|_{L^2(K)}\right)\,dt \\
&\quad\leq C_{3;2}^3C_S^2h_{\max}^{2-d/2}\|\nabla\tilde{\mathbf{u}}_h^1-\nabla\tilde{\mathbf{u}}_h^2\|_{L^\infty(L^2)}\|\nabla\tilde{\mathbf{z}}_h\|_{L^2(Q_T)}\|\nabla\tilde{\mathbf{v}}_h\|_{L^2(Q_T)} \\
&\quad\leq (C_{1,2;T}C_{3;2})^3C_S^2h_{\max}^{2-d/2}\|\tilde{\mathbf{u}}_h^1-\tilde{\mathbf{u}}_h^2\|_{L^\infty_T(H_0^1)}\|\tilde{\mathbf{z}}_h\|_{L^2_T(H_0^1)}\|\tilde{\mathbf{v}}_h\|_{L^2_T(H_0^1)} \\
&\quad\leq (C_{1,2;T}C_{3;2})^3C_S^2h_{\max}^{2-d/2}\Delta t^{-1/2}\|\tilde{U}_h^1-\tilde{U}_h^2\|_{\tilde{X}}\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}},
\end{aligned}$$

where we have used the Local Inverse Inequality (5.9).

And, for the last term, we use that

$$\left|\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2\frac{\nabla\tilde{\mathbf{w}}_h^1:\nabla\tilde{\mathbf{z}}_h}{|\nabla\tilde{\mathbf{w}}_h^1|}(\nabla\tilde{\mathbf{w}}_h^1:\nabla\tilde{\mathbf{v}}_h)\,dKdt\right.$$

$$\left.-\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2\frac{\nabla\tilde{\mathbf{w}}_h^2:\nabla\tilde{\mathbf{z}}_h}{|\nabla\tilde{\mathbf{w}}_h^2|}(\nabla\tilde{\mathbf{w}}_h^2:\nabla\tilde{\mathbf{v}}_h)\,dKdt\right|$$

$$=\left|\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2\left[\frac{\nabla\tilde{\mathbf{w}}_h^1:\nabla\tilde{\mathbf{z}}_h}{|\nabla\tilde{\mathbf{w}}_h^1|}(\nabla(\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2):\nabla\tilde{\mathbf{v}}_h\right.\right.$$

$$\left.+\frac{\nabla(\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2):\nabla\tilde{\mathbf{z}}_h}{|\nabla\tilde{\mathbf{w}}_h^2|}(\nabla\tilde{\mathbf{w}}_h^2:\nabla\tilde{\mathbf{v}}_h)\right]dKdt$$

$$\left.+\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2\frac{(|\nabla\tilde{\mathbf{w}}_h^2|-|\nabla\tilde{\mathbf{w}}_h^1|)\nabla\tilde{\mathbf{w}}_h^1:\nabla\tilde{\mathbf{z}}_h}{|\nabla\tilde{\mathbf{w}}_h^1||\nabla\tilde{\mathbf{w}}_h^2|}(\nabla\tilde{\mathbf{w}}_h^2:\nabla\tilde{\mathbf{v}}_h)\,dKdt\right|$$

$$\leq 2\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2|\nabla\tilde{\mathbf{z}}_h||\nabla(\tilde{\mathbf{w}}_h^1-\tilde{\mathbf{w}}_h^2)||\nabla\tilde{\mathbf{v}}_h|\,dKdt$$

$$+\sum_{K\in\mathcal{T}_h}\int_0^{T_f}\int_K (C_S h_K)^2\,\big||\nabla\tilde{\mathbf{w}}_h^1|-|\nabla\tilde{\mathbf{w}}_h^2|\big|\,|\nabla\tilde{\mathbf{z}}_h||\nabla\tilde{\mathbf{v}}_h|\,dKdt$$

$$\leq 3(C_{1,2;T}C_{3;2})^3C_S^2h_{\max}^{2-d/2}\Delta t^{-1/2}\|\tilde{U}_h^1-\tilde{U}_h^2\|_{\tilde{X}}\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}},$$

where the last estimate is obtained similarly to the previous one.

We obtain the result with

$$\rho_T=\Delta t^{-1/2}(2C_{1,2;T}C_{4;T}^2+4(C_{1,2;T}C_{3;2})^3C_S^2h_{\max}^{2-d/2}).$$

□

The Lipschitz constant grows as a negative power of the time step Δt. In this case, the power of Δt is the lowest with respect to the obtained for the continuity $\tilde{\gamma}_h$ and the Stability factor β_h.

In order to guarantee the well-posedness of the Reduced Basis Problem (5.5) in the same way as in (5.1), we consider the inf-sup and continuity factors.

Definition 5.1. *The Stability Factor $\beta_N(\mu)$ and the continuity factor $\gamma_N(\mu)$ are defined by*

$$\beta_N(\mu)\equiv\inf_{\tilde{Z}_h\in\tilde{X}_h}\sup_{\tilde{V}_h\in\tilde{X}_h}\frac{\partial_1\tilde{A}(\tilde{U}_N(\mu),\tilde{V}_h;\mu)(\tilde{Z}_h)}{\|\tilde{Z}_h\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}}},\qquad(5.19)$$

$$\gamma_N(\mu) \equiv \sup_{\tilde{Z}_h \in \tilde{X}_h} \sup_{\tilde{V}_h \in \tilde{X}_h} \frac{\partial_1 \tilde{A}(\tilde{U}_N(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)}{\|\tilde{Z}_h\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}}}. \tag{5.20}$$

for all $\tilde{U}_N(\mu) \in \tilde{X}_N$.

Note that from Proposition 5.2, $\beta_N(\mu) > 0$ whenever the data are small enough and the pair of spaces (Y_h, Q_h) satisfies the discrete inf-sup condition. This condition implies that the tangent operator $\partial_1 \tilde{A}(\tilde{U}_N(\mu), \cdot; \mu)$ is an isomorphism of $\tilde{X}_h$ into its self, for all $\mu \in \mathcal{D}$.

Moreover, note that both the continuity factor $\gamma_N(\mu)$ and the Stability Factor $\beta_N(\mu)$ depends on the time-step Δt from Proposition 5.1 and 5.2.

Now we introduce the *supremizer* operator in order to facilitate the proof of an *a posteriori* estimator.

Definition 5.2. *The supremizer operator* $T_N^\mu : \tilde{X}_h \longrightarrow \tilde{X}_h$ *as a supremizer operator, by*

$$T_N^\mu \tilde{Z}_h = \arg \sup_{\tilde{V}_h \in \tilde{X}_h} \frac{\partial_1 \tilde{A}(\tilde{U}_N(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)}{\|\tilde{V}_h\|_{\tilde{X}}}. \tag{5.21}$$

This *supremizer* exists because $\partial_1 \tilde{A}$ is continuous and $\tilde{X}_h$ is a finite-dimensional space.

Remark 5.5. *From the definitions* (5.19) *and* (5.20) *we have that*

$$\beta_N(\mu) \equiv \inf_{\tilde{Z}_h \in \tilde{X}_h} \frac{\|T_N^\mu \tilde{Z}_h\|_{\tilde{X}}}{\|\tilde{Z}_h\|_{\tilde{X}}}, \quad \gamma_N(\mu) \equiv \sup_{\tilde{Z}_h \in \tilde{X}_h} \frac{\|T_N^\mu \tilde{Z}_h\|_{\tilde{X}}}{\|\tilde{Z}_h\|_{\tilde{X}}}.$$

From this, we have that

$$\beta_N(\mu) \le \frac{\|T_N^\mu \tilde{Z}_h\|_{\tilde{X}}}{\|\tilde{Z}_h\|_{\tilde{X}}}, \ \forall \tilde{Z}_h \in \tilde{X}_h,$$

and from (5.21)*,*

$$\beta_N(\mu) \le \frac{\partial_1 \tilde{A}(\tilde{U}_N(\mu), T_N^\mu \tilde{Z}_h; \mu)(\tilde{Z}_h)}{\|T_N^\mu \tilde{Z}_h\|_{\tilde{X}} \|\tilde{Z}_h\|_{\tilde{X}}}, \ \forall \tilde{Z}_h \in \tilde{X}_h. \tag{5.22}$$

We are now in a position to state our first result leading to an *a posteriori* error estimate:

Theorem 5.1 (Uniqueness)**.** *Let* $\mu \in \mathcal{D}$*, and assume that* $\beta_N(\mu) > 0$*. If problem* (5.1) *admits a solution* $\tilde{U}_h(\mu)$ *such that*

$$\|\tilde{U}_h(\mu) - \tilde{U}_N(\mu)\|_{\tilde{X}} \le \frac{\beta_N(\mu)}{\rho_T},$$

then this solution is unique in the ball $B_{\tilde{X}}\left(\tilde{U}_N(\mu), \dfrac{\beta_N(\mu)}{\rho_T}\right)$.

For the proof of this theorem and the next one, we define the following operators:

- The residual $\mathscr{R}(\cdot;\mu):\tilde{X}_h \longrightarrow \tilde{X}_h'$ by

$$\langle \mathscr{R}(\tilde{Z}_h;\mu),\tilde{V}_h\rangle = \tilde{A}(\tilde{Z}_h,\tilde{V}_h;\mu) - F(\tilde{V}_h;\mu),\ \forall \tilde{Z}_h,\tilde{V}_h \in \tilde{X}_h. \tag{5.23}$$

- The derivative of the operator A, $\mathcal{DA}(\tilde{U}_h(\mu);\mu):\tilde{X}_h \longrightarrow \tilde{X}_h'$, defined, for $\tilde{U}_h(\mu) \in \tilde{X}_h$, as

$$\langle \mathcal{DA}(\tilde{U}_h(\mu);\mu)\tilde{Z}_h,\tilde{V}_h\rangle = \partial_1 \tilde{A}(\tilde{U}_h(\mu),\tilde{V}_h;\mu)(\tilde{Z}_h),\ \forall \tilde{Z}_h,\tilde{V}_h \in \tilde{X}_h. \tag{5.24}$$

  This operator is invertible since $\beta_N(\mu) > 0$ for all $\mu \in \mathcal{D}$.

- And finally the mapping $H:\tilde{X}_h \longrightarrow \tilde{X}_h$ defined as,

$$H(\tilde{Z}_h;\mu) = \tilde{Z}_h - \mathcal{DA}(\tilde{U}_N(\mu);\mu)^{-1}\mathscr{R}(\tilde{Z}_h;\mu),\ \forall \tilde{Z}_h \in \tilde{X}_h. \tag{5.25}$$

Proof. The strategy is to proof that $H(\cdot;\mu)$ is a contraction in the sense of Definition A.3. If there exists a fixed point $\tilde{U}_h$, this point is a solution of problem (5.1) because of definition (5.25).

We have that

$$H(\tilde{Z}_h^1;\mu) - H(\tilde{Z}_h^2;\mu) = (\tilde{Z}_h^1 - \tilde{Z}_h^2) - \mathcal{DA}(\tilde{U}_N(\mu);\mu)^{-1}(\mathscr{R}(\tilde{Z}_h^1;\mu) - \mathscr{R}(\tilde{Z}_h^2;\mu)). \tag{5.26}$$

It holds

$$\mathscr{R}(\tilde{Z}_h^1;\mu) - \mathscr{R}(\tilde{Z}_h^2;\mu) = \mathcal{DA}(\tilde{\xi};\mu)(\tilde{Z}_h^1 - \tilde{Z}_h^2), \tag{5.27}$$

where $\tilde{\xi} = s\tilde{Z}_h^1 + (1-s)\tilde{Z}_h^2$, for some $s \in (0,1)$. In order to prove this, we define $T:[0,1] \longrightarrow \mathbb{R}$, by

$$T(s) = \langle \mathscr{R}(s\tilde{Z}_h^1 + (1-s)\tilde{Z}_h^2;\mu),\tilde{V}_h\rangle,\ \forall \tilde{V}_h \in \tilde{X}_h.$$

From this definition,

$$T'(s) = \langle \mathcal{DA}(s\tilde{Z}_h^1 + (1-s)\tilde{Z}_h^2;\mu)(\tilde{Z}_h^1 - \tilde{Z}_h^2),\tilde{V}_h\rangle.$$

If we apply the mean value theorem to T we have (5.27). Now, multiplying (5.26) by $\mathcal{DA}(\tilde{U}_N(\mu);\mu)$ and applying (5.27), we can write

$$\mathcal{DA}(\tilde{U}_N(\mu);\mu)(H(\tilde{Z}_h^1;\mu) - H(\tilde{Z}_h^2;\mu)) = [\mathcal{DA}(\tilde{U}_N(\mu);\mu) - \mathcal{DA}(\tilde{\xi};\mu)](\tilde{Z}_h^1 - \tilde{Z}_h^2).$$

Then, thanks to (5.18), we can write

$$\langle \mathcal{DA}(\tilde{U}_N(\mu);\mu)(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu)),\tilde{V}_h\rangle \leq \rho_T\|\tilde{U}_N(\mu)-\tilde{\xi}\|_{\tilde{X}}\|\tilde{Z}_h^1-\tilde{Z}_h^2\|_{\tilde{X}}\|\tilde{V}_h\|_{\tilde{X}} \tag{5.28}$$

From (5.22)

$$\begin{aligned}\langle \mathcal{DA}(\tilde{U}_N(\mu);\mu)(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu)),T_N^\mu(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu))\rangle\\ \geq \beta_N(\mu)\|H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu)\|_{\tilde{X}}\|T_N^\mu(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu))\|_{\tilde{X}}\end{aligned}$$

And applying (5.28) to $\tilde{V}_h = T_N^\mu(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu))$ we have that,

$$\begin{aligned}\beta_N(\mu)\|H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu)\|_{\tilde{X}}\|T_N^\mu(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu))\|_{\tilde{X}}\\ \leq \rho_T\|\tilde{U}_N(\mu)-\tilde{\xi}\|_{\tilde{X}}\|\tilde{Z}_h^1-\tilde{Z}_h^2\|_{\tilde{X}}\|T_N^\mu(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu))\|_{\tilde{X}}.\end{aligned}$$

Note that $T_N^\mu\tilde{Z}_h = 0$ if and only if $\tilde{Z}_h = 0$. Therefore, $T_N^\mu(H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu)) \neq 0$ if $H(\tilde{Z}_h^1;\mu) \neq H(\tilde{Z}_h^2;\mu)$. Simplifying

$$\|H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu)\|_{\tilde{X}} \leq \frac{\rho_T}{\beta_N(\mu)}\|\tilde{U}_N(\mu)-\tilde{\xi}\|_{\tilde{X}}\|\tilde{Z}_h^1-\tilde{Z}_h^2\|_{\tilde{X}}.$$

If $\tilde{Z}_h^1$ and $\tilde{Z}_h^2$ are in $B_{\tilde{X}}(\tilde{U}_N(\mu),\alpha)$, then $\|\tilde{U}_N(\mu)-\tilde{\xi}\|_{\tilde{X}} \leq \alpha$ and

$$\|H(\tilde{Z}_h^1;\mu)-H(\tilde{Z}_h^2;\mu)\|_{\tilde{X}} \leq \frac{\rho_T}{\beta_N(\mu)}\alpha\|\tilde{Z}_h^1-\tilde{Z}_h^2\|_{\tilde{X}}.$$

Then, $H(\cdot;\mu)$ is a contraction if $\alpha < \dfrac{\beta_N(\mu)}{\rho_T}$. So it follows that there can exist at most one fixed point of $H(\cdot;\mu)$ inside $B_{\tilde{X}}\left(\tilde{U}_N(\mu),\dfrac{\beta_N(\mu)}{\rho_T}\right)$, and hence, at most one solution $\tilde{U}_h(\mu)$ of (5.1) in this ball. □

Now we are in a position to define an *a posteriori* error bound estimator by

$$\Delta_N(\mu) = \frac{\beta_N(\mu)}{2\rho_T}\left[1-\sqrt{1-\tau_N(\mu)}\right],\ \forall\mu\in\mathcal{D}, \tag{5.29}$$

where $\beta_N(\mu)$ is defined in (5.19), ρ_T is defined in (5.18) and

$$\tau_N(\mu) = \frac{4\varepsilon_N(\mu)\rho_T}{\beta_N(\mu)^2}, \tag{5.30}$$

with

$$\varepsilon_N(\mu) = \|\mathscr{R}(\tilde{U}_N(\mu);\mu)\|_{\tilde{X}'} = \sup_{\tilde{V}_h\in\tilde{X}_h}\frac{\langle\mathscr{R}(\tilde{U}_N(\mu);\mu),\tilde{V}_h\rangle}{\|\tilde{V}_h\|_{\tilde{X}}}. \tag{5.31}$$

We observe that $\tau_N(\mu)$ in (5.30) is a re-scaling of the residual (5.31). Note that $\Delta_N(\mu)$ in (5.29) is only defined if $\tau_N(\mu)\leq 1$, which holds only if the residual is sufficiently small.

The error estimator is given by the following theorem.

Theorem 5.2 (Existence). *Assume that $\beta_N(\mu) > 0$ and $\tau_N(\mu) \leq 1$ for all $\mu \in \mathcal{D}$. Then there exists a unique solution $\tilde{U}_h(\mu)$ of* (5.1) *such that the error with respect $\tilde{U}_N(\mu)$, solution of* (5.5)*, is bounded by the a posteriori error bound estimator, i.e.,*

$$\|\tilde{U}_h(\mu) - \tilde{U}_N(\mu)\|_{\tilde{X}} \leq \Delta_N(\mu), \tag{5.32}$$

with effectivity,

$$\Delta_N(\mu) \leq \left[\frac{2\gamma_N(\mu)}{\beta_N(\mu)} + \tau_N(\mu)\right] \|\tilde{U}_h(\mu) - \tilde{U}_N(\mu)\|_{\tilde{X}}. \tag{5.33}$$

Proof. The main goal of this proof is to show that there exists a unique $\tilde{U}_h(\mu) \in \tilde{X}_h$ close to $\tilde{U}_N(\mu) \in \tilde{X}_N$ solution of (5.1) and (5.5) (resp.), i.e.,

$$\langle R(\tilde{U}_h(\mu);\mu), \tilde{V}_h\rangle = 0, \ \forall \tilde{V}_h \in \tilde{X}_h. \tag{5.34}$$

To do this, it is enough to prove that H is a contraction in a certain compact subset K of $\tilde{X}_h$ such that $H(K) \subset K$. Therefore, H has a fixed point thanks to the fixed-point theorem [9]. The operator H is continuous by the Proposition 5.1, then we only need to prove that it is contractive.

We consider the definition at the beginning of the previous proof. Then,

$$H(\tilde{Z}_h;\mu) - \tilde{U}_N(\mu) = \tilde{Z}_h - \tilde{U}_N(\mu) - \mathcal{DA}(\tilde{U}_N(\mu);\mu)^{-1}\mathscr{R}(\tilde{Z}_h;\mu).$$

Multiplying by $\mathcal{DA}(\tilde{U}_N(\mu);\mu)$, we obtain

$$\begin{aligned}\langle \mathcal{DA}(\tilde{U}_N(\mu);\mu)(H(\tilde{Z}_h;\mu) - \tilde{U}_N(\mu)), \tilde{V}_h\rangle &= \langle \mathcal{DA}(\tilde{U}_N(\mu);\mu)(\tilde{Z}_h - \tilde{U}_N(\mu)), \tilde{V}_h\rangle \\ &- \langle \mathscr{R}(\tilde{Z}_h;\mu) - \mathscr{R}(\tilde{U}_N(\mu);\mu), \tilde{V}_h\rangle - \langle \mathscr{R}(\tilde{U}_N(\mu);\mu), \tilde{V}_h\rangle, \ \forall \tilde{V}_h \in \tilde{X}_h\end{aligned}$$

As in the proof of Theorem 5.1, it holds

$$\mathscr{R}(\tilde{Z}_h;\mu) - \mathscr{R}(\tilde{U}_N(\mu);\mu) = \mathcal{DA}(\tilde{\xi}(\mu);\mu)(\tilde{Z}_h - \tilde{U}_N(\mu)),$$

where $\tilde{\xi}(\mu) = s\tilde{Z}_h + (1-s)\tilde{U}_N(\mu)$, $s \in (0,1)$. By this way, and thanks to Lemma 5.1, we obtain:

$$\begin{aligned}\langle \mathcal{DA}(\tilde{U}_N(\mu);\mu)&(H(\tilde{Z}_h;\mu) - \tilde{U}_N(\mu)), \tilde{V}_h\rangle \\ &= \langle \mathcal{DA}(\tilde{U}_N(\mu);\mu)(\tilde{Z}_h - \tilde{U}_N(\mu)), \tilde{V}_h\rangle \\ &- \langle \mathcal{DA}(\tilde{\xi}(\mu);\mu)(\tilde{Z}_h - \tilde{U}_N(\mu)), \tilde{V}_h\rangle - \langle \mathscr{R}(\tilde{U}_N(\mu);\mu), \tilde{V}_h\rangle \\ &= \langle (\mathcal{DA}(\tilde{U}_N(\mu);\mu) - \mathcal{DA}(\tilde{\xi}(\mu);\mu))(\tilde{Z}_h - \tilde{U}_N(\mu)), \tilde{V}_h\rangle \\ &- \langle \mathscr{R}(\tilde{U}_N(\mu);\mu), \tilde{V}_h\rangle \\ &\leq \rho_T \|\tilde{U}_N(\mu) - \tilde{\xi}(\mu)\|_{\tilde{X}} \|\tilde{Z}_h - \tilde{U}_N(\mu)\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}} + \varepsilon_N(\mu)\|\tilde{V}_h\|_{\tilde{X}} \\ &\leq (\rho_T \|\tilde{Z}_h - \tilde{U}_N(\mu)\|_{\tilde{X}}^2 + \varepsilon_N(\mu))\|\tilde{V}_h\|_{\tilde{X}}.\end{aligned}$$

Then, using (5.22)

$$\begin{aligned}\beta_N(\mu)\|H(\tilde{Z}_h;\mu)-\tilde{U}_N(\mu)\|_{\tilde{X}}\|T_N^\mu(H(\tilde{Z}_h;\mu)-\tilde{U}_N(\mu))\|_{\tilde{X}}\\ \leq \langle \mathcal{DA}(\tilde{U}_N(\mu);\mu)(H(\tilde{Z}_h;\mu)-\tilde{U}_N(\mu)),T_N^\mu(H(\tilde{Z}_h;\mu)-\tilde{U}_N(\mu))\rangle\\ \leq (\rho_T\|\tilde{Z}_h-\tilde{U}_N(\mu)\|_{\tilde{X}}^2+\varepsilon_N(\mu))\|T_N^\mu(H(\tilde{Z}_h;\mu)-\tilde{U}_N(\mu))\|_{\tilde{X}}\end{aligned}$$

Then as $\tilde{Z}_h \in B_{\tilde{X}}(\tilde{U}_N(\mu),\alpha)$ and simplifying,

$$\|H(\tilde{Z}_h;\mu)-\tilde{U}_N(\mu)\|_{\tilde{X}} \leq \frac{\rho_T}{\beta_N(\mu)}\alpha^2+\frac{\varepsilon_N(\mu)}{\beta_N(\mu)}$$

In order to ensure that H maps $B_{\tilde{X}}(\tilde{U}_N(\mu),\alpha)$ into a part of itself, we are seeking the values of α such that

$$\frac{\rho_T}{\beta_N(\mu)}\alpha^2+\frac{\varepsilon_N(\mu)}{\beta_N(\mu)} \leq \alpha$$

This holds if α is between the two roots of the second order equation

$$\rho_T\alpha^2-\beta_N(\mu)\alpha+\varepsilon_N(\mu)=0$$

which are,

$$\alpha_\pm=\frac{\beta_N(\mu)\pm\sqrt{\beta_N(\mu)^2-4\rho_T\varepsilon_N(\mu)}}{2\rho_T}=\frac{\beta_N(\mu)}{2\rho_T}\left[1\pm\sqrt{1-\frac{4\rho_T\varepsilon_N(\mu)}{\beta_N(\mu)^2}}\right].$$

Observe that as $\tau_N(\mu)\leq 1$, then $\alpha_-\leq\alpha_+\leq\dfrac{\beta_N(\mu)}{\rho_T}$. Consequently, if $\alpha_-\leq\alpha\leq\alpha_+$, there exists a unique solution $\tilde{U}_h(\mu)$ to (5.1) in the ball $B_{\tilde{X}}(\tilde{U}_N(\mu),\alpha)$. To obtain the estimator, we take $\alpha=\alpha_-=\Delta_N(\mu)$.

To prove the efficiency, let us define the error $\tilde{E}_h(\mu)=\tilde{U}_h(\mu)-\tilde{U}_N(\mu)$. From the definition of the residual and applying the mean value theorem, for some $s\in(0,1)$ we have that

$$\begin{aligned}\langle\mathscr{R}(\tilde{U}_N(\mu);\mu),\tilde{V}_h\rangle &= \tilde{A}(\tilde{U}_N(\mu),\tilde{V}_h;\mu)-F(\tilde{V}_h;\mu)\\ &= \tilde{A}(\tilde{U}_N(\mu),\tilde{V}_h;\mu)-\tilde{A}(\tilde{U}_h(\mu),\tilde{V}_h;\mu)\\ &= \partial_1\tilde{A}(s\tilde{U}_h(\mu)+(1-s)\tilde{U}_N(\mu),\tilde{V}_h;\mu)(\tilde{E}_h(\mu))\\ &= \langle\mathcal{DA}(s\tilde{U}_h(\mu)+(1-s)\tilde{U}_N(\mu);\mu)\tilde{E}_h(\mu),\tilde{V}_h\rangle\\ &= \langle(\mathcal{DA}(s\tilde{U}_h(\mu)+(1-s)\tilde{U}_N(\mu);\mu)-\mathcal{DA}(\tilde{U}_N(\mu);\mu))\tilde{E}_h(\mu),\tilde{V}_h\rangle\\ &\quad+\langle\mathcal{DA}(\tilde{U}_N(\mu);\mu)\tilde{E}_h(\mu),\tilde{V}_h\rangle\end{aligned}$$

Thanks to the Lemma 5.1, and taking into account the definition of $\gamma_N(\mu)$ and $\beta_N(\mu)$ in (5.19) and (5.20) respectively, we obtain,

$$\langle \mathscr{R}(\tilde{U}_N(\mu);\mu), \tilde{V}_h\rangle \leq \rho_T \|s(\tilde{U}_N(\mu) - \tilde{U}_h(\mu))\|_{\tilde{X}} \|\tilde{E}_h(\mu)\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}} + \gamma_N(\mu)\|\tilde{E}_h(\mu)\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}}.$$

Dividing by $\|\tilde{V}_h\|_{\tilde{X}}$, taking supremum in $\tilde{V}_h \in \tilde{X}_h$ and using the definition of (5.31), it holds

$$\varepsilon_N(\mu) \leq \rho_T \|\tilde{E}_h(\mu)\|^2_{\tilde{X}} + \gamma_N(\mu)\|\tilde{E}_h(\mu)\|_{\tilde{X}}.$$

Since $0 \leq \tau_N(\mu) \leq 1$ and $1 - \sqrt{1-\tau_N(\mu)} \leq \tau_N(\mu)$, we have that

$$\frac{2\rho_T}{\beta_N(\mu)}\Delta_N(\mu) \leq \tau_N(\mu)$$

and then

$$\Delta_N(\mu) \leq \frac{2\varepsilon_N(\mu)}{\beta_N(\mu)}.$$

It follows that

$$\Delta_N(\mu) \leq \frac{2\rho_T}{\beta_N(\mu)}\|\tilde{E}_h(\mu)\|^2_{\tilde{X}} + \frac{2\gamma_N(\mu)}{\beta_N(\mu)}\|\tilde{E}_h(\mu)\|_{\tilde{X}}$$

Thanks to (5.32), we know that $\|\tilde{E}_h(\mu)\|_{\tilde{X}} \leq \Delta_N(\mu)$, then

$$\frac{2\rho_T}{\beta_N(\mu)}\|\tilde{E}_h(\mu)\|_{\tilde{X}} \leq \tau_N(\mu).$$

It follows that

$$\Delta_N(\mu) \leq \left[\frac{2\gamma_N(\mu)}{\beta_N(\mu)} + \tau_N(\mu)\right]\|\tilde{E}_h(\mu)\|_{\tilde{X}}.$$

□

5.4 Stabiliy Factor approximation for unsteady problems

In Theorems 5.1-5.2, we have obtained the *a posteriori* error bound estimator $\Delta_N(\mu)$ to apply the RB method through the Greedy Algorithm. Unfortunately, the construction of the *a posteriori* error bound estimator is expensive, specifically, due to the computation of the Stability Factor. We outline the application of the same technique as in Section 1.5 in [14] for the computation of $\beta_h(\mu)$ for the unsteady case hereunder.

Let us define the Stability Factor $\beta_h(\mu)$ for the full-order problem (5.1) by

$$\beta_h(\mu) := \inf_{\tilde{Z}_h \in \tilde{X}_h} \sup_{\tilde{V}_h \in \tilde{X}_h} \frac{\partial_1 \tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)}{\|\tilde{Z}_h\|_{\tilde{X}} \|\tilde{V}_h\|_{\tilde{X}}} \tag{5.35}$$

and the so called *supremizer* operator $T_h^\mu : \tilde{X}_h \longrightarrow \tilde{X}_h$ by

$$T_h^\mu \tilde{Z}_h = \arg \sup_{\tilde{V}_h \in \tilde{X}_h} \frac{\partial_1 \tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h)}{\|\tilde{V}_h\|_{\tilde{X}}} \tag{5.36}$$

where $\tilde{U}_h(\mu) = (\tilde{\mathbf{u}}_h(\mu), \tilde{p}_h(\mu))$ is the solution of problem (5.1). Then,

$$\beta_h(\mu) = \inf_{\tilde{Z}_h \in \tilde{X}_h} \frac{\|T_h^\mu \tilde{Z}_h\|_{\tilde{X}}}{\|\tilde{Z}_h\|_{\tilde{X}}}. \tag{5.37}$$

We recall that the $\tilde{X}$-norm was introduced in (5.4) and it is associated to the inner product

$$(\tilde{V}_h, \tilde{Z}_h)_{\tilde{X}} = (\tilde{\mathbf{v}}_h, \tilde{\mathbf{z}}_h)_{L^2_T(H^1_0)} + (\tilde{q}_h, \tilde{r}_h)_{L^2(Q_T)}. \tag{5.38}$$

where $\tilde{V}_h = (\tilde{\mathbf{v}}_h, \tilde{q}_h)$ and $\tilde{Z}_h = (\tilde{\mathbf{z}}_h, \tilde{r}_h)$.

The key is to express (5.37) in an algebraic form and transform the computation of $\beta_h(\mu)$ into an eigenvalue problem. This is, we look after two matrices $\tilde{\mathbb{X}}, \tilde{\mathbb{F}}(\mu) \in \mathscr{M}^{N_h L \times N_h L}(\mathbb{R})$ such that

$$\begin{aligned} &\underline{\tilde{V}_h}^T \tilde{\mathbb{X}} \underline{\tilde{Z}_h} = (\tilde{V}_h, \tilde{Z}_h)_{\tilde{X}}, && \forall \tilde{V}_h, \tilde{Z}_h \in \tilde{X}_h, \\ &\underline{\tilde{V}_h}^T \tilde{\mathbb{F}}(\mu) \underline{\tilde{Z}_h} = \partial_1 \tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h), && \forall \tilde{V}_h, \tilde{Z}_h \in \tilde{X}_h, \ \forall \mu \in \mathcal{D}, \end{aligned}$$

where $N_h = \dim(\tilde{X}_h)$ and vectors $\underline{\tilde{V}_h}, \underline{\tilde{Z}_h} \in \mathbb{R}^{N_h L}$ are the values of the degrees of freedom of the representation of $\tilde{V}_h$ and $\tilde{Z}_h$ on $\tilde{X}_h$. For all $k = 1, \dots, L$ let us define

$$\underline{V_h}^k = \begin{pmatrix} \underline{\mathbf{v}_h^k} \\ \underline{q_h^k} \end{pmatrix}, \quad \underline{Z_h}^k = \begin{pmatrix} \underline{\mathbf{z}_h^k} \\ \underline{r_h^k} \end{pmatrix},$$

and

$$\underline{\tilde{V}_h} = \begin{pmatrix} \underline{V_h^1} \\ \underline{V_h^2} \\ \vdots \\ \underline{V_h^L} \end{pmatrix}, \quad \underline{\tilde{Z}_h} = \begin{pmatrix} \underline{Z_h^1} \\ \underline{Z_h^2} \\ \vdots \\ \underline{Z_h^L} \end{pmatrix}.$$

Now, we rewrite (5.37) by

$$\beta_h(\mu) = \inf_{\tilde{Z}_h \in \tilde{X}_h} \frac{\underline{\tilde{Z}_h}^T \tilde{\mathbb{F}}(\mu)^T \tilde{\mathbb{X}}^{-1} \tilde{\mathbb{F}}(\mu) \underline{\tilde{Z}_h}}{\underline{\tilde{Z}_h}^T \tilde{\mathbb{X}} \underline{\tilde{Z}_h}}. \tag{5.39}$$

This quotient is the minimum of a Rayleigh quotient. Then, for a given $\mu \in \mathcal{D}$, we solve the following eigenvalue problem

$$\left\{ \begin{array}{l} \text{Find } (\alpha, \tilde{Z}_h) \in \mathbb{R} \times \tilde{X}_h, \ \tilde{Z}_h \neq 0, \text{ such that} \\ \tilde{\mathbb{F}}(\mu)^T \tilde{\mathbb{X}}^{-1} \tilde{\mathbb{F}}(\mu) \underline{\tilde{Z}_h} = \alpha \tilde{\mathbb{X}} \underline{\tilde{Z}_h}, \quad \forall \tilde{Z}_h \in \tilde{X}_h. \end{array} \right. \tag{5.40}$$

and from (5.39), we obtain that $\beta_h(\mu) = (\alpha_{\min})^{1/2}$ with $\alpha_{\min}$ the minimum eigenvalue of (5.40). To solve the eigenvalue problem (5.40), we perform a Power Iteration method, consequently, we shall compute the largest eigenvalue of matrix $\tilde{\mathbb{F}}(\mu)^{-1}\tilde{\mathbb{X}}\tilde{\mathbb{F}}(\mu)^{-T}\tilde{\mathbb{X}}$.

We know that

$$(\tilde{V}_h, \tilde{Z}_h)_{\tilde{X}} = \int_0^{T_f} (\tilde{\mathbf{v}}_h, \tilde{\mathbf{z}}_h)_T + (\tilde{q}_h, \tilde{r}_h)_{L^2(\Omega)}\, dt = \sum_{k=1}^{L} \Delta t (\mathbf{v}_h^k, \mathbf{z}_h^k)_T + \Delta t (q_h^k, r_h^k)_{L^2(\Omega)}$$

Let us define the matrices $\mathbb{L}$ and $\mathbb{T}$ by

$$(q_h, r_h)_{L^2(\Omega)} = \underline{q_h}^T \mathbb{L} \underline{r_h}, \ (\mathbf{v}_h, \mathbf{z}_h)_T = \underline{\mathbf{v}_h}^T \mathbb{T} \underline{\mathbf{z}_h}, \tag{5.41}$$

for all $\mathbf{v}_h, \mathbf{z}_h \in Y_h$ and $q_h, r_h \in Q_h$. Then

$$(\tilde{V}_h, \tilde{Z}_h)_{\tilde{X}} = \sum_{k=1}^{L} \Delta t (\underline{\mathbf{v}_h^k})^T \mathbb{T} \underline{\mathbf{z}_h^k} + \Delta t (\underline{q_h^k})^T \mathbb{L} \underline{r_h^k}.$$

Then,

$$(\tilde{V}_h, \tilde{Z}_h)_{\tilde{X}} = \sum_{k=1}^{L} \Delta t \left[(\underline{V_h^k})^T \begin{pmatrix} \mathbb{T} & 0 \\ 0 & \mathbb{L} \end{pmatrix} \underline{Z_h^k} \right].$$

Finally, we obtain that $(\tilde{V}_h, \tilde{Z}_h)_{\tilde{X}} = \underline{\tilde{V}_h}^T \tilde{\mathbb{X}} \underline{\tilde{Z}_h}$, where

$$\tilde{\mathbb{X}} = \Delta t \begin{pmatrix} \mathbb{T} & & & & \\ & \mathbb{L} & & & \\ & & \ddots & & \\ & & & \mathbb{T} & \\ & & & & \mathbb{L} \end{pmatrix}. \tag{5.42}$$

To obtain the $\tilde{\mathbb{F}}(\mu)$ matrix, we recall that we have obtained the decomposition (5.15),

$$\partial_1 \tilde{A}(\tilde{U}_h(\mu), \tilde{V}_h; \mu)(\tilde{Z}_h) = \sum_{k=1}^{L} \Delta t \partial_1 A(U_h^k(\mu), V_h^k; \mu)(Z_h^k).$$

Then, we are able to define $\mathbb{F}_k(\mu)$ for $k = 1, \ldots, L$, such that for all $\tilde{V}_h, \tilde{Z}_h \in \tilde{X}_h$ and $\mu \in \mathcal{D}$ it holds

$$(\underline{V_h^k})^T \mathbb{F}_k(\mu) \underline{Z_h^k} = \partial_1 A(U_h^k(\mu), V_h^k; \mu)(Z_h^k),$$

and we finally define $\tilde{\mathbb{F}}(\mu)$ by

$$\tilde{\mathbb{F}}(\mu) = \Delta t \begin{pmatrix} \mathbb{F}_1(\mu) & & & \\ & \mathbb{F}_2(\mu) & & \\ & & \ddots & \\ & & & \mathbb{F}_L(\mu) \end{pmatrix}. \tag{5.43}$$

We observe that matrix $\tilde{\mathbb{X}}$ in (5.42) is a block-diagonal matrix, whose elements are composed by the matrices $\mathbb{L}$ and $\mathbb{T}$ defined in (5.41) and also $\tilde{\mathbb{F}}(\mu)$ whose elements are the matrices $\mathbb{F}_k(\mu)$ for $k = 1,\dots,L$.

Our goal is to compute the largest eigenvalue of the matrix $\tilde{\mathbb{F}}(\mu)^{-1}\tilde{\mathbb{X}}\tilde{\mathbb{F}}(\mu)^{-T}\tilde{\mathbb{X}}$ which turns out to be a matrix with very large dimension. The computation of the largest eigenvalue of this matrix does not seen to be affordable with the current techniques in the literature.

To sum up, the problem of building low cost techniques for the approximation of $\beta_h(\mu)$ remains open.

We intend to use hyper-reduction techniques to afford the computation of the eigenvalue problem (5.40). We could use only some time steps in the building of $\tilde{\mathbb{F}}(\mu)$ or/and we could use a coarser mesh in order to decrease N_h.

6

A posteriori error estimation based upon Kolmogórov turbulence theory

The aim of this chapter is to introduce an alternative *a posteriori* error estimation. This estimator is based upon the Kolmogórov turbulence theory, which introduces the idea of energy cascade and an expression for the energy spectrum.

In Section 6.1, we introduce an energy balance analysis of the Navier-Stokes equations to see how the energy is dissipated. In Section 6.2, we introduce the Fourier Transform and thanks to which, we are able to study the eddy energy by its size. In sections 6.3 and 6.4, we introduce the idea behind energy cascades and the energy spectrum. Finally, in Section 6.5, we introduce an *a posteriori* error estimator and we develop an academic test for a RB problem in Section 6.6.

6.1 System energy study

In this section, we study the basics of the energy balance of the Navier-Stokes (NS) equations for three cases. Let Ω be a bounded polyhedral domain in $\mathbb{R}^d$, with $d = 2,3$ and $T_f > 0$ be a chosen final finite time. We denote the time interval as $I_f = (0, T_f)$ and the time-space domain as $Q_T = I_f \times \Omega$. The NS equations in differential form are:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{f}, & \text{in } Q_T, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } Q_T. \end{cases} \tag{6.1}$$

We complement this problem with homogeneous or periodic boundary conditions and $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x})$ as the initial condition.

The NS equations determine any flow by the velocity field ($[\mathbf{u}] = \mathrm{m\,s^{-1}}$) and pressure per mass density ($[p] = \mathrm{m^2\,s^{-2}}$). This flow can be seen as a laminar or turbulent flow, depending on the Reynolds number associated with the problem, see (A.22). The Reynolds number establishes a relation between inertial and viscous forces, understanding that we have a turbulent flow hen inertial forces are important and laminar when viscous forces are relevant. That said, since the velocity and characteristic length are difference along the domain, we can find different Reynolds numbers within the domain.

Now, we define the total kinetic energy of the system as

$$\mathbb{E} = \frac{1}{2}\langle|\mathbf{u}|^2\rangle$$

where we denote $\langle\cdot\rangle$ a suitable average operator. This average verifies the following properties:

- $\langle u+v\rangle = \langle u\rangle + \langle v\rangle$,
- $\langle au\rangle = a\langle u\rangle$ for all constant a,
- $\langle\partial_t u\rangle = \partial_t\langle u\rangle$.

From now on, for simplicity, we will suppose that $|\Omega| = 1$, and we choose a spacial average as follows

$$\langle\mathbf{u}\rangle = \int_\Omega \mathbf{u}\,d\mathbf{x}.$$

We study the energy of equations (6.1) in difference situations, depending on the values of $\mathbf{f}$ and ν.

Case 1: $\mathbf{f} = \mathbf{0}$ and $\nu = 0$

We multiply (6.1) by $\mathbf{u}$, and applying the average, we obtain that

$$\frac{1}{2}\partial_t\langle|\mathbf{u}|^2\rangle = \partial_t\mathbb{E} = 0.$$

This is, the energy is time-constant since we are removing the source and viscosity terms, indeed, $\mathbb{E} = \frac{1}{2}\langle|\mathbf{u}_0|^2\rangle$.

Case 2: $\mathbf{f} = 0$ and $\nu \neq 0$

We repeat the process and we obtain that

$$\partial_t \mathbb{E} = -\varepsilon_\nu$$

where $\varepsilon_\nu = 2\nu\langle|D\mathbf{u}|^2\rangle$ is the turbulent dissipation, that we will introduce with more detail below, with $D\mathbf{u} = (1/2)(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$.

Note that by definition, $\varepsilon_\nu \geq 0$, then, $\partial_t \mathbb{E} \leq 0$. This is, the energy is decreasing over time, then we can deduce that the energy dissipation is caused by the viscosity forces.

Case 3: $\mathbf{f} \neq \mathbf{0}$ and $\nu \neq 0$

We obtain that

$$\partial_t \mathbb{E} = \varepsilon_f - \varepsilon_\nu$$

where $\varepsilon_f = \langle \mathbf{f} \cdot \mathbf{u} \rangle$.

In this case, we can not ensure the behavior of energy since the force terms could add or remove energy to the system, depending on the sign of ε_f.

6.2 Fourier space representation

One of the most notorious features in turbulence flow is the variability of eddies induced by the fluid motion. Each eddy could be classified by its size r, and also by its wavenumber $k = 1/r$. In this section, we express the energy and the turbulent dissipation over the wavenumber. This will allow us to describe the energy cascade and energy spectrum concepts in the following sections.

Thanks to the Fourier transform, we are able to decompose the velocity field in Fourier modes $\hat{\mathbf{u}}(t,\mathbf{k})$ as follows

$$\hat{\mathbf{u}}(t,\mathbf{k}) = \int_\Omega \mathbf{u}(t,\mathbf{x})\overline{\phi}_\mathbf{k}(\mathbf{x})\, d\mathbf{x}$$

and on the contrary,

$$\mathbf{u}(t,\mathbf{x}) = \int_{\mathbb{R}^d} \hat{\mathbf{u}}(t,\mathbf{k})\phi_\mathbf{k}(\mathbf{x})\, d\mathbf{k}$$

where $\phi_\mathbf{k}(\mathbf{x}) = e^{2\pi i\mathbf{k}\cdot\mathbf{x}}$, $\overline{\phi}_\mathbf{k}(\mathbf{x}) = e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}$ is the complex conjugate, $\mathbf{k} = k\mathbf{n}$, $\mathbf{n}$ a unit vector in $\mathbb{Z}^d$. We assume that $\hat{\mathbf{u}}(t,\mathbf{k}) = 0$ for $\mathbf{k} = 0$, this is, the mean flow is zero. From now on, to simplify notation, we consider $\mathbf{u}(\mathbf{x}) = \mathbf{u}(t,\mathbf{x})$ and $\hat{\mathbf{u}}(k) = \hat{\mathbf{u}}(t,k)$.

Note that, if $\mathbf{u}(\cdot,t) \in L^2(\Omega)$ for all $t \in I_f$, we can apply the Parseval Identity as follows,

$$\begin{aligned}\int_\Omega |\mathbf{u}(\mathbf{x})|^2\, d\mathbf{x} &= \int_\Omega \mathbf{u}(\mathbf{x})\overline{\mathbf{u}}(\mathbf{x})\, d\mathbf{x} \\ &= \int_\Omega \mathbf{u}(\mathbf{x})\overline{\int_{\mathbb{R}^d} \hat{\mathbf{u}}(\mathbf{k})\phi_{\mathbf{k}}(\mathbf{x})\, d\mathbf{k}}\, d\mathbf{x} \\ &= \int_\Omega \mathbf{u}(\mathbf{x}) \int_{\mathbb{R}^d} \overline{\hat{\mathbf{u}}}(\mathbf{k})\overline{\phi}_{\mathbf{k}}(\mathbf{x})\, d\mathbf{k}\, d\mathbf{x} \\ &= \int_{\mathbb{R}^d} \overline{\hat{\mathbf{u}}}(\mathbf{k}) \int_\Omega \mathbf{u}(\mathbf{x})\overline{\phi}_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x}\, d\mathbf{k} \\ &= \int_{\mathbb{R}^d} \overline{\hat{\mathbf{u}}}(\mathbf{k})\hat{\mathbf{u}}(\mathbf{k})\, d\mathbf{k} = \int_{\mathbb{R}^d} |\hat{\mathbf{u}}(\mathbf{k})|^2\, d\mathbf{k},\end{aligned} \tag{6.2}$$

therefore,

$$\mathbb{E} = \frac{1}{2}\int_{\mathbb{R}^d} |\hat{\mathbf{u}}(\mathbf{k})|^2\, d\mathbf{k}, \tag{6.3}$$

this is, we can express the energy system through the Fourier modes.

Henceforth, we assume that the flow is isotropic, which supposes that the statistical properties of the mean flow are invariant under rotations. We refer to Section 5.2.1 in [12] and Section 3.7 in [36] for a more specific description of isotropy.

Theorem 6.1. *There exists a measurable integrable function $E(k) = E(t,k)$, defined over $\mathbb{R}^+$, and such that*

$$\mathbb{E} = \int_0^\infty E(k)dk, \tag{6.4}$$

where $\mathbb{E}$ is the total kinetic energy.

Proof. See Theorem 5.3 in [12]. □

The function $E(k)$ for $k \in (0,+\infty)$ is the energy associated to the eddies of size $r = 1/k$ and it is well known as the energy spectrum. It can be described as

$$E(k) = \int_{k=|\mathbf{k}|} E(\mathbf{k})\, d\mathbf{k} \tag{6.5}$$

where $E(\mathbf{k})$ represents the kinetic energy part on the Fourier mode $\mathbf{k}$.

Now, we introduce the concept of turbulent dissipation as

$$\varepsilon = 2\nu\langle|D\mathbf{u}|^2\rangle \tag{6.6}$$

which appears as ε_ν in Case 2 in Section 6.1. Actually, since $\nabla\cdot\mathbf{u} = 0$, then $\varepsilon = \nu\langle|\nabla\mathbf{u}|^2\rangle$. As its name suggest, it is related to the energy dissipation through viscosity.

Following the same idea as for the energy $\mathbb{E}$, it is possible to link the turbulent dissipation ε to the energy spectrum E. This is the key for the derivation of Subgrid-Scale Modelling (SGM), in particular for the Smagorinsky model, as we will see in Section 6.4.

Lemma 6.1. *The turbulent dissipation ε can be expressed by a measurable function $E(k) = E(t,k)$, defined over $\mathbb{R}^+$ as*

$$\varepsilon = \nu \int_0^\infty 8\pi^2 k^2 E(k)\, dk. \tag{6.7}$$

Proof. See Lemma 5.2 in [12]. □

Corollary 6.1. *According to* (6.6)*, we also have*

$$\langle |D\boldsymbol{u}|^2 \rangle = \int_0^\infty 4\pi^2 k^2 E(k)\, dk.$$

6.3 A brief introduction to energy cascades

It was first introduced by L. F. Richardson in 1922 [40]. As we have already explained, turbulence is composed by eddies of different sizes. Each eddy is also described through their velocity and time scales. Then, for a certain eddy scale r, we could define a local Reynolds number

$$\mathrm{Re}(r) = \frac{u_r r}{\nu}$$

where u_r is the local velocity.

Richardson states that the largest eddies have large Reynolds numbers. Thus, they are unstable and they end up breaking, transferring their energy to smaller eddies. This phenomenon occurs across all scales until the viscosity effects are greater than the inertia and the energy is dissipated by heat. This is called the direct energy cascade.

Richardson summarizes this process with:

"*Big whorls have little whorls, that feed on their velocity, and little whorls have lesser whorls and so on to viscosity.*"

Richardson theory matches with the system energy studied in Section 6.1. We already saw that if the fluid is inviscid, the energy is conserved, by contrast, if $\nu \neq 0$ energy is dissipated by viscosity effects. This viscosity effects are important when the Reynolds number is low and this is linked to small scales. Moreover, at large scales, a high Reynolds number means the inertia effects are significant, and we can assume that large scales have large energy.

6.4 Energy spectrum

Andréi Kolmogórov stated in 1941 that under the following hypothesis:

- similarity, which assumes that the physical properties of the flow are invariant under scales changed;
- and isotropy;

there exists an inertial range $[k_1, k_2]$ where the energy spectrum $E(k)$ can be expressed by the wavenumber k and the turbulent dissipation ε, this is, $E(k) \sim \varepsilon^{\alpha} k^{\beta}$ for some α, β to be determined.

The inertial range $[k_1, k_2]$ is defined by two wavenumbers,

- k_1 is associated to the largest scale of the problem.
- k_2 is associated to the smaller scale r_0 under which the viscosity starts taking an active part.

Remark 6.1. *Kolmogórov also established three hypotheses that support the "Universal Equilibrium Theory" (cf. [17]). The first hypothesis says that if the Reynolds number is sufficiently high and $k_2 >> k_1$, then, for all wavenumber $k \sim k_2$ the turbulence is in statistical equilibrium. Moreover, the statistical properties of these eddies depend on two dimensionless independent magnitudes, the viscosity ν and the energy dissipation rate ε.*

This hypothesis allows us to obtain an estimation of the size of the smallest eddies, called the Kolmogórov length scale, followed by the velocity and time scale, defined by

$$r_0 = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \quad u_0 = (\nu\varepsilon)^{1/4}, \quad \tau_0 = \left(\frac{\nu}{\varepsilon}\right)^{1/2}, \tag{6.8}$$

respectively.

From this assumption, we can suppose that

$$\mathbb{E} = \int_{k_1}^{k_2} E(k) dk$$

and

$$\langle |D\mathbf{u}|^2 \rangle = \int_{k_1}^{k_2} 4\pi^2 k^2 E(k) \, dk.$$

Apart from that, solving the NS equations directly using a numerical approximation is impracticable for real problems, due to the dimension of the resulting system and the sensitivity to the initial conditions. A good solution could be to consider the Reynolds decomposition:

$$\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}', \quad p = \langle p \rangle + p', \tag{6.9}$$

where $\mathbf{u}'$ and p' are the fluctuation of velocity and pressure. Then, the main idea is to find the equations satisfied by the average fields $\langle \mathbf{u} \rangle$ and $\langle p \rangle$. The selection of the average $\langle \cdot \rangle$ is not unique and it generates a "closure problem" which derives into different models to use in practice.

If the strategy is to solve large grid scales and to model the subgrid scales, we obtain a Large Eddy Simulation (LES) model (see Chapter 5 in [12] for more details). In the LES modeling, large scales are solved and small scales are approximated by eddy diffusion term in the NS equations, whereby the mesh size is involved. From decomposition (6.9), we also formally have

$$\langle |D\mathbf{u}|^2 \rangle = |D\langle \mathbf{u} \rangle|^2 + \langle |D\mathbf{u}'|^2 \rangle.$$

Thanks to this, we can choose a $k_c = \delta^{-1}$ where δ is the mesh size, and $k_c \in (k_1, k_2)$ such that

$$|D\langle \mathbf{u} \rangle|^2 = \frac{1}{2} \int_{k_1}^{k_c} k^2 E(k)\, dk, \;\; \langle |D\mathbf{u}'|^2 \rangle = \frac{1}{2} \int_{k_c}^{k_2} k^2 E(k)\, dk.$$

The $[k_1, k_c]$ range represents the solved scales, then we can assume that $k_c >> k_1$.

Remark 6.2. *The average used to obtain a LES model can be identified with the spatial average. This is also called the ergodic hypothesis.*

We deduce the energy spectrum expression by the Kolmogórov theory, applying a dimensional analysis. To do this, let us define L and T that determine the spatial and time dimension.

We know that the units of energy and wavenumber are

$$[\mathbb{E}] = L^2/T^2, \quad [k] = L^{-1},$$

then by (6.4), $[E] = L^3/T^2$.

Furthermore, when viscosity is relevant, we know that the turbulent dissipation is described in terms of viscosity and the gradient of velocity as in (6.6). We know that $[\nu] = L^2/T$ and $[|D\mathbf{u}|] = T^{-1}$, then $[\varepsilon] = L^2/T^3$.

Finally, from Kolmogórov Theory, we have that $E(k) \sim \varepsilon^{\alpha} k^{\beta}$, therefore

$$\frac{L^3}{T^2} = \left(\frac{L^2}{T^3}\right)^{\alpha} \cdot \left(\frac{1}{L}\right)^{\beta} \quad \Leftrightarrow \quad \left.\begin{array}{l} 3 = 2\alpha - \beta \\ 2 = 3\alpha \end{array}\right\} \quad \Leftrightarrow \quad \begin{array}{l} \alpha = 2/3 \\ \beta = -5/3 \end{array}$$

and we finally obtain

$$E(k) = C\varepsilon^{2/3} k^{-5/3} \tag{6.10}$$

where C is a constant. This is the so-called Kolmogórov $-5/3$ law.

Remark 6.3. *From the Kolmogórov $-5/3$ law, we deduce that*

$$|D\langle \boldsymbol{u} \rangle|^2 \;=\; \frac{1}{2}\int_{k_1}^{k_c} k^2 E(k)dk \;=\; \frac{1}{2} C\varepsilon^{2/3} \int_{k_1}^{k_c} k^{1/3} dk \;=\; \frac{3}{8} C\varepsilon^{2/3} (k_c^{4/3} \;-\; k_1^{4/3}) \tag{6.11}$$

As $k_c >> k_1$, we can approximate (6.11) *by*

$$|D\langle \boldsymbol{u} \rangle|^2 \sim C\varepsilon^{2/3} k_c^{4/3}$$

and recalling that $k_c \sim \delta^{-1}$, we deduce that $\varepsilon = C|D\langle \boldsymbol{u} \rangle|^3 \delta^2$.

On the other hand, to deduce the eddy viscosity ν_t, we replace r_0 by δ in (6.8) *and we obtain that $\delta = \nu_t^{3/4} \varepsilon^{-1/4}$ or $\nu_t = \varepsilon^{1/3} \delta^{4/3}$.*

Finally, we get that $\nu_t = C\delta^2 |D\langle \boldsymbol{u} \rangle|$, this is the term added to the NS equations and both constitute the Smagorinsky model, used along this dissertation.

6.5 *A posteriori* error estimation

The main goal of this section is to use the Kolmogórov theory to obtain an *a posteriori* error estimator to apply the RB method to the Smagorinsky Model. However, this theory can be extended to any other model for which an inertial energy spectrum is known.

We already saw that the Smagorinsky model solves a part of the inertial range since it is a LES model. To avoid repetition, we directly describe the Smagorinsky Model FE approximation.

Let $Y_h \subset W^{1,3}(\Omega)$ and $Q_h \subset L^2(\Omega)$ be two finite element approximation spaces associated to the mesh $\{\mathcal{T}_h\}_{h>0}$. Then, for all $\mu \in \mathcal{D}$ and $t \in I_f$, find $(\mathbf{u}_h(t), p_h(t)) = (\mathbf{u}_h(t;\mu), p_h(t;\mu)) \in Y_h \times Q_h$ solution of

$$\begin{cases} \displaystyle\int_\Omega \partial_t \mathbf{u}_h(t) \cdot \mathbf{v}_h \, d\Omega + \frac{1}{\mathrm{Re}} \int_\Omega \nabla \mathbf{u}_h(t) : \nabla \mathbf{v}_h \, d\Omega & \\ \displaystyle + \int_\Omega \nu_t(\mathbf{u}_h(t)) \nabla \mathbf{u}_h(t) : \nabla \mathbf{v}_h \, d\Omega + \int_\Omega (\mathbf{u}_h(t) \cdot \nabla) \mathbf{u}_h(t) \cdot \mathbf{v}_h \, d\Omega & \\ \displaystyle - \int_\Omega (\nabla \cdot \mathbf{v}_h) p_h(t) \, d\Omega = \langle \mathbf{f}, \mathbf{v}_h \rangle_\Omega, & \forall \mathbf{v}_h \in Y_h, \\ \displaystyle \int_\Omega (\nabla \cdot \mathbf{u}_h(t)) q_h \, d\Omega = 0, & \forall q_h \in Q_h. \end{cases} \tag{6.12}$$

We apply the RB method considering the Reynolds number as the parameter. Then, let Y_N and Q_N be the reduced spaces built from solutions of (6.12) for all $\mu \in \mathcal{D}$ and $t \in I_f$, then we are able to deduce the RB problem.

For all $\mu \in \mathcal{D}$ and $t \in I_f$, find $(\mathbf{u}_N(\mu), p_N(\mu)) = (\mathbf{u}_N(t;\mu), p_N(t;\mu)) \in Y_N \times Q_N$ solution of

$$\begin{cases} \displaystyle\int_\Omega \partial_t \mathbf{u}_N(\mu)\cdot \mathbf{v}_N \, d\Omega + \frac{1}{\mathrm{Re}}\int_\Omega \nabla \mathbf{u}_N(\mu) : \nabla \mathbf{v}_N \, d\Omega \\ \displaystyle + \int_\Omega \nu_t(\mathbf{u}_N(\mu))\nabla \mathbf{u}_N(\mu) : \nabla \mathbf{v}_N \, d\Omega + \int_\Omega (\mathbf{u}_N(\mu)\cdot\nabla)\mathbf{u}_N(\mu)\cdot \mathbf{v}_N \, d\Omega \\ \displaystyle - \int_\Omega (\nabla\cdot \mathbf{v}_N) p_N(\mu) \, d\Omega = \langle \mathbf{f}, \mathbf{v}_N\rangle_\Omega, & \forall \mathbf{v}_N \in Y_N, \\ \displaystyle\int_\Omega (\nabla \cdot \mathbf{u}_N(\mu)) q_N \, d\Omega = 0, & \forall q_N \in Q_N. \end{cases} \tag{6.13}$$

As we saw in Chapter 1, the Greedy Algorithm requires the computation of the error between the *high fidelity* solution $U_h(\mu) = (\mathbf{u}_h(\mu), p_h(\mu))$ of (6.12) and the RB solution $U_N(\mu) = (\mathbf{u}_N(\mu), p_N(\mu))$ of (6.13) for all $\mu \in \mathcal{D}$ and $t \in I_f$ in each Greedy iteration.

The computation of $U(\mu)$ solution of the NS equation (6.1) is mostly impracticable, this is why, we usually consider $U_h(\mu) = (\mathbf{u}_h(\mu), p_h(\mu))$ as the *high fidelity* solution. In this section, we keep $U(\mu)$ as the *high fidelity* solution and U_h as the FE solution.

Then, our goal is to replace this error by an estimation whose computation is affordable.

Since we consider the Smagorinsky model, we are solving the scales in the inertial subrange $[k_1, k_c]$ which is much smaller than the inertial range. The mesh $\mathcal{T}_h$ should be carefully chosen in order to solve a part of the inertial range, this is, $k_c \in [k_1, k_2]$.

We have already seen that the energy spectrum in the inertial range $[k_1, k_2]$ has a specific expression depending on the wavenumber, this is, from (6.10)

$$E(k;\mu) = a(\mu) k^{-5/3} \tag{6.14}$$

where $a(\mu) > 0$ depends on $\varepsilon(\mu)$.

The energy spectrum of the *high fidelity* solution $\mathbf{u}(\mu)$ should be as (6.14) for a suitable $a(\mu)$ and for all $\mu \in \mathcal{D}$.

If the FE solution $\mathbf{u}_h(\mu)$ of (6.12) is a good approximation of $\mathbf{u}(\mu)$, we should expect a similar energy spectrum. The RB problem is built from FE solution $\mathbf{u}_h(\mu)$, then, if the RB solution $\mathbf{u}_N(\mu)$ is a good approximation of the FE solution $\mathbf{u}_h(\mu)$ for all $\mu \in \mathcal{D}$, then, it is razonable to think that the energy spectrum for the RB solution $\mathbf{u}_N(\mu)$ should approximate the energy spectrum in (6.14), for a suitable $a(\mu)$ for all $\mu \in \mathcal{D}$.

Then, let $E_N(k;\mu)$ be the energy spectrum associated to $\mathbf{u}_N(\mu)$. We define an *a posteriori* error estimator as follows

$$\Delta_N(\mu) = \min_a \left(\int_{k_1}^{k_c} |E_N(k;\mu) - a(\mu)k^{-5/3}|^2 \, dk \right)^{1/2}. \tag{6.15}$$

This measures how close is a given solution (either reduced or FOM-obtained) to the theoretical Kolmogórov spectrum, in the range of inertial wavenumbers $[k_1, k_c]$ which is solved by the Smagorinsky model.

We look for designing an academic test to use this *a posteriori* estimator. For simplicity in the computation, we board a 2D problem.

Discrete Fourier Transform

In practice, we use the Discrete Fourier Transform (DFT) as an approximation of the Fourier Transform, as well as its inverse, defined in Section 6.2.

To do this, we suppose that Ω is square of length s, this is $\Omega = [-s/2, s/2]^2$. We divide each edge in $\mathcal{N}$ intervals of length $h = \frac{s}{\mathcal{N}}$, with $\mathcal{N} \in \mathbb{N}$ pair, generating a mesh $\hat{\mathcal{T}}_h$.

Let us consider $u : \Omega \longrightarrow \mathbb{C}$ a continuous periodic function. Then, for a Fourier mode $\mathbf{k} = (k_1, k_2)$, $k_1, k_2 = -\mathcal{N}/2, -\mathcal{N}/2+1, \dots, \mathcal{N}/2$,

$$\hat{u}_{k_1 k_2} = \frac{1}{(\mathcal{N}+1)^d} \sum_{j_1, j_2 = -\mathcal{N}/2}^{\mathcal{N}/2} u_{j_1 j_2} \overline{\phi}_{k_1 k_2}(j_1 h, j_2 h), \tag{6.16}$$

where we recall that $\phi_{k_1 k_2}(j_1 h, j_2 h) = e^{2\pi i h (k_1 j_1 + k_2 j_2)}$ and

$$u_{j_1 j_2} = u(j_1 h, j_2 h), \; j_1, j_2 = -\mathcal{N}/2, -\mathcal{N}/2+1, \dots, \mathcal{N}/2. \tag{6.17}$$

On the other side, $j_1, j_2 = -\mathcal{N}/2, -\mathcal{N}/2+1, \dots, \mathcal{N}/2$, it holds

$$u_{j_1 j_2} = \sum_{k_1, k_2 = \mathcal{N}/2}^{\mathcal{N}/2} \hat{u}_{k_1 k_2} \phi_{k_1 k_2}(j_1 h, j_2 h). \tag{6.18}$$

Remark 6.4. *For a faster computation of the Discrete Fourier Transform, it is usual to use a decomposition in terms of a Vandermonde matrix* $\mathbb{F}_\varepsilon \in \mathscr{M}^{(\mathcal{N}+1)\times(\mathcal{N}+1)}(\mathbb{C})$. *This* *matrix is defined as follows*

$$\mathbb{F}_\varepsilon = \begin{bmatrix} w_\varepsilon^{(-\mathcal{N}/2)\cdot\mathcal{N}/2} & \cdots & w_\varepsilon^{0\cdot\mathcal{N}/2} & \cdots & w_\varepsilon^{\mathcal{N}/2\cdot\mathcal{N}/2} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ w_\varepsilon^{(-\mathcal{N}/2)\cdot 0} & \cdots & w_\varepsilon^{0\cdot 0} & \cdots & w_\varepsilon^{\mathcal{N}/2\cdot 0} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ w_\varepsilon^{(-\mathcal{N}/2)\cdot(-\mathcal{N}/2)} & \cdots & w_\varepsilon^{0\cdot(-\mathcal{N}/2)} & \cdots & w_\varepsilon^{\mathcal{N}/2\cdot(-\mathcal{N}/2)} \end{bmatrix} \tag{6.19}$$

where $\varepsilon = -1, 1$*, depending on the direct or inverse Fourier Transform computation. Moreover, we build the Vandermonde matrix in parallel, which accelerates the computation.*

Discrete energy spectrum

For the computation of the energy spectrum and taking into account (6.4), we can approximate the total energy system by

$$\mathbb{E} = \int_0^\infty E(k)\, dk \approx \sum_{l=0}^{L_\mathcal{N}} \hat{E}(l)$$

where $L_\mathcal{N}$ is the integer part of $\sqrt{2}\mathcal{N}$ and taking into account (6.5), we approximate the energy spectrum by

$$\hat{E}(l) = \sum_{l \le |\mathbf{k}| < l+1} \frac{1}{2} |\hat{\mathbf{u}}_\mathbf{k}|^2.$$

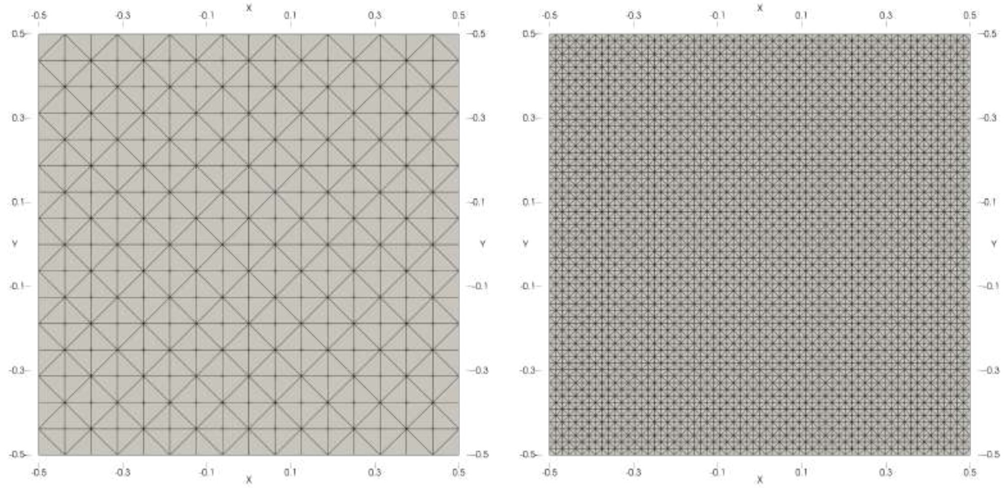
6.6 Academic test

The aim of this test is to obtain an energy spectrum following the Kolmogórov $-5/3$ law defined in (6.10) solving the Smagorinsky model. Then, we look for applying the estimator $\Delta_N(\mu)$ defined in (6.15) to build a RB Smagorinsky model.

Problem statement

We solve the Smagorinsky model (6.12) in its dimensionless version for all $t \in (0, T_f)$ with $T_f = 15$, over the unit square $\Omega = [-1/2, 1/2]^2$ with periodic boundary conditions. We do not consider any source, then $\mathbf{f} = 0$. We select the Reynolds number Re as the only parameter μ and we fix the parameter set as $\mathcal{D} = [1\,000, 16\,000]$.

We have supposed an spatial isotropy hypothesis over the flow. We should be able to reproduce this characteristic numerically, in order to reproduce the energy spectrum



(a) $\mathscr{N} = 16$, $h = 8.839 \cdot 10^{-2}$, 512 triangles, 289 vertices. **(b)** $\mathscr{N} = 64$, $h = 2.209 \cdot 10^{-2}$, 8 192 triangles, 4 225 vertices.

Figure 6.1: Triangular meshes

desired. In Chapter 4 in [20], D. Franco reproduces the energy spectrum for the Euler equation. In this case, for a finite number of Fourier modes, they obtain a constant energy inertial spectrum, departing from an invariant mesh under rotation that remains the unit cube. This gives us an idea that the first step should be the construction of this kind of mesh.

We divide each edge of the square Ω into $\mathscr{N}$ intervals, generating a triangular mesh as we see in Figure 6.1. This mesh is invariant under rotations that keep invariant the square and it is symmetric with respect to the axis OX, OY, XY and $-XY$.

As usual, we consider a FE approximation with the Taylor-Hood finite element, i.e., we consider $\mathbb{P}2-\mathbb{P}1$ finite elements for velocity-pressure discretization.

We board the time discretization as in the previous chapters, using a semi-implicit linearized Euler scheme as in (1.20), with $\Delta t = 20/\mathscr{N}$.

Remark 6.5. *We have tried also Explicit Euler, Backward differencing (BDF2), explicit pseudo-Crank-Nicolson and Crank-Nicolson schemes, obtaining the same results for the energy spectrum. We retain the semi-implicit Euler scheme since it is the simplest and it has good stability properties.*

In order to build the initial condition, we look for a velocity field with an inertial energy spectrum as in (6.14). We consider a velocity field $\mathbf{w}_h^0 = (v, v)$, where v is defined

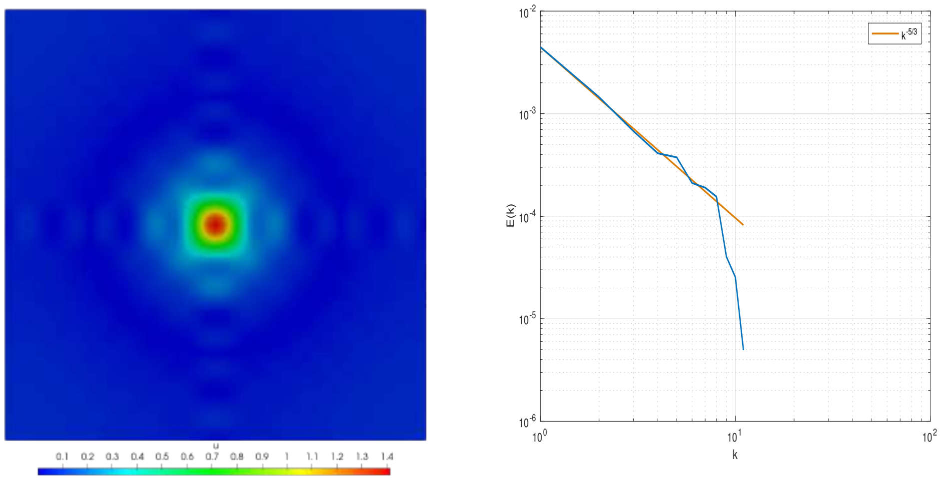


Figure 6.2: Representation of the module and energy spectrum of $\mathbf{w}_h^0$.

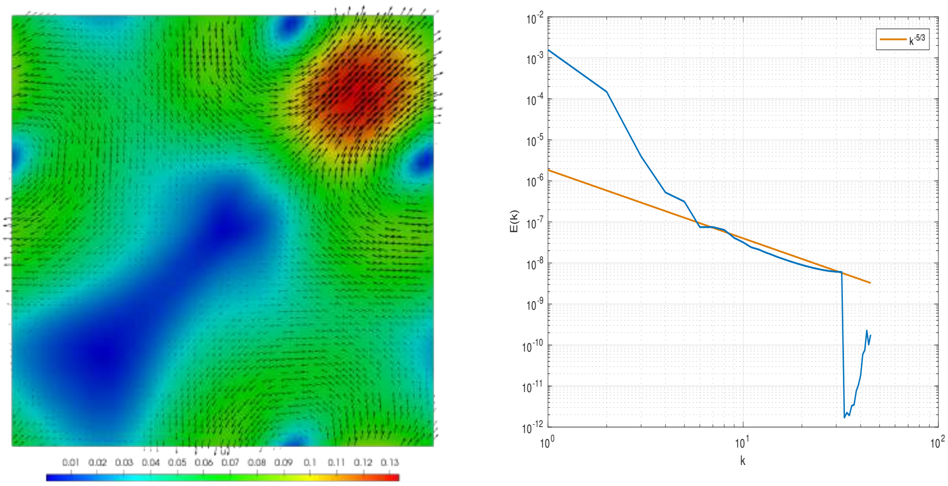


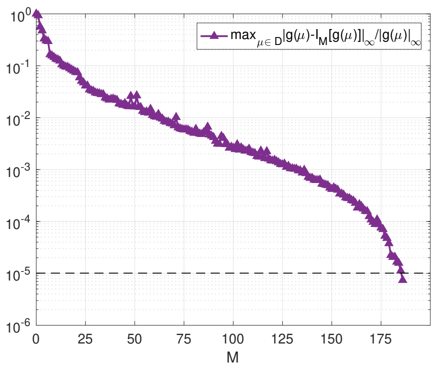
Figure 6.3: Initial condition $\mathbf{u}_h^0$ and its energy spectrum.

through its Fourier transform:

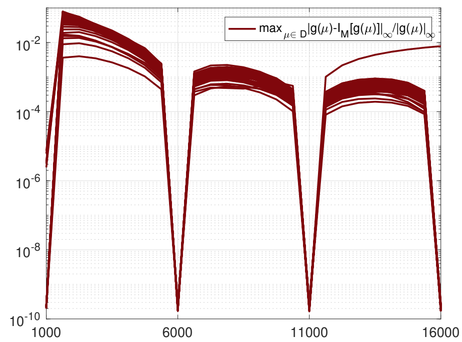
$$\hat{v}(\mathbf{k}) = \begin{cases} k^{-(5/3+1)/2} & \text{if } 0 < k \leq \mathcal{N}/4, \\ 0 & \text{other case} , \end{cases}$$

The energy spectrum of $\mathbf{w}_h^0$ is shown in Figure 6.2. We build an initial condition with a physical inertial spectrum. To do this, we solve the Smagorinsky model taking $\mathbf{w}_h^0$ as the initial state for $\mu = 8\,500$, the intermediate Reynolds number. We stop for $t = 15$, this is, $n = 48$ and we take the velocity field at this time as the initial condition.

In Figure 6.3, we show the initial condition $\mathbf{u}_h^0$ and its energy spectrum. For wavenumbers between 5 and 32, we obtain a good approximation of the inertial spectrum. For $k > 32$,



(a) EIM history convergence error.



(b) EIM error, each line correspond to a time step t_n for $n = 1,\ldots,L$.

Figure 6.4: EIM applied to unsteady Smagorinsky Model.

we observe an abrupt decay of the energy. This is produced by the wavenumbers that are out of the circle of the largest radio inside the square. As we can see, we start from a well-developed energy spectrum.

Empirical Interpolation Method

As we saw in Chapters 1 and 2, the eddy viscosity term requires to be linearized with respect to the parameter. With this purpose, it is necessary to compute some solutions for different parameters, and then to apply Algorithm 5 for an unsteady problem.

We compute the finite element solution $(\mathbf{u}_h^n(\mu), p_h^n(\mu))$ for all $n = 1,\ldots,L$, $\mu = \{1\,000, 6\,000, 11\,000, 16\,000\}$ and we apply the EIM. We stop the algorithm on 186 basis when the error is below $\varepsilon_{EIM} = 10^{-5}$. The convergence error is shown in Figure 6.4a, while in Figure 6.4b, we show the error and each line represents a time step t_n for $n = 1,\ldots,48$. The maximum error is $7.929 \cdot 10^{-2}$.

POD+Greedy strategy

We use a POD+Greedy strategy as it was described in Algorithm 4. We recall that we follow a POD strategy considering the time t as a parameter and the Greedy algorithm for the Reynolds number, with the purpose of building a reduced basis space that takes into account time and parameter variability.

For the POD, we use a separate strategy in the sense that we apply POD to a velocity snapshot matrix $\mathbb{S}^u$ and a pressure snapshot matrix $\mathbb{S}^p$. To build the correlation matrices, it is necessary to establish spatial norms for velocity and pressure, since the time should be considered as a parameter. In this case, we use the inner product

$$(\mathbf{v}_h, \mathbf{z}_h)_T = \int_\Omega \frac{1}{\mathrm{Re}_f} \nabla \mathbf{v}_h : \nabla \mathbf{z}_h \, d\Omega, \quad \mathbf{v}_h, \mathbf{z}_h \in Y_h \tag{6.20}$$

where $\mathrm{Re}_f = 8\,500$ for velocity which drift into a T-norm, and L^2-norm for the pressure, for the spaces Y_h and Q_h, respectively. Let define the matrices $\mathbb{Y}_h$ and $\mathbb{Q}_h$ which are related to the velocity and pressure norms.

Again, the reduced spaces Y_N and Q_N must be inf-sup stable, therefore, we propose the use of an inner pressure *supremizer* operator $T_N^\mu : Q_h \longrightarrow Y_h$ defined as follows

$$(T_N^\mu p_h, \mathbf{v}_h)_T = -\int_\Omega (\nabla \cdot \mathbf{v}_h) p_h \, d\Omega, \quad \forall \mathbf{v}_h \in Y_h$$

for a given $p_h \in Q_h$. In [47], G. Stabile and G. Rozza propose two different strategies to implement the *supremizer* when POD is applied over a parameter. We stay with the *exact supremizer enrichment*, this is, we compute the *supremizer* basis from the pressure basis obtained in the POD procedure.

Algorithm 7 POD+Greedy with *supremizer*

Set $\varepsilon_{1,tol}, \varepsilon_{2,tol} > 0$, $N_{max} \in \mathbb{N}$, $\mu^* \in \mathcal{D}_{train}$, $\mathbb{Z}^u = [\,]$, $\mathbb{Z}^p = [\,]$ and $S = \{\,\}$;
while $N < N_{\max}$ **do**
  $S = S \cup \{\mu^*\}$;
  Compute $U_h^n(\mu^*) = (\mathbf{u}_h^n(\mu^*), p_h^n(\mu^*))$ for $n = 1, \dots, L$;
  Build $\mathbb{S}^u = [\underline{\mathbf{u}}_h^1(\mu^*), \underline{\mathbf{u}}_h^2(\mu^*), \dots, \underline{\mathbf{u}}_h^L(\mu^*)]$, $\mathbb{S}^p = [\underline{p}_h^1(\mu^*), \underline{p}_h^2(\mu^*), \dots, \underline{p}_h^L(\mu^*)]$;
  $[\underline{\xi}_1^u, \dots, \underline{\xi}_{M^u}^u]$=POD$(\mathbb{S}^u, \mathbb{Y}_h, \varepsilon_{1,tol})$;
  $[\underline{\xi}_1^p, \dots, \underline{\xi}_{M^p}^p]$=POD$(\mathbb{S}^p, \mathbb{Q}_h, \varepsilon_{1,tol})$;
  $\mathbb{Z}^u = [\mathbb{Z}^u, \underline{\xi}_1^u, \dots, \underline{\xi}_{M^u}^u]$;
  $\mathbb{Z}^p = [\mathbb{Z}^p, \underline{\xi}_1^p, \dots, \underline{\xi}_{M^p}^p]$;
  $[\underline{\varphi}_1^u, \dots, \underline{\varphi}_{N^u}^u]$=POD$(\mathbb{Z}^u, \mathbb{Y}_h, \varepsilon_{2,tol})$;
  $[\underline{\varphi}_1^p, \dots, \underline{\varphi}_{N^p}^p]$=POD$(\mathbb{Z}^p, \mathbb{Q}_h, \varepsilon_{2,tol})$;
  Compute $\varphi_{N^u+i}^u = T_N^\mu \varphi_i^p$ for $i = 1, \dots, N^p$;
  $N = N^u + 2N^p$;
  $Y_N = \{\varphi_i^u\}_{i=1}^{N^u+N^p}$, $Q_N = \{\varphi_i^p\}_{i=1}^{N^p}$,
  $\mu^* = \arg\max_{\mu \in \mathcal{D}_{train}} \Delta_N(\mu, L)$;
  $\varepsilon_N = \Delta_N(\mu^*, L)$;
  if $\varepsilon_N \leq \varepsilon_{tol}$ **then**
    $N_{max} = N$;
  end if
end while

Finally, we apply the algorithm described in Algorithm 7. We give a general idea:

1. For a given μ^*, we solve the Smagorinsky Model for any time t_n for $n = 1, \ldots, L$, and we save the result in the snapshot matrices $\mathbb{S}^u$ for velocity and $\mathbb{S}^p$ for pressure.

2. We apply the POD procedure separately for velocity and pressure, for a given tolerance $\varepsilon_{1,tol}$ and we add the results in the matrices $\mathbb{Z}^u$ and $\mathbb{Z}^p$.

3. Finally, we apply a second POD to $\mathbb{Z}^u$ and $\mathbb{Z}^p$ for a given tolerance $\varepsilon_{2,tol}$. This procedure avoid repetition in the basis.

4. We compute the *supremizer* for the pressure basis resulting above and we add it to the velocity basis, obtaining Y_N and Q_N.

5. Lastly, we apply the Greedy Algorithm to the RB problem associated to the spaces Y_N and Q_N, obtaining the new parameter μ^*. We use the estimate $\Delta_N(\mu)$ at the last time since it is assumable that the energy spectrum is well-developed at that time.

To compute the estimator, we should know the resolved part of the inertial range in the Smagorinsky model $[k_1, k_c]$. Since we already compute $(\mathbf{u}_h^n(\mu), p_h^n(\mu))$ for $n = 1, \ldots, L$, $\mu = \{1\,000, 6\,000, 11\,000, 16\,000\}$ for the EIM, we know the energy spectrum associated to each parameter in $n = L$. In Figure 6.5, we show the energy spectrum and we see clearly that we could assume $k_1 = 5$ and $k_c = 32$. We also show the velocity field $\mathbf{u}_h^L(\mu)$ in Figure 6.6.

Results

We compare the use of the estimate $\Delta_N(\mu)$ introduced in (6.15) versus the use of the exact error at the final time

$$\varepsilon_N(\mu) = \|\mathbf{u}_h^L(\mu) - \mathbf{u}_N^L(\mu)\|_T, \tag{6.21}$$

for the parameter selection in Algorithm 7.

Tables 6.1-6.2 shows the comparison using the estimator $\Delta_N(\mu)$ (upper table) and the exact error $\varepsilon_N(\mu)$ (lower table) for the selection of the parameter μ. We set $\varepsilon_{2,tol} = \sqrt{\varepsilon_{1,tol}}$, with $\varepsilon_{1,tol} = 10^{-10}$ in Table 6.1 and $\varepsilon_{1,tol} = 10^{-13}$ in Table 6.2. In both tables, using $\Delta_N(\mu)$, we stop the algorithm in the third iteration, since the next Reynolds number has been already select and we remain with the same number of basis, $N = 98$ in Table 6.1 and $N = 143$ in Table 6.2. The exact error and the number of RB basis are similar whether we use the estimator $\Delta_N(\mu)$ or the exact error $\varepsilon_N(\mu)$, which tell us that the use of the estimator $\Delta_N(\mu)$ is quasi-optimum.

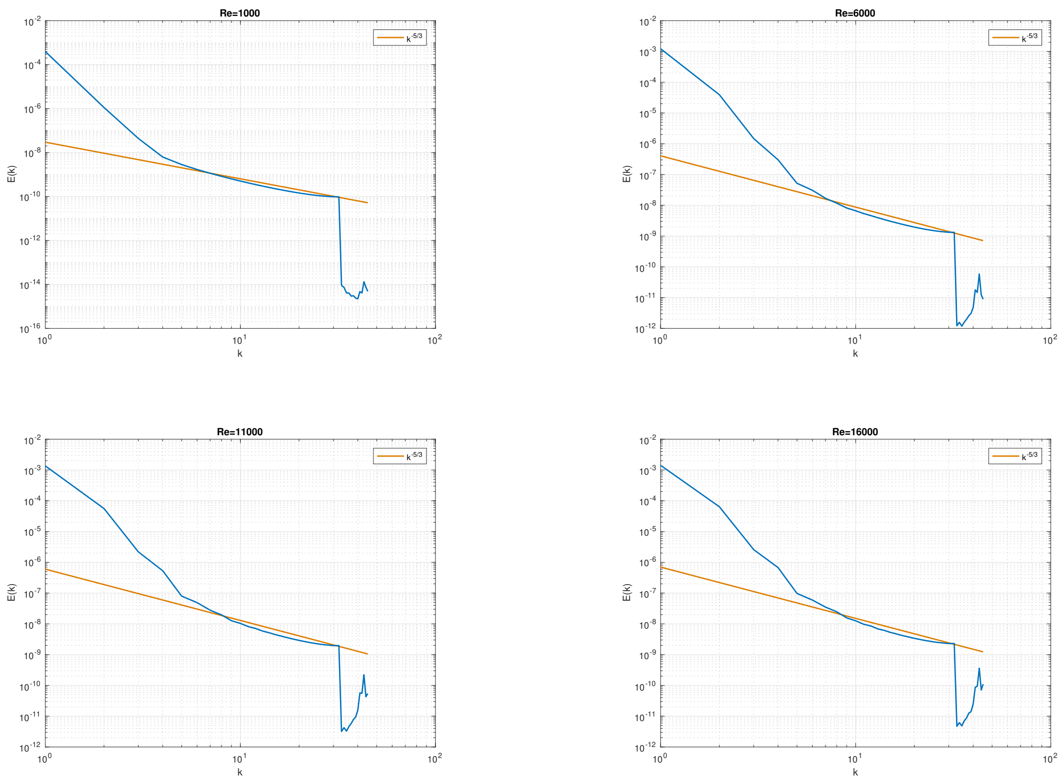


Figure 6.5: Energy Spectra for $\mu = \{1\,000,\, 6\,000,\, 11\,000,\, 16\,000\}$ at final time.

Comparing both tables, the exact errors are similar, therefore, we deduce that a very small tolerance in the POD steps does not ensure a decrease in error.

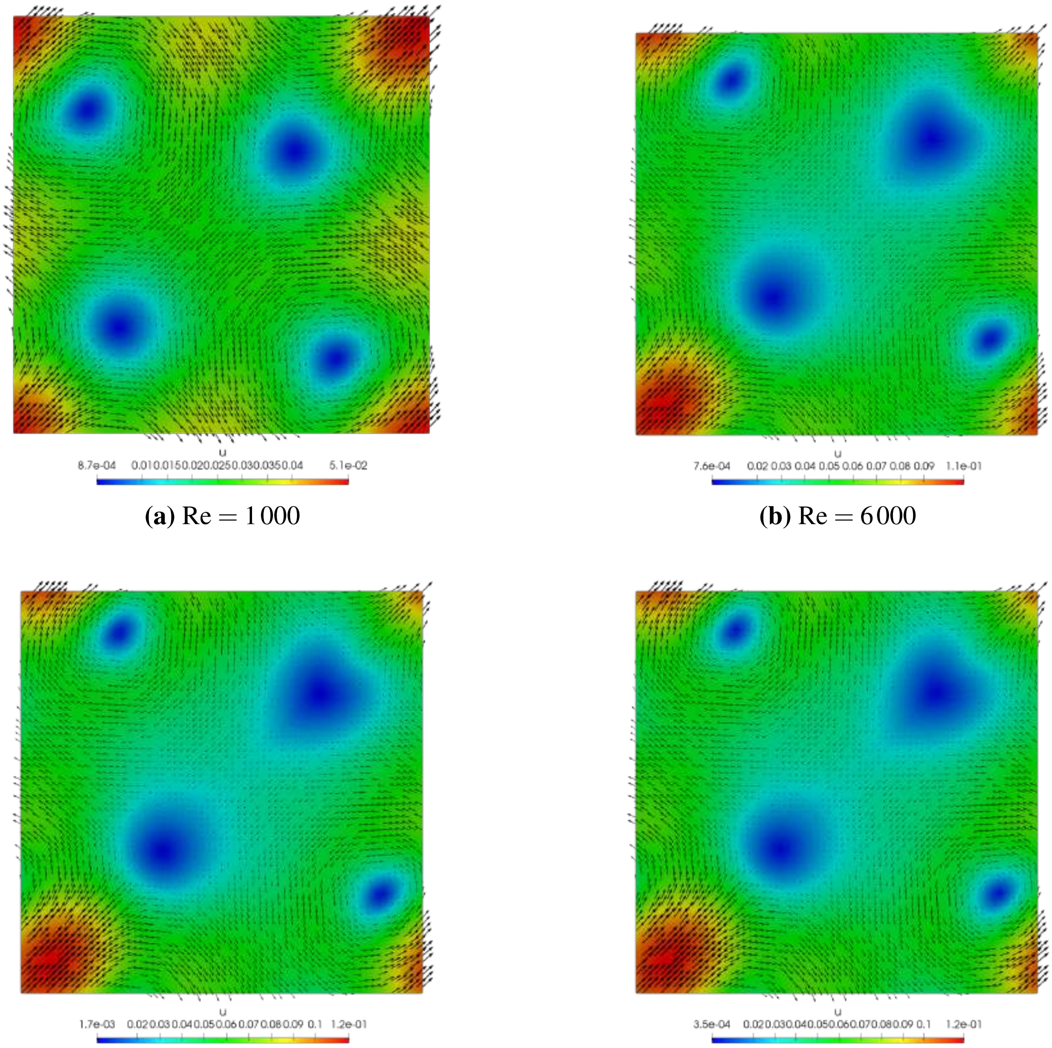
In Figure 6.7a, we show the comparison of the estimate $\Delta_N(\mu)$ in each iteration of the POD+Greedy algorithm described in Table 6.1 versus

$$\Delta_h(\mu) = \min_a \left(\int_{k_1}^{k_c} |E_h(k;\mu) - a(\mu) k^{-5/3}|^2 \, dk \right)^{1/2}$$

and $E_h(k;\mu)$ for $k \in (k_1, k_c)$ represents the energy spectrum of $\mathbf{u}_h^L(\mu)$.

Since the reduced solution is built from the FE approximation, we should not expect that $\Delta_N(\mu)$ tends to 0 when $N \longrightarrow \infty$, it should rather converge to Δ_h. We observe in Figure 6.7a that indeed $\Delta_N(\mu)$ approaches Δ_h as N increases and Δ_h needs not be zero as the finite element solution is just an approximation of the physical flow. Therefore, we depend on the error committed by the FE approximation to approximate the inertial spectrum.

In Figure 6.7b, we show the error $\varepsilon_N(\mu)$ for $\mu = 1\,000, 1\,625, \ldots, 16\,000$ at each POD+Greedy algorithm iteration. In the last iteration, the error is smaller than 10^{-4}.



(a) Re = 1 000 (b) Re = 6 000 (c) Re = 11 000 (d) Re = 16 000

Figure 6.6: Module of velocity at final time $|\mathbf{u}_h^L(\mu)|$.

Online phase

To test the Reduced Basis spaces obtained by the previous procedure using $\Delta_N(\mu)$, we select five random Reynolds values different from the selected ones by the Greedy Algorithm, and we solve the RB and FE problems. The results below have been computed on only one processor in a cluster with CPUs AMD EPYC 7542 2.9 GHz.

In Table 6.3, we show the computational time, the values of the estimates, and the error between the RB and FE solution at the final time $T_f = 15$.

We obtain speed-ups ratio close to 20, what is satisfying for an evolution turbulence model. We already saw that considering $\varepsilon_N(\mu)$ instead of $\Delta_N(\mu)$ does not reduce the

It.	Re	N	$\max_\mu \Delta_N(\mu)$	$\max_\mu \varepsilon_N(\mu)$
1	1 000	30	$7.92 \cdot 10^{-1}$	$2.57 \cdot 10^{-3}$
2	16 000	72	$5.27 \cdot 10^{-1}$	$2.37 \cdot 10^{-4}$
3	2 250	**98**	$3.51 \cdot 10^{-1}$	$\mathbf{6.04 \cdot 10^{-5}}$
4	16 000	98		

It.	Re	N	$\max_\mu \varepsilon_N(\mu)$
1	1 000	30	$2.57 \cdot 10^{-3}$
2	16 000	72	$2.37 \cdot 10^{-4}$
3	3 500	**100**	$\mathbf{3.52 \cdot 10^{-5}}$
4	7 250	119	$3.53 \cdot 10^{-5}$

Table 6.1: Step by step of the POD+Greedy algorithm for $\varepsilon_{1,tol} = 10^{-10}$, $\varepsilon_{2,tol} = 10^{-5}$, using $\Delta_N(\mu)$ (upper table) and $\varepsilon_N(\mu)$ (lower table) for the parameter selection.

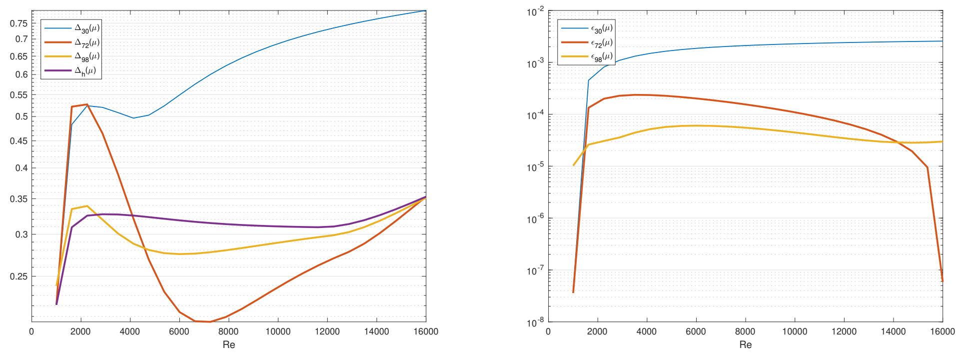
It.	Re	N	$\max_\mu \Delta_N(\mu)$	$\max_\mu \varepsilon_N(\mu)$
1	1 000	39	5.52	$2.6 \cdot 10^{-3}$
2	16 000	105	$4.03 \cdot 10^{-1}$	$2.17 \cdot 10^{-3}$
3	1 625	**143**	$3.53 \cdot 10^{-1}$	$\mathbf{7.09 \cdot 10^{-5}}$
4	16 000	143		

It.	Re	N	$\max_\mu \varepsilon_N(\mu)$
1	1 000	39	$2.6 \cdot 10^{-3}$
2	16 000	105	$2.17 \cdot 10^{-4}$
3	3 500	**152**	$\mathbf{3.25 \cdot 10^{-5}}$
4	7 250	197	$2.2 \cdot 10^{-5}$

Table 6.2: Step by step of the POD+Greedy algorithm for $\varepsilon_{1,tol} = 10^{-13}$, $\varepsilon_{2,tol} = 3.16 \cdot 10^{-7}$, using $\Delta_N(\mu)$ (upper table) and $\varepsilon_N(\mu)$ (lower table) for the parameter selection.

number of basis functions, thus, we can not blame the estimator.

Finally, in Figures 6.8-6.9 we show velocity and pressure for Re = 11 757, along a time step selection.



(a) Estimate $\Delta_N(\mu)$ in each iteration and $\Delta_h(\mu)$.

(b) Error $\varepsilon_N(\mu)$ in each iteration.

Figure 6.7: Convergence of POD+Greedy algorithm.

	Re $= 1\,825$	Re $= 4\,804$	Re $= 11\,757$	Re $= 13\,605$	Re $= 14\,027$
T_{FE}	55.63s	58.67s	58.3s	58.09s	57.94s
T_{RB}	2.95s	2.99s	2.83s	2.92s	3.06s
Speedup	19	20	21	20	19
$\Delta_N(\mu)$	$3.42 \cdot 10^{-1}$	$2.8 \cdot 10^{-1}$	$2.97 \cdot 10^{-1}$	$3.11 \cdot 10^{-1}$	$3.17 \cdot 10^{-1}$
$\Delta_h(\mu)$	$3.18 \cdot 10^{-1}$	$3.23 \cdot 10^{-1}$	$3.09 \cdot 10^{-1}$	$3.2 \cdot 10^{-1}$	$3.25 \cdot 10^{-1}$
$\varepsilon_N(\mu)$	$2.86 \cdot 10^{-5}$	$5.73 \cdot 10^{-5}$	$3.56 \cdot 10^{-5}$	$2.94 \cdot 10^{-5}$	$2.88 \cdot 10^{-5}$

Table 6.3: Validation of RB model.

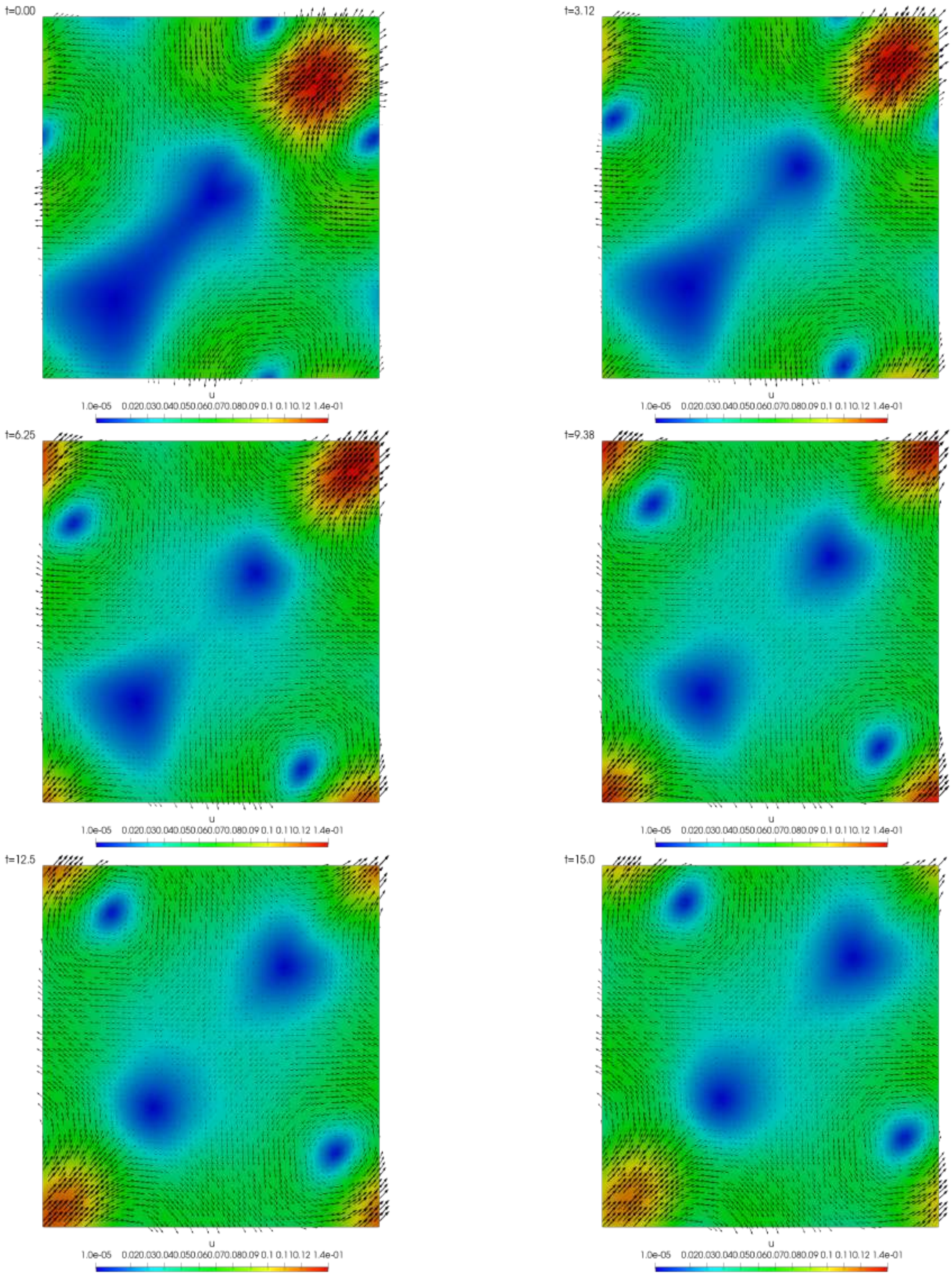


Figure 6.8: Velocity field for Re = 11 757.

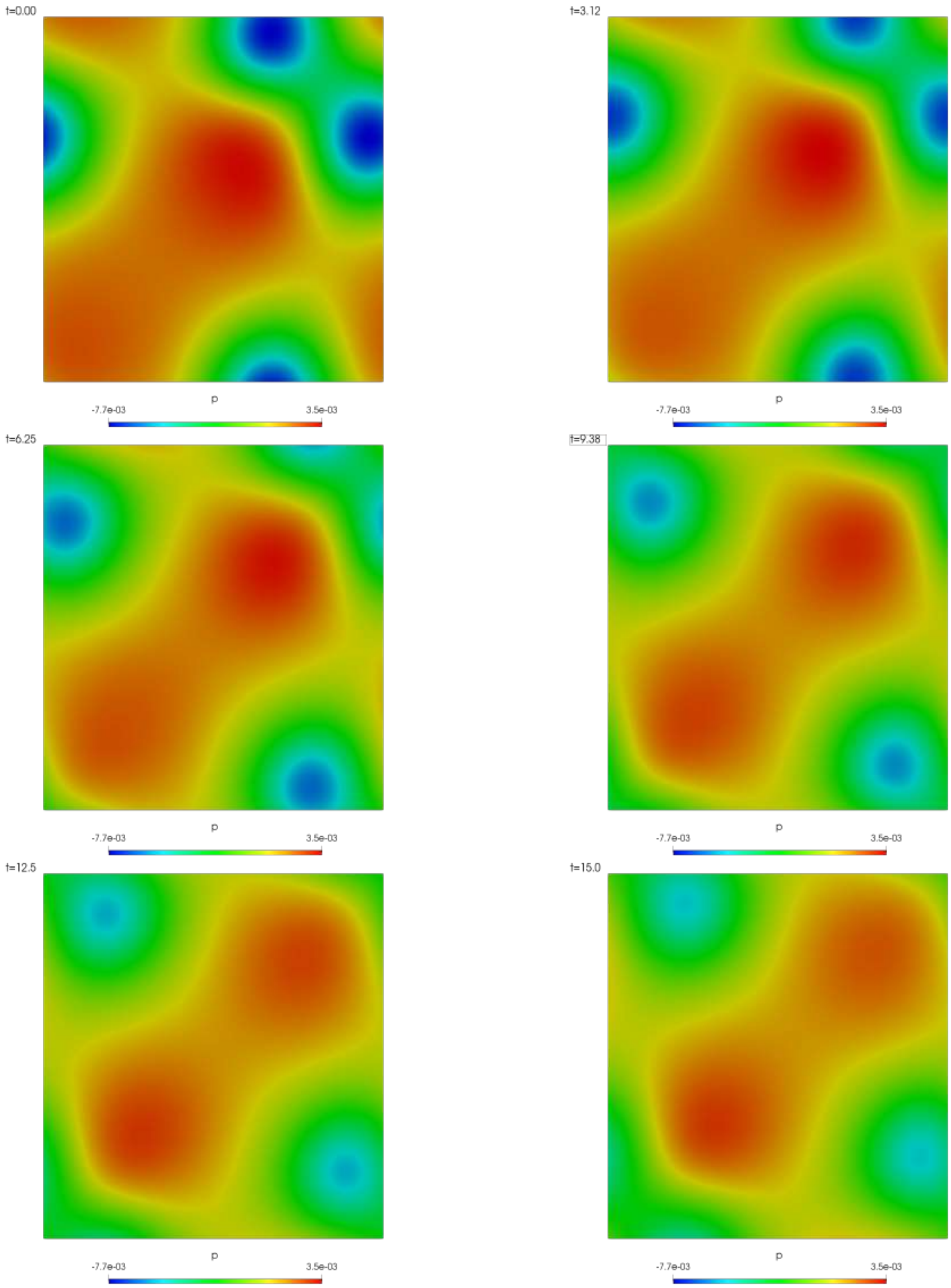


Figure 6.9: Pressure for Re = 11 757.

Open problems

As possible ways to continue this research, we set some open problems related to the results presented in this part:

- ☐ Computation of the Stability Factor $\beta_h(\mu)$ for $\mu \in \mathcal{D}$ in Chapter 5, using hyper-reduction techniques in space (with coarser meshes) and time (using only some time iterations instead of all).

- ☐ Extend the academic test in Section 6.6 to other definitions of the Kolmogórov estimate. Instead of using the last time, we purpose the use of mean value, maximum value, or some time values in an interval of time.

- ☐ Extension of the academic test in Section 6.6 to the 3D case.

- ☐ Application to industry problems thanks to the low cost Kolmogórov estimate.

- ☐ Extension to another space discretization, such as finite volume or finite difference method since the Kolmogórov estimate is independent of the used method.

- ☐ Find an estimate for the exact error using the Kolmogórov estimation.

- ☐ Application to Variational Multi-scale (VMS) Smagorinsky model in order to consider less diffusive Smagorinsky models.

Appendix

A

Basic notations and auxiliary results

The goal of this section is to introduce the difference properties and basic notations that we use along this document.

A.1 Basic notations

Notations for Matrix-Vector Operations

Let $\mathbf{x} = (x_i)_{i=1}^d$ and $\mathbf{y} = (y_i)_{i=1}^d$ be two vectors and $A = (a_{ij})_{i,j=1}^d$ and $B = (b_{ij})_{i,j=1}^d$ two $d \times d$ matrices. Then,

- The dot product of two vectors:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{d} x_i y_i$$

- The dot product of two matrices:

$$A : B = \sum_{i,j=1}^{d} a_{ij} b_{ij}$$

- The Frobenius norm of the matrix A:

$$|A| = \left(\sum_{i,j=1}^{d} a_{i,j}^2 \right)^{1/2} = (A : A)^{1/2} \tag{A.1}$$

Lebesgue Spaces

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a domain and let $p \in [1,\infty]$. The Banach spaces $L^p(\Omega)$ are formed by measurable functions on Ω with the norm

- if $p \in [1,\infty)$,
$$\|\mathbf{v}\|_{L^p(\Omega)} := \left(\int_\Omega |\mathbf{v}|^p \, d\Omega \right)^{1/p},$$
- if $p = \infty$,
$$\|\mathbf{v}\|_{L^\infty(\Omega)} := \sup_{x \in \Omega} |\mathbf{v}(x)|.$$

In addition, $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(\mathbf{v},\mathbf{w})_\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{w} \, d\Omega$$

If $p \in (1,\infty)$, then the dual space of $L^p(\Omega)$ is $L^q(\Omega)$ where $p^{-1} + q^{-1} = 1$ with the dual pairing

$$\langle \mathbf{v},\mathbf{w} \rangle_\Omega = \int_\Omega \mathbf{v} \cdot \mathbf{w} \, d\Omega, \; \mathbf{v} \in L^p(\Omega), \; \mathbf{w} \in L^q(\Omega).$$

For more details, see [1].

Sobolev Spaces

Let p and r be two non-negative integers. The Sobolev spaces are defined by

$$W^{r,p}(\Omega) = \{\mathbf{f} \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega), \; \forall \alpha \in \mathbb{N}^d : |\alpha| \leq r\}, \tag{A.2}$$

where $\alpha = (\alpha_1,\ldots,\alpha_d)$ with $\alpha_i \geq 0$ for any $i = 1,\ldots,d$ is a multi-index such that $|\alpha| = \sum_{i=1}^d \alpha_i$ and

$$D^\alpha \mathbf{v} := \frac{\partial^\alpha \mathbf{v}}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_d}}.$$

This spaces are equipped with the norm

- If $p \in [1,\infty)$,
$$\|\mathbf{v}\|_{W^{r,p}(\Omega)} := \left(\sum_{0 \leq |\alpha| \leq r} \|D^\alpha \mathbf{v}\|^p_{L^p(\Omega)} \right)^{1/p},$$

- if $p = \infty$,

$$\|\mathbf{v}\|_{W^{r,\infty}(\Omega)} := \max_{0 \le |\alpha| \le r} \|D^{\alpha}\mathbf{v}\|_{L^{\infty}(\Omega)}.$$

In general, the Sobolev Spaces are Banach Spaces and in particular, the spaces $W^{r,2}(\Omega)$ are Hilbert Spaces, denoted by $H^r(\Omega)$.

The dual space of $W^{r,p}(\Omega)$ is the space $\mathscr{L}(W^{r,p}(\Omega),\mathbb{R})$ such that for $F \in \mathscr{L}(W^{r,p}(\Omega),\mathbb{R})$,

$$|F(v)| \le C\|v\|_{W^{r,p}(\Omega)},\ \forall v \in W^{r,p}(\Omega).$$

Particularly, for $p = 2$, the dual of $H^r(\Omega)$ is $H^{-r}(\Omega)$. For more details, see [1].

Bochner Spaces

Let (a,b) be a time interval and $\mathbf{v}$ be a function defined on $(a,b)\times\Omega$. Let $(V,\|\cdot\|_V)$ a Banach space on Ω. We denote by $L^p(a,b;V)$ for $p \in [1,\infty]$, the Bochner space endowed with the norm

- If $p \in [1,\infty)$,

$$\|\mathbf{v}\|_{L^p(V)} := \left(\int_a^b \|\mathbf{v}(t)\|_V^p\, dt\right)^{1/p};$$

- if $p = \infty$,

$$\|\mathbf{v}\|_{L^{\infty}(V)} := \sup_{t\in(a,b)} \|\mathbf{v}(t)\|_V.$$

A.2 Auxiliary results

For this section, let $(X,\|\cdot\|_X)$ be a Hilbert space on $\Omega \subset \mathbb{R}^d$.

Property A.1. *For any $a,b \in X$ then,*

$$2(a-b,a)_{\Omega} = \|a\|_X^2 - \|b\|_X^2 + \|a-b\|_X^2. \tag{A.3}$$

Proof.

$$\begin{aligned}
2(a-b,a)_{\Omega} &= (a-b,a)_{\Omega} + (a,a-b)_{\Omega}\\
&= (a-b,a)_{\Omega} + (a,a)_{\Omega} - (a,b)_{\Omega} + (b,b)_{\Omega} - (b,b)_{\Omega}\\
&= (a-b,a)_{\Omega} + \|a\|_X^2 - (a-b,b)_{\Omega} - \|b\|_X^2\\
&= (a-b,a-b)_{\Omega} + \|a\|_X^2 - \|b\|_X^2\\
&= \|a-b\|_X^2 + \|a\|_X^2 - \|b\|_X^2
\end{aligned}$$

□

Property A.2. *For any $a,b \in X$ then,*

$$(a,b)_\Omega \leq \frac{1}{2}\|a\|_X^2 + \frac{1}{2}\|b\|_X^2 \tag{A.4}$$

Proof.

$$\|a-b\|_X^2 = (a-b,a-b)_\Omega = \|a\|_X^2 + \|b\|_X^2 - 2(a,b)_\Omega \geq 0$$

□

Property A.3. *For any $a,b \in X$ and $c \in \mathbb{R}$,*

$$(a,b)_\Omega \leq \frac{c}{4}\|a\|_X^2 + \frac{1}{c}\|b\|_X^2 \tag{A.5}$$

Proof.

$$\left\|\frac{\sqrt{c}}{2}a - \frac{1}{\sqrt{c}}b\right\|_X^2 = \frac{c}{4}\|a\|_X^2 + \frac{1}{c}\|b\|_X^2 - (a,b)_\Omega \geq 0$$

□

Property A.4. *For any $a,b \in \mathbb{R}$,*

$$|a|a(a-b) \geq \frac{1}{3}|a|^3 - \frac{1}{3}|b|^3 \tag{A.6}$$

Proof. Taking into account the Taylor series for the function $f(x) = \frac{1}{3}|x|^3$ ($f'(x) = |x|x$), we obtain that

$$\frac{1}{3}|b|^3 = \frac{1}{3}|a|^3 + |a|a(b-a) + |c|(b-a)^2$$

for $a < b \in \mathbb{R}$ and $c \in (a,b)$. As the last term is non-negative,

$$\frac{1}{3}|b|^3 - \frac{1}{3}|a|^3 + |a|a(a-b) = |c|(b-a)^2 \geq 0$$

and this completes the proof. □

Definition A.1. *Let X be a Banach space and $A(\cdot,\cdot;\mu) : X \times X \longrightarrow \mathbb{R}$. We define the directional derivative of $A(\cdot,\cdot;\mu)$ with respect to the first variable, in the direction of $Z \in X$, for all $U,V \in X$, as*

$$\partial_1 A(U,V;\mu)(Z) = \lim_{\alpha\to 0} \frac{A(U+\alpha Z,V;\mu) - A(U,V;\mu)}{\alpha}, \tag{A.7}$$

if this limit exists.

Theorem A.1 (Hölder's inequality). *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *for* $p, q \in [1,\infty]$ *with* $p^{-1} + q^{-1} = 1$.. *Then* $fg \in L^1(\Omega)$ *since*

$$(f,g)_\Omega \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \tag{A.8}$$

Proof. See [7], page 92. □

Remark A.1. *For* $p = q = 2$, *the Hölder's inequality is known as the Cauchy-Schwarz inequality.*

Theorem A.2 (Sobolev embedding). *Let* $1 \leq p \leq \infty$. *We have*

- *If* $p < d$,
$$W^{1,p}(\Omega) \subset L^q(\Omega), \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{d}.$$
- *If* $p = d$,
$$W^{1,p}(\Omega) \subset L^q(\Omega), \quad \forall q \in [p, +\infty).$$
- *If* $p > d$,
$$W^{1,p}(\Omega) \subset L^\infty(\Omega).$$

All these injections are continuous.

Proof. See [7], page 285. □

Remark A.2. *Thanks to the Sobolev embedding Theorem A.2, we obtain the injection from* $H^1(\Omega)$ *to* $L^4(\Omega)$. *Because of the equivalence between the* $H^1(\Omega)$ *and* $H_0^1(\Omega)$ *norms, there exists a constant* $C_{4;1,2} > 0$ *depending on* Ω *such that*

$$\|\boldsymbol{v}\|_{L^4(\Omega)} \leq C_{4;1,2} \|\boldsymbol{v}\|_{H_0^1(\Omega)}, \quad \forall \boldsymbol{v} \in H^1(\Omega). \tag{A.9}$$

This inequality is used along this dissertation.

For each $h > 0$, let $\mathcal{T}_h$ be a triangulation of $\bar{\Omega}$ made of closed triangles K with diameters bounded by $h_{\max}$. In other words:

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$$

where any two triangles K_1 and K_2 are either disjoint or share at most one side or one vertex. The size and shape of each triangle K are specified by two quantities

- h_K the diameter of K,
- ρ_K the diameter of the inscribed circle in K.

The regularity of a triangle K is measured by the ratio $\sigma_K = h_K/\rho_K$.

Definition A.2. *A family $\{\mathcal{T}_h\}_{h>0}$ of triangulations of $\bar{\Omega}$ is said to be regular as h tends to zero if there exists a number $\sigma > 0$, independent of h and K, such that*

$$\sigma_K \leq \sigma, \ \forall K \in \mathcal{T}_h.$$

In addition, $\mathcal{T}_h$ is said to be uniformly regular as h tends to zero if there exists another constant $\tau > 0$ such that

$$\tau h \leq h_K \leq \sigma \rho_K, \ \forall K \in \mathcal{T}_h.$$

Theorem A.3 (Local Inverse inequality for polynomial functions)**.** *Let q_1, q_2 be two real numbers such that $1 \leq q_1, \ q_2 \leq +\infty$. Let k_1, k_2 be two non-negative integer numbers. Assume that $k_2 \leq k_1$ and $k_2 - \frac{d}{q_2} \leq k_1 - \frac{d}{q_1}$. For any non-negative integer l there exists a constant $C > 0$ such that*

$$\|\boldsymbol{v}\|_{W^{k_1,q_1}(K)} \leq C\rho_k^{k_2-k_1-\frac{d}{q_2}} h_K^{\frac{d}{q_1}} \|\boldsymbol{v}\|_{W^{k_2,q_2}(K)}, \ \forall \boldsymbol{v} \in P_l(K)$$

If in addition, the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is regular, then for all $\mathcal{T}_h$

$$\|\boldsymbol{v}\|_{W^{k_1,q_1}(K)} \leq C h_K^{k_2-k_1-\frac{d}{q_2}+\frac{d}{q_1}} \|\boldsymbol{v}\|_{W^{k_2,q_2}(K)}, \ \forall K \in \mathcal{T}_h, \ \forall \boldsymbol{v} \in P_l(K) \tag{A.10}$$

where h_K is the diameter of a triangle K, ρ_K is the diameter of the largest circle be inscribed in K and the constant C only depends on q_1, q_2, k_1, k_2, d, l and the aspect ratio of the family of triangulations.

Proof. See VII.4 in [6]. □

Remark A.3. *Thanks to the application of the Local Inverse Inequality A.3 to the spaces $L^3(K)$ and $L^2(K)$ for $K \in \mathcal{T}_h$, there exists a constant $C_{3;2} > 0$ such that*

$$\|\boldsymbol{v}\|_{L^3(K)} \leq C_{3;2} h_K^{-d/6} \|\boldsymbol{v}\|_{L^2(K)}, \quad \forall K \in \mathcal{T}_h, \forall \boldsymbol{v} \in P_l(K) \tag{A.11}$$

This inequality is used along this dissertation.

Theorem A.4 (Global Inverse inequality)**.** *Let q_1, q_2 be two real numbers such that $1 \leq q_1, \ q_2 \leq +\infty$. Let k_1, k_2 be two non-negative integer numbers. Assume that $k_2 \leq k_1$ and $k_2 - \frac{d}{q_2} \leq k_1 - \frac{d}{q_1}$. Supposing that for all h, the FE space X_h is in $W^{k_1,q_1}(\Omega)$. There exists $C > 0$ depending on the polynomial degree in each element $K \in \mathcal{T}_h$ such that*

- *for* $q_2 \leq q_1$,

$$\|\boldsymbol{v}\|_{W^{k_1,q_1}(\Omega)} \leq Ch_{\min}^{k_2-k_1-\frac{d}{q_2}+\frac{d}{q_1}}\|\boldsymbol{v}\|_{W^{k_2,q_2}(\Omega)},\ \forall \boldsymbol{v} \in X_h$$

- *for* $q_1 < q_2$,

$$\|\boldsymbol{v}\|_{W^{k_1,q_1}(\Omega)} \leq Ch_{\min}^{k_2-k_1}\|\boldsymbol{v}\|_{W^{k_2,q_2}(\Omega)},\ \forall \boldsymbol{v} \in X_h. \tag{A.12}$$

where $h_{\min} = \min_{K\in\mathcal{T}_h} h_K$.

Proof. See VIII.5 in [6]. □

Theorem A.5 (Riesz representation theorem)**.** *Let* X' *denote the dual space of* X. *Let* $f \in X'$, *then there exists a unique* $x_f \in X$ *such that*

$$f(y) = (x_f, y) \quad \forall y \in X.$$

Moreover,

$$\|f\|_{X'} = \|x_f\|_X.$$

Proof. See [7]. □

Lemma A.1 (Lax-Milgram lemma)**.** *Assume that* $a(\cdot,\cdot)$ *is a bilinear form on* X *such that*

$$|a(u,v)| \leq \gamma \|u\|_X \|v\|_X, \quad \forall u,v \in X$$

and

$$a(u,u) \geq \beta \|u\|_X^2 \quad \forall u \in X.$$

for some $\gamma > 0$ *and* $\beta > 0$. *Then, given* $f : X \longrightarrow \mathbb{R}$, *there exists a unique element* $u \in X$ *such that*

$$a(u,v) = \langle f, v\rangle, \quad v \in X$$

that satisfies

$$\|u\|_X \leq \frac{1}{\beta}\|f\|_{X'}.$$

Proof. See [7]. □

Lemma A.2 (Poincaré)**.** *Let* $1 \leq p < +\infty$ *and let* Ω *be a bounded open set. Then, there exists* $C_{p;1,p} > 0$ *such that*

$$\|v\|_{L^p(\Omega)} \leq C_{p;1,p}\|\nabla v\|_{L^p(\Omega)}, \quad \forall v \in W_0^{1,p}(\Omega) \tag{A.13}$$

where $W_0^{1,p}(\Omega)$ *is the space formed by* $v \in W^{1,p}(\Omega)$ *with zero trace.*

Theorem A.6 (Trace theorem)**.** *Let Ω be a domain of $\mathbb{R}^d$ provided with a Lipschitz continuous boundary $\partial\Omega$. There exists one and only one linear and continuous application*

$$\gamma_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega),$$

such that $\gamma_0 v = v_{|\partial\Omega}$, for all $v \in H^1 \cap C^0(\bar{\Omega})$; and

$$\|\gamma_0 v\|_{L^2(\Gamma)} \leq C_\Gamma \|v\|_{H^1(\Omega)}$$

for some $C_\Gamma > 0$. The result still holds if we consider the trace operator $\gamma_\Gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ where Γ is a sufficiently regular portion of the boundary of Ω with positive measure.

Proof. See [19], Section 5.5. □

Lemma A.3 (Discrete Gronwall's Lemma)**.** *Let k, B, and a_n, b_n, c_n, γ_n, for integers $n \geq 0$, be nonnegative numbers such that*

$$a_m + k\sum_{n=0}^{m} b_n \leq k\sum_{n=0}^{m} \gamma_n a_n + k\sum_{n=0}^{m} c_n + B, \text{ for } n \geq 0. \tag{A.14}$$

Suppose that $k\gamma_n < 1$, for all n, and set $\sigma_n \equiv (1-k\gamma_n)^{-1}$. Then,

$$a_m + k\sum_{n=0}^{m} b_n \leq \exp\left(k\sum_{n=0}^{m} \gamma_n\sigma_n\right)\left(k\sum_{n=0}^{m} c_n + B\right), \text{ for } n \geq 0. \tag{A.15}$$

Proof. See [24], page 369. □

Definition A.3. *Let $H : X \longrightarrow X$ with X be a Hilbert space. We say that H is a contraction if there exists some $L \in (0,1)$ such that*

$$\|H(u) - H(v)\|_X \leq L\|u - v\|_X, \quad \forall u, v \in X. \tag{A.16}$$

Theorem A.7 (Schauder Fixed-Point Theorem)**.** *Let K be a nonempty, compact, convex subset of a space X, and suppose $H : K \longrightarrow K$ is a continuous operator. Then, H has a fixed point.*

Proof. See [50], Chapter 2. □

Remark A.4. *It can be proved that if a contractive function has a fixed point, this point is unique. Moreover, if we are under the conditions of theorem A.7, we can ensure the existence and uniqueness.*

A.3 Well-posedness of saddle point problem

Let Y and Q be two real Hilbert spaces on Ω with norm $\|\cdot\|_Y$ and $\|\cdot\|_Q$ respectively. Let us define the non-linear form $a(\mathbf{u};\mathbf{w},\mathbf{v})$ on $Y\times Y\times Y$ such that the mapping $(\mathbf{w},\mathbf{v})\longmapsto a(\mathbf{u};\mathbf{w},\mathbf{v})$ is a bilinear and continuous form on $Y\times Y$ and the continuous bilinear form $b(\mathbf{v},q)$ on $Y\times Q$.

Then, for a given $\mathbf{f}\in Y'$, we consider the following problem:

$$\begin{cases} \text{Find } (\mathbf{u},p)\in Y\times Q \text{ such that} \\ a(\mathbf{u};\mathbf{u},\mathbf{v})+b(\mathbf{v},p)=\langle \mathbf{f},\mathbf{v}\rangle_\Omega, & \forall \mathbf{v}\in Y \\ b(\mathbf{u},q)=0 & \forall q\in Q \end{cases} \tag{A.17}$$

Then, we introduce the operators $A(\mathbf{u})\in\mathscr{L}(Y;Y')$ for $\mathbf{u}\in Y$ and $B\in\mathscr{L}(Y;Q')$ defined by

$$\langle A(\mathbf{u})\mathbf{w},\mathbf{v}\rangle_\Omega = a(\mathbf{u};\mathbf{w},\mathbf{v}),\ \forall \mathbf{w},\mathbf{v}\in Y,$$

$$\langle B\mathbf{v},q\rangle_\Omega = b(\mathbf{v},q),\ \forall \mathbf{v}\in Y,\ \forall q\in Q.$$

We can rewrite the problem (A.17) into:

$$\begin{cases} \text{Find } (\mathbf{u},p)\in Y\times Q \text{ such that} \\ A(\mathbf{u})\mathbf{u}+B'p=\mathbf{f}, & \text{in } Y' \\ B\mathbf{u}=0, & \text{in } Q' \end{cases} \tag{A.18}$$

Then, we set $X=\ker(B)$ in Y, then we can define a problem associated to (A.17) such that

$$\begin{cases} \text{Find } \mathbf{u}\in X \text{ such that} \\ a(\mathbf{u};\mathbf{u},\mathbf{v})=\langle \mathbf{f},\mathbf{v}\rangle_\Omega, & \forall \mathbf{v}\in X \end{cases} \tag{A.19}$$

Theorem A.8 (Inf-sup Condition)**.** *Let Ω be a bounded, connected, Lipschitz-continuous domain in $\mathbb{R}^d$ and let p_1, p_2 be two real numbers such that $p_1^{-1}+p_2^{-1}=1$. Then, there is a constant $\alpha>0$ such that*

$$\alpha\|q\|_{L^{p_1}(\Omega)} \le \sup_{v\in W^{1,p_2}(\Omega)} \frac{b(v,q)}{\|v\|_{W^{1,p_2}(\Omega)}},\ \forall q\in L^{p_1}(\Omega) \tag{A.20}$$

Proof. See Corollary 3.2. in [2]. □

Remark A.5. *The pair of spaces $([H_0^1(\Omega)]^d, L^2(\Omega))$ verifies the inf-sup condition* (A.20).

Theorem A.9 (Uniqueness problem (A.19))**.** *Let us make the following hypotheses:*

1. *The form $a(\cdot;\cdot,\cdot)$ is uniformly elliptic with respect to the first variable, that is, there exists a constant $\beta > 0$ such that:*

$$a(\boldsymbol{u};\boldsymbol{v},\boldsymbol{v}) \geq \beta \|\boldsymbol{v}\|_Y^2, \ \forall \boldsymbol{v}, \boldsymbol{u} \in X.$$

2. *The mapping $\boldsymbol{u} \longmapsto A(\boldsymbol{u})$ is locally Lipschitz continuous in X; that is, there exists a continuous and monotonically increasing function $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\forall \varepsilon > 0$:*

$$|a(\boldsymbol{u}_1;\boldsymbol{w},\boldsymbol{v}) - a(\boldsymbol{u}_2;\boldsymbol{w},\boldsymbol{v})| \leq L(\varepsilon)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_Y \|\boldsymbol{w}\|_Y \|\boldsymbol{v}\|_Y, \ \forall \boldsymbol{w}, \boldsymbol{v} \in X$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in B_\varepsilon$ where $B_\varepsilon = \{\boldsymbol{v} \in X : \|\boldsymbol{v}\|_Y \leq \varepsilon\}$.

Then, problem (A.19) *has a unique solution $\boldsymbol{u} \in X$.*

Theorem A.10 (Uniqueness problem (A.17))**.** *Suppose that the form $b(\cdot,\cdot) : Y \times Q \longrightarrow \mathbb{R}$ satisfies the inf-sup condition* (A.20) *then, for each solution $\boldsymbol{u}$ of problem* (A.19)*, there exists a unique $p \in Q$ such that the pair $(\boldsymbol{u}, p)$ satisfies problem* (A.17).

Proof. For the proof of theorems A.9 and A.10 we refer to Chapter IV in [21]. □

A.4 Dimensionless numbers

Prandlt

It depends only on the fluid and the fluid state since defines the ratio of momentum diffusivity (ν) and thermal diffusivity (κ). It is around 0.71 for air and 7.56 for water at 18°C.

$$Pr = \frac{\nu}{\kappa} = \frac{c_p \mu}{k} \tag{A.21}$$

Reynolds

It is the ratio of inertial forces to viscous forces. At low Reynolds numbers, flows tend to be dominated by laminar flow, while at high Reynolds numbers, flows tend to be turbulent.

$$Re = \frac{UL}{\nu} = \frac{\rho UL}{\mu} \tag{A.22}$$

Péclet

It is the ratio of the rate of advection transport rate and diffusive transport rate. For heat transfer, the Péclet number is equivalent to the product of the Reynolds number and the Prandlt number.

$$\mathrm{Pe} = \frac{LU}{\kappa} = \frac{LU\rho c_p}{k} = \mathrm{Re}Pr \tag{A.23}$$

Nusselt

It is the ratio of convective to conductive heat transfer at a boundary in a fluid.

$$\mathrm{Nu} = \frac{\alpha L}{k} \tag{A.24}$$

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