Programa de doctorado "Matemáticas"

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Spaces of analytic functions with average radial integrability and integration operators

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Para mi familia.
Para Marina.

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Resumen

En esta memoria, introducimos la familia de espacios de funciones holomorfas en el disco unidad con integrabilidad radial media $RM(p,q)$, $0 < p, q \leq \infty$. Esta familia contiene los espacios clásicos de Hardy H^q (cuando $p = \infty$) y los espacios de Bergman A^p (cuando $p = q$). Caracterizamos la inclusión entre $RM(p_1, q_1)$ y $RM(p_2, q_2)$ en función de los parámetros. Para $1 < p, q < \infty$, proporcionamos una descripción de los espacios duales de $RM(p,q)$ por medio de la acotación de la proyección de Bergman. Mostramos que $RM(p,q)$ es separable si y sólo si $q < \infty$. De hecho, damos un método para construir copias isomorfas de ℓ^∞ en $RM(p,\infty)$.

En la segunda mitad, estudiamos los operadores de integración

$$T_g(f)(z) = \int_0^z f(w)g'(w)\, dw$$

actuando sobre los espacios $RM(p,q)$. Cuando consideramos el operador T_g entre el mismo espacio $RM(p,q)$, proporcionamos una caracterización de la acotación, la compacidad y la compacidad débil. Al considerar la acción de T_g entre diferentes espacios, debido a la complejidad técnica de la situación, sólo caracterizamos su acotación. Para el primer caso, desarrollamos diferentes herramientas como una descripción del bidual de $RM(p,0)$ y estimaciones de la norma de estos espacios utilizando la derivada de las funciones, una familia de resultados que llamamos desigualdades de tipo Littlewood-Paley. Para el segundo caso, resolvemos un problema de medidas de tipo Carleson para los espacios tienda de funciones analíticas AT_p^q en el disco unidad. Estos espacios consisten en las funciones analíticas de los espacios tienda T_p^q introducidos por Coifman, Meyer y Stein, y resulta que en muchos casos se tiene que $RM(p,q)$ coincide con AT_p^q. Este problema de tipo Carleson fue planteado originalmente por Luecking.

Abstract

In this thesis, we introduce the family of spaces of holomorphic functions in the unit disc with average radial integrability $RM(p,q)$, $0 < p, q \leq \infty$. This family contains the classical Hardy spaces H^q (when $p = \infty$) and Bergman spaces A^p (when $p = q$). We characterize the inclusion between $RM(p_1, q_1)$ and $RM(p_2, q_2)$ depending on the parameters. For $1 < p, q < \infty$, we provide a description of the dual spaces of $RM(p,q)$ by means of the boundedness of the Bergman projection. We show that $RM(p,q)$ is separable if and only if $q < \infty$. In fact, we provide a method to build isomorphic copies of ℓ^∞ in $RM(p,\infty)$.

In the second half, we study integration operators

$$T_g(f)(z) = \int_0^z f(w)g'(w)\, dw$$

acting on $RM(p,q)$ spaces. When we consider the operator T_g between the same $RM(p,q)$ space, we provide a characterization of the boundedness, compactness, and weak compactness. When considering the action of T_g between different spaces, which is already an involved situation, we only characterize its boundedness. For the first case, we develop different tools such as a description of the bidual of $RM(p,0)$ and estimates of the norm of these spaces using the derivative of the functions, a family of results that we call Littlewood-Paley type inequalities. For the second case, we solve a problem of Carleson type measures for tent spaces of analytic functions AT_p^q in the unit disc. These spaces consist of those analytic functions of the tent spaces spaces T_p^q introduced by Coifman, Meyer, and Stein, and it turns out that in many cases $RM(p,q) = AT_p^q$. This Carleson type problem was originally posed by Luecking.

Glossary

$\otimes$	tensor product
A^c	the complement of the set A
$\#A$	the cardinality of the set A
$\mathbb{T}$	the unit circle
$\mathbb{D}$	the unit disc
$\mathbb{C}_\infty$	the Riemann sphere
$\mathsf{Re}\,(z)$	the real part of $z \in \mathbb{C}$
$\mathsf{Im}\,(z)$	the imaginary part of $z \in \mathbb{C}$
$\mathrm{Arg}(z)$	the principal argument of $z \in \mathbb{C} \setminus \{0\}$. $\mathrm{Arg}(z) \in (-\pi, \pi]$
$\arg(z)$	an argument of $z \in \mathbb{C} \setminus \{0\}$
$\mathrm{co}\{e_1, e_2, \dots\}$	the finite convex combinations
A	the normalized area measure on the unit disc $\mathbb{D}$
m_n	Lebesgue measure of dimension n
$⨍_A f\ dm$	denote the mean $\frac{1}{m(A)} \int_A f\ dm$ given a measurable set A of positive measure and $f \in L^1(A)$
$C = C(\cdot)$	denote an absolute constant whose value depends on the parameters indicated in the parenthesis
$A \asymp B$	means that there is a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$
$A \lesssim B$	means that there is a constant $C > 0$ such that $A \leq CB$
$A \gtrsim B$	means that there is a constant $C > 0$ such that $A \geq CB$
p'	the conjugate exponent of p, that is, the number such that $\frac{1}{p} + \frac{1}{p'} = 1$, whenever $1 \leq p \leq \infty$. We understand that if $p = 1$ then $p' = \infty$ and if $p = \infty$ then $p' = 1$

Contents

Introducción y conclusiones

En 1923, los espacios clásicos de Hardy H^p fueron introducidos por F. Riesz [58]. Puso el nombre a esos espacios por el artículo de G.H. Hardy [38]. Posteriormente, los espacios de Bergman A^p aparecieron hacia 1950 en un trabajo de S. Bergman [14] centrado en los espacios de funciones analíticas que son cuadrado-integrables sobre un dominio dado con respecto de la medida de área de Lebesgue. Desde entonces, se han hecho grandes progresos en el estudio de estos y otros espacios de funciones analíticas en el disco unidad. En la mayoría de los casos, la pertenencia de una función al espacio viene dada en términos de acotación (o integrabilidad) de una determinada media de la función sobre círculos centrados en el origen o en términos de integrabilidad con respecto de la medida de área de Lebesgue, quizás con un cierto peso. Hay bastantes libros sobre estos espacios, pero nosotros destacamos [27, 34, 40, 43].

En otros casos menos estudiados, la pertenencia viene dada por la integrabilidad radial media. Quizás el espacio más conocido en esta situación es el espacio de variación radial acotada BRV, un tema que se remonta a Zygmund y en el que muchos autores han trabajado (véase, por ejemplo, los artículos de Bourgain [17], Rudin [60] y Zygmund [69]). El espacio BRV de funciones analíticas con variación radial acotada consiste en aquellas funciones holomorfas $g \in \mathcal{H}(\mathbb{D})$ tales que

$$\sup_\theta \int_0^1 |g'(te^{i\theta})|\, dt < \infty.$$

Otra situación diferente donde la integrabilidad radial juega un papel importante es en el teorema de Féjer-Riesz que afirma que si f pertenece al espacio de Hardy H^p entonces

$$\sup_\theta \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right) \leq \frac{1}{2} \|f\|_{H^p}^p. \tag{A}$$

En el Capítulo 1 introducimos la familia de espacios de integrabilidad radial media, $RM(p,q)$ (Definición 1.2.1). Estos espacios están formados por las funciones analíticas tales que

$$\left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{q/p} d\theta \right)^{1/q} < +\infty$$

para $0 < p, q < \infty$. Si p or q es infinito, cambiamos la integral por el supremo esencial, respectivamente. Además, mostramos una serie de ejemplos de funciones que pertenecen a esta familia de espacios. Entre ellos, destacamos la Proposición 1.4.1 donde caracterizamos las series lacunares pertenecientes a $RM(p,q)$. Analizamos otras propiedades como la acotación del funcional de evaluación (Proposición 1.5.2), completitud (Proposición 1.5.8) y separabilidad (Proposición 1.5.12 y

Teorema 1.5.18). Demostramos que $RM(p,q)$ es separable si y sólo si $q < +\infty$. De hecho, el espacio $RM(p,\infty)$ siempre contiene un espacio isomorfo a ℓ^∞ (Teorema 1.5.18).

En el Capítulo 2 proporcionamos una caracterización completa de cuándo uno de esos espacios está incluido en otro (Teorema 2.1.3) y, en tal caso, caracterizamos cuando la inclusión es compacta (Teorema 2.2.3). Como consecuencia de dicha caracterización, obtenemos que la desigualdad inversa de (A) no se satisface, esto es, existen funciones holomorfas f en $\mathbb{D}$ tales que

$$\sup_\theta \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{1/p} < +\infty,$$

pero que no pertenecen a H^p.

En el Capítulo 3 demostramos la acotación de la proyección de Bergman de $L^{(p,q)}([0,1) \times [0,2\pi))$ sobre nuestros espacios $RM(p,q)$ cuando $1 < p,q < +\infty$ (Teorema 3.2.2). Esto nos permite identificar el espacio dual de $RM(p,q)$ como el espacio $RM(p',q')$, para $1 < p,q < +\infty$ (Teorema 3.1.7). Las técnicas y herramientas empleadas en la demostración de la acotación de la proyección de Bergman proceden del Análisis Armónico. En particular, usamos un resultado clásico de C. Fefferman y E. Stein (Teorema 3.2.7), que proporciona la acotación de la versión vectorial de la función maximal. El caso $p = q$ nos da la conocida acotación de la proyección de Bergman de $L^p(\mathbb{D})$ sobre el espacio de Bergman A^p, que se suele demostrar con diferentes técnicas que no funcionan en nuestra situación. Por otro lado, para los casos en los que o $\min\{p,q\} = 1$ o bien $\max\{p,q\} = +\infty$ demostramos que la proyección de Bergman no envía $L^{(p,q)}([0,1) \times [0,2\pi))$ en el espacio $RM(p,q)$ (Teorema 3.2.8).

Los resultados de dualidad anteriores nos premiten simplificar el estudio de las relaciones de contención entre los espacios $RM(p,q)$ y el caso particular de los espacios de norma mixta $H^{q,p,1/p}$, esto es, el espacio de functiones holomorfas f sobre $\mathbb{D}$ tal que

$$\|f\|_{H^{q,p,1/p}} = \left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \, \frac{d\theta}{2\pi} \right)^{p/q} dr \right)^{1/p} < +\infty.$$

En la última sección de este capítulo, presentamos una identificación de forma natural del bidual de $RM(p,0)$ con $RM(p,\infty)$ siguiendo el esquema de la prueba de K.-M. Perfekt en [54] para una situación similar.

La última parte de esta tesis está dedicada a estudiar los operadores de integración que actúan sobre la familia de espacios $RM(p,q)$. Estos operadores se definen de la siguiente manera. Sea X un espacio de Banach de funciones analíticas sobre el disco unidad $\mathbb{D}$. El operador de integración T_g viene dado por

$$T_g(f)(z) = \int_0^z f(w)g'(w)\,dw, \quad f \in X,$$

donde $g : \mathbb{D} \to \mathbb{C}$ es una función analítica.

En [55] Pommerenke probó que este operador está acotado en el espacio de Hardy H^2 si y sólo si g pertenece al espacio de funciones analíticas de oscilación media acotada $BMOA$. Además, su prueba puede ser adaptada para obtener que T_g es compacto si y sólo si g pertenece al espacio de funciones analíticas de oscilación media evanescente $VMOA$. Posteriormente, Aleman y Siskakis extendieron estos resultados a los espacios de Hardy H^p con $1 \leq p < +\infty$ [5]. Además, también mostraron en [6] que el operador de integración T_g está acotado en el espacio Bergman A^p, con $1 \leq p < +\infty$, si y sólo si g pertenece al espacio de Bloch. Además, obtuvieron un resultado similar para la compacidad de T_g pero en este caso la función g debe pertenecer al espacio pequeño de Bloch $\mathcal{B}_0$.

En 2014, Anderson, Jovovic y Smith [8] conjeturaron que este operador está acotado en H^∞ si y sólo si el símbolo g pertenece al espacio BRV de funciones holomorfas en el disco unidad de variación radial acotada. En 2017, Smith, Stolyarov y Volberg publicaron un artículo [63] en el que mostraban un contraejemplo a esta conjetura. En la actualidad, la acotación del operador de integración $T_g : H^\infty \to H^\infty$ es un problema que sigue abierto y que continúa despertando el interés de un gran número de matemáticos, como Volberg, Smith, Peláez, Rättyä, etc.

En el Capítulo 5 proporcionamos una caracterización de cuándo el operador de integración T_g está acotado, compacto o débilmente compacto sobre los espacios de integrabilidad radial media $RM(p,q)$, para $p < +\infty$. Demostramos que T_g está acotado (respectivamente, compacto) sobre los espacios $RM(p,q)$ si y sólo si g pertenece al espacio de Bloch $\mathcal{B}$ (respectivamente, espacio pequeño de Bloch $\mathcal{B}_0$). Nótese que si tomamos $p = q$ se recupera la caracterización de la acotación y compacidad del operador de integración para los espacios de Bergman.

La cuestión principal en este capítulo es la compacidad débil del operador de integración $T_g : RM(p,q) \to RM(p,q)$, por supuesto cuando $RM(p,q)$ no es reflexivo. En este contexto demostramos:

Teorema. *Sea $g \in \mathcal{B}$. Entonces*

(1) Si $1 < p < +\infty$, el operador $T_g : RM(p,\infty) \to RM(p,\infty)$ es débilmente compacto si y sólo si $T_g(RM(p,\infty)) \subset RM(p,0)$ (Teorema 5.2.2).

(2) Si $1 < q < +\infty$, el operador $T_g : RM(1,q) \to RM(1,q)$ es débilmente compacto si y sólo si $T_g : RM(1,q) \to RM(1,1)$ es compacto y si y sólo si g pertenece al

espacio débilmente pequeño de Bloch, esto es, $g \in \mathcal{B}$ *y*

$$\lim_{r \to 1} (1 - r^2)|g'(re^{i\theta})| = 0$$

para casi todo $e^{i\theta} \in \mathbb{T}$ *(Proposición 5.2.5 y Teorema 5.2.11).*

(3) Si $1 \leq p < +\infty$*, el operador* $T_g : RM(p, 1) \to RM(p, 1)$ *es débilmente compacto si y sólo si* $T_g : RM(p, 1) \to RM(p, 1)$ *es compacto y si y sólo si g pertenece al espacio pequeño de Bloch (Corolario 5.2.21).*

El resultado análogo al enunciado *(1)* en el teorema anterior para $p = +\infty$ ya había sido demostrado en [23] por Contreras, Peláez, Pommerenke y Rättyä. Ellos obtuvieron que el operador $T_g : H^\infty \to H^\infty$ es débilmente compacto si y sólo si $T_g : H^\infty \to A$ está acotado, donde A es el álgebra del disco, que resulta ser la clausura de los polinomios en H^∞. La afirmación *(2)* se basa en una caracterización (Teorema 5.2.10) de la compacidad débil de un cierto operador en $L^1([0, 1] \times \mathbb{T})$ que depende de un resultado clásico de Fefferman and Stein sobre la función maximal en funciones con valores en ℓ^p.

También señalamos que la compacidad débil de los operadores de integración está estrechamente relacionada con la propiedad de fijar una copia de algunos espacios clásicos de sucesiones. En concreto, para $1 < q < +\infty$, $T_g : RM(1, q) \to RM(1, q)$ no es débilmente compacto si y sólo si fija una copia de ℓ^1 (Proposición 5.2.5) mientras que para $1 \leq p < +\infty$, $T_g : RM(p, 1) \to RM(p, 1)$ no es débilmente compacto si y sólo si su adjunto fija una copia de c_0 (Corolarios 5.2.20 y 5.2.21). Este tipo de singularidad de ℓ^p ya se ha estudiado en el contexto de los operadores de integración en los espacios de Hardy (véase, por ejemplo, [49]).

Una herramienta importante para el estudio anterior del operador de integración, que presentamos en el Capítulo 4, son las desigualdades de tipo Littelwood-Paley. Esta familia de estimaciones, para $1 \leq p < +\infty$, $1 \leq q < +\infty$, es la siguiente:

$$\rho_{p,q}(f) \leq p \cdot \rho_{p,q}(f'(z)(1 - |z|)) + |f(0)|$$

para $f \in RM(p, q)$ (Proposición 4.1.3). Para $p = +\infty$, estas desigualdades no se satisfacen. También demostramos que la desigualdad inversa se cumple (Proposición 4.2.7) para los casos $1 < p, q < +\infty$, $(1, q)$ con $1 \leq q < +\infty$ y (∞, q) con $1 \leq q \leq +\infty$.

En el último capítulo de esta tesis, presentamos parte del trabajo sobre los espacios tienda realizado en colaboración con el Profesor P. Galanopoulos (véase [4]). Esta colaboración comenzó durante mi estancia de investigación en la Universidad Aristóteles de Tesalónica a principios de 2020. Analizamos la relación entre los espacios de integrabilidad radial media, los espacios tienda AT_p^q (véanse las Definiciones 6.1.1 y 6.1.4), medidas de tipo Carleson y operadores de integración.

En la Sección 6.2 demostramos que $RM(p,q) = AT_p^q$ cuando $1 \leq p,q < +\infty$ o $1 \leq q < +\infty$ y $p = +\infty$ (Teoremas 6.2.1 y 6.2.3). Este es en realidad un resultado (probablemente) conocido en la teoría de los espacios de Triebel-Lizorkin (véase [21, p. 321-322]), pero no hemos podido encontrar una referencia para su demostración. Así que hemos decidido incluirla.

En la Sección 6.4 consideramos un problema de medidas de tipo Carleson. En [20] Carleson caracterizó las medidas (p, H^p)-Carleson, esto es, las medidas de Borel positivas μ en el disco unidad tales que existe una constante positiva C que satisface

$$\int_{\mathbb{D}} |f(w)|^p \, d\mu(w) \leq C\|f\|_{H^p}^p$$

para toda $f \in H^p$. Demostró que las medidas de (p, H^p)-Carleson son descritas por la condición geométrica

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|} < +\infty,$$

donde el supremo se toma sobre todos los arcos I de $\mathbb{T}$,

$$S(I) = \left\{ z \in \mathbb{D} \ : \ 1 - |I| \leq |z| < 1 \ \text{ and } \ \frac{z}{|z|} \in I \right\}$$

es el cuadrado de Carleson con base I e $|I|$ es la longitud de arco de I.

En [48] Luecking consideró y resolvió una versión del problema de Carleson en el semiplano superior para espacios tienda de funciones analíticas y para la derivada n-ésima de las funciones en esos espacios (Teorema 6.4.2). Su demostración puede adaptarse para el caso del disco unidad. Además, planteó el siguiente problema más profundo:

Sea $0 < p, q, t, s < \infty$. Caracterizar las medidas de Borel positivas μ en $\mathbb{D}$ para las que existe una constante positiva C tales que

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f(z)|^t \, d\mu(z) \right)^{s/t} |d\xi| \leq C\|f\|_{T_p^q}^s \tag{B}$$

para todo $f \in AT_p^q$, donde $\Gamma(\xi) = \left\{ r\xi e^{i\theta} : |\theta| < 1 - r, \ 0 < r < 1 \right\}$.

En este capítulo, proporcionamos una caracterización de aquellas medidas de Borel positivas que satisfacen (B) (Teorema 6.4.1).

Como consecuencia de este resultado presentamos dos aplicaciones. Caracterizamos la acotación del operador de integración

$$T_g : RM(p,q) \to RM(t,s),$$

para $1 \leq p, q, s, t < +\infty$ (Teorema 6.5.1).

Una función f pertenece al espacio de Hardy H^p si y sólo si

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(w)|^2 \, dm_2(w) \right)^{p/2} |d\xi| < +\infty$$

(véase [70]). Esta caracterización fue utilizada por Wu para estudiar los operadores de integración desde un espacio de Bergman a un espacio de Hardy [67]. Así pues, es natural preguntarse sobre la acción de T_g desde $RM(p,q)$ a H^s. De hecho, concluimos esta memoria proporcionando una respuesta es esta pregunta (Teorema 6.5.3).

A lo largo de la disertación, hemos proporcionado una serie de ejemplos al lector para comprender el alcance de nuestros resultados o la imposibilidad de extender ciertos resultados a un rango diferente de valores. Ejemplos de la primera situación pueden verse en la Sección 1.4. La segunda situación puede ilustrarse con el Ejemplo 5.2.3 que muestra que el Teorema 5.2.11 no es válido para $q = +\infty$ (véase Observación 5.2.14).

La mayoría de los resultados de esta tesis se recogen en los siguientes trabajos:

- T. Aguilar-Hernández, M.D. Contreras, and L. Rodríguez-Piazza, *Average radial integrability spaces of analytic functions.* Preprint. ArXiv:2002.12264.

- T. Aguilar-Hernández, M.D. Contreras, and L. Rodríguez-Piazza, *Integration operators in average radial integrability spaces of analytic functions*, Mediterr. J. Math. **18** (2021), Article number: 117.

- T. Aguilar-Hernández and P. Galanopoulos, *Average radial integrability spaces, tent spaces and integration operators.* Preprint. ArXiv:2105.10054.

Introduction and conclusions

In 1923, the classical Hardy spaces H^p were introduced by F. Riesz [58]. He named those spaces after the article of G.H. Hardy [38]. Subsequently, the Bergman spaces A^p appeared around 1950 in a work of S. Bergman [14] focused on the spaces of analytic functions that are square-integrable over a given domain with respect to the Lebesgue area measure. Since then, great progress has been made in the study of these and other spaces of analytic functions in the unit disc. In most of the cases, the belonging of a function to the space is given in terms of boundedness (or integrability) of a certain average of the function on circles centered at the origin or in terms of the integrability with respect to the Lebesgue area measure, maybe with a certain weight. There are many good books about these spaces, but we stand out [27, 34, 40, 43].

In other less studied cases, the belonging is determined by the average radial integrability. Maybe the most well-known space in this situation is the space of bounded radial variation BRV, a topic that goes back to Zygmund and where many different authors have worked (see, i.e., the papers of Bourgain [17], Rudin [60], and Zygmund [69]). The space BRV of analytic functions with bounded radial variation consists of those holomorphic functions $g \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_\theta \int_0^1 |g'(te^{i\theta})|\, dt < \infty.$$

Other different situation where the radial integrability plays an important role is in the Féjer-Riesz theorem which says that if f belongs to the Hardy space H^p then

$$\sup_\theta \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right) \leq \frac{1}{2} \|f\|_{H^p}^p. \tag{A}$$

In Chapter 1 we introduce the family of spaces of average radial integrability, $RM(p,q)$ (Definition 1.2.1). These spaces are formed by the analytic functions such that

$$\left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{q/p} d\theta \right)^{1/q} < +\infty$$

for $0 < p, q < \infty$. If either p or q is infinity, we change the integral by the essential supremum, respectively. In addition, we show a range of examples of functions that belong to this family of spaces. Among them, we point out Proposition 1.4.1 where we characterize lacunary series belonging to $RM(p,q)$. We analyse other properties such as boundedness of evaluation functional (Proposition 1.5.2), completeness (Proposition 1.5.8) and separability (Proposition 1.5.12 and Theorem 1.5.18). We show that $RM(p,q)$ is separable if and only if $q < +\infty$. Indeed, the space $RM(p,\infty)$ always contains a subspace isomorphic to ℓ^∞ (Theorem 1.5.18).

In Chapter 2 we provide a complete characterization of when one of such spaces is included in another one (Theorem 2.1.3) and, in such case, we characterize when the inclusion mapping is compact (Theorem 2.2.3). As a consequence of such characterization, we obtain that the converse of the inequality (A) does not hold, that is, there are holomorphic functions f in $\mathbb{D}$ such that

$$\sup_{\theta} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{1/p} < +\infty,$$

but they do not belong to H^p.

In Chapter 3 we show the boundedness of the Bergman projection from $L^{(p,q)}([0,1)\times[0,2\pi))$ onto our spaces $RM(p,q)$ when $1 < p, q < +\infty$ (Theorem 3.2.2). This allows us to identify the dual space of $RM(p,q)$ as the $RM(p',q')$ space, for $1 < p, q < +\infty$ (Theorem 3.1.7). The techniques and tools employed in the proof of the boundedness of the Bergman projection come from Harmonic Analysis. In particular, we use a classical result of C. Fefferman and E. Stein (Theorem 3.2.7), that provides the boundedness of the vector version of the maximal function. The case $p = q$ gives the well-known boundedness of the Bergman projection from $L^p(\mathbb{D})$ onto the Bergman space A^p, which is usually proved with different techniques not working in our situation. On the other hand, for the cases when either $\min\{p,q\} = 1$ or $\max\{p,q\} = +\infty$ we prove that the Bergman projection does not send $L^{(p,q)}([0,1) \times [0,2\pi))$ into the $RM(p,q)$ space (Theorem 3.2.8).

The previous duality results allow us to simplify the study of the containment relationship between the $RM(p,q)$ spaces and the particular case of the mixed norm spaces $H^{q,p,1/p}$, that is, the space of holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^{q,p,1/p}} = \left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \, \frac{d\theta}{2\pi} \right)^{p/q} dr \right)^{1/p} < +\infty.$$

In the last section of this chapter, we present an identification in a natural way of the bidual of $RM(p,0)$ with $RM(p,\infty)$ following the scheme of the proof in a parallel situation by K.-M. Perfekt in [54].

The last part of this thesis is devoted to study the integration operators acting on the family of spaces $RM(p,q)$. These operators are defined in the following way. Let X be a Banach space of analytic functions on the unit disc $\mathbb{D}$. The integration operator T_g is given by

$$T_g(f)(z) = \int_0^z f(w) g'(w) \, dw, \quad f \in X,$$

where $g : \mathbb{D} \to \mathbb{C}$ is an analytic function.

In [55] Pommerenke proved that this operator is bounded on the Hardy space H^2

if and only if g belongs to the space of analytic functions of bounded mean oscillation $BMOA$. Moreover, his proof can be adapted to obtain that T_g is compact if and only if g belongs to the space of analytic functions of vanishing mean oscillation $VMOA$. Later on, Aleman and Siskakis extended these results to Hardy spaces H^p with $1 \leq p < +\infty$ [5]. Moreover, they also showed in [6] that the integration operator T_g is bounded on the Bergman space A^p, with $1 \leq p < +\infty$, if and only if g belongs to the Bloch space. Additionally, they obtained a similar result for the compactness of T_g but in this case the function g must belong to the little Bloch space $\mathcal{B}_0$.

In 2014, Anderson, Jovovic, and Smith [8] conjectured that this operator is bounded on H^∞ if and only if the symbol g belongs to the space BRV of holomorphic functions in the unit disc of bounded radial variation. In 2017, Smith, Stolyarov, and Volberg published an article [63] in which they showed a counterexample to this conjecture. At present, the boundedness of the integration operator $T_g : H^\infty \to H^\infty$ is a problem that is still open and which continues to arouse the interest of a large number of mathematicians, such as Volberg, Smith, Peláez, Rättyä, etc.

In Chapter 5 we provide a characterization of when the integration operator T_g is bounded, compact or weakly compact over the average radial integrability spaces $RM(p,q)$, for $p < +\infty$. We show that T_g is bounded (respectively, compact) over the $RM(p,q)$ spaces if and only if g belongs to the Bloch space $\mathcal{B}$ (respectively, little Bloch space $\mathcal{B}_0$). Notice that if we take $p = q$ then the characterization of the boundedness and compactness of the integration operator for Bergman spaces is recovered.

The main issue in this chapter is the weak compactness of the integration operator $T_g : RM(p,q) \to RM(p,q)$, of course when $RM(p,q)$ is not reflexive. In this setting we prove:

Theorem. *Let $g \in \mathcal{B}$. Then*

(1) If $1 < p < +\infty$, the operator $T_g : RM(p,\infty) \to RM(p,\infty)$ is weakly compact if and only if $T_g(RM(p,\infty)) \subset RM(p,0)$ (Theorem 5.2.2).

(2) If $1 < q < +\infty$, the operator $T_g : RM(1,q) \to RM(1,q)$ is weakly compact if and only if $T_g : RM(1,q) \to RM(1,1)$ is compact and if and only if g belongs to the weakly little Bloch space, that is, $g \in \mathcal{B}$ and

$$\lim_{r\to 1}(1-r^2)|g'(re^{i\theta})| = 0$$

for almost every $e^{i\theta} \in \mathbb{T}$ (Proposition 5.2.5 and Theorem 5.2.11).

(3) If $1 \leq p < +\infty$, the operator $T_g : RM(p,1) \to RM(p,1)$ is weakly compact if and only if $T_g : RM(p,1) \to RM(p,1)$ is compact and if and only if g belongs to the little Bloch space (Corollary 5.2.21).

The analogous result to the statement *(1)* in above theorem for $p = +\infty$ had already been proved in [23] by Contreras, Peláez, Pommerenke, and Rättyä. They obtained that the operator $T_g : H^\infty \to H^\infty$ is weakly compact if and only if $T_g : H^\infty \to A$ is bounded, where A is the disc algebra, that turns out to be the closure of polynomials in H^∞. The statement *(2)* relies on a characterization (Theorem 5.2.10) of the weak compactness of a certain operator into $L^1([0,1] \times \mathbb{T})$ that depend on a classical result of Fefferman and Stein about the maximal function on ℓ^p-valued functions.

We also point out that the weak compactness of integration operators is closely related to the property of fixing copies of some classical spaces of sequences. Namely, for $1 < q < +\infty$, $T_g : RM(1,q) \to RM(1,q)$ is not weakly compact if and only if it fixes a copy of ℓ^1 (Proposition 5.2.5) whereas for $1 \leq p < +\infty$, $T_g : RM(p,1) \to RM(p,1)$ is not weakly compact if and only if its adjoint fixes a copy of c_0 (Corollaries 5.2.20 and 5.2.21). This type of ℓ^p-singularity has been already studied in the setting of integration operators on Hardy spaces (see, for instance, [49]).

An important tool for the above study of the integration operator, that we present in Chapter 4, is Littlewood-Paley type inequalities. This family of estimates, for $1 \leq p < +\infty$, $1 \leq q < +\infty$, is the following:

$$\rho_{p,q}(f) \leq p \cdot \rho_{p,q}(f'(z)(1-|z|)) + |f(0)|$$

for $f \in RM(p,q)$ (Proposition 4.1.3). For $p = +\infty$, these inequalities are not satisfied. We also proved that the converse inequality holds (Proposition 4.2.7) for the cases $1 < p, q < +\infty$, $(1,q)$ with $1 \leq q < +\infty$ and (∞, q) with $1 \leq q \leq +\infty$.

In the last chapter of this dissertation, we present part of the work about tent spaces carried out in collaboration with Prof. P. Galanopoulos (see [4]). This collaboration began during my research stay at the Aristotle University of Thessaloniki at the beginning of 2020. We analyse the relationship between the spaces of average radial integrability, the tent spaces of analytic functions AT_p^q (Definitions 6.1.1 and 6.1.4), Carleson-type measures, and integration operators.

In Section 6.2 we prove that $RM(p,q) = AT_p^q$ when either $1 \leq p, q < +\infty$ or $1 \leq q < +\infty$ and $p = +\infty$ (Theorems 6.2.1 and 6.2.3). This is actually a (probably) known result in the theory of the Triebel-Lizorkin spaces (see [21, p. 321-322]), but we could not find a reference for its proof. So that we have decided to include it.

In Section 6.4 we consider a problem of Carleson-type measures. In [20] Carleson characterized the (p, H^p)-Carleson measures, that is, the positive Borel measures μ on the unit disc such that there is a positive constant C satisfying

$$\int_{\mathbb{D}} |f(w)|^p \, d\mu(w) \leq C \|f\|_{H^p}^p$$

for all $f \in H^p$. He proved that the (p, H^p)-Carleson measures are described by the geometric condition

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|} < +\infty,$$

where the supremum is taken over all arcs I of $\mathbb{T}$,

$$S(I) = \left\{ z \in \mathbb{D} \ : \ 1 - |I| \leq |z| < 1 \ \text{ and } \frac{z}{|z|} \in I \right\}$$

is the Carleson square based on I, and $|I|$ is the arc length of I.

In [48] Luecking considered and solved a version of the Carleson problem on the upper half-plane for tent spaces of analytic functions and for the n-derivatives of the functions in those spaces (Theorem 6.4.2). His proof can be adapted for the case of the unit disc. Moreover, he posed the following deeper problem:

Let $0 < p, q, t, s < \infty$. Characterize the positive Borel measures μ on $\mathbb{D}$ for which there is a positive constant C such that

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f(z)|^t \, d\mu(z) \right)^{s/t} |d\xi| \leq C \|f\|^s_{T^q_p} \tag{B}$$

for all $f \in AT^q_p$, where $\Gamma(\xi) = \left\{ r\xi e^{i\theta} : |\theta| < 1 - r, \ 0 < r < 1 \right\}$.

In this chapter, we provide a characterization of those positive Borel measures satisfying (B) (Theorem 6.4.1).

As a consequences of this result we present two applications. We characterize boundedness of the integration operator

$$T_g : RM(p, q) \to RM(t, s),$$

for $1 \leq p, q, s, t < +\infty$ (Theorem 6.5.1).

A function f belongs to the Hardy space H^p if and only if

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f'(w)|^2 \, dm_2(w) \right)^{p/2} |d\xi| < +\infty$$

(see [70]). This characterization was used by Wu to study integration operators from a Bergman space to a Hardy space [67]. So that, it is natural to wonder about the action of T_g from $RM(p, q)$ to H^s. Indeed, we conclude this memoir by providing an answer to this question (Theorem 6.5.3).

Throughout the dissertation, we have provided a number of examples in order to help the reader to understand either the scope of our results or the impossibility to extend certain results to a different range of values. Examples of the first situation can be seen in Section 1.4. The second situation can be illustrated with Example 5.2.3 which provides that Theorem 5.2.11 is not valid for $q = +\infty$ (see Remark 5.2.14).

Most of the results of this thesis are contained in the following papers:

- T. Aguilar-Hernández, M.D. Contreras, and L. Rodríguez-Piazza, *Average radial integrability spaces of analytic functions.* Preprint. ArXiv:2002.12264.
- T. Aguilar-Hernández, M.D. Contreras, and L. Rodríguez-Piazza, *Integration operators in average radial integrability spaces of analytic functions*, Mediterr. J. Math. **18** (2021), Article number: 117.
- T. Aguilar-Hernández and P. Galanopoulos, *Average radial integrability spaces, tent spaces and integration operators.* Preprint. ArXiv:2105.10054.

Chapter 1

Definition and first properties

1.1 Classical spaces of analytic functions

“Banach spaces of analytic functions” is a vast area of Analysis where ideas from Complex Analysis, Functional Analysis, Harmonic Analysis, and Operator Theory converge. Throughout this Phd Thesis, it will appear results from all these fields.

The complete list of interesting Banach spaces of analytic functions is too big to be presented here. Below we introduce only those which will play a role hereinafter. In this introductory section, we only mention the definitions, their properties and their corresponding references will be given for the right moment they will be used.

Hardy spaces

The theory of Hardy spaces had its inception in Privalov's study of the boundary behavior of bounded holomorphic functions, some years before Hardy defined the spaces which go under its name. It was at the beginning of the XXth century. Mathematicians as Littlewood, F. and M. Riesz, Smirnov or Szegö appear associated to the beginning of the study of properties of individual functions of the Hardy spaces. Later on, Functional Analysis comes into play and Hardy spaces where studied as linear spaces with results as Beurling's theorem on invariant subspaces, extremal problems, interpolation theory, ...

For many years Hardy spaces H^q and operators acting on them were studied in great depth, and an elegant and profound theory was developed.

Without further ado, we introduce the definition:

Definition 1.1.1. Let $0 < q \leq +\infty$. The holomorphic function f on $\mathbb{D}$ is said to belong to the Hardy space H^q if $\|f\|_{H^q} < +\infty$, where

$$\|f\|_{H^q} = \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^q \, \frac{d\theta}{2\pi} \right)^{1/q}, \quad \text{if} \quad 0 < q < +\infty,$$
$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|.$$

When $q \in [1, +\infty]$, H^q is a Banach space, separable if $q < +\infty$. Namely, polynomials are dense in H^q. The closure of the polynomials in H^∞ is the disc algebra:

Definition 1.1.2. The disc algebra A is the set of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ that extends to a continuous function on the closure of $\mathbb{D}$.

A beautiful reference for the results we need on Hardy spaces is the classical monograph of Duren [27].

Bergman spaces

Around 1950, Bergman developed an elegant theory of Hilbert spaces in planar domains (and in higher-dimensions, but this is not our *task*) relying reproducing kernels that become nowadays known as the Bergman kernel function. Bergman's work focussed on spaces of analytic functions that are square-integrable over a given domain with respect to the Lebesgue measure. When the *order of integrability* was changed to L^p, it was natural to call them Bergman spaces. It soon became apparent that Bergman spaces are in many issues much more complicated than Hardy spaces. The reason behind this problem could be that functions in Bergman spaces may have wild boundary behavior and many natural tools in Hardy spaces –non-tangential limits, "inner-outer" factorization,...– do not have their accompanying implement. Thus, the theory of Bergman spaces had their own development.

Definition 1.1.3. For $0 < p < +\infty$, the Bergman space A^p is formed by all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{1/p} < +\infty,$$

where $dA(z)$ is the normalized area measure.

As in the case of Hardy spaces, the space A^p is a separable Banach space whenever $1 \leq p < +\infty$.

Two recent books about Bergman spaces are [28] and [40].

Bloch space

The Bloch space has been studied by many authors because of its intrinsic interest and because it is the meeting place of several areas of analysis.

In 1924, Bloch proved that there is a universal constant B such that if $f : \mathbb{D} \to \mathbb{C}$ is holomorphic, $z_0 \in \mathbb{D}$ and $f'(z_0) \neq 0$, then there is a domain $\Omega \subset \mathbb{D}$ such that f is one-to-one in Ω and $D(f(z_0), B|f'(z_0)|(1 - |z_0|^2)) \subset f(\Omega)$. The biggest value of such number B is called the Bloch's constant. The seek of the Bloch's constant has been one of the key problems in geometric function theory with contributions of significant mathematicians. The expression $|f'(z_0)|(1 - |z_0|^2)$ leads to the definition of the Bloch space:

Definition 1.1.4. An analytic function f on $\mathbb{D}$ is said to belong to the Bloch space $\mathcal{B}$ if the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|)$$

is finite. The little Bloch space $\mathcal{B}_0$ is the closed subspace of the Bloch space formed by the functions satisfying

$$\lim_{|z|\to 1^-} |f'(z)|(1-|z|) = 0.$$

It is clear that the condition in the definition of $\mathcal{B}$ arises naturally when one applies the Cauchy estimates to a bounded analytic function to bound its derivative.

$\mathcal{B}$ is a Banach space and it can be identified as the dual of the Bergman space A^1. Weirdly, Bloch space has no natural containment relationship with any of the Bergman and Hardy spaces except for H^∞. It is easily proved that $H^\infty \subset \mathcal{B}$ but it is larger. Moreover, by taking the closure of the polynomials in $\mathcal{B}$ we obtain the little Bloch space $\mathcal{B}_0$.

A standard reference for Banach space properties of the Bloch space is the paper of Anderson, Clunie, and Pommerenke [7].

BMOA space

The notion of bounded mean oscillation goes back to the sixties where it was introduced by John and Niremberg.

A notable breakthrough was Fefferman discovery, in 1971, that the dual of the real Hardy space H^1 is the space BMO of functions having bounded mean oscillations. In the setting of analytic functions, this space is given by:

Definition 1.1.5. A function $f \in H^2$ is said to belong to the space of analytic functions of bounded mean oscillation $BMOA$ if the norm

$$\|f\|_{BMOA} = |f(0)| + \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|}\int_{R(I)} |f'(z)|^2(1-|z|^2)\,dA(z) < +\infty$$

where I runs the arcs of $\partial\mathbb{D}$, $|I|$ denotes its length, and $R(I)$ is the Carleson rectangle

$$R(I) = \{re^{i\theta} \in \mathbb{D} : 1 - \frac{|I|}{2\pi} < r < 1,\, e^{i\theta} \in I\}.$$

The space of analytic functions of vanishing mean oscillation is given by

$$VMOA := \{f \in BMOA : \lim_{|I|\to 0} \frac{1}{|I|}\int_{R(I)} |f'(z)|^2(1-|z|^2)\,dA(z) = 0\}.$$

The closure of all polynomials in $BMOA$ is nothing but $VMOA$ and both are Banach spaces. The analytic counterpart of Fefferman's theorem shows that the dual of the (complex) Hardy space H^1 is BMOA. Girela's paper [35] is a nice survey about the properties of these spaces.

1.2 Definition of spaces of average radial integrability

We start with the definition of the spaces that will be the main subject of our memoir, the spaces $RM(p,q)$. They are formed by those holomorphic functions in $\mathbb{D}$ such that taking the p-norm in every radius and then the q-norm, the result is a finite number. More rigorously we have the following definition.

Definition 1.2.1. Let $0 < p, q \leq +\infty$. We define the spaces of analytic functions

$$RM(p,q) = \{f \in \mathcal{H}(\mathbb{D}) : \rho_{p,q}(f) < +\infty\}$$

where

$$\rho_{p,q}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{it})|^p \, dr \right)^{q/p} dt \right)^{1/q}, \quad \text{if } p, q < +\infty,$$

$$\rho_{p,\infty}(f) = \operatorname*{ess\,sup}_{t \in [0,2\pi)} \left(\int_0^1 |f(re^{it})|^p \, dr \right)^{1/p}, \quad \text{if } p < +\infty,$$

$$\rho_{\infty,q}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\sup_{r \in [0,1)} |f(re^{it})| \right)^q dt \right)^{1/q}, \quad \text{if } q < +\infty, \quad \text{and}$$

$$\rho_{\infty,\infty}(f) = \|f\|_{H^\infty}.$$

Remark 1.2.2. It is easy to check that $\rho_{p,q}$ is a norm when $p, q \geq 1$ and otherwise it is a quasinorm.

Remark 1.2.3. In the definition of $\rho_{p,\infty}$ the essential supremum can be replaced by the supremum. Let us check this fact. Fix $\theta \in [0, 2\pi]$. Since the set of $t \in [0, 2\pi]$ such that $\left(\int_0^1 |f(re^{it})|^p \, dr\right)^{1/p} \leq \rho_{p,\infty}(f)$ is dense in $[0, 2\pi]$, we can extract a sequence $\{t_n\}$ in this set such that $t_n \to \theta$. Using Fatou's lemma it follows that

$$\left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{1/p} \leq \liminf_n \left(\int_0^1 |f(re^{it_n})|^p \, dr \right)^{1/p} \leq \rho_{p,\infty}(f).$$

Remark 1.2.4. If in the radial integration of $\rho_{p,q}$ we consider rdr instead of dr, we obtain the same sets of analytic functions. This can be seen by decomposing an analytic function f as the sum of functions with support in $D(0,1/2)$ and functions with support in $\mathbb{D} \setminus D(0,1/2)$:

$$f = f\chi_{D(0,1/2)} + f\chi_{\mathbb{D} \setminus D(0,1/2)}.$$

The term $f\chi_{D(0,1/2)}$ is bounded so that it is integrable with dr and rdr and in the second term $f\chi_{\mathbb{D}\setminus D(0,1/2)}$ both integrals are equivalent. The fact that the expressions for dr and rdr are equivalent on $\mathbb{D}$ follows from some facts that we will prove later.

For certain parameters p, q these spaces $RM(p,q)$ are well known classical spaces. Indeed, it is easy to see that $RM(p,p)$ is nothing but the Bergman space A^p, for $0 < p \leq +\infty$.

Another classical spaces appear when $p = +\infty$, $0 < q \leq +\infty$. One can easily check that $RM(\infty, q)$ is contained in the Hardy space H^q, because, for $f \in RM(\infty, q)$,

$$\|f\|_{H^q} = \sup_{r\in[0,1)} \left(\int_0^{2\pi} |f(re^{it})|^q \, dt \right)^{1/q} \leq \left(\int_0^{2\pi} \sup_{r\in[0,1)} |f(re^{it})|^q \, dt \right)^{1/q} = \rho_{\infty,q}(f).$$

Thus $RM(\infty, q) \subset H^q$. For the other inclusion we use the following result concerning the nontangential maximal function.

Theorem 1.2.5. *[61, Theorem 17.11(a), p. 340] Let $0 < q < +\infty$ and $f \in H^q$. Then for any $C > 1$, the nontangential maximal function*

$$H_f(e^{i\theta}) = \sup_{z\in S_C(e^{i\theta})} |f(z)|$$

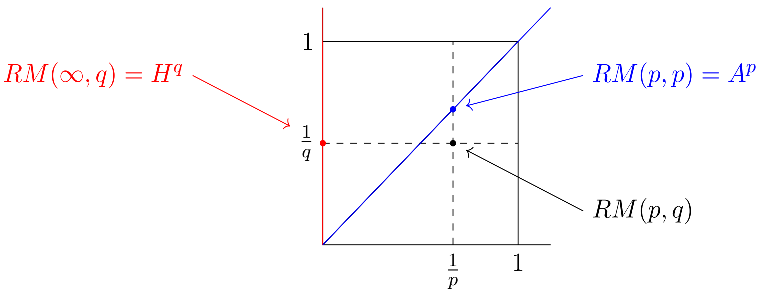
belongs to $L^q(\mathbb{T})$, where $S_C(e^{i\theta}) := \{z \in \mathbb{D} \, : \, |z - e^{i\theta}| \leq C(1-|z|)\}$. Moreover, there is a constant $\gamma = \gamma(C)$ such that $\|H_f\|^q_{L^q} \leq \gamma \|f\|^q_{H^q}$

Thus, by Theorem 1.2.5, there is a constant C such that, if $f \in H^q$, then

$$\int_0^{2\pi} \sup_{r\in[0,1]} |f(re^{i\theta})|^q \, d\theta \leq \int_0^{2\pi} \sup_{z\in S_3(e^{i\theta})} |f(z)|^q \, d\theta \leq C\|f\|^q_{H^q}, \tag{1.2.1}$$

so that we get that $RM(\infty, q) = H^q$ for all $q \in (0, +\infty]$.

For the sake of clearness we will represent this biparametric family of spaces in the first quadrant. The $RM(p,q)$ space is represented as the point with coordinates $(1/p, 1/q)$. Notice that in the diagonal of the quadrant we find Bergman spaces and in the left vertical line we have Hardy spaces.



This picture will help us to clarify in several situations the scope of our results.

We are mainly interested in the cases $p, q \geq 1$, which are represented in the closed unit square.

1.3 First examples

In this section we present some functions in $RM(p,q)$ that we will use in what follows.

Example 1.3.1. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq +\infty$ with either p or q finite, the function $f_\alpha(z) = (1-z)^{-\alpha}$, where we are using the main branch of the logarithm to define $w^{-\alpha}$, belongs to $RM(p,q)$ if and only if $\alpha < \frac{1}{p} + \frac{1}{q}$. In the particular case $p = q = +\infty$, f_α belongs to H^∞ if and only if $\alpha \leq 0$.

Proof. If $\alpha \leq 0$, then the function $|f_\alpha|$ is bounded so that it belongs to $RM(p,q)$ for all p and q. Thus, in what follows we will only consider the case $\alpha > 0$.

Assume now that $0 < p, q < \infty$. Write $I(t) = \int_0^1 |f_\alpha(re^{it})|^p \, dr$. Since I is even and decreasing in $[0,\pi]$, we have

$$\int_0^{\pi/4} I(t)^{q/p} \, dt \leq \rho_{p,q}^q(f_\alpha) = 2\int_0^{\pi} I(t)^{q/p} \, dt \leq 8\int_0^{\pi/4} I(t)^{q/p} \, dt.$$

In addition, for $t \in [0, \pi/4]$, we have that $1 - \cos(t) \asymp t^2/2$. Therefore,

$$\rho_{p,q}^q(f_\alpha) \asymp \int_0^{\pi/4} \left[\int_0^1 \frac{1}{((1-r)^2 + rt^2)^{\alpha p/2}} \, dr \right]^{q/p} dt.$$

With a similar argument, we can reduce the integral in r to the interval $[1/2, 1]$ and using that, when r runs this interval, the function rt^2 is equivalent to t^2 we have

$$\rho_{p,q}^q(f_\alpha) \asymp \int_0^{\pi/4} \left[\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} \, dr \right]^{q/p} dt. \tag{1.3.1}$$

If $\alpha \geq \frac{1}{p} + \frac{1}{q}$, and $t \in [0, 1/2]$, then

$$\begin{aligned} \int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} \, dr &\geq \int_{1-t}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} \, dr \\ &\geq \int_{1-t}^1 \frac{1}{(2t^2)^{\alpha p/2}} \, dr = \frac{1}{2^{\alpha p/2}} \frac{1}{t^{\alpha p - 1}}. \end{aligned}$$

Thus

$$\int_0^{\pi/4} \left[\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} \, dr \right]^{q/p} dt \geq \frac{1}{2^{\alpha q/2}} \int_0^{1/2} \left[\frac{1}{t^{\alpha p - 1}} \right]^{q/p} dt = +\infty$$

and so, by (1.3.1), f_α does not belong to $RM(p,q)$.

Assume now that $\alpha < \frac{1}{p} + \frac{1}{q}$. If $\alpha p < 1$, then

$$\int_{1/2}^{1} \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}}\, dr \le \int_{1/2}^{1} \frac{1}{(1-r)^{\alpha p}}\, dr < +\infty,$$

so that, by (1.3.1), $f_\alpha \in RM(p,q)$. If $\alpha p = 1$, then we obtain

$$\int_{1/2}^{1} \frac{1}{((1-r)^2 + t^2)^{\frac{\alpha p}{2}}}\, dr \le \int_{1-t}^{1} \frac{1}{t}\, dr + \int_{1/2}^{1-t} \frac{1}{1-r}\, dr \le \ln\left(\frac{e}{2t}\right).$$

Integrating with respect to t it follows

$$\int_{0}^{\pi/4} \left(\int_{1/2}^{1} \frac{1}{((1-r)^2 + t^2)^{\frac{\alpha p}{2}}}\, dr\right)^{q/p} dt \le \int_0^{\pi/4} \ln^{q/p}\left(\frac{e}{2t}\right) dt < +\infty.$$

It remains to see what happens if $1 < \alpha p < 1 + \frac{p}{q}$. In this case, if $t \in [0, \pi/4]$, we have

$$\int_{1/2}^{1} \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}}\, dr \le \int_{1/2}^{1-t} \frac{1}{(1-r)^{\alpha p}}\, dr + \int_{1-t}^{1} \frac{1}{t^{\alpha p}}\, dr \le \frac{\alpha p}{\alpha p - 1} \frac{1}{t^{\alpha p - 1}}.$$

Therefore, by (1.3.1),

$$\rho_{p,q}^q(f_\alpha) \lesssim \left(\frac{\alpha p}{\alpha p - 1}\right)^{q/p} \int_0^{\pi/4} \frac{1}{t^{\alpha q - \frac{q}{p}}}\, dt < +\infty.$$

Summing up, the result holds if both p and q are finite. For $p = \infty$, since $RM(\infty, q) = H^q$, the result is well-known (see, i.e., [27, Page 13]).

For $q = \infty$, arguing as above we have

$$\rho_{p,\infty}^p(f_\alpha) \asymp \sup_{0 \le t \le \pi/2} \int_{1/2}^{1} \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}}\, dr = \int_{1/2}^{1} \frac{dr}{(1-r)^{\alpha p}} < +\infty \qquad (1.3.2)$$

if and only if $\alpha < \frac{1}{p}$.

Finally it is clear that if $\alpha > 0$, the function f_α does not belong to H^∞. □

Given $\alpha \in \mathbb{D}$ and $\beta \ge 0$ the function $f(z) = (1 - \overline{\alpha} z)^{-\beta}$, $z \in \mathbb{D}$, is bounded, so it is clear that it belongs to $RM(p,q)$ for all p, q. In the next proposition, we estimate $\rho_{p,q}$ for certain values of β.

Proposition 1.3.2. *Let $0 < p, q \le +\infty$. Let $\alpha \in \mathbb{D}$ and $\beta > \frac{1}{p} + \frac{1}{q}$ then*

$$\rho_{p,q}((1 - \overline{\alpha} z)^{-\beta}) \asymp (1 - |\alpha|)^{\frac{1}{p} + \frac{1}{q} - \beta},$$

where we are using the main branch of the logarithm to define $w^{-\beta}$. We underline *that the equivalent constants depend on p,q and β, but not on α.*

Proof. Let $0 < p, q < +\infty$. We can assume without loss of generality that $\alpha \in [0,1)$. Moreover, we can assume that $1/2 \leq \alpha < 1$.

Let us estime the quantity $|1-\alpha re^{i\theta}|^2$ for points around 1. If $0 < \theta < 1-\alpha$ and $1/2 < r < 1$, then $|1-\alpha re^{i\theta}|^2 \asymp (1-r\alpha)^2$. If $1/2 > \theta > 1-\alpha$, then

$$|1-\alpha re^{i\theta}|^2 \asymp \begin{cases} \theta^2, & 1 > r \geq \frac{1-\theta}{\alpha}, \\ (1-r\alpha)^2, & 1/2 < r \leq \frac{1-\theta}{\alpha}. \end{cases}$$

First of all, let us see that $\rho_{p,q}((1-\overline{\alpha}z)^{-\beta}) \lesssim (1-|\alpha|)^{\frac{1}{p}+\frac{1}{q}-\beta}$ if $\beta > \frac{1}{p}+\frac{1}{q}$. Using the symmetry in θ and the monotonicity in θ and r, we have

$$\int_0^{2\pi}\left(\int_0^1 \frac{dr}{|1-\alpha re^{i\theta}|^{\beta p}}\right)^{q/p} d\theta \leq 2^{2+q/p}\pi \int_0^{1/2}\left(\int_{1/2}^1 \frac{dr}{|1-\alpha re^{i\theta}|^{\beta p}}\right)^{q/p} d\theta$$

Therefore,

$$\begin{aligned}
&\int_0^{2\pi}\left(\int_0^1 \frac{dr}{|1-\alpha re^{i\theta}|^{\beta p}}\right)^{q/p} d\theta \\
&\lesssim \left(\int_0^{1-\alpha}\left(\int_{1/2}^1 \frac{dr}{|1-\alpha re^{i\theta}|^{\beta p}}\right)^{q/p} d\theta + \int_{1-\alpha}^{1/2}\left(\int_{1/2}^1 \frac{dr}{|1-\alpha re^{i\theta}|^{\beta p}}\right)^{q/p} d\theta\right) \\
&\leq \int_{1-\alpha}^{1/2}\left(\int_{\frac{1-\theta}{\alpha}}^1 \frac{dr}{\theta^{\beta p}} + \int_{1/2}^{\frac{1-\theta}{\alpha}} \frac{dr}{(1-r\alpha)^{\beta p}}\right)^{q/p} d\theta + \int_0^{1-\alpha}\left(\int_{1/2}^1 \frac{dr}{(1-r\alpha)^{\beta p}}\right)^{q/p} d\theta \\
&\leq \int_{1-\alpha}^{1/2}\left(\frac{\theta-(1-\alpha)}{\alpha\theta^{\beta p}} + \frac{1}{\alpha(\beta p-1)}\left(\frac{1}{\theta^{\beta p-1}} - \frac{1}{(1-\alpha/2)^{\beta p-1}}\right)\right)^{q/p} d\theta \\
&\quad + \int_0^{1-\alpha}\left(\frac{1}{\alpha(\beta p-1)}\left(\frac{1}{(1-\alpha)^{\beta p-1}} - \frac{1}{(1-\alpha/2)^{\beta p-1}}\right)\right)^{q/p} d\theta \\
&\leq \left(\frac{\beta p}{\alpha(\beta p-1)}\right)^{q/p}\left(\int_{1-\alpha}^{1/2} \frac{1}{\theta^{\beta q-q/p}}\, d\theta + \frac{1}{(1-\alpha)^{\beta q-q/p-1}}\right) \\
&\leq \left(\frac{\beta p}{\alpha(\beta p-1)}\right)^{q/p}\left(\frac{\beta q-q/p}{\beta q-q/p-1}\right)(1-\alpha)^{1+q/p-\beta q}.
\end{aligned}$$

Now, we will show that $\rho_{p,q}((1-\overline{\alpha}z)^{-\beta}) \gtrsim (1-|\alpha|)^{\frac{1}{p}+\frac{1}{q}-\beta}$ if $\beta > \frac{1}{p}+\frac{1}{q}$. Since for $0 < \theta < 1-\alpha$ and $0 \leq r < 1$ one have that $|1-\alpha re^{i\theta}| \lesssim (1-\alpha r)$, we have

$$\int_0^{2\pi}\left(\int_0^1 \frac{dr}{|1-\alpha re^{i\theta}|^{\beta p}}\right)^{q/p} d\theta \gtrsim \int_0^{1-\alpha}\left(\int_0^1 \frac{dr}{(1-\alpha r)^{\beta p}}\right)^{q/p} d\theta$$
$$\geq \frac{1}{(\beta p-1)^{q/p}\alpha^{q/p}}\int_0^{1-\alpha}\left(\frac{1}{(1-\alpha)^{\beta p-1}}-1\right)^{q/p} d\theta$$
$$\geq \left(\frac{1-(1/2)^{\beta p-1}}{(\beta p-1)\alpha}\right)^{q/p}(1-\alpha)^{1+q/p-\beta q}.$$

If p or q is ∞, the proof follows similarly. □

Example 1.3.3. Let $0<p,q<\infty$, $n\geq 1$ and take α such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{\alpha}$. The $\rho_{p,q}$ value of the holomorphic function

$$f_{n,\alpha}(z)=\left(\sum_{k=0}^{n} z^k\right)^{1/\alpha}=\left(\frac{1-z^{n+1}}{1-z}\right)^{1/\alpha},$$

where we are using the main branch of the logarithm to define $w^{1/\alpha}$, can be estimated as

$$\rho_{p,q}(f_{n,\alpha}) \lesssim \left(\frac{p}{p-\alpha}\right)^{1/p} \ln^{1/q}(n+1). \tag{1.3.3}$$

Proof. Clearly, $f_{n,\alpha}$ is well-defined. Since $\rho_{p,q}(f_{n,\alpha})=\rho^{1/\alpha}_{p/\alpha,q/\alpha}(f_{n,1})$, it is not difficult to see that the proof of (1.3.3) can be reduced to the case $\alpha=1$ and $\frac{1}{p}+\frac{1}{q}=1$. Notice that, in this case, $p>1$.

Since

$$\int_{\pi/4}^{\pi}\left(\int_0^1 |f_{n,1}(re^{i\theta})|^p\,dr\right)^{q/p} d\theta \leq \int_{\pi/4}^{\pi}\left(\int_0^1 |4|^p\,dr\right)^{q/p} d\theta = \frac{3\pi}{4}4^q,$$

we have

$$2\pi\rho_{p,q}(f_{n,1})^q \leq 2\int_0^{\pi/4}\left(\int_0^1 |f_{n,1}(re^{i\theta})|^p\ dr\right)^{q/p} d\theta + \frac{3\pi}{4}4^q.$$

If $1-\theta\leq r\leq 1$ and $\theta\in[0,\pi/4]$, arguing as in Example 1.3.1, we obtain

$$|f_{n,1}(re^{i\theta})| \lesssim \frac{2}{\sqrt{(1-r)^2+r\theta^2}} \leq \frac{2}{\theta\sqrt{1-\theta}} \leq \frac{2}{\theta\sqrt{1-\frac{\pi}{4}}} < \frac{5}{\theta}.$$

Therefore, there is a constant $C > 2$ such that

$$\int_{\frac{1}{n+1}}^{\pi/4}\left(\int_0^1 |f_{n,1}(re^{i\theta})|^p\, dr\right)^{q/p} d\theta =$$
$$= \int_{\frac{1}{n+1}}^{\pi/4}\left(\int_0^{1-\theta} |f_{n,1}(re^{i\theta})|^p\, dr + \int_{1-\theta}^1 |f_{n,1}(re^{i\theta})|^p\, dr\right)^{q/p} d\theta$$
$$\leq \int_{\frac{1}{n+1}}^{\pi/4}\left(\int_0^{1-\theta} \frac{2^p}{(1-r)^p}\, dr + \int_{1-\theta}^1 \frac{C^p}{\theta^p}\, dr\right)^{q/p} d\theta$$
$$\leq C^q \int_{\frac{1}{n+1}}^{\pi/4}\left(\frac{p}{p-1}\theta^{-p+1}\right)^{q/p} d\theta = C^q\left(\frac{p}{p-1}\right)^{q/p}(\ln(\pi/4) + \ln(n+1))$$

and

$$\int_0^{\frac{1}{n+1}}\left(\int_0^1 |f_{n,1}(re^{i\theta})|^p\, dr\right)^{q/p} d\theta =$$
$$= \int_0^{\frac{1}{n+1}}\left(\int_0^{1-\frac{1}{n+1}} |f_{n,1}(re^{i\theta})|^p\, dr + \int_{1-\frac{1}{n+1}}^1 |f_{n,1}(re^{i\theta})|^p\, dr\right)^{q/p} d\theta$$
$$\leq 2^q \int_0^{\frac{1}{n+1}}\left(\int_0^{1-\frac{1}{n+1}} \frac{1}{(1-r)^p}\, dr + \int_{1-\frac{1}{n+1}}^1 (n+1)^p\, dr\right)^{q/p} d\theta \leq 2^q\left(\frac{p}{p-1}\right)^{q/p}.$$

Putting altogether, we get the estimation of $\rho_{p,q}(f_{n,\alpha})$. □

1.4 Lacunary series and $RM(p,q)$

We say that a sequence of positive numbers $\{x_k\}$ is a lacunary sequence if there is a constant λ such that $\frac{x_{k+1}}{x_k} \geq \lambda > 1$. Given $\{n_k\}$ a lacunary sequence of positive integers, the function $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ is called a lacunary series (or Hadamard gap series).

A classical result due to Paley (see [43, Theorem 6.2.2]) states that, given $1 \leq q < +\infty$, a lacunary series $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ belongs to H^q if and only if the sequence of Taylor coefficientes $\{a_k\}$ belongs to ℓ^2. Moreover, Sidon showed that $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ belongs to H^∞ if and only if $\{a_k\}$ belongs to ℓ^1 (see [70, Vol. I, p. 247]). The next result provides a characterization of lacunary series belonging to $RM(p,q)$ for p finite (that is, when the space $RM(p,q)$ is not a Hardy space).

Proposition 1.4.1. *Let $\{n_k\}_{k=0}^{\infty}$ be a lacunary sequence of positive integer numbers, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then*

$$f(z) = \sum_{k=0}^{\infty} \alpha_k z^{n_k}$$

belongs to $RM(p,q)$ *if and only if*

$$\sum_{k=0}^{\infty} \frac{|\alpha_k|^p}{n_k} < +\infty.$$

Moreover, it holds

$$\rho_{p,q}(f) \asymp \left(\sum_{k=0}^{\infty} \frac{|a_k|^p}{n_k} \right)^{1/p}. \tag{1.4.1}$$

Remark 1.4.2. Notice that the second part of the expression (1.4.1) does not depend on q.

Proof. Notice that $\left\{ n_k + \frac{1}{p} \right\}_{k \geq 0}$ is also a lacunary sequence. The proof of this result is based on a characterization of bases on $L^p[0,1]$ due to Gurariĭ and Macaev [37]. Namely they proved that, fixed $p \in [1, +\infty)$, if a sequence $\{n_k\}_{k \geq 0}$ is lacunary then there exist two positive constants A and B such that

$$A \left(\sum_{k=0}^{\infty} |\beta_k|^p \right)^{1/p} \leq \left\| \sum_{k=0}^{\infty} \beta_k \sqrt[p]{n_k + 1/p}\, t^{n_k} \right\|_{L^p} \leq B \left(\sum_{k=0}^{\infty} |\beta_k|^p \right)^{1/p}, \tag{1.4.2}$$

for every $\{\beta_k\} \in \ell^p$.

Take now $f(z) = \sum_{k=0}^{\infty} \alpha_k z^{n_k}$ a holomorphic function in the unit disc. Fix $\theta \in [0, 2\pi]$ and write $\beta_k := \frac{\alpha_k}{\sqrt[p]{n_j + 1/p}} e^{i\theta n_k}$ if $k \geq 0$. By (1.4.2),

$$\begin{aligned} A \left(\sum_{k=0}^{\infty} \frac{|\alpha_k|^p}{n_k + 1/p} \right)^{1/p} &\leq \left\| \sum_{k=0}^{\infty} \alpha_k\, r^{n_k} e^{i\theta n_k} \right\|_{L^p} = \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{1/p} \\ &\leq B \left(\sum_{k=0}^{\infty} \frac{|\alpha_k|^p}{n_k + 1/p} \right)^{1/p}. \end{aligned}$$

Now, looking at the very definition of $\rho_{p,q}$ we get the result. □

For Bergman spaces the above result is due to Buckley, Koskela, and Vukotic [19, Proposition 2.1]. It is worth pointing out that their proof follows a completely different argument.

Remark 1.4.3. The constants A and B in (1.4.2) can be chosen arbitrarily close to 1 provided that all the quotients $\frac{n_{k+1}}{n_k}$ are big enough. Namely, if the sequence $\{n_k\}$ is super-lacunary (that is, $\lim_{k \to +\infty} \frac{n_{k+1}}{n_k} = +\infty$), the map sending $\{\beta_k\} \in \ell^p$ to $\sum_{k=0}^{\infty} \beta_k \sqrt[p]{n_k + 1/p}\, t^{n_k}$ is an almost isometric embedding of ℓ^p into $L^p[0,1]$ [33, Theorem 2.1, p. 3]. In the same way we will have an almost isometric embedding of ℓ^p into our space $RM(p,q)$.

1.5 $RM(p,q)$ as function spaces

In this section, we study $RM(p,q)$ as (quasi)normed spaces. We analyse the evaluating functionals, the completeness and the separability of these spaces.

1.5.1 Evaluating functionals

This subsection is devoted to the functionals $f \mapsto f(z)$ and $f \mapsto f'(z)$. We prove that both of them are bounded and estimate their norms. We will need the following inclusion.

Proposition 1.5.1. *Let $0 < s \leq +\infty$. Then $H^s \subset RM(p,q)$ if and only if $\frac{1}{p}+\frac{1}{q} \geq \frac{1}{s}$.*

Proof. The result is clear for $s = +\infty$ since H^∞ is a subspace of $RM(p,q)$ for every p, q. Thus, from now on we consider the case $s < +\infty$. Assume that $H^s \subset RM(p,q)$ and suppose that $\frac{1}{s} > \frac{1}{p} + \frac{1}{q}$. Take $\frac{1}{s} > \alpha > \frac{1}{p} + \frac{1}{q}$. Then the function f_α defined in Example 1.3.1 belongs to $RM(\infty, s) = H^s$ and not to $RM(p,q)$. A contradiction. Thus $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{s}$.

To see the converse implication we claim that $H^s \subset RM(p_1, q_1)$ whenever $0 < s < +\infty$ and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{s}$. Assume for the moment that the claim holds. Fix p and q such that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{s}$. We consider $p_1 \geq p$ and $q_1 \geq q$ such as $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{s}$. By the claim $H^s \subset RM(p_1, q_1)$. Moreover, it is easy to prove that $RM(p_1, q_1) \subset RM(p,q)$ using Hölder's inequality twice (for $p_1 \geq p$ and $q_1 \geq q$).

Thus it remains to prove the claim. By Féjer-Riesz theorem [27, Theorem 3.13, p. 46], we have that for each $f \in H^s$ and θ,

$$\int_0^1 |f(re^{i\theta})|^s \, dr \leq \frac{1}{2} \|f\|_{H^s}^s$$

(notice that, in particular, this implies that $H^s \subset RM(s, \infty)$). Now, since $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{s}$ we can take $\lambda \in [0,1]$ such that $\frac{1}{p_1} = \frac{\lambda}{s}$ and $\frac{1}{q_1} = \frac{1-\lambda}{s}$. Then

$$\begin{aligned}
\rho_{p_1,q_1}(f) &= \left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^{p_1(1-\lambda)} |f(re^{i\theta})|^{p_1\lambda} \, dr \right)^{q_1/p_1} \frac{d\theta}{2\pi} \right)^{1/q_1} \\
&\leq \left(\int_0^{2\pi} \sup_r |f(re^{i\theta})|^{q_1(1-\lambda)} \left(\int_0^1 |f(re^{i\theta})|^s \, dr \right)^{q_1\lambda/s} \frac{d\theta}{2\pi} \right)^{1/q_1} \\
&\leq \left(\int_0^{2\pi} \sup_r |f(re^{i\theta})|^s \left(\int_0^1 |f(re^{i\theta})|^s \, dr \right)^{q_1\lambda/s} \frac{d\theta}{2\pi} \right)^{1/q_1} \\
&\lesssim \|f\|_{H^s}^\lambda \left(\int_0^{2\pi} \sup_r |f(re^{i\theta})|^s \frac{d\theta}{2\pi} \right)^{1/q_1} = \|f\|_{H^s}^\lambda \rho_{\infty,s}(f)^{1-\lambda} \lesssim \|f\|_{H^s},
\end{aligned}$$

where in the last inequality we have used (1.2.1). Hence, we have proved the claim and we are done. □

Proposition 1.5.2. *Let* $0 < p, q \leq \infty$ *and* $z \in \mathbb{D}$. *The functional* $\delta_z : RM(p,q) \to \mathbb{C}$ *given by* $\delta_z(f) := f(z)$, *for all* $f \in RM(p,q)$, *is continuous and*

$$\|\delta_z\|_{(RM(p,q))^*} \asymp \frac{1}{(1-|z|)^{\frac{1}{p}+\frac{1}{q}}},$$

where the underlying constants depend on p *and* q.

Proof. Assume $p, q < +\infty$. Given $p_0 > 0$, the subharmonicity of the function $|f|^{p_0}$ shows that for all $z \in \mathbb{D}$,

$$|f(z)|^{p_0} \leq \frac{1}{\pi r^2} \int_{D(z,r)} |f(w)|^{p_0} \, dA(w)$$

where $r = 1 - |z|$, $D(z,r)$ is the disc centered at z with radius r and $dA(w)$ means integration with respect to the Lebesgue measure on the unit disc $\mathbb{D}$.

Due to the rotational invariance of the space $RM(p,q)$ we can assume that z belongs to the interval $[0,1)$. Take $f \in RM(p,q)$. Fix $p_0 > 0$. To prove the result we may assume that $\frac{1}{2} \leq z < 1$. Set $r = 1 - |z|$. Bearing in mind that

$$\arcsin\left(\frac{1-z}{z}\right) \leq \pi(1-z)$$

for $\frac{1}{2} \leq z < 1$, we have $|\text{Arg}(w)| \leq \pi r$ for $w \in D(z,r)$ (see Figure 1.1).

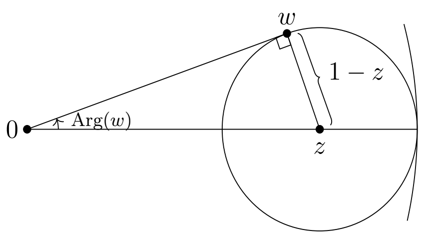


Figure 1.1

It follows

$$\frac{1}{\pi r^2} \int_{D(z,r)} |f(w)|^{p_0} \, dA(w) \leq \frac{1}{\pi r^2} \int_{-\pi r}^{\pi r} \left(\int_{1-2r}^{1} |f(\rho e^{i\theta})|^{p_0} \, d\rho \right) \, d\theta.$$

If $p, q \geq p_0$, applying Hölder's inequality twice we get

$$\frac{1}{\pi r^2}\int_{-\pi r}^{\pi r}\left(\int_{1-2r}^{1}|f(\rho e^{i\theta})|^{p_0}\,d\rho\right)\,d\theta \leq \frac{2}{r^2}\int_{-\pi r}^{\pi r}\left(\int_{1-2r}^{1}|f(\rho e^{i\theta})|^{p}\,d\rho\right)^{\frac{p_0}{p}}(2r)^{1-\frac{p_0}{p}}\,\frac{d\theta}{2\pi}$$
$$\leq \frac{2^{2-\frac{p_0}{p}}}{r^{1+\frac{p_0}{p}}}\left(\int_{-\pi r}^{\pi r}\left(\int_{1-2r}^{1}|f(\rho e^{i\theta})|^{p}\,d\rho\right)^{\frac{q}{p}}\frac{d\theta}{2\pi}\right)^{\frac{p_0}{q}} r^{1-\frac{p_0}{q}} \leq \frac{2^{2-\frac{p_0}{p}}}{(1-|z|)^{p_0\left(\frac{1}{p}+\frac{1}{q}\right)}}\rho_{p,q}^{p_0}(f).$$

So that

$$|f(z)|^{p_0} \leq \frac{2^{2-\frac{p_0}{p}}}{(1-|z|)^{\frac{p_0}{p}+\frac{p_0}{q}}}\rho_{p,q}(f)^{p_0}. \tag{1.5.1}$$

Hence δ_z is continuous and $\|\delta_z\| \lesssim 1/(1-|z|)^{\frac{1}{p}+\frac{1}{q}}$.

This argument can be adapted if p or q is infinite.

To see the converse inequality, take s such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}$. Assume $s<+\infty$. By Proposition 1.5.1, $RM(\infty, s) = H^s \subset RM(p,q)$ and thus

$$\|\delta_z\|_{(RM(p,q))^*} \gtrsim \|\delta_z\|_{(RM(\infty,s))^*} \asymp \|\delta_z\|_{(H^s)^*} = \frac{1}{(1-|z|^2)^{1/s}},$$

where we have used [43, Exercise 2, p. 86] or [34, Exercise 5, p. 85]. If $s=+\infty$, the result follows using constant functions. □

Corollary 1.5.3. *Let $0<p,q\leq+\infty$. Convergence in the $\rho_{p,q}$-(quasi)norm implies uniform convergence on compact sets. Moreover, boundedness in $RM(p,q)$ implies uniform boundedness on compact sets.*

Proof. Let $\{f_n\}$ be a sequence that converges to f in $RM(p,q)$. If we consider a compact set K, it is clear that there is a constant $\gamma>0$ such that

$$d(K,\partial\mathbb{D}) > \gamma > 0.$$

So, applying Proposition 1.5.2 it follows that

$$|f_n(z)-f(z)| \lesssim \frac{\rho_{p,q}(f_n-f)}{(1-|z|)^{\frac{1}{p}+\frac{1}{q}}} \leq \gamma^{-\frac{1}{p}-\frac{1}{q}}\rho_{p,q}(f_n-f)$$

for all $z\in K$. So, $\{f_n\}$ converges uniformly to f on compact sets. The same argument works for bounded sequences. □

Lemma 1.5.4. *Let $0<p,q\leq+\infty$. Let $\{f_n\}$ be a bounded sequence in $RM(p,q)$ that converges uniformly on compacta to $h\in\mathcal{H}(\mathbb{D})$. Then $h\in RM(p,q)$ and $\rho_{p,q}(h)\leq \liminf_n \rho_{p,q}(f_n)$.*

Proof. Assume $p,q < +\infty$. Let $\{f_n\}$ and h as in the statement. Using Fatou's lemma twice

$$\begin{aligned}\rho_{p,q}(h) &= \left(\int_0^{2\pi}\left(\int_0^1 \lim_n |f_n(re^{i\theta})|^p\, dr\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q}\\ &\leq \left(\int_0^{2\pi}\liminf_n\left(\int_0^1 |f_n(re^{i\theta})|^p\, dr\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q}\\ &\leq \liminf_n\left(\int_0^{2\pi}\left(\int_0^1 |f_n(re^{i\theta})|^p\, dr\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q}\\ &= \liminf_n \rho_{p,q}(f_n) < +\infty.\end{aligned}$$

A similar argument works in the remaining cases, so that we are done. □

Another important consequence of the previous proposition is the boundedness of the multiplication operator $M_g : RM(p,q) \to RM(p,q)$ defined, for a certain analytic function g, as $M_g(f) := g \cdot f$ for every $f \in RM(p,q)$.

Corollary 1.5.5. *Let $1 \leq p,q \leq +\infty$. The operator $M_g : RM(p,q) \to RM(p,q)$ is bounded if and only if $g \in H^\infty$. Moreover, $\|M_g\| = \|g\|_{H^\infty}$.*

Proof. Assume that $M_g : RM(p,q) \to RM(p,q)$ is bounded. Let $z \in \mathbb{D}$ and $f \in RM(p,q)$, then $|f(z)g(z)| = |\delta_z(fg)| = |\delta_z(M_g(f))|$. By Proposition 1.5.2 and boundedness of M_g, it follows that

$$|f(z)g(z)| \leq \|\delta_z\|_{(RM(p,q))^*}\rho_{p,q}(M_g f) \leq \|\delta_z\|_{(RM(p,q))^*}\|M_g\|\rho_{p,q}(f).$$

Taking supremum over all functions f of $B_{RM(p,q)}$, we have

$$\|\delta_z\|_{(RM(p,q))^*}|g(z)| \leq \|\delta_z\|_{(RM(p,q))^*}\|M_g\|.$$

Therefore, $|g(z)| \leq \|M_g\|$ for all $z \in \mathbb{D}$. So that, $g \in H^\infty$.

Conversely, assume that $g \in H^\infty$. Let $f \in RM(p,q)$, then

$$\rho_{p,q}(M_g(f)) = \left(\int_0^{2\pi}\left(\int_0^1 |f(re^{i\theta})g(re^{i\theta})|^p\, dr\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q} \leq \rho_{p,q}(f)\|g\|_{H^\infty}.$$

Hence, it follows that $\|M_g\| \leq \|g\|_{H^\infty}$. □

Proposition 1.5.6. *Let $0 < p, q \leq \infty$ and $z \in \mathbb{D}$. The functional $\delta'_z : RM(p,q) \to \mathbb{C}$ given by $\delta'_z(f) := f'(z)$, for all $f \in RM(p,q)$, is continuous and*

$$\|\delta'_z\|_{(RM(p,q))^*} \asymp \frac{1}{(1-|z|)^{\frac{1}{p}+\frac{1}{q}+1}},$$

where the underlying constants depend on p and q.

Proof. Again we assume that $z \in [0,1)$. Fix $z \in [0,1)$ and denote by C the boundary of the disc centered at z and with radius $(1-|z|)/2$. The Cauchy's integral formula and the estimate of the evaluation functional given in Proposition 1.5.2 show

$$\begin{aligned}|f'(z)| &\leq \frac{1}{\pi}\int_0^{2\pi} \frac{|f(z+\frac{1-z}{2}e^{i\theta})|}{1-z}\, d\theta \lesssim \frac{1}{\pi(1-z)}\int_0^{2\pi} \frac{\rho_{p,q}(f)}{(1-|z+\frac{1-z}{2}e^{i\theta}|)^{\frac{1}{p}+\frac{1}{q}}}\, d\theta \\ &\leq \frac{2^{\frac{1}{p}+\frac{1}{q}+1}}{(1-z)^{\frac{1}{p}+\frac{1}{q}+1}}\rho_{p,q}(f).\end{aligned}$$

To prove the converse inequality, we will use a similar argument to the one given in Proposition 1.5.2. Using Proposition 1.5.1 we have that $RM(\infty, s) = H^s \subset RM(p,q)$ for $\frac{1}{p}+\frac{1}{q} = \frac{1}{s}$. So, it follows

$$\|\delta'_z\|_{(RM(p,q))^*} \gtrsim \|\delta'_z\|_{(RM(\infty,s))^*} \asymp \|\delta'_z\|_{(H^s)^*}.$$

On the one hand, if $s < +\infty$, since for the Hardy space H^s it is known that $\|\delta'_z\|_{(H^s)^*} \asymp \frac{1}{(1-|z|)^{\frac{1}{s}+1}}$ [34, Exercise 5, p. 85], we obtain

$$\|\delta'_z\|_{(RM(p,q))^*} \gtrsim \frac{1}{(1-|z|)^{\frac{1}{s}+1}}.$$

On the other hand, if $s = +\infty$, take the function $\varphi(w) := \frac{w-z}{1-\bar{z}w}$, $w \in \mathbb{D}$. Since φ is an automorphism of the unit disc, we have that $||\varphi||_{H^\infty} = 1$ and

$$||\delta'_z||_{(H^\infty)^*} \geq |\varphi'(z)| = \frac{1}{1-|z|^2} \geq \frac{1}{2(1-|z|)}.$$

And we end with a similar argument. □

Combining Propositions 1.5.2 and 1.5.6, the next corollary follows.

Corollary 1.5.7. *Let $0 < p, q \leq +\infty$. If $z \in \mathbb{D}$, then*

$$\|\delta'_z\|_{(RM(p,q))^*} \asymp \frac{\|\delta_z\|_{(RM(p,q))^*}}{1-|z|}.$$

1.5.2 Completeness

Proposition 1.5.8. *Let $1 \le p, q \le +\infty$. Then $RM(p,q)$ is a Banach space.*

Proof. As we have already mentioned that $\rho_{p,q}$ is a norm, let us check that $RM(p,q)$ is complete.

Let $\{f_n\}$ be a Cauchy sequence. Thus $\{f_n\}$ is bounded. Given $\varepsilon > 0$, there is $\nu > 0$ such that $\rho_{p,q}(f_n - f_m) < \varepsilon$ for all $m, n > \nu$. Using Proposition 1.5.2 one can see that $\{f_n\}$ is a uniformly Cauchy sequence on compact sets of $\mathbb{D}$. Hence, the sequence $\{f_n\}$ is uniformly convergent on compact sets of $\mathbb{D}$ to a holomorphic function f. Using Lemma 1.5.4 one obtains that the function f belongs to $RM(p,q)$.

Moreover, applying Fatou's lemma twice we can see that $\{f_n\}$ is a convergent sequence to f in $RM(p,q)$: given $n > \nu$,

$$\begin{aligned}
\rho_{p,q}(f - f_n) &= \left(\int_0^{2\pi} \left(\int_0^1 \lim_m |f_m(re^{i\theta}) - f_n(re^{i\theta})|^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q} \\
&\le \left(\int_0^{2\pi} \liminf_m \left(\int_0^1 |f_m(re^{i\theta}) - f_n(re^{i\theta})|^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q} \\
&\le \liminf_m \left(\int_0^{2\pi} \left(\int_0^1 |f_m(re^{i\theta}) - f_n(re^{i\theta})|^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q} \\
&= \liminf_m \rho_{p,q}(f_n - f_m) \le \varepsilon
\end{aligned}$$

Therefore, we conclude that $RM(p,q)$ is a Banach space for $p, q < +\infty$. A similar argument works in the remaining cases, so that we are done. □

Remark 1.5.9. If p or q is less than 1, then $RM(p,q)$ is a quasi-Banach space, because $\rho_{p,q}$ define a quasinorm and also one can check that $RM(p,q)$ is a complete quasinormed space.

1.5.3 Density of polynomials and separability

Next lemma is obvious if f is continuous on $[0,1]$ (and then uniformly continuous) and by density of such functions we extend to the whole space:

Lemma 1.5.10. *Let $0 < p < +\infty$ and $f \in L^p([0,1])$. Then*

$$\lim_{\rho \to 1} \int_0^1 |f(x) - f(\rho x)|^p \, dx = 0. \tag{1.5.2}$$

Given a holomorphic function f in the unit disc and $0 < r < 1$, we define $f_r(z) := f(rz)$, for all $z \in \mathbb{D}$.

Proposition 1.5.11. *Let $0 < p, q \le +\infty$. Then*

(1) There is a constant $C = C(p,q)$ such that $\rho_{p,q}(f_r) \leq C\rho_{p,q}(f)$ for all $f \in RM(p,q)$ and for $r \in [0,1)$.

(2) If, in addition $q < +\infty$, then $\rho_{p,q}(f - f_r) \to 0$ when $r \to 1^-$ for all $f \in RM(p,q)$.

Proof. For Hardy spaces the statement *(1)* follows immediately and the second one is known [27, Theorem 2.6, p. 21]. Therefore, we can assume that p is finite.

Fix $f \in RM(p,q)$. We consider

$$R_p(\theta, f) = \left(\int_0^1 |f(ue^{i\theta})|^p \, du \right)^{1/p},$$

what it is well-defined for almost every θ.

First, let us check that *(1)* holds. If we take $r > 1/2$, then we have $\rho_{p,q}(f_r) \leq 2^{1/p}\rho_{p,q}(f)$, because

$$\begin{aligned} R_p(\theta, f_r)^p = \int_0^1 |f(rue^{i\theta})|^p \, du = \int_0^r |f(ue^{i\theta})|^p \frac{du}{r} \\ \leq \frac{1}{r} R_p(\theta, f)^p < 2R_p(\theta, f)^p. \end{aligned} \tag{1.5.3}$$

For $0 \leq r \leq 1/2$, the inequality $\rho_{p,q}(f_r) \lesssim \rho_{p,q}(f)$ follows by Proposition 1.5.2, because

$$\rho_{p,q}(f_r) \leq \|f_r\|_{H^\infty} \lesssim \frac{\rho_{p,q}(f)}{(1-r)^{\frac{1}{p}+\frac{1}{q}}} \leq 2^{\frac{1}{p}+\frac{1}{q}} \rho_{p,q}(f).$$

So that, we have concluded the proof of the statement *(1)*.

Now, we proceed by proving *(2)*. Easily we can see that there is $C_p > 0$ such that

$$R_p(\theta, f - f_r) \leq C_p \left(R_p(\theta, f_r) + R_p(\theta, f) \right).$$

Hence, using (1.5.3) we have that $R_p(\theta, f - f_r) \leq (1 + 2^{1/p})C_p R_p(\theta, f)$ for $r > 1/2$. By Lemma 1.5.10, $R_p(\theta, f - f_r) \to 0$, when $r \to 1$, for a.e. θ. Since the function $[0, 2\pi] \ni \theta \mapsto (R_p(\theta, f))^q$ is integrable, using the dominated convergence theorem we conclude the proof of *(2)* for $p < +\infty$. □

Proposition 1.5.12. *Let $0 < p \leq +\infty$, $0 < q < +\infty$. Polynomials are dense in $RM(p,q)$. In particular, $RM(p,q)$ is a separable space.*

Proof. We will study the cases $0 < p, q < \infty$ since it is well-known that polynomials are dense in Hardy spaces $H^q = RM(\infty, q)$ for $0 < q < \infty$. Let $f \in RM(p,q)$. Let us fix $r < 1$. The function f_r is holomorphic on $\frac{1}{r}\mathbb{D}$. Since $\overline{\mathbb{D}} \subset \frac{1}{r}\mathbb{D}$ with $r \in (0,1)$, the sequence of partial sums $\{P_n\}_n$ of the Taylor expansion of f_r converges uniformly to f_r in $\overline{\mathbb{D}}$. Therefore, polynomials $\{P_n\}_n$ converges in the topology of $RM(p,q)$ to f_r and together with Proposition 1.5.11 we obtain that polynomials are dense in $RM(p,q)$. This is enough to show the separability. □

It is well-known that $H^\infty = RM(\infty,\infty)$ is a non-separable Banach space. In order to study the non-separability of $RM(p,\infty)$, for $p<+\infty$, we introduce:

Definition 1.5.13. Let $0<p<+\infty$. We define the subspace $RM(p,0)$ of $RM(p,\infty)$

$$RM(p,0) := \left\{ f\in\mathcal{H}(\mathbb{D}) : \lim_{\rho\to 1}\sup_\theta \left(\int_\rho^1 |f(re^{i\theta})|^p dr\right)^{1/p} = 0\right\}.$$

Let us see that $RM(p,0)$ is a closed subspace of $RM(p,\infty)$, so that it is a Banach space if $p\geq 1$ (and quasi-Banach if $p<1$). Take $C_p = 2^{-1+1/p}$ if $p<1$ and $C_p=1$ if $p\geq 1$. Assume that the sequence $\{f_n\}$ converges to f with $f_n\in RM(p,0)$, for all n and $f\in RM(p,\infty)$. Fix $\varepsilon>0$. There is n such that $\rho_{p,\infty}(f_n-f)<\varepsilon/(2C_p)$. In addition, there is ρ_0 such that

$$\sup_\theta \left(\int_\rho^1 |f_n(re^{i\theta})|^p\, dr\right)^{1/p} < \varepsilon/(2C_p), \text{ for } \rho_0<\rho<1.$$

Therefore,

$$\begin{aligned}&\sup_\theta \left(\int_\rho^1 |f(re^{i\theta})|^p\, dr\right)^{1/p}\\ &\leq C_p\left(\sup_\theta\left(\int_\rho^1 |f(re^{i\theta})-f_n(re^{i\theta})|^p\, dr\right)^{1/p} + \sup_\theta\left(\int_\rho^1 |f_n(re^{i\theta})|^p\, dr\right)^{1/p}\right) < \varepsilon.\end{aligned}$$

Thus, $f\in RM(p,0)$. In particular, $RM(p,0)$ is a Banach space for $p\geq 1$. We will show later on that $RM(p,\infty)\neq RM(p,0)$ (see Theorem 1.5.18).

Remark 1.5.14. Notice that every function f of H^∞ belongs to $RM(p,0)$. Indeed, for $\rho<1$, we have

$$\sup_\theta \left(\int_\rho^1 |f(ue^{\theta i})|^p\, du\right)^{1/p} \leq (1-\rho)^{1/p}\sup_{z\in\mathbb{D}}|f(z)| \leq (1-\rho)^{1/p}\|f\|_{H^\infty}.$$

Thus

$$\lim_{\rho\to 1}\sup_\theta \left(\int_\rho^1 |f(ue^{\theta i})|^p\, du\right)^{1/p} = 0.$$

Now, we can provide an analogous to Proposition 1.5.11 for $q=\infty$:

Proposition 1.5.15. *Let $0 < p < +\infty$ and $f \in RM(p,\infty)$. Then $f_r \in RM(p,0)$. Moreover, $f \in RM(p,0)$ if and only if*

$$\rho_{p,\infty}(f - f_r) \to 0 \tag{1.5.4}$$

when $r \to 1$.

Proof. Assume that $f \in RM(p,\infty)$. Since $f_r \in H^\infty$, we have $f_r \in RM(p,0)$. Since $RM(p,0)$ is closed in $RM(p,\infty)$, we get $f \in RM(p,0)$ if (1.5.4) holds.

Assume now that $f \in RM(p,0)$, we have to see that $\rho_{p,\infty}(f - f_r) \to 0$. Fix $\varepsilon > 0$. Then there is $\rho_0 < 1$ such that

$$\sup_\theta \left(\int_\rho^1 |f(se^{i\theta})|^p \, ds \right)^{1/p} \leq \varepsilon \tag{1.5.5}$$

for all $\rho_0 \leq \rho < 1$. Take $\rho = (\rho_0 + 1)/2$ and $r < 1$ such that $r\rho > \rho_0$. Since f_r converges to f uniformly on the $\rho\overline{\mathbb{D}}$, we have

$$\lim_{r\to 1} \sup_\theta \int_0^\rho |f(se^{\theta i}) - f_r(se^{\theta i})|^p \; ds = 0.$$

Bearing in mind (1.5.5), for each θ, we have

$$\begin{aligned} \left(\int_\rho^1 |f(se^{\theta i}) - f_r(se^{\theta i})|^p \, ds \right)^{1/p} &\leq C \left(\int_\rho^1 |f(se^{\theta i})|^p \, ds \right)^{1/p} + C \left(\int_\rho^1 |f_r(se^{\theta i})|^p \, ds \right)^{1/p} \\ &\leq C\varepsilon + \frac{C}{r^{1/p}} \left(\int_{r\rho}^r |f(se^{\theta i})|^p \, ds \right)^{1/p} \leq C\varepsilon + \frac{C}{r^{1/p}}\varepsilon, \end{aligned}$$

where C is a constant that depends on p (being 1 if $p \geq 1$). This implies that $\limsup_{r\to 1} \rho_{p,\infty}(f - f_r) \leq 2C\varepsilon$. Thus $\lim_{r\to 1} \rho_{p,\infty}(f - f_r) = 0$. □

A density argument similar to the one used in Proposition 1.5.12 shows that:

Corollary 1.5.16. *Let $0 < p < +\infty$. Polynomials are dense in $RM(p,0)$. In particular, $RM(p,0)$ is a separable space.*

Corollary 1.5.17. *Let $0 < p < +\infty$. If $z \in \mathbb{D}$, then*

$$\|\delta_z\|_{(RM(p,0))^*} = \|\delta_z\|_{(RM(p,\infty))^*} \quad \textit{and} \quad \|\delta'_z\|_{(RM(p,0))^*} = \|\delta'_z\|_{(RM(p,\infty))^*}.$$

In particular,

$$\|\delta'_z\|_{(RM(p,0))^*} \asymp \frac{\|\delta_z\|_{(RM(p,0))^*}}{1 - |z|^2}.$$

Proof. Since $RM(p,0) \subset RM(p,\infty)$, we have that $\|\delta_z\|_{(RM(p,0))^*} \leq \|\delta_z\|_{(RM(p,\infty))^*}$ and $\|\delta'_z\|_{(RM(p,0))^*} \leq \|\delta'_z\|_{(RM(p,\infty))^*}$.

Let us see that $\|\delta_z\|_{(RM(p,0))^*} \geq \|\delta_z\|_{(RM(p,\infty))^*}$ and $\|\delta'_z\|_{(RM(p,0))^*} \geq \|\delta'_z\|_{(RM(p,\infty))^*}$. If $f \in RM(p,\infty)$ with $\rho_{p,\infty}(f) = 1$, by Proposition 1.5.15, $f_r \in RM(p,0)$ and

$$\rho_{p,\infty}(f_r) = \sup_\theta \left(\int_0^1 |f(rue^{i\theta})|^p \, du\right)^{1/p} = \sup_\theta \left(\int_0^r |f(ue^{i\theta})|^p \, \frac{du}{r}\right)^{1/p} \leq \frac{1}{r^{1/p}}\rho_{p,\infty}(f)$$

Moreover, it is easy to see that $\delta_z(f_r) \to \delta_z(f)$ and $\delta'_z(f_r) \to \delta'_z(f)$, when $r \to 1^-$ for a fixed $z \in \mathbb{D}$.

Fixing $\varepsilon > 0$, there exists $f \in RM(p,\infty)$ with $\rho_{p,q}(f) = 1$ such that

$$|\delta_z(f)| \geq \|\delta_z\|_{(RM(p,\infty))^*} - \varepsilon.$$

In addition, we know that

$$\begin{aligned} |\delta_z(f)| &= |f(z)| = \lim_{r\to 1} |f(rz)| \\ &= \lim_{r\to 1} \frac{|\delta_z(f_r)|}{\rho_{p,\infty}(f_r)} \rho_{p,\infty}(f_r) \leq \lim_{r\to 1} \frac{\|\delta_z\|_{(RM(p,0))^*}}{r^{1/p}} = \|\delta_z\|_{(RM(p,0))^*}. \end{aligned}$$

Therefore, it is satisfied, for all $\varepsilon > 0$,

$$\|\delta_z\|_{(RM(p,\infty))^*} \leq \|\delta_z\|_{(RM(p,0))^*} + \varepsilon,$$

that is, $\|\delta_z\|_{(RM(p,\infty))^*} \leq \|\delta_z\|_{(RM(p,0))^*}$. The proof for δ'_z can be done in a similar way. □

The non-separability of $RM(p,\infty)$ is an easy consequence of the following much deepest result.

Theorem 1.5.18. *Let* $0 < p < \infty$. *Then* $RM(p,\infty)$ *has a subspace isomorphic to* ℓ^∞. *Namely, there is a sequence* $\{f_k\}$ *of functions in* $RM(p,0)$ *such that for every* $\{\alpha_k\} \in \ell^\infty$ *the series* $\sum_{k=0}^\infty \alpha_k f_k$ *converges uniformly on compact subsets of* $\mathbb{D}$ *and the operator*

$$T : \ell^\infty \to RM(p,\infty) \quad \text{defined by} \quad T(\{\alpha_k\}) := \sum_{k=0}^\infty \alpha_k f_k$$

establishes an isomorphism between ℓ^∞ *and* $T(\ell^\infty)$. *Moreover,* $T(\{\alpha_k\}) \in RM(p,0)$ *if and only if* $\{\alpha_k\} \in c_0$. *In particular,* $\sum_{k=0}^\infty f_k \in RM(p,\infty) \setminus RM(p,0)$.

Proof. Take $C_p = 2^{-1+1/p}$ if $p < 1$ and $C_p = 1$ if $p \geq 1$. For each $k = 0, 1, 2, \ldots,$

take $r_k = 2^{-(k+1)}$, $a_k = 1 + 14^{-(k+1)}$, and

$$\varepsilon_k = \frac{1}{2^{k+1}(7^{k+1}-1)}.$$

It is clear that $\sum_{k=0}^{\infty} r_k = 1$, there is N such that

$$\begin{aligned}\sum_{k=N}^{\infty}\left(\frac{\varepsilon_k}{r_k^2}\right)^{1/p} &= \sum_{k=N}^{\infty}\left(\frac{2^{k+1}}{7^{k+1}-1}\right)^{1/p} \leq \left(\frac{7}{6}\right)^{1/p}\sum_{k=N}^{\infty}\left(\frac{2^{k+1}}{7^{k+1}}\right)^{1/p} \\ &= \left(\frac{7}{6}\right)^{1/p}\frac{(2/7)^{\frac{N+1}{p}}}{1-(2/7)^{1/p}} < \frac{1}{C_p} \leq 1,\end{aligned}$$

and

$$\int_{a_k-r_k}^{1} \frac{\varepsilon_k}{|a_k - r|^2}\,dr = \varepsilon_k\left(\frac{1}{(a_k-1)} - \frac{1}{r_k}\right) = 1. \tag{1.5.6}$$

In addition we can find a sequence $\{\theta_k\}$ such that the discs $D(a_k e^{\theta_k i}, r_k)$ are pairwise disjoint. Indeed, we can consider

$$\theta_k = \arcsin(r_k) + 2\sum_{n=0}^{k-1}\arcsin(r_n).$$

It is easy to see that $D(a_k e^{i\theta_k}, r_k) \cap D(a_{k+1}e^{i\theta_{k+1}}, r_{k+1}) = \emptyset$, because

$$\theta_{k+1} - \theta_k = \arcsin(r_{k+1}) + \arcsin(r_k).$$

Moreover, it is also obtained that

$$|\theta_k| \leq \frac{\pi}{2} r_k + \pi\sum_{n=0}^{k-1} r_n < \pi\sum_{n=0}^{k} r_n < \pi.$$

Finally, take $g_k(z) := \frac{\varepsilon_k^{1/p}}{(a_k - ze^{-i\theta_k})^{2/p}}$, $z \in \mathbb{D}$, where we use the main branch of the logarithm to define $w^{2/p}$. Since g_k is bounded in $\mathbb{D}$, it belongs to $RM(p,0)$. In addition, we have that $|g_k(z)| \leq \frac{\varepsilon_k^{1/p}}{r_k^{2/p}}$ if $z \notin D(a_k e^{i\theta_k}, r_k)$.

The functions we are looking for are $f_k = g_{k+N}$.

Since $\sum_{k=N}^{\infty}\left(\frac{\varepsilon_k}{r_k^2}\right)^{1/p} < \infty$, it is easy to see that, given a bounded sequence $\{\alpha_k\}$, the sequence $\{\sum_{n=0}^{k}\alpha_n\, g_{n+N}(z)\}_k$ converges uniformly on compacta of $\mathbb{D}$ to $T(\{\alpha_k\}) := \sum_{k=0}^{\infty}\alpha_k\, g_{k+N}$, so that $T(\{\alpha_k\})$ is holomorphic in $\mathbb{D}$.

Let us see that $g = T(\{\alpha_k\}) \in RM(p,\infty)$. By construction, every radius $L_\theta = \{te^{i\theta} : t \in [0,1)\}$ cuts at most one of the open discs $D(a_k e^{i\theta_k}, r_k)$. On the one hand, if $\theta \in [0,2\pi]$ is such that there is k_0 with $L_\theta \cap D(a_{k_0}e^{i\theta_{k_0}}, r_{k_0}) \neq \emptyset$. Then,

$$\begin{aligned}\left(\int_0^1 |g(re^{i\theta})|^p\, dr\right)^{1/p} &\leq C_p|\alpha_{k_0-N}|\left(\int_0^1 |g_{k_0}(re^{i\theta})|^p\, dr\right)^{1/p} + C_p\sum_{j=N}^{\infty}|\alpha_j|\left(\frac{\varepsilon_j}{r_j^2}\right)^{1/p}\\ &\leq C_p||\{\alpha_k\}||_{\ell^\infty}\left(\left(\int_0^1 |g_{k_0}(re^{i\theta_{k_0}})|^p\, dr\right)^{1/p}+1\right)\\ &\leq C_p||\{\alpha_k\}||_{\ell^\infty}\left(\left(\int_0^1 \frac{\varepsilon_{k_0}}{(a_{k_0}-r)^2}\, dr\right)^{1/p}+1\right)\\ &\leq C_p(1+2^{1/p})||\{\alpha_k\}||_{\ell^\infty}.\end{aligned}$$

On the other hand, if $\theta \in [0,2\pi]$ is such that $e^{i\theta} \notin D(a_k e^{i\theta_k}, r_k)$ for all k, then

$$\left(\int_0^1 |g(re^{i\theta})|^p\, dr\right)^{1/p} \leq ||\{\alpha_k\}||_{\ell^\infty}\sum_{k=N}^{\infty}\left(\frac{\varepsilon_k}{r_k^2}\right)^{1/p} \leq ||\{\alpha_k\}||_{\ell^\infty}.$$

That is, $g = T(\{\alpha_k\}) \in RM(p,\infty)$ and, in particular, $T : \ell^\infty \to RM(p,\infty)$ is bounded.

Let us see that T is open from ℓ^∞ to $T(\ell^\infty)$ so that it establishes an isomorphism between ℓ^∞ and $T(\ell^\infty)$. For each n, using (1.5.6), it follows

$$\begin{aligned}\rho_{p,\infty}(T(\{\alpha_k\})) &\geq \left(\int_0^1 |T(\{\alpha_k\})(re^{i\theta_n})|^p\, dr\right)^{1/p} \geq \left(\int_{a_n-r_n}^1 |T(\{\alpha_k\})(re^{i\theta_n})|^p\, dr\right)^{1/p}\\ &\geq \frac{|\alpha_{n-N}|}{C_p} - \|\{\alpha_k\}\|_{\ell^\infty}\left(\sum_{j=N}^{\infty}\left(\frac{\varepsilon_j}{r_j^2}\right)^{1/p}\right)(1-a_n+r_n)^{1/p}\\ &\geq \frac{|\alpha_{n-N}|}{C_p} - \|\{\alpha_k\}\|_{\ell^\infty}\left(\sum_{j=N}^{\infty}\left(\frac{\varepsilon_j}{r_j^2}\right)^{1/p}\right).\end{aligned}$$

Therefore, $\frac{|\alpha_{n-N}|}{C_p} \leq \|\{\alpha_k\}\|_{\ell^\infty}\left(\sum_{j=N}^{\infty}\left(\frac{\varepsilon_j}{r_j^2}\right)^{1/p}\right)+\rho_{p,\infty}(T(\{\alpha_k\}))$ and taking supremum in n we obtain

$$\rho_{p,\infty}(T(\{\alpha_k\})) \geq \left(\frac{1}{C_p} - \sum_{j=N}^{\infty}\left(\frac{\varepsilon_j}{r_j^2}\right)^{1/p}\right)\|\{\alpha_k\}\|_{\ell^\infty}.$$

Since $\sum_{j=N}^{\infty}\left(\frac{\varepsilon_j}{r_j^2}\right)^{1/p} < \frac{1}{C_p}$, we get that T establishes an isomorphism between ℓ^∞ and $T(\ell^\infty)$.

To end the proof, we show that $T(\{\alpha_k\}) \in RM(p,0)$ if and only if $\{\alpha_k\} \in c_0$.

Let $T(\{\alpha_k\}) \in RM(p,0)$. Then, for each n,

$$\sup_\theta \left(\int_{a_n - r_n}^1 |T(\{\alpha_k\})(re^{i\theta})|^p \, dr \right)^{1/p} \geq \left(\int_{a_n - r_n}^1 |T(\{\alpha_k\})(re^{i\theta_n})|^p \, dr \right)^{1/p}$$
$$\geq \frac{|\alpha_{n-N}|}{C_p} - \|\{\alpha_k\}\|_{\ell^\infty} \left(\sum_{j=N}^\infty \left(\frac{\varepsilon_j}{r_j^2} \right)^{1/p} \right) (1 - a_n + r_n)^{1/p}.$$

Since $1 - a_n + r_n \to 0$ and $T(\{\alpha_k\}) \in RM(p,0)$, it follows that $\{\alpha_k\} \in c_0$.

Conversely, let $\alpha = \{\alpha_k\}_k \in c_0$ and let us prove that $T(\alpha) \in RM(p,0)$. Since $g_k \in RM(p,0)$, then $\sum_{k=0}^n \alpha_k \, g_{k+N} \in RM(p,0)$ for all $n \in \mathbb{N}$. Moreover, $\sum_{k=0}^n \alpha_k \, g_{k+N} \to T(\alpha)$ because T is continuous and $(\alpha_N, \dots, \alpha_{n+N}, 0, 0, \dots) \to \alpha$ in ℓ^∞. Finally, $T(\alpha) \in RM(p,0)$ since $RM(p,0)$ is a closed subspace of $RM(p,\infty)$. $\square$

Chapter 2

Containment relationships

After introducing this biparametric family of spaces $RM(p,q)$ and studying their main properties in Chapter 1, it is natural to wonder about the containment relationships between them.

Therefore, this chapter is dedicated to provide a characterization of the inclusions between the $RM(p,q)$ spaces and later on to study the compactness of such inclusions.

2.1 Inclusions

In this section we give a characterization for the containment relationships between our spaces. To do this, we recall the notion of the Marcinkiewicz spaces $L^{p,\infty}$, also called the weak L^p spaces.

Definition 2.1.1. Let $0 < p < \infty$. We define the weak L^p space of measurable functions

$$L^{p,\infty} = \left\{ f : X \mapsto \mathbb{C} \text{ measurable} : \|f\|_{p,\infty} := \sup_{t>0} t\lambda_f^{1/p}(t) < \infty \right\}$$

where

$$\lambda_f(t) = \mu\left(\{x \in X \ : |f(x)| > t\}\right).$$

A simple change of variable allows us to use the function λ_f to estimate the norm of a function of L^p. Namely (see [36, Proposition 1.1.4, p. 4]), given a measurable function f,

$$\|f\|_p^p = p\int_0^\infty t^{p-1}\lambda_f(t)\, dt. \tag{2.1.1}$$

Obtaining an equality that we will use throughout the text. In particular, we get $L^p \subset L^{p,\infty}$ (see [36, Proposition 1.1.6, p. 5]). As usual, we will also define $L^{\infty,\infty}$ as L^∞. In order to facilitate the reading, we have included the following well-known result.

Lemma 2.1.2. *[36, Proposition 1.1.14, p. 8] Let* $f \in L^{p_0,\infty} \cap L^{p_1,\infty}$ *with* $p_0 \neq p_1$. *Then* $f \in L^p$ *for* $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$, $\lambda \in (0,1)$. *Moreover, there exists a constant* $C(p_0,p_1,\lambda) > 0$ *such that*

$$\|f\|_p \leq C(p_0,p_1,\lambda)\|f\|_{p_0,\infty}^{1-\lambda}\|f\|_{p_1,\infty}^{\lambda},$$

for $\lambda \in (0,1)$.

Proof. Without loss of generality, we consider $0 < p_0 < p_1 \leq +\infty$. Assume firstly that $p_1 < +\infty$. We know that

$$\lambda_f(t) \le \min\left\{\frac{\|f\|_{p_0,\infty}^{p_0}}{t^{p_0}}, \frac{\|f\|_{p_1,\infty}^{p_1}}{t^{p_1}}\right\} \quad \text{for all} \quad t > 0. \tag{2.1.2}$$

Set

$$\beta = \left(\frac{\|f\|_{p_1,\infty}^{p_1}}{\|f\|_{p_0,\infty}^{p_0}}\right)^{\frac{1}{p_1-p_0}}. \tag{2.1.3}$$

By (2.1.1), (2.1.2), and (2.1.3), we estimate the L^p norm of f:

$$\begin{aligned}\|f\|_p^p &= p\int_0^\infty t^{p-1}\lambda_f(t)\,dt \le p\int_0^\infty t^{p-1}\min\left\{\frac{\|f\|_{p_0,\infty}^{p_0}}{t^{p_0}}, \frac{\|f\|_{p_1,\infty}^{p_1}}{t^{p_1}}\right\}\,dt\\ &= p\int_0^\beta t^{p-p_0-1}\|f\|_{p_0,\infty}^{p_0}\,dt + p\int_\beta^{+\infty} t^{p-p_1-1}\|f\|_{p_0,\infty}^{p_1}\,dt.\end{aligned}$$

Since $p-p_0>0$ and $p-p_1<0$, the above integrals converge, so that

$$\begin{aligned}\|f\|_p^p &\le \frac{p}{p-p_0}\beta^{p-p_0}\|f\|_{p_0,\infty}^{p_0} + \frac{p}{p_1-p}\beta^{p-p_1}\|f\|_{p_1,\infty}^{p_1}\\ &= \left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)\left(\|f\|_{p_0,\infty}^{p_0}\right)^{\frac{p_1-p}{p_1-p_0}}\left(\|f\|_{p_1,\infty}^{p_1}\right)^{\frac{p-p_0}{p_1-p_0}}.\end{aligned}$$

Moreover, if $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$ for $\lambda \in (0,1)$, it can be written as follows:

$$\|f\|_p^p \le \left(\frac{\lambda p_0 + (1-\lambda)p_1}{\lambda(1-\lambda)(p_1-p_0)}\right)\|f\|_{p_0,\infty}^{(1-\lambda)p}\|f\|_{p_1,\infty}^{p\lambda}.$$

The case $p_1 = +\infty$ follows in a similar way. Since $\lambda_f(t) = 0$ for $t > \|f\|_\infty$, we only use the inequality

$$\lambda_f(t) \le \frac{\|f\|_{p_0,\infty}^{p_0}}{t^{p_0}}$$

for $t \le \|f\|_\infty$. So, we can estimate the L^p norm of f as follows:

$$\begin{aligned}\|f\|_p^p &= p\int_0^\infty t^{p-1}\lambda_f(t)\,dt \le p\int_0^{\|f\|_\infty} t^{p-p_0-1}\|f\|_{p_0,\infty}^{p_0}\,dt\\ &= \frac{p}{p-p_0}\|f\|_{p_0,\infty}^{p_0}\|f\|_\infty^{p-p_0} = \frac{1}{\lambda}\|f\|_{p_0,\infty}^{p_0}\|f\|_\infty^{p\lambda}\end{aligned}$$

where $\frac{1}{p} = \frac{1-\lambda}{p_0}$. □

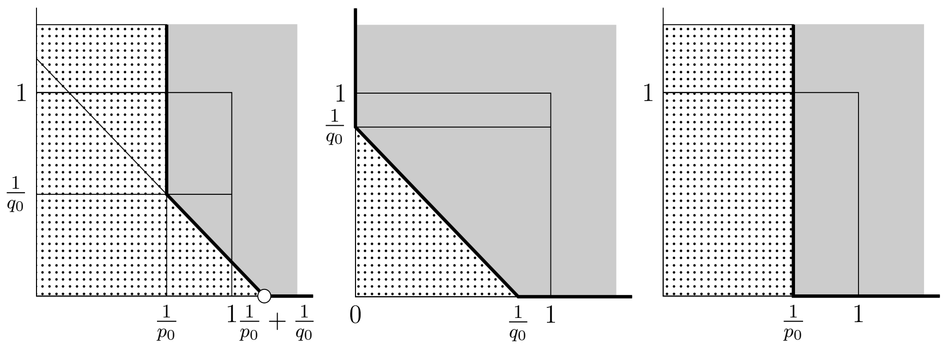
Now we are ready to state the main theorem.

Theorem 2.1.3. *Let $0 < p_0, q_0 \le \infty$ and set*

$$A(p_0,q_0) = \left\{(p,q) \in (0,+\infty] \times (0,+\infty] : \frac{1}{p}+\frac{1}{q} \ge \frac{1}{p_0}+\frac{1}{q_0}, \quad p_0 \ge p\right\}.$$

1. *If* $p_0, q_0 < +\infty$*, then* $RM(p_0,q_0) \subset RM(p,q)$ *if and only if* $(p,q) \in A(p_0,q_0) \setminus \{(\beta,\infty)\}$*, where* $\beta = \frac{p_0 q_0}{p_0+q_0}$*.*
2. *If either* p_0 *or* q_0 *are* $+\infty$*, then* $RM(p_0,q_0) \subset RM(p,q)$ *if and only if* $(p,q) \in A(p_0,q_0)$*.*

Before proving the result, it is worth showing a picture of the set $A(p_0,q_0)$. If $p_0, q_0 < +\infty$ then the set $\{(\frac{1}{p},\frac{1}{q}) : \ (p,q) \in A(p_0,q_0)\}$ is the grey region (including its boundary) in Figure 2.1a while if either p_0 or q_0 are $+\infty$, the set $\{(\frac{1}{p},\frac{1}{q}) : \ (p,q) \in A(p_0,q_0)\}$ is the grey region (including its boundary) in Figure 2.1b or Figure 2.1c, respectively.



(a) $p_0, q_0 < +\infty$ (b) $p_0 = +\infty$ (c) $q_0 = +\infty$

Figure 2.1

Proof. Bearing in mind that $H^{q_0} = RM(\infty, q_0)$, Proposition 1.5.1 is nothing but the case $p_0 = +\infty$. Therefore, from now on, we will assume that $p_0 < +\infty$. To clarify the exposition, we split the proof in several steps.

Step 1. If $p_0, q_0 < +\infty$ (Figure 2.1a) and (p,q) is such that $(1/p, 1/q)$ belongs to the open segment with end points $(1/p_0, 1/q_0)$ and $(\frac{1}{p_0}+\frac{1}{q_0}, 0)$, then $RM(p_0,q_0) \subset RM(p,q)$.

Write $(\frac{1}{p},\frac{1}{q}) = \lambda(\frac{1}{p_0},\frac{1}{q_0}) + (1-\lambda)(\frac{1}{p_0}+\frac{1}{q_0}, 0)$ for some $\lambda \in (0,1)$. Take $f \in RM(p_0,q_0)$ with $\rho_{p_0,q_0}(f) \leq 1$. For each θ, define $f_\theta(r) := f(re^{i\theta})$. Let us see that $f_\theta \in L^{p_0,\infty}([0,1]) \cap L^{\alpha,\infty}([0,1])$ for almost every $\theta \in [0,2\pi]$, where $\frac{1}{p_0}+\frac{1}{q_0} = \frac{1}{\alpha}$. Since $f \in RM(p_0,q_0)$, by the very definition, we have that $f_\theta \in L^{p_0}([0,1))$ for almost every θ and $\|f_\theta\|_{p_0,\infty} \leq \|f_\theta\|_{p_0}$. Moreover, by Proposition 1.5.2, there is a constant $C > 0$

such that $|f(z)| \leq C \frac{1}{(1-|z|^2)^{\frac{1}{p_0}+\frac{1}{q_0}}}$, for all z, and thus

$$\begin{aligned}
\|f_\theta\|_{\alpha,\infty} &= \sup_{t\geq 0} t\ m_1(\{r\in[0,1): |f_\theta(r)|>t\})^{1/\alpha} \\
&\leq \sup_{t\geq 0} t\ m_1\left(\left\{r\in[0,1): 1-r \leq \frac{C^\alpha}{t^\alpha}\right\}\right)^{1/\alpha} = \sup_{t\geq 0}\min\{t, C\} \leq C,
\end{aligned}$$

so that $f_\theta \in L^{\alpha,\infty}([0,1])$ for all θ. Hence, applying Lemma 2.1.2 we have

$$\|f_\theta\|_p \leq C(p_0,\alpha,\lambda)\cdot \|f_\theta\|_{p_0,\infty}^{\lambda}\cdot\|f_\theta\|_{\alpha,\infty}^{1-\lambda}.$$

Thus

$$\begin{aligned}
\rho_{p,q}(f) &\leq C(p_0,\alpha,\lambda)\left(\int_0^{2\pi}\|f_\theta\|_{p_0,\infty}^{\lambda q}\cdot\|f_\theta\|_{\alpha,\infty}^{(1-\lambda)q}\ d\theta\right)^{1/q} \\
&\leq C(p_0,\alpha,\lambda)\ C^{1-\lambda}\left(\int_0^{2\pi}\|f_\theta\|_{p_0}^{\lambda q}\ d\theta\right)^{1/q} = C(p_0,\alpha,\lambda)\ C^{1-\lambda}\left(\int_0^{2\pi}\|f_\theta\|_{p_0}^{q_0}\ d\theta\right)^{1/q} \\
&= C(p_0,\alpha,\lambda)\ C^{1-\lambda}\rho_{p_0,q_0}(f)^{\lambda} \leq C(p_0,\alpha,\lambda)\ C^{1-\lambda}.
\end{aligned}$$

Step 2. If $\frac{1}{p} > \frac{1}{p_0}+\frac{1}{q_0}$, then $RM(p_0,q_0)\subset RM(p,\infty)$.

Take $f\in RM(p_0,q_0)$. By Proposition 1.5.2, there is $C>0$ such that

$$\begin{aligned}
\rho_{p,\infty}(f) &= \operatorname*{ess\,sup}_{\theta\in[0,2\pi)}\left(\int_0^1 |f(re^{i\theta})|^p\ dr\right)^{1/p} \\
&\leq C \operatorname*{ess\,sup}_{\theta\in[0,2\pi)}\left(\int_0^1 \frac{\rho_{p_0,q_0}(f)^p}{(1-r)^{\left(\frac{1}{p_0}+\frac{1}{q_0}\right)p}}\ dr\right)^{1/p} < +\infty.
\end{aligned}$$

Step 3. If $p_0\geq p$ and $q_0\geq q$ then $RM(p_0,q_0)\subset RM(p,q)$.

This inclusion is a direct consequence of Hölder's inequality

$$\begin{aligned}
\rho_{p,q}(f) &= \left(\int_0^{2\pi}\left(\int_0^1 |f(re^{i\theta})|^p\ dr\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q} \\
&\leq \left(\int_0^{2\pi}\left(\int_0^1 |f(re^{i\theta})|^{p_0}\ dr\right)^{q/p_0}\frac{d\theta}{2\pi}\right)^{1/q} \leq \rho_{p_0,q_0}(f).
\end{aligned}$$

Denote by $B(p_0,q_0)=\{(p,q)\in\mathbb{R}^+\times\mathbb{R}^+ : p_0\geq p,\ q_0\geq q\}$ and (p_λ,q_λ) the couple such that $\left(\frac{1}{p_\lambda},\frac{1}{q_\lambda}\right)=\lambda\left(\frac{1}{p_0},\frac{1}{q_0}\right)+(1-\lambda)\left(\frac{1}{p_0}+\frac{1}{q_0},0\right)$. Since

$$A(p_0,q_0)\setminus\left\{\left(\frac{p_0q_0}{p_0+q_0},\infty\right)\right\} = \cup_{\lambda\in(0,1]}B(p_\lambda,q_\lambda)\cup\left\{(p,\infty): \frac{1}{p}>\frac{1}{p_0}+\frac{1}{q_0}\right\},$$

Steps 1, 2 and 3 give that if $p_0, q_0 < +\infty$ and $(p,q) \in A(p_0,q_0) \setminus \left\{\left(\frac{p_0q_0}{p_0+q_0}, \infty\right)\right\}$ then $RM(p_0,q_0) \subset RM(p,q)$. Clearly, Step 3 implies that if $(p,q) \in A(p_0,\infty)$ then $RM(p_0,\infty) \subset RM(p,q)$.

Step 4. If $RM(p_0,q_0) \subset RM(p,q)$ then $p_0 \geq p$.

By the closed graph theorem (see Theorem A.1.2 and Theorem A.1.3) there is a constant $C > 0$ such that $\rho_{p,q}(f) \leq C\rho_{p_0,q_0}(f)$ for all $f \in RM(p_0,q_0)$. Taking $f_n(z) = z^n$, a simple calculation shows that

$$\rho_{p,q}(f_n) = \left(\int_0^1 r^{np}\, dr\right)^{1/p} = (1+np)^{-\frac{1}{p}},$$

so that

$$(1+np)^{-\frac{1}{p}} = \rho_{p,q}(f_n) \leq C\rho_{p_0,q_0}(f_n) = C(1+np_0)^{-\frac{1}{p_0}}$$

and this inequality holds for all n if and only if $p_0 \geq p$.

Step 5. If $\frac{1}{p_0} + \frac{1}{q_0} > \frac{1}{p_1} + \frac{1}{q_1}$ then $RM(p_0,q_0) \not\subseteq RM(p_1,q_1)$

We consider a function f_α of Example 1.3.1 such that $\frac{1}{p_1} + \frac{1}{q_1} < \alpha < \frac{1}{p_0} + \frac{1}{q_0}$. Hence, we have a function f_α such that $f_\alpha \in RM(p_0,q_0) \setminus RM(p_1,q_1)$.

Step 6. If $p_0, q_0 < +\infty$, then $RM(p_0,q_0) \not\subseteq RM(\beta,\infty)$, where $\beta = \frac{p_0q_0}{p_0+q_0}$.

Assume that $RM(p_0,q_0) \subset RM(\beta,\infty)$. By the closed graph theorem (see Theorem A.1.2 and Theorem A.1.3) there is a positive constant $C > 0$ such that $\rho_{\beta,\infty}(f) \leq C\rho_{p_0,q_0}(f)$. For each n, consider the function $f_{n,\beta}$ introduced in Example 1.3.3. Then

$$\rho_{\beta,\infty}(f_{n,\beta}) \geq \left(\int_0^1 \sum_{k=0}^{n} r^k\, dr\right)^{\frac{1}{\beta}} = \left(\sum_{k=0}^{n} \frac{1}{k+1}\right)^{\frac{1}{\beta}} \geq \ln^{\frac{1}{\beta}}(n+1).$$

Thus, Example 1.3.3 would imply

$$\ln^{1/\beta}(n+1) \leq C\left(\frac{p_0}{p_0-\beta}\right)^{1/p_0} \ln^{1/q_0}(n+1),$$

what is not possible if n is large enough. So $RM(p_0,q_0) \not\subseteq RM(\beta,\infty)$.

Clearly, Steps 4, 5 and 6 imply that if $RM(p_0,q_0) \subset RM(p,q)$ then $(p,q) \in A(p_0,q_0) \setminus \{(\beta,\infty)\}$. In addition, Steps 4 and 5 give that if $RM(p_0,\infty) \subset RM(p,q)$ then $(p,q) \in A(p_0,\infty)$. Therefore, statements (1) and (2) are proved. □

A simple argument shows that if $q < +\infty$, the density of the polynomials in $RM(p,q)$ (see Proposition 1.5.12) implies the following result.

Proposition 2.1.4. *Let* $0 < p \leq +\infty$, $0 < p_0, q < +\infty$. *Then* $RM(p,q) \subset RM(p_0,\infty)$ *if and only if* $RM(p,q) \subset RM(p_0,0)$.

The analysis of when $RM(p_0,0)$ is contained in $RM(p,q)$ is not so immediate. To characterize it, we need the following lemma.

Lemma 2.1.5. *Let $0<p,q\le+\infty$. If $\{f_n\}$ is a bounded sequence in $RM(p,q)$ that converges uniformly on compact subsets of the unit disc to f. Then $f\in RM(p,q)$.*

Proof. Clearly the function f is holomorphic. Assume that $p,q<+\infty$. By Fatou's Lemma, for each θ we have

$$\int_0^1 |f(re^{i\theta})|^p\,dr \le g(\theta) := \liminf_n g_n(\theta),$$

where, for each n, $g_n(\theta) := \int_0^1 |f_n(re^{i\theta})|^p\,dr$. Repeating again the argument, we have

$$\begin{aligned}\rho_{p,q}(f)^q &\le \frac{1}{2\pi}\int_0^{2\pi} (g(\theta))^{q/p}\,d\theta \le \liminf_n \frac{1}{2\pi}\int_0^{2\pi} (g_n(\theta))^{q/p}\,d\theta \\ &= \liminf_n \rho^q_{p,q}(f_n) \le \sup_n \rho^q_{p,q}(f_n) < +\infty.\end{aligned}$$

Assume now that $p=+\infty$. Taking supremum, we have that

$$\sup_{r\in[0,1)} |f(re^{i\theta})|^q = \sup_{r\in[0,1)} \liminf_n |f_n(re^{i\theta})|^q \le \liminf_n \sup_{r\in[0,1)} |f_n(re^{i\theta})|^q.$$

By Fatou's lemma, it follows that

$$\begin{aligned}\rho^q_{\infty,q}(f) &= \int_0^{2\pi} \sup_{r\in[0,1)} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} \le \int_0^{2\pi} \liminf_n \sup_{r\in[0,1)} |f_n(re^{i\theta})|^q \frac{d\theta}{2\pi} \\ &\le \liminf_n \int_0^{2\pi} \sup_{r\in[0,1)} |f_n(re^{i\theta})|^q \frac{d\theta}{2\pi} \le \liminf_n \rho^q_{\infty,q}(f_n) \le \sup_n \rho^q_{p,q}(f_n) < +\infty.\end{aligned}$$

A similar argument works in the remaining cases, so that we are done. □

Proposition 2.1.6. *Let $0<p_0\le+\infty$. Then $RM(p_0,\infty)\subset RM(p,q)$ if and only if $RM(p_0,0)\subset RM(p,q)$.*

Proof. Assume that $RM(p_0,0)\subset RM(p,q)$. Take $f\in RM(p_0,\infty)$. For each $r<1$, the function f_r belongs to $RM(p_0,0)$ and then to $RM(p,q)$. Since $\{f_r : r<1\}$ is bounded in $RM(p_0,0)$ (see the proof of Proposition 1.5.15), it is also bounded in $RM(p,q)$. Since f_r converges uniformly on compact subset of $\mathbb{D}$ to f, Lemma 2.1.5 guarantees that $f\in RM(p,q)$. □

2.2 Compactness of the inclusions

Once the containment relationships of these spaces have been determined, we study when such inclusions are compact. For this research, we will only consider the case when $RM(p,q)$ is a Banach space, that is, when $1 \leq p, q \leq +\infty$.

A standard argument shows the following characterization of compactness. The idea behind the argument in the proof of the next lemma will appear later on in a more general setting. For the sake of clearness, we include here this weak version.

Lemma 2.2.1. *Let $1 \leq p_0, q_0 \leq +\infty$ and $1 \leq p, q \leq +\infty$. Then $i : RM(p_0, q_0) \to RM(p,q)$ is compact if and only if every bounded sequence $\{f_n\}$ in $RM(p_0, q_0)$ that converges to zero uniformly on compact subsets of the unit disc satisfies that* $\lim_n \rho_{p,q}(f_n) = 0$.

Proof. Suppose that $i : RM(p_0, q_0) \to RM(p,q)$ is a compact operator and assume that there is a constant $\gamma > 0$ and a bounded sequence $\{f_n\}$ in $RM(p_0, q_0)$ that converges to zero uniformly on compact subsets of the unit disc such that $\lim_n \rho_{p,q}(f_n) \geq \gamma > 0$. By the compactness of the inclusion, we have that there are a holomorphic function f on $\mathbb{D}$ and a subsequence $\{f_{n_k}\}$ such that $\rho_{p,q}(i(f_{n_k}) - f) = \rho_{p,q}(f_{n_k} - f) \to 0$ when $k \to \infty$. Since $\{f_{n_k}\}$ converges to zero uniformly on compact subsets of the unit disc, we have that $f = 0$ by Fatou's lemma. But, we obtain a contradiction because $\rho_{p,q}(f_{n_k}) \to 0$ when $k \to \infty$.

Conversely, assume that every bounded sequence $\{f_n\}$ in $RM(p_0, q_0)$ that converges to zero uniformly on compact subsets of $\mathbb{D}$ satisfies $\lim_n \rho_{p,q}(f_n) = 0$. By Corollary 1.5.3, $\{f_n\}$ is uniformly bounded on each compact subset of $\mathbb{D}$. Therefore, Montel's theorem gives us that there are a holomorphic function f on $\mathbb{D}$ and a subsequence $\{f_{n_k}\}$ of holomorphic functions that converges uniformly to f on compact sets. Moreover, it is easy to see that $\rho_{p_0,q_0}(f) \leq \sup_n \rho_{p_0,q_0}(f_n)$ by Fatou's lemma. Thus, the sequence $\{h_k\} := \left\{\frac{f_{n_k} - f}{2}\right\}$ is a bounded sequence in $RM(p_0, q_0)$ that converges to zero uniformly on compact subsets of $\mathbb{D}$. So that f_{n_k} converge to f in $RM(p,q)$. Hence, $i : RM(p_0, q_0) \to RM(p,q)$ is compact. □

We will use this lemma several times in the proof of the next theorem. We also need the following result.

Proposition 2.2.2. *Let $1 \leq p < +\infty$, $f \in RM(p, \infty)$, and $\sigma \in \partial\mathbb{D}$. Then for the non-tangential limit we have* $\angle \lim_{z \to \sigma} f(z)(1 - \overline{\sigma} z)^{1/p} = 0$.

Proof. Without loss of generality we assume that $\sigma = 1$. Suppose that $\rho_{p,\infty}(f) \leq 1$ and consider the holomorphic function $h(z) = f(z)(1-z)^{1/p}$. Fix $R > 1$ and the Stolz region $S_R(1) = \{z \in \mathbb{D} : |1 - z| < R(1 - |z|)\}$. Looking at (1.5.1) in the proof of Proposition 1.5.2, we see that there is a constant C such that

$$|h(z)| \leq R^{1/p} |f(z)| (1 - |z|)^{1/p} \leq C R^{1/p}, \quad z \in S_R(1).$$

That is, the function h is bounded on $S_R(1)$. Therefore, by Lindelöf's theorem (see Appendix A) , it is enough to prove that $\lim_{r\to 1^-} |f(r)|(1-r)^{1/p} = 0$.

Assume by contradiction that there is a constant $c_1 > 0$ and a sequence $\{r_k\}$ where $r_k \to 1^-$ such that $c_1 \leq |f(r_k)|(1-|r_k|)^{1/p}$ for all k. Write $\delta_k := 1 - r_k$. By Proposition 1.5.6, there is a constant C such that $|f'(x)| \leq \frac{C}{(1-x)^{1+\frac{1}{p}}}$ for all $x \in (0,1)$. Choose $\varepsilon < \frac{c_1}{2C}$. Then, for $1-(1+\varepsilon)\delta_k < x < 1 - \delta_k$,

$$|f(x) - f(r_k)| \leq C|x - r_k| \frac{1}{(1-r_k)^{1+1/p}} \leq C\varepsilon\delta_k \frac{1}{\delta_k^{1+1/p}} = \frac{C\varepsilon}{\delta_k^{1/p}} < \frac{c_1}{2\delta_k^{1/p}}.$$

Thus

$$|f(x)| \geq |f(r_k)| - |f(x) - f(r_k)| \geq \frac{c_1}{(1-r_k)^{1/p}} - \frac{c_1}{2\delta_k^{1/p}} = \frac{c_1}{2\delta_k^{1/p}}$$

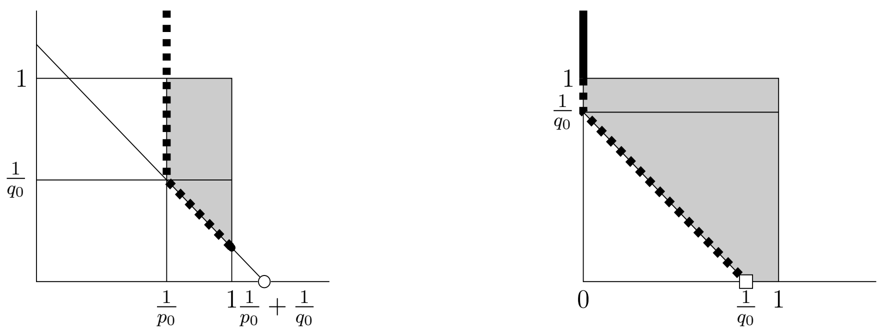
and

$$\left(\int_{1-(1+\varepsilon)\delta_k}^{1-\delta_k} |f(x)|^p \, dx \right)^{1/p} \geq (\varepsilon\delta_k)^{1/p} \frac{c_1}{2\delta_k^{1/p}} = \frac{c_1 \varepsilon^{1/p}}{2}.$$

Notice that this lower bound does not depend on k. But, this is impossible because $\int_0^1 |f(x)|^p \, dx < +\infty$ (see Remark 1.2.3). □

Theorem 2.2.3. *Let $1 \leq p_0, q_0 \leq +\infty$ and $1 \leq p, q \leq +\infty$. Then $i : RM(p_0, q_0) \to RM(p,q)$ is compact if and only if $\frac{1}{p} + \frac{1}{q} > \frac{1}{p_0} + \frac{1}{q_0}$ and $p < p_0$.*

As we can see in the Figure 2.2, the grey region, removing this time the dotted lines, represents the spaces $RM(p,q)$ such that $i : RM(p_0,q_0) \to RM(p,q)$ is compact when $p_0 < +\infty$ in Figure 2.2a and when $p_0 = +\infty$ in Figure 2.2b.



(a) $p_0 < +\infty$ (b) $p_0 = +\infty$

Figure 2.2

Proof. Bearing in mind Theorem 2.1.3, we have to prove that the inclusion is compact if $\frac{1}{p}+\frac{1}{q} > \frac{1}{p_0}+\frac{1}{q_0}$ and $p<p_0$ and it is not compact if either $\frac{1}{p}+\frac{1}{q} = \frac{1}{p_0}+\frac{1}{q_0}$ or $p=p_0$.

Let us start by showing that it is not compact if $p=p_0$. For each n, consider the function $f_n(z)=(np_0+1)^{1/p_0}z^n$, $z\in\mathbb{D}$. One can see that

$$\rho_{p_0,q_0}(f_n) = (np_0+1)^{1/p_0}\left(\int_0^1 r^{np_0}\,dr\right)^{1/p_0} = 1$$

and that the sequence $\{f_n\}$ converges uniformly to zero on compact subsets of the unit disc. Assume that $i: RM(p_0,q_0)\to RM(p_0,q)$ is compact, then there exists a subsequence $\{f_{n_k}\}$ such that $\rho_{p_0,q}(f_{n_k})$ must go to 0 as k goes to ∞. But this is not possible because $\rho_{p_0,q}(f_n)=1$ for all n.

Now take p and q such that $\frac{1}{p}+\frac{1}{q} = \frac{1}{p_0}+\frac{1}{q_0}$. Assume that $i: RM(p_0,q_0)\to RM(p,q)$ is compact. Then $i^*: (RM(p,q))^*\to (RM(p_0,q_0))^*$ is also a compact operator.

Assume firstly that $q<+\infty$. Let us see that $\frac{\delta_z}{\|\delta_z\|_{(RM(p,q))^*}}$ w^*-converges to 0 when $|z|\to 1$. Taking p a polynomial we obtain

$$\frac{|\delta_z(p)|}{\|\delta_z\|_{(RM(p,q))^*}} \asymp |p(z)|(1-|z|^2)^{\frac{1}{p}+\frac{1}{q}} \le \|p\|_\infty (1-|z|^2)^{\frac{1}{p}+\frac{1}{q}},$$

which clearly goes to 0 when $|z|\to 1$. Since $q<+\infty$, by Proposition 1.5.12, polynomials are dense in $RM(p,q)$ and then $\frac{\delta_z}{\|\delta_z\|_{(RM(p,q))^*}}$ w^*-converges to 0 when $|z|\to 1$. Therefore, the compactness of i^* gives

$$\lim_{|z|\to 1}\left\| i^*\left(\frac{\delta_z}{\|\delta_z\|_{(RM(p,q))^*}}\right)\right\|_{(RM(p_0,q_0))^*} = 0.$$

However, this is impossible because such norm must be greater than a certain positive constant. Indeed, since $i^*(\delta_z)=\delta_z$, by Proposition 1.5.2,

$$\left\| i^*\left(\frac{\delta_z}{\|\delta_z\|_{(RM(p,q))^*}}\right)\right\|_{(RM(p_0,q_0))^*} \ge \frac{\|\delta_z\|_{(RM(p_0,q_0))^*}}{\|\delta_z\|_{(RM(p,q))^*}} \asymp 1.$$

For the case $q=+\infty$, since the identity is not compact then we have the case $p_0=q=+\infty$. It follows in a similar way. Assume by contradiction that $i: RM(\infty,q_0)\to RM(p,\infty)$ is compact, then $i: RM(\infty,q_0)\to RM(p,0)$ is compact bearing in mind Proposition 2.1.4. But, using the previous argument one can show that the inclusion $i: RM(\infty,q_0)\to RM(p,0)$ is not compact.

Assume now that $\frac{1}{p_1} > \frac{1}{p_0}+\frac{1}{q_0}$ and take a sequence $\{f_n\}$ in $RM(p_0,q_0)$ such that $\rho_{p_0,q_0}(f_n)\le 1$ for all n and it converges to zero uniformly on compact subsets of $\mathbb{D}$.

We claim that $\lim_n \rho_{p_1,\infty}(f_n) = 0$. Otherwise, there is $\epsilon > 0$ and a subsequence (that we denote equal) such that $\rho_{p_1,\infty}(f_n) > \epsilon$ for all n. Thus, we find $\{\theta_n\}$ such that

$$\int_0^1 |f_n(re^{i\theta_n})|^{p_1} dr \geq \epsilon^{p_1}, \tag{2.2.1}$$

for all $n \in \mathbb{N}$. For each n, we write $g_n(r) := f_n(re^{i\theta_n})$, $r \in [0,1)$. Since $\rho_{p_0,q_0}(f_n) \leq 1$, by Proposition 1.5.2 there is a constant $C > 0$ such that

$$|g_n(r)| = |f_n(re^{i\theta_n})| \leq \frac{C}{(1-r^2)^{\frac{1}{p_0}+\frac{1}{q_0}}}.$$

Since the map $r \mapsto \frac{C}{(1-r^2)^{\frac{1}{p_0}+\frac{1}{q_0}}}$ belongs to $L^{p_1}([0,1])$ and $\{g_n\}$ converges pointwise to zero, we get that it converges to zero in the norm of $L^{p_1}([0,1])$ which contradicts (2.2.1). So that the claim holds.

Take now p, q such that there is $\lambda \in (0,1)$ with $\left(\frac{1}{p}, \frac{1}{q}\right) = \lambda\left(\frac{1}{p_0}, \frac{1}{q_0}\right) + (1 - \lambda)\left(\frac{1}{p_1}, 0\right)$. Then, for each $f \in RM(p,q)$, applying Hölder's inequality we have

$$\begin{aligned}\int_0^1 |f(re^{i\theta})|^p \, dr &= \int_0^1 |f(re^{i\theta})|^{\lambda p} |f(re^{i\theta})|^{(1-\lambda)p} \, dr \\ &\leq \left(\int_0^1 |f(re^{i\theta})|^{p_0} \, dr\right)^{\lambda p/p_0} \left(\int_0^1 |f(re^{i\theta})|^{p_1} \, dr\right)^{(1-\lambda)p/p_1}.\end{aligned}$$

So that

$$\rho_{p,q}(f) \leq \rho_{p_0,q_0}(f)^{\lambda} \rho_{p_1,\infty}(f)^{1-\lambda}.$$

This inequality, the above claim and Lemma 2.2.1 show that $i\colon RM(p_0,q_0) \to RM(p,q)$ is compact whenever $\frac{1}{p} + \frac{1}{q} > \frac{1}{p_0} + \frac{1}{q_0}$ and $q_0 < q$.

Take now p, q such that $\frac{1}{p} + \frac{1}{q} > \frac{1}{p_0} + \frac{1}{q_0}$, $p < p_0$ and $q_0 \geq q$. Fix $\tilde{q} < q$ such that $\frac{1}{p_0} + \frac{1}{\tilde{q}} < \frac{1}{p} + \frac{1}{q}$. By the above argument, the inclusion map $\tilde{i}$ from $RM(p_0, \tilde{q})$ into $RM(p,q)$ is compact. Since $i : RM(p_0,q_0) \to RM(p,q)$ factorizes through $\tilde{i}$, we get that i is compact. □

Remark 2.2.4. The argument of the case $\frac{1}{p} + \frac{1}{q} = \frac{1}{p_0} + \frac{1}{q_0}$ with $q < +\infty$ can be adapted for the case $p_0 = q = \infty$. However, we must consider a sequence $\{z_n\}$ in the Stolz region such that $|z_n| \to 1$. In this way, we obtain the w^*-convergence bearing in mind Proposition 2.2.2.

Remark 2.2.5. Bearing in mind Propositions 2.1.4 and 2.1.6, one can see that the inclusion $i : RM(p_0,q_0) \to RM(p,0)$ is compact if and only if $i : RM(p_0,q_0) \to RM(p,\infty)$ is compact and the inclusion $i : RM(p_0,0) \to RM(p,q)$ is compact if and only if $i : RM(p_0,\infty) \to RM(p,q)$ is compact, respectively.

Chapter 3

Bergman projection and duality

This chapter is dedicated to showing several results linked to the duality of the $RM(p,q)$ spaces. The main result of the chapter is the characterization of the boundedness of the Bergman projection in terms of p and q. We provide a characterization of the dual of $RM(p,q)$, for $1 < p,q < +\infty$, applying the boundedness of the Bergman projection. Then we differentiate between our spaces $RM(p,q)$ and the classical mixed norm spaces: our study of the duality allows us to cover some cases. Finally, we study the bidual space of $RM(p,0)$, for $1 < p < +\infty$, following the argument of K.-M. Perfekt for some related spaces in [54].

3.1 Duality

In the theory of Banach spaces of analytic functions, a useful integral operator is the Bergman projection

$$P(f)(z) := \int_{\mathbb{D}} K(z,w)f(w)\, dA(w), \quad z \in \mathbb{D},$$

with kernel

$$K(z,w) := (1 - z\overline{w})^{-2}, \quad z,w \in \mathbb{D}, \tag{3.1.1}$$

which is called the Bergman kernel. Such function is the reproducing kernel for the Bergman space A^2.

Proposition 3.1.1. *[28, p. 32; Theorem 5, p. 34]*

1. *The Bergman projection is well-defined on $L^1(\mathbb{D})$, mapping each function of $L^1(\mathbb{D})$ to an analytic function and mapping each function of the Bergman space A^1 into itself.*

2. *For $1 < p < \infty$, the Bergman projection P is a bounded operator from $L^p(\mathbb{D})$ onto A^p.*

These properties allow to describe the dual of Bergman spaces A^p. We give the proof of the following theorem to introduce the method that we will use in the study of the duality of $RM(p,q)$.

Theorem 3.1.2. *[28, Theorem 6, p. 35] For $1 < p < +\infty$, the dual space of A^p can be identified with $A^{p'}$, where p' is the conjugated index, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Namely, every functional $\phi \in (A^p)^*$ has a unique representation*

$$\phi(f) = \phi_g(f) = \int_{\mathbb{D}} f(w)\overline{g(w)}\, dA(w), \quad f \in A^p,$$

for some $g \in A^{p'}$. *Conversely, for every* $g \in A^{p'}$, *the map*

$$f \mapsto \int_{\mathbb{D}} f(w)\overline{g(w)}\, dA(w)$$

is an element of $(A^p)^*$. *Moreover,* $\|\phi_g\|_{(A^p)^*} \asymp \|g\|_{A^{p'}}$.

Proof. One part of the proof follows immediately. Indeed, applying the Hölder's inequality, one has that the functional defined by

$$\phi_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z), \quad f \in A^p,\ g \in A^{p'},$$

is bounded and $\|\phi_g\|_{(A^p)^*} \leq \|g\|_{p'}$. The map $g \mapsto \phi_g$ is antilinear and injective, since $\phi_g(z^n) = \frac{\overline{a_n}}{n+1}$, where a_n is the n-th Taylor coefficient of g.

Now, let ϕ be a functional in $(A^p)^*$. We have to show that there exists $g \in A^{p'}$ such that

$$\phi(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z), \quad \text{for every} \quad f \in A^p.$$

Using the Hahn-Banach theorem, this functional can be extended to a certain $\Phi \in (L^p(\mathbb{D}))^*$ such that $\|\phi\|_{(A^p)^*} = \|\Phi\|_{(L^p)^*}$. Now, by Riesz representation theorem there is a function $h \in L^{p'}$ such that

$$\Phi(f) = \int_{\mathbb{D}} f(z)\overline{h(z)}\, dA(z), \quad \text{for every} \quad f \in L^p$$

and $\|\Phi\|_{(A^p)^*} = \|h\|_{p'}$.

Let $g = Ph$, where P is the Bergman projection, and notice that, using Proposition 3.1.1(2), $g \in A^{p'}$. So, by Fubini's theorem and Proposition 3.1.1(1) we have, for $f \in A^p$,

$$\begin{aligned}
\phi(f) = \Phi(f) = \int_{\mathbb{D}} f(z)\overline{h(z)}\, dA(z) &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^2}\, dA(w)\overline{h(z)} \right) dA(z) \\
&= \int_{\mathbb{D}} f(w) \int_{\mathbb{D}} \frac{\overline{h(z)}}{(1-z\overline{w})^2}\, dA(z)\, dA(w) = \int_{\mathbb{D}} f(w)\overline{Ph(w)}\, dA(w) \\
&= \int_{\mathbb{D}} f(w)\overline{g(w)}\, dA(w) = \phi_g(f).
\end{aligned}$$

Also, we obtain that $\|g\|_p \leq C\|h\|_{p'} = C\|\phi_g\|_{(A^p)^*}$ by Proposition 3.1.1(2), so that $\|g\|_p \asymp \|\phi_g\|_{(A^p)^*}$. □

Replacing the Proposition 3.1.1(2) in above argument by the deeper related result in $RM(p,q)$ and mimicking this proof for Bergman spaces, in this section we describe the dual of $RM(p,q)$, where $1 < p, q < \infty$.

This related result is the boundedness of the Bergman projection from $Z_{p,q}$ or $Y_{p,q}$ (see the definition below) onto $RM(p,q)$, that we will prove in the next section.

Definition 3.1.3. Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be σ-finite measure spaces and $1 \leq p,q \leq +\infty$. We define the spaces $L^{(p,q)}(X\times Y)$ as those formed by (the class of) measurable functions in the product space $(X\times Y, \mathcal{A}\otimes\mathcal{B}, \mu\times\nu)$ such that $\|f\|_{L^{(p,q)}(X\times Y)} < +\infty$, where

$$\|f\|_{L^{(p,q)}(X\times Y)} = \left(\int_Y \left(\int_X |f(x,y)|^p \, d\mu(x)\right)^{q/p} d\nu(y)\right)^{1/q}, \quad \text{if } p,q<+\infty,$$
$$\|f\|_{L^{(p,\infty)}(X\times Y)} = \operatorname*{ess\,sup}_Y \left(\int_X |f(x,y)|^p \, d\mu(x)\right)^{1/p}, \quad \text{if } p<+\infty,$$
$$\|f\|_{L^{(\infty,q)}(X\times Y)} = \left(\int_Y \left(\operatorname*{ess\,sup}_X |f(x,y)|\right)^{q} d\nu(y)\right)^{1/q}, \quad \text{if } q<+\infty,$$
$$\|f\|_{L^{(\infty,\infty)}(X\times Y)} = \|f\|_{L^\infty(X\times Y)}.$$

Notation 3.1.4. With the aim of simplifying the notation to facilitate the reading, we denote by $Y_{p,q}$ the space formed by (the class of) measurable functions in the product space $\left([0,1)\times[0,2\pi), \mathcal{B}_1\otimes\mathcal{B}_2, dr\,\frac{d\theta}{2\pi}\right)$ such that

$$\|f\|_{Y_{p,q}} = \|f\|_{L^{(p,q)}([0,1)\times[0,2\pi))} < +\infty,$$

where $\mathcal{B}_1$ (and $\mathcal{B}_2$) is the Borel σ-algebra of $[0,1)$ (and $[0,2\pi)$ respectively). Analogously, we also define by $Z_{p,q}$ the space formed by (the class of) measurable functions in the product space $\left([0,1)\times[0,2\pi), \mathcal{B}_1\otimes\mathcal{B}_2, rdr\,\frac{d\theta}{\pi}\right)$ such that

$$\|f\|_{Z_{p,q}} = \|f\|_{L^{(p,q)}([0,1)\times[0,2\pi))} < +\infty.$$

Notice that the identification $(r,\theta)\mapsto re^{i\theta}$ allows us to consider $RM(p,q)$ as a subspace of $Y_{p,q}$. Recall that if $f\in RM(p,q)$, then $\rho_{p,q}(f) = \|f\|_{Y_{p,q}}$ and $\rho_{p,q}(f)\asymp \|f\|_{Z_{p,q}}$. So that we can also consider $RM(p,q)$ as a closed subspace of $Z_{p,q}$.

To study the duality of $RM(p,q)$ spaces, the following theorem will be important because it provides a characterization of the dual space of $L^{(p,q)}(X\times Y)$, for $1\leq p,q<+\infty$.

Theorem 3.1.5. *[13, Theorem 1, p. 303-304] Let $1\leq p,q\leq+\infty$.*

1. *If* $1 \leq p, q \leq +\infty$ *and* $f \in L^{(p,q)}(X \times Y)$*, then*

$$\begin{aligned}\|f\|_{L^{(p,q)}(X\times Y)} &= \sup_{g\in B_{L^{(p',q')}(X\times Y)}} \left|\int_{X\times Y} f(x,y)g(x,y)\ d\mu(x)\ d\nu(y)\right| \\ &= \sup_{g\in B_{L^{(p',q')}(X\times Y)}} \int_{X\times Y} |f(x,y)g(x,y)|\ d\mu(x)\ d\nu(y),\end{aligned}$$

where $B_{L^{(p',q')}(X\times Y)}$ *denotes the unit ball of* $L^{(p',q')}(X \times Y)$*. If* $1 \leq p, q < +\infty$ *and* $f \in L^{(p,q)}(X \times Y)$*, then there exists* $g \in B_{L^{(p',q')}(X\times Y)}$ *such that*

$$\begin{aligned}\|f\|_{L^{(p,q)}(X\times Y)} &= \int_{X\times Y} f(x,y)g(x,y)\ d\mu(x)\ d\nu(y) \\ &= \int_{X\times Y} |f(x,y)g(x,y)|\ d\mu(x)\ d\nu(y).\end{aligned}$$

2. *Let* $1 \leq p, q < +\infty$*.* ϕ *is a continuous functional on the normed space* $L^{(p,q)}(X \times Y)$ *if and only if it can be represented by*

$$\phi(f) = \int_{X\times Y} h(x,y)f(x,y)\ d\mu(x)\ d\nu(y)$$

where h *is a uniquely determined (class of) function(s) in* $L^{(p',q')}(X \times Y)$ *and also* $\|\phi\| = \|h\|_{L^{(p',q')}(X\times Y)}$*.*

In order to show clearly the proof of the duality of $RM(p, q)$ spaces, we will devote the Section 3.2 to study the boundedness of the Bergman projection.

Theorem 3.1.6 (See Theorem 3.2.2)**.** *Let* $1 < p, q < +\infty$*. The Bergman projection* P *is bounded from the space* $Y_{p,q}$ *(and* $Z_{p,q}$*) onto* $RM(p, q)$*.*

Now we can state and prove the main result of this section.

Theorem 3.1.7. *Let* $1 < p, q < \infty$*. Then* $(RM(p,q))^* \cong RM(p', q')$*, where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$*. The antiisomorphism between* $RM(p', q')$ *and* $(RM(p, q))^*$ *is given by the operator*

$$g \mapsto \lambda_g$$

where λ_g *is defined by*

$$\lambda_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}\ dA(z), \quad f \in RM(p,q).$$

Proof. Applying the Hölder's inequality, one has that the functional defined by

$$\lambda_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}\ dA(z), \quad f \in RM(p,q),\ g \in RM(p',q'),$$

is bounded and $\|\lambda_g\|_{(RM(p,q))^*} \leq 2\rho_{p',q'}(g)$. The map $g \mapsto \lambda_g$ is antilinear and injective, since $\lambda_g(z^n) = \frac{\overline{a_n}}{n+1}$, where a_n is the n-th Taylor coefficient of g.

Now, let λ be a functional in $(RM(p,q))^*$. We have to show that there exists $g \in RM(p',q')$ such that

$$\lambda(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z) \quad \text{for every} \quad f \in RM(p,q).$$

Using the Hahn-Banach theorem, this functional can be extended to a certain $\Lambda \in (Z_{p,q})^*$ such that $\|\lambda\|_{(RM(p,q))^*} = \|\Lambda\|_{(Z_{p,q})^*}$. Now, by means of Theorem 3.1.5(2) there is a function $h \in Z_{p',q'}$ such that

$$\Lambda(f) = \int_{\mathbb{D}} f(z)\overline{h(z)}\, dA(z) \quad \text{for every} \quad f \in Z_{p,q}$$

and $\|\Lambda\|_{(Z_{p,q})^*} = \|h\|_{Z_{p',q'}}$.

Let $g = Ph$, where P is the Bergman projection, and notice that, using Theorem 3.1.6, $g \in RM(p',q')$. So, by Fubini's theorem we have, for $f \in RM(p,q)$,

$$\begin{aligned}
\lambda(f) = \Lambda(f) &= \int_{\mathbb{D}} f(z)\overline{h(z)}\, dA(z) = \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^2}\, dA(w)\overline{h(z)} \right) dA(z) \\
&= \int_{\mathbb{D}} f(w) \int_{\mathbb{D}} \frac{\overline{h(z)}}{(1-z\overline{w})^2}\, dA(z)\, dA(w) = \int_{\mathbb{D}} f(w)\overline{Ph(w)}\, dA(w) \\
&= \int_{\mathbb{D}} f(w)\overline{g(w)}\, dA(w) = \lambda_g(f).
\end{aligned}$$

Also, we obtain that $\rho_{p,q}(g) \leq C\|h\|_{Z_{p',q'}} = C\|\lambda_g\|_{(RM(p,q))^*}$ by Theorem 3.1.6, so that $\rho_{p,q}(g) \asymp \|\lambda_g\|_{(RM(p,q))^*}$. □

3.2 The Bergman projection

Once the duality of the $RM(p,q)$ spaces has been studied in the previous section, we proceed with the proof of Theorem 3.1.6 that we had left pending for the sake of clearness.

Proposition 3.2.1. *Let $1 \leq p,q \leq +\infty$. The Bergman projection P from the space $Y_{p,q}$ onto $RM(p,q)$ is bounded if and only if the Bergman projection P from the space $Z_{p,q}$ onto $RM(p,q)$ is bounded.*

Proof. As we have seen the norms $\|\cdot\|_{Z_{p,q}}$ and $\|\cdot\|_{Y_{p,q}}$ are equivalent for analytic functions. For the proof of this proposition, we decompose the space $Y_{p,q}$ (or $Z_{p,q}$ respectively) as the sum of functions with support in $D(0,1/2)$ and functions with support in $\mathbb{D} \setminus D(0,1/2)$. Since in the first space the Bergman projection is bounded into $H^\infty \subset RM(p,q)$ and in the second space the norms of $Y_{p,q}$ and $Z_{p,q}$ are equivalent, we conclude the result. □

Theorem 3.2.2. *Let $1 < p, q < +\infty$. The Bergman projection P is bounded from the space $Y_{p,q}$ (and $Z_{p,q}$) onto $RM(p,q)$.*

Bearing in mind Proposition 3.1.1 and Proposition 3.2.1, since the restriction of P to $RM(p,q)$ is the identity (because $RM(p,q) \subset A^1$) and Pf is analytic for all $f \in Y_{p,q}$ (because $Y_{p,q} \subset L^1(\mathbb{D})$), in order to prove above theorem it is enough to show that P is bounded from $Y_{p,q}$ into itself.

Before going into the proof of this result, we introduce some necessary terminology. In general, given a measurable function $M : \mathbb{D} \times \mathbb{D} \mapsto \mathbb{C}$ we can define the integral operator

$$T_M(f)(re^{i\theta}) = \int_0^{2\pi}\int_0^1 M(re^{i\theta}, \rho e^{i\varphi})\ f(\rho e^{i\varphi})\ \frac{d\rho d\varphi}{2\pi}, \quad r \in [0,1),\ \theta \in [0, 2\pi],$$

whenever such integral exists. Observe that the Bergman projection P coincide with the case

$$M(re^{i\theta}, \rho e^{i\varphi}) = 2\rho K(re^{i\theta}, \rho e^{i\varphi}) = \frac{2\rho}{(1 - r\rho e^{i(\theta-\varphi)})^2}.$$

Notice that if M is a bounded functions, then T_M is bounded. Moreover, if $T_{|M|}$ is bounded then so is T_M.

From now on, with a little abuse of notation, $|\theta - \varphi|$ will denote the distance between θ and ϕ in the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, that is, $\min_{k\in\mathbb{Z}} |\theta - \varphi + 2k\pi|$. Notice also that in order to prove the boundedness of P, it is sufficient to check the boundedness of $T_{\tilde{K}}$, where

$$\tilde{K}(re^{i\theta}, \rho e^{i\varphi}) := |K(re^{i\theta}, \rho e^{i\varphi})|\chi_{\{|\theta-\varphi|\le 1\}} = \left|1 - re^{i\theta}\overline{\rho e^{i\varphi}}\right|^{-2} \chi_{\{|\theta-\varphi|\le 1\}},$$

because $|K| - \tilde{K} = |K|\chi_{\{|\theta-\varphi|>1\}}$ is a bounded function.

It will be very convenient for us to consider the product $[0, 2\pi)^2 \times [0, 1)^2$. Moreover, by showing that $T_D : Y_{p,q} \to Y_{p,q}$ is bounded, where

$$T_D(f)(\theta, r) = \int_0^{2\pi}\int_0^1 D(\theta, \varphi, r, \rho)\ f(\varphi, \rho)\ \frac{d\rho d\varphi}{2\pi}, \quad r \in [0, 1),\ \theta \in [0, 2\pi],$$

with

$$D(\theta, \varphi, r, \rho) = \begin{cases} 0, & \text{if } \ |\theta - \varphi| > 1, \\ \frac{1}{|\varphi-\theta|^2}, & \text{if } \ 1 \ge |\theta - \varphi| \ge 1 - r\rho, \\ \frac{1}{(1-r\rho)^2}, & \text{if } \ |\theta - \varphi| \le 1 - r\rho, \end{cases}$$

we obtain the boundedness of $P : Y_{p,q} \to Y_{p,q}$ since

$$\begin{aligned}|K(re^{i\theta},\rho e^{i\varphi})|\chi_{\{|\theta-\varphi|\le 1\}} &= \left((1-r\rho)^2 + 2r\rho(1-\cos(\theta-\varphi))\right)^{-1}\chi_{\{|\theta-\varphi|\le 1\}}\\ &\le 2\left(2(1-r\rho)^2 + r\rho|\theta-\varphi|^2\right)^{-1}\chi_{\{|\theta-\varphi|\le 1\}} \le 4D(\theta,\varphi,r,\rho).\end{aligned}$$

Bearing in mind the change of variables $x = 1-r$, $y = 1-\rho$, which preserves the measure, it follows that

$$\frac{\tilde{H}(\theta,\varphi,x,y)}{4} \le D(\theta,\varphi,1-x,1-y) \le \tilde{H}(\theta,\varphi,x,y), \quad x,y\in[0,1],$$

with

$$\tilde{H}(\theta,\varphi,x,y) = \begin{cases} 0, & \text{if } \ |\theta-\varphi| > 1,\\ \frac{1}{|\theta-\varphi|^2}, & \text{if } \ 1 \ge |\theta-\varphi| \ge \max\{x,y\},\\ \frac{1}{(\max\{x,y\})^2}, & \text{if } \ \max\{x,y\} \ge |\theta-\varphi|,\end{cases}$$

because $\max\{x,y\} \le 1-r\rho \le 2\max\{x,y\}$.

Finally, Lemma 3.2.4 shows that the boundedness of $T_{\tilde{H}} : Y_{p,q} \to Y_{p,q}$ is equivalent to the boundedness of $T_H : Y_{p,q} \to Y_{p,q}$, where

$$H(\theta,\varphi,x,y) = \begin{cases} 0, & \text{if } \ |\varphi-\theta| > 1 \quad \text{or} \quad \max\{x,y\} > |\varphi-\theta|,\\ \frac{1}{|\varphi-\theta|^2}, & \text{if } \ 1 \ge |\varphi-\theta| \ge \max\{x,y\}.\end{cases}$$

A preliminary lemma is needed.

Lemma 3.2.3. *Let $1 \le p,q < +\infty$ and $a,b \in (0,1]$. If we have the following relation $J(\theta,\varphi,x,y) = M(\theta,\varphi,ax,by)$ between the kernels J and M, then*

$$\|T_J\|_{Y_{p,q}} \le \frac{b^{1/p}}{ba^{1/p}}\|T_M\|_{Y_{p,q}}.$$

Proof. It is sufficient to prove the result for the two particular cases: $a=1$, $b<1$ and $a<1$, $b=1$.

Set $f \in Y_{p,q}$. If $b=1$, $a<1$, it is clear that $T_J f(\theta,x) = T_M f(\theta,ax)$. Hence, by a change of variable it follows

$$\begin{aligned}\|T_J f\|_{Y_{p,q}} &= \left(\int_0^{2\pi}\left(\int_0^1 |T_M f(\theta,ax)|^p \ dx\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q}\\ &= \left(\int_0^{2\pi}\left(\int_0^a |T_M f(\theta,x)|^p \ \frac{dx}{a}\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q}\\ &\le a^{-1/p}\|T_M\|\|f\|_{Y_{p,q}}.\end{aligned}$$

If $a = 1$, $b < 1$ and $f \in Y_{p,q}$, we consider the function

$$\tilde{f}(\varphi, y) = \begin{cases} f\left(\varphi, \frac{y}{b}\right), & \text{if} \quad y < b, \\ 0, & \text{if} \quad b \leq y \leq 1, \end{cases}$$

which satisfies that

$$\begin{aligned} \|\tilde{f}\|_{Y_{p,q}} &= \left(\int_0^{2\pi} \left(\int_0^b |f\left(\varphi, \frac{y}{b}\right)|^p \, dy \right)^{q/p} \frac{d\varphi}{2\pi} \right)^{1/q} \\ &= b^{1/p} \left(\int_0^{2\pi} \left(\int_0^1 |f\left(\varphi, y\right)|^p \, dy \right)^{q/p} \frac{d\varphi}{2\pi} \right)^{1/q} = b^{1/b} \|f\|_{Y_{p,q}}. \end{aligned}$$

Moreover, using also a change of variable it holds that

$$\begin{aligned} T_J f(\theta, x) &= \int_0^{2\pi} \int_0^1 M(\theta, \varphi, x, by) f(\varphi, y) \, dy \frac{d\varphi}{2\pi} \\ &= b^{-1} \int_0^{2\pi} \int_0^b M(\theta, \varphi, x, y) f\left(\varphi, \frac{y}{b}\right) \, dy \frac{d\varphi}{2\pi} = b^{-1} T_M \tilde{f}(\theta, x) \end{aligned}$$

for $(\theta, x) \in [0, 2\pi) \times [0, 1)$. Then

$$\|T_J f\|_{Y_{p,q}} = b^{-1} \|T_M \tilde{f}\|_{Y_{p,q}} \leq b^{-1} \|T_M\| \|\tilde{f}\|_{Y_{p,q}} = b^{\frac{1}{p}-1} \|T_M\| \|f\|_{Y_{p,q}}.$$

Combining the two cases we obtain the general case. □

Lemma 3.2.4. *Let $1 \leq p, q < +\infty$. Then $T_H : Y_{p,q} \to Y_{p,q}$ is bounded if and only if the operator $T_{\tilde{H}} : Y_{p,q} \to Y_{p,q}$ is bounded.*

Proof. Clearly, the boundedness of $T_{\tilde{H}}$ implies the boundedness of T_H because $0 \leq H \leq \tilde{H}$. Now, we show the converse implication. First of all, we define the dilated kernels $H_n(\theta, \varphi, x, y) := 2^{-2n} H(\theta, \varphi, 2^{-n}x, 2^{-n}y)$. Using Lemma 3.2.3 and denoting by $\|\cdot\|$ the operator norm from $Y_{p,q}$ into itself, we have

$$\|T_{H_n}\| \leq 2^{-n} \|T_H\|. \tag{3.2.1}$$

Recall

$$\tilde{H}(\theta, \varphi, x, y) = \begin{cases} 0, & \text{if} \quad |\theta - \varphi| > 1, \\ \frac{1}{|\theta - \varphi|^2}, & \text{if} \quad 1 \geq |\theta - \varphi| \geq \max\{x, y\}, \\ \frac{1}{(\max\{x,y\})^2}, & \text{if} \quad \max\{x, y\} \geq |\theta - \varphi|. \end{cases}$$

Let us prove that for $\theta \neq \varphi$

$$\tilde{H}(\theta, \varphi, x, y) \leq 3 \sum_{n=0}^{\infty} H_n(\theta, \varphi, x, y). \tag{3.2.2}$$

Clearly we can assume $1 \geq |\theta - \varphi| > 0$. We take

$$N = \min\{n \geq 0 \ : \ |\theta - \varphi| \geq \max\{2^{-n}x, 2^{-n}y\}\}.$$

If $N = 0$, then $1 \geq |\theta - \varphi| \geq \max\{x, y\}$. Moreover, one has that

$$\tilde{H}(\theta, \varphi, x, y) = \frac{1}{|\theta - \varphi|^2}$$

and

$$H_n(\theta, \varphi, x, y) = 2^{-2n} \frac{1}{|\theta - \varphi|^2}$$

for all $n \geq 0$. In particular, $\tilde{H}(\theta, \varphi, x, y) = H_0(\theta, \varphi, x, y)$ and we get (3.2.2) in case $N = 0$.

If $N \geq 1$, it is clear that $2^{N-1}|\theta - \varphi| < \max\{x, y\}$ and $|\theta - \varphi| \geq \max\{2^{-n}x, 2^{-n}y\}$ for all $n \geq N$. In particular

$$\tilde{H}(\theta, \varphi, x, y) = \frac{1}{(\max\{x, y\})^2}.$$

and

$$H_n(\theta, \varphi, x, y) = 2^{-2n} \frac{1}{|\theta - \varphi|^2}$$

for all $n \geq N$. Thus, it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(\theta, \varphi, x, y) &\geq \frac{1}{|\theta - \varphi|^2} \sum_{n=N}^{\infty} 2^{-2n} = \frac{4}{3} \frac{2^{-2N}}{|\theta - \varphi|^2} = \frac{1}{3} \frac{1}{(2^{N-1}|\theta - \varphi|)^2} \\ &> \frac{1}{3} \frac{1}{(\max\{x, y\})^2} = \frac{1}{3} H(\theta, \varphi, x, y). \end{aligned}$$

Notice that $\big\{(\theta_0, \varphi_0, x_0, y_0) \in [0, 2\pi]^2 \times (0, 1]^2 \ : \ \theta_0 = \varphi_0\big\}$ is set of measure zero. Therefore, by (3.2.2) and (3.2.1), we conclude

$$\|T_{\tilde{H}}\| \leq 3 \sum_{n=0}^{\infty} \|T_{H_n}\| \leq 3 \sum_{n=0}^{\infty} 2^{-n} \|T_H\| \leq 6 \|T_H\|,$$

and we are done. □

The following two results will be important in the proof of Theorem 3.2.2.

Lemma 3.2.5. *Let f, g be positive measurable functions on $[0,1)\times[0,2\pi)$. Then*

$$\int_0^{2\pi}\int_0^1 (T_H f)\ g\ dx\ d\theta \leq \int_0^{2\pi}\int_0^{2\pi} Rf(\theta,|\varphi-\theta|)Rg(\varphi,|\varphi-\theta|)\ d\theta\ d\varphi,$$

where $Rf(\theta,x)=\begin{cases}\sup_{1\geq t\geq x}\frac{1}{t}\int_0^t f(\theta,u)\ du, & \text{if } x<1,\\ 0, & \text{if } x\geq 1.\end{cases}$

Proof. Using the definition of the kernel H and grouping terms, it follows

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 T_H f(\theta,x)\ g(\theta,x)\ dx\ d\theta\\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^1\int_0^1 H(\theta,\varphi,x,y)f(\varphi,y)g(\theta,x)\ dx\ dy\ \frac{d\varphi}{2\pi}\ d\theta\\
&\leq \iiiint_{0\leq x,y\leq|\theta-\varphi|\leq 1}\frac{1}{|\theta-\varphi|^2}f(\varphi,y)g(\theta,x)\ dx\ dy\ d\varphi\ d\theta\\
&= \iint_{|\varphi-\theta|\leq 1}\left(\frac{1}{|\theta-\varphi|}\int_0^{|\theta-\varphi|} f(\varphi,y)\ dy\right)\left(\frac{1}{|\theta-\varphi|}\int_0^{|\theta-\varphi|} g(\theta,x)\ dx\right)\ d\theta\ d\varphi\\
&\leq \int_0^{2\pi}\int_0^{2\pi} Rf(\varphi,|\varphi-\theta|)Rg(\theta,|\varphi-\theta|)\ d\theta\ d\varphi.
\end{aligned}$$

□

Remark 3.2.6. Let $1<p<\infty$. Notice that if $0\leq x\leq x_1\leq 1$ then $Rf(\theta,x)\geq Rf(\theta,x_1)$. Moreover, for $e^{\theta i}\in\mathbb{T}$ fixed we define $f_\theta(x):=f(\theta,x)$. Therefore, since $Rf(\theta,x)\leq Mf_\theta(x)$, where M is the Hardy-Littlewood maximal function, there is a constant $C_p>0$ such that $\|Rf(\theta,\cdot)\|_{L^p[0,1]}\leq C_p\|f_\theta\|_{L^p[0,1]}$ (see [61, Theorem 8.18, p. 173]).

The last result we need to study the boundedness of the Bergman projection is the next theorem due to Fefferman and Stein.

Theorem 3.2.7. *[29, Theorem 1, p. 107] Let $1<p,q<+\infty$. There is a constant $C_{p,q}>0$ such that if $\{f_k\}_k$ is a sequence of measurable functions on $\mathbb{R}^n$, then*

$$\left\|\left(\sum_{k=0}^{\infty}(Mf_k(\cdot))^p\right)^{1/p}\right\|_{L^q(\mathbb{R}^n)}\leq C_{p,q}\left\|\left(\sum_{k=0}^{\infty}|f_k(\cdot)|^p\right)^{1/p}\right\|_{L^q(\mathbb{R}^n)}.$$

Proof of Theorem 3.2.2. Bearing in mind the notation of the previous lemma, for $f\in Y_{p,q}$ and $g\in Y_{p',q'}$ such that $\|f\|_{Y_{p,q}}\leq 1$ and $\|g\|_{Y_{p',q'}}\leq 1$ we consider the functions $F=R|f|$ and $G=R|g|$. Moreover, we define the following sequences of functions $f_k(\varphi)=F(\varphi,2^{-k})$ and $g_k(\varphi)=G(\varphi,2^{-k})$, $\varphi\in\mathbb{T}$ and $k\in\mathbb{N}\cup\{0\}$. Notice that for all $x \in I_k = [2^{-k}, 2^{-k+1})$ we have that $f_{k-1}(\varphi) \leq F(\varphi, x) \leq f_k(\varphi)$ and $g_{k-1}(\varphi) \leq G(\varphi, x) \leq g_k(\varphi)$. Thus, it follows

$$\sum_{k=1}^{\infty} f_{k-1}(\varphi)\chi_{I_k}(x) \leq F(\varphi, x) \leq \sum_{k=1}^{\infty} f_k(\varphi)\chi_{I_k}(x),$$
$$\sum_{k=1}^{\infty} g_{k-1}(\varphi)\chi_{I_k}(x) \leq G(\varphi, x) \leq \sum_{k=1}^{\infty} g_k(\varphi)\chi_{I_k}(x).$$

Using Remark 3.2.6 and these inequalities, we obtain

$$\int_0^{2\pi} \left(\sum_{k=1}^{\infty} f_{k-1}^p(\varphi)\, 2^{-k} \right)^{q/p} d\varphi \leq \int_0^{2\pi} \left(\int_0^1 |F(\varphi, x)|^p \, dx \right)^{q/p} d\varphi \leq 2\pi C_p^q$$

and therefore

$$\int_0^{2\pi} \left(\sum_{k=0}^{\infty} f_k^p(\varphi) 2^{-k} \right)^{q/p} d\varphi \leq 2^{1+\frac{q}{p}} \pi C_p^q. \tag{3.2.3}$$

Following the same argument, we obtain the inequality

$$\int_0^{2\pi} \left(\sum_{k=0}^{\infty} g_k^{p'}(\varphi) 2^{-k} \right)^{q'/p'} d\varphi \leq 2^{1+\frac{q'}{p'}} \pi C_{p'}^{q'}. \tag{3.2.4}$$

Hence, by Lemma 3.2.5 we have

$$\begin{aligned}
&\int_0^{2\pi}\!\!\int_0^1 (T_H f)\; g \; dx \; d\theta \leq \int_0^{2\pi}\!\!\int_0^{2\pi} F(\theta, |\varphi - \theta|) G(\varphi, |\varphi - \theta|) \; d\theta \; d\varphi \\
&\leq \int_0^{2\pi} \left(\sum_{k=1}^{\infty} f_k(\theta) \int_0^{2\pi} g_k(\varphi) \chi_{I_k}(|\theta - \varphi|) \; d\varphi \right) d\theta \\
&\leq \int_0^{2\pi} \sum_{k=1}^{\infty} f_k(\theta) \; 2^{2-k} \left(\frac{1}{2^{2-k}} \int_{\theta - 2^{-k+1}}^{\theta + 2^{-k+1}} g_k(\varphi) \; d\varphi \right) \; d\theta \\
&\leq \int_0^{2\pi} \sum_{k=1}^{\infty} f_k(\theta) \; 2^{2-k} M g_k(\theta) \; d\theta.
\end{aligned}$$

Applying Hölder's inequality it follows

$$\begin{aligned}&\int_0^{2\pi}\int_0^1 (T_H f(\theta,x))\ g(\theta,x)\ dx\ d\theta\\ &\leq 4\int_0^{2\pi}\left(\sum_{k=1}^{\infty} f_k^p(\theta)\ 2^{-k}\right)^{1/p}\left(\sum_{k=1}^{\infty}(Mg_k)^{p'}(\theta)\ 2^{-k}\right)^{1/p'} d\theta\\ &\leq 4\left(\int_0^{2\pi}\left(\sum_{k=1}^{\infty} f_k^p(\theta)\ 2^{-k}\right)^{q/p} d\theta\right)^{1/q}\left(\int_0^{2\pi}\left(\sum_{k=1}^{\infty}(Mg_k)^{p'}(\theta)\ 2^{-k}\right)^{q'/p'} d\theta\right)^{1/q'}.\end{aligned}$$

Hence, by Theorem 3.2.7 and the inequalities (3.2.3) and (3.2.4), we get

$$\begin{aligned}&\int_0^{2\pi}\int_0^1 (T_H f(\theta,x))\ g(\theta,x)\ dx\ d\theta\\ &\leq 2^{2+\frac{1}{p}+\frac{1}{q}}\pi^{1/q}\ C_p\ A_{p',q'}\left(\int_0^{2\pi}\left(\sum_{k=1}^{\infty} g_k^{p'}(\theta)\ 2^{-k}\right)^{q'/p'} d\theta\right)^{1/q'} \leq 16\pi\ C_p\ C_{p'}\ A_{p',q'}.\end{aligned}$$

Finally, we conclude the proof of the boundedness of the Bergman projection using Theorem 3.1.5(1). □

For the cases not covered by Theorem 3.1.6, we have

Theorem 3.2.8. *Let* $1 \leq p, q \leq +\infty$. *If* $\max\{p,q\} = +\infty$ *or* $\min\{p,q\} = 1$, *then the Bergman projection* P *does not send* $Y_{p,q}$ *(nor* $Z_{p,q}$*) into* $RM(p,q)$.

Before starting with the proof of the theorem, we state the following elementary lemma.

Lemma 3.2.9. *If* $z, w \in \Omega := \{re^{i\theta} : 0 < \theta < 1/2,\ 0 < r < 1-2\theta\}$, *then*

1. $|1-z| \asymp 1-|z|$,
2. $\left|\operatorname{Arg}\left(\frac{1-z}{1-w}\right)\right| \leq \arctan\left(\frac{1}{2}\right) < \frac{\pi}{4}$,
3. $\mathsf{Re}\left[\left(\frac{1-z}{1-w}\right)^2\right] \geq \frac{3}{5}\left|\frac{1-z}{1-w}\right|^2$.

Proof. The first identity follows immediately using the triangle inequality and the definition of the set Ω:

$$1-r \leq |1-re^{i\theta}| = \sqrt{(1-r)^2 + 2r(1-\cos(\theta))} \leq \sqrt{(1-r)^2+\theta^2} \leq \sqrt{\frac{5}{4}}(1-r).$$

To prove the second one it is enough to show that $\tan(\operatorname{Arg}(1-\overline{z})) \leq \frac{1}{2}$ for $z \in \Omega$, because we have that $\operatorname{Arg}(1-\overline{z}) \in (0, \arctan(1/2))$ and $\operatorname{Arg}(1-z) \in (-\arctan(1/2), 0)$.

Clearly, one can see, for $re^{i\theta} \in \Omega$, that

$$\tan(\text{Arg}(1-\overline{z})) = \frac{r\sin(\theta)}{1-r\cos(\theta)} \leq \frac{(1-2\theta)\sin(\theta)}{1-(1-2\theta)\cos(\theta)}.$$

To finish the proof of (2), we have to show that $\frac{(1-2\theta)\sin(\theta)}{1-(1-2\theta)\cos(\theta)} \leq \frac{1}{2}$. But this is clear because the function $f(\theta) = \frac{1}{2}(1-(1-2\theta)\cos(\theta)) - (1-2\theta)\sin(\theta)$ for $\theta \in \left[0, \frac{1}{2}\right)$ satisfies that $f(0) = 0$ and $f'(\theta) = 2\theta\cos(\theta) + \frac{1}{2}(5-2\theta)\sin(\theta) \geq 0$ for $\theta \in \left(0, \frac{1}{2}\right)$.

The last inequality follows from (2), since for $\left|\text{Arg}\left(\frac{1-z}{1-w}\right)\right| \leq \arctan\left(\frac{1}{2}\right) < \frac{\pi}{4}$ we have that

$$\frac{\text{Re}\left[\left(\frac{1-z}{1-w}\right)^2\right]}{\left|\frac{1-z}{1-w}\right|^2} = \cos\left(2\text{Arg}\left(\frac{1-z}{1-w}\right)\right) \geq \cos\left(2\arctan\left(\frac{1}{2}\right)\right)$$
$$= \frac{1-\tan^2(\arctan(1/2))}{1+\tan^2(\arctan(1/2))} = \frac{3}{5}.$$

□

Proof of Theorem 3.2.8. Bearing in mind Proposition 3.2.1 it is enough to show the result for $Y_{p,q}$ (or $Z_{p,q}$).

The case $p = +\infty$. Let us recall that the Bergman projection P is a bounded operator from $L^\infty(\mathbb{D})$ onto the Bloch space $\mathcal{B}$ (see [28, p. 47, Theorem 7] or [68, p. 102, Theorem 5.2]). Moreover, using lacunary sequences, it is possible to find functions in $\mathcal{B}$ whose Taylor coefficients do not go to zero (see [7, Lemma 2.1]). Therefore, $\mathcal{B} \nsubseteq H^q$, $1 \leq q \leq +\infty$. Thus, since $L^\infty(\mathbb{D}) \subset Y_{\infty,q}$, the Bergman projection P is not bounded from $Y_{\infty,q}$ to $RM(\infty, q) = H^q$.

The case $q = +\infty$**,** $1 \leq p < +\infty$. We show that there exists a function $f \in Y_{p,\infty}$ such that

$$|P(f)(a)| \gtrsim (1-a)^{-1/p}, \text{ for every } a \in \left(\frac{3}{4}, 1\right),$$

so that $P(f) \notin RM(p, \infty)$ (see Remark 1.2.3). To prove this, take the set

$$\Omega = \left\{re^{i\theta} \ : \ 0 < \theta < 1/2, 0 < r < 1-2\theta\right\}.$$

Given $\alpha \in \mathbb{R}$, consider the function

$$f(re^{i\theta}) := \begin{cases} 0, & re^{i\theta} \notin \Omega, \\ \theta^\alpha K(1-\theta, re^{-i\theta}), & re^{i\theta} \in \Omega, \end{cases}$$

where, as usual, K is the Bergman kernel. Taking $\alpha = 2 - \frac{1}{p} = 1 + \frac{1}{p'}$, we have $f \in Y_{p,\infty}$. Indeed, for $0 < \theta < 1/2$,

$$\begin{aligned} \int_0^1 |f(re^{i\theta})|^p \, dr &= \theta^{p\alpha} \int_0^{1-2\theta} |K(1-\theta, re^{-i\theta})|^p \, dr \le \theta^{p\alpha} \int_0^1 \frac{dr}{(1-(1-\theta)r)^{2p}} \\ &= \frac{\theta^{p\alpha}}{2p-1} \frac{\theta^{1-2p}-1}{1-\theta} \le \frac{2}{2p-1} \theta^{p\alpha+1-2p} = \frac{2}{2p-1} < +\infty. \end{aligned}$$

Now let us see that this function f satisfies that $|P(f)(a)| \gtrsim (1-a)^{-1/p}$ for every $a \in \left(\frac{3}{4}, 1\right)$. We have that the Bergman projection of the function f, for $a \in (0,1)$, is

$$\begin{aligned} P(f)(a) &= \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \frac{rdr}{(1-are^{-i\theta})^2 (1-(1-\theta)re^{i\theta})^2} \right) d\theta \\ &= \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \left(\frac{1-(1-\theta)re^{-i\theta}}{1-are^{-i\theta}} \right)^2 \frac{rdr}{|1-(1-\theta)re^{-i\theta}|^4} \right) d\theta. \end{aligned}$$

By Lemma 3.2.9 (applying first (3) and then (1)), we obtain

$$\begin{aligned} |P(f)(a)| &\ge \mathsf{Re}\,[P(f)(a)] \\ &\ge \frac{3}{5} \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \left| \frac{1-(1-\theta)re^{i\theta}}{1-are^{i\theta}} \right|^2 \frac{rdr}{|1-(1-\theta)re^{-i\theta}|^4} \right) d\theta \\ &\asymp \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \frac{rdr}{(1-ar)^2(1-(1-\theta)r)^2} \right) d\theta \\ &\ge \int_0^{1-a} \theta^\alpha \left(\int_0^{1-2\theta} \frac{rdr}{(1-ar)^4} \right) d\theta \ge \frac{1}{4} \int_0^{1-a} \theta^\alpha \left(\int_{1/4}^{1-2\theta} \frac{dr}{(1-ar)^4} \right) d\theta \\ &= \frac{1}{12a} \int_0^{1-a} \theta^\alpha \left((1-a(1-2\theta))^{-3} - (1-a/4)^{-3} \right) d\theta. \end{aligned}$$

Using that $\theta < 1-a$ and $3/4 \le a < 1$ we deduce $(1-a(1-2\theta))^{-3} - (1-a/4)^{-3} \ge (1-a(1-2\theta))^{-3}/2$ and $1-a(1-2\theta) < 3(1-a)$. Hence

$$\begin{aligned} &\frac{1}{12a} \int_0^{1-a} \theta^\alpha \left((1-a(1-2\theta))^{-3} - (1-a/4)^{-3} \right) d\theta \\ &\quad \ge \frac{1}{24a} \int_0^{1-a} \theta^\alpha (1-a(1-2\theta))^{-3} d\theta \ge \frac{1}{24a} \frac{1}{27} \frac{1}{(1-a)^3} \int_0^{1-a} \theta^\alpha \, d\theta \\ &\quad \asymp (1-a)^{\alpha-2} = (1-a)^{-1/p}. \end{aligned}$$

Thus, for $a > 3/4$, we have $|P(f)(a)| \gtrsim (1-a)^{-1/p}$ and the function $P(f)$ does not belong to $RM(p,\infty)$.

The remaining cases. For the remaining cases, we use the fact that if the Bergman projection $P : Z_{p',q'} \to RM(p',q')$ is bounded then $P : Z_{p,q} \to RM(p,q)$ is bounded since, for $f \in Z_{p,q}$,

$$\begin{aligned} \|P(f)\|_{Z_{p,q}} &= \sup_{g\in B_{Z_{p',q'}}} \left| \int_0^{2\pi}\int_0^1 P(f)(re^{i\theta})\,\overline{g(re^{i\theta})}\, rdr\frac{d\theta}{\pi}\right| \\ &= \sup_{g\in B_{Z_{p',q'}}} \left| \int_0^{2\pi}\int_0^1 f(re^{i\theta})\,\overline{P(g)(re^{i\theta})}\, rdr\frac{d\theta}{\pi}\right| \\ &\leq \|f\|_{Z_{p,q}} \sup_{g\in B_{Z_{p',q'}}} \|P(g)\|_{Z_{p',q'}} \leq C\|f\|_{Z_{p,q}}. \end{aligned}$$

where C is the norm of the operator $P : Z_{p',q'} \to RM(p',q')$ and, as usual, $B_{Z_{p',q'}}$ denotes the unit ball of $Z_{p',q'}$. □

3.3 Mixed norm spaces

In this section we present the containment relationship between the $RM(p,q)$ spaces and a particular case of the mixed norm spaces, which we will define below. In some cases we will use the duality result of the first section of this chapter.

Definition 3.3.1. Let $0 < p,q < +\infty$, $0 < \alpha < +\infty$. We define the mixed norm spaces

$$H^{q,p,\alpha} = \{f \in \mathcal{H}(\mathbb{D}) \ : \ \|f\|_{H^{q,p,\alpha}} < +\infty\}$$

where

$$\|f\|_{H^{q,p,\alpha}} = \left(\int_0^1 (1-r)^{p\alpha-1}\left(\int_0^{2\pi} |f(re^{i\theta})|^q\,\frac{d\theta}{2\pi}\right)^{p/q} dr\right)^{1/p}, \quad \text{if } p,q < +\infty.$$

Although these integral expressions appeared firstly in the Hardy and Littlewood's paper on properties of the integral mean [39], these spaces were defined explicitly by Flett in [30, 31]. Since then, these spaces have been studied by many authors (see [43]).

Notice that for $\alpha = 1/p$,

$$\|f\|_{H^{q,p,1/p}} = \left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q\,\frac{d\theta}{2\pi}\right)^{p/q} dr\right)^{1/p}$$

and, by Fubini's theorem, it is clear that

$$\|f\|_{H^{p,p,1/p}} = \rho_{p,p}(f).$$

Thus, it is a natural question to analyse what happens when $p \neq q$. This is the aim of this section.

Bearing in mind the standard argument of Theorem 3.1.7, one can prove the following duality result for the mixed norm spaces by means of the boundedness of the weighted Bergman projection:

Proposition 3.3.2. *[43, Corollary 7.3.4, p. 153] Let* $1 \leq p, q < +\infty$ *and* $0 < \alpha < \gamma + 1$*. Then the operator*

$$P_\gamma f(z) = (\gamma + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\gamma f(w)}{(1 - z\overline{w})^{2+\gamma}} \, dA(w)$$

is a bounded projection mapping $L^{q,p,\alpha}$ *onto* $H^{q,p,\alpha}$*, where* $L^{q,p,\alpha}$ *is the corresponding space of equivalence classes of measurable functions. In particular (*$\gamma = 0$*), we have the Bergman projection* P *maps* $L^{q,p,1/p}$ *onto* $H^{q,p,1/p}$*, when* $1 < p < +\infty$*.*

In order to study the duality of $H^{q,p,1/p}$, it will be useful to define the space $W_{p,q}$, that is, the space formed by the measurable functions in the product space $\left([0, 2\pi) \times [0, 1), \mathcal{B}_1 \otimes \mathcal{B}_2, \frac{d\theta}{\pi} \, rdr\right)$, where $\mathcal{B}_1$ (and $\mathcal{B}_2$) is the Borel σ-algebra of $[0, 2\pi)$ (and $[0, 1)$ respectively), such that $\|f\|_{W_{p,q}} = \|f\|_{L^{(q,p)}([0,2\pi)\times[0,1))} < +\infty$ (see Definition 3.1.3).

The proofs of the next two results follow the same scheme as for the spaces $RM(p, q)$.

Proposition 3.3.3. *Let* $1 < p, q < +\infty$*. The Bergman projection* P *from the space* $L^{q,p,1/p}$ *onto* $H^{q,p,1/p}$ *is bounded if and only if the Bergman projection* P *from the space* $W_{p,q}$ *onto* $H^{q,p,1/p}$ *is bounded.*

Proposition 3.3.4. *Let* $1 < p, q < +\infty$*. Then* $(H^{q,p,1/p})^* \cong H^{q',p',1/p'}$*, where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$*. The antiisomorphism between* $H^{q',p',1/p'}$ *and* $(H^{q,p,1/p})^*$ *is given by the operator*

$$g \mapsto \lambda_g$$

where λ_g *is defined by*

$$\lambda_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z), \quad f \in H^{q,p,1/p}.$$

The main result of this section shows the containment relationships between the $RM(p, q)$ spaces and the mixed norm spaces $H^{q,p,1/p}$ for $1 < p, q < +\infty$.

Theorem 3.3.5. *Let $1 < p, q < +\infty$.*

a) If $p > q$, then $RM(p,q) \subsetneq H^{q,p,1/p}$.

b) If $q > p$, then $H^{q,p,1/p} \subsetneq RM(p,q)$.

Proof. a) If $p \geq q$ then $RM(p,q) \subset H^{q,p,1/p}$, because using Minkowski's integral inequality we have

$$\left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi}\right)^{p/q} dr\right)^{1/p} \leq \left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr\right)^{q/p} \frac{d\theta}{2\pi}\right)^{1/q}.$$

Let us see that $RM(p,q) \subsetneq H^{q,p,1/p}$ if $p > q$.

Consider the functions

$$u_\delta(z) = \frac{\delta}{(1+\delta-z)^{1+\frac{1}{p}+\frac{1}{q}}}, \quad z \in \mathbb{D},$$

where $0 < \delta < 1/2$. One can see that $\|u_\delta\|_{H^{q,p,1/p}} \asymp \rho_{p,q}(u_\delta) \asymp 1$, because, for $\alpha \in \mathbb{D}$,

$$\|(1-\overline{\alpha}z)^{-1-\frac{1}{p}-\frac{1}{q}}\|_{H^{q,p,1/p}} \asymp (1-|\alpha|)^{-1}$$

(see [9, Proposition 2, p. 947]) and

$$\rho_{p,q}((1-\overline{\alpha}z)^{-1-\frac{1}{p}-\frac{1}{q}}) \asymp (1-|\alpha|)^{-1}$$

(see Proposition 1.3.2).

Let $\{\delta_n\}$ be a sequence of positive numbers such that $n^2\delta_n < \frac{1}{4}$ for all $n \geq 1$, $\delta_n/n^{2p} > (n+1)\delta_{n+1}^{1/2}$ for all $n \geq 1$, and $\sum_{j=1}^{\infty} j^2\delta_j < 2$. Define the sets

$$A_n := \left\{re^{i\theta} : |\theta - \theta_n| \leq n^2\delta_n \ , \ r \in [1 - n\delta_n^{1/2}, 1 - \delta_n/n^{2p}]\right\},$$

where $\theta_1 = \delta_1$ and $\theta_n - \theta_{n-1} = \frac{1}{n^2} + (n-1)^2\delta_{n-1} + n^2\delta_n$, for $n \geq 2$. Observe that $A_n = \{re^{i\theta} \ : \ r \in I_n, \ \theta \in J_n\}$ where the sets $\{I_n\}$ are pairwise disjoint and so are the sets $\{J_n\}$.

Let us check that $\rho_{p,q}(u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)) \lesssim \frac{1}{n^2}$. Firstly, notice that

$$\rho_{p,q}\left(u_{\delta_n}(ze^{-i\theta_n})\chi_{\{0<|w|<1-n\delta_n^{1/2}\}}(z)\right) \tag{3.3.1}$$
$$= \left(\int_0^{2\pi} \left(\int_0^{1-n\delta_n^{1/2}} \frac{\delta_n^p}{|1+\delta_n - re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \, dr\right)^{q/p} \frac{d\theta}{2\pi}\right)^{1/q}$$

$$\leq \left(\int_0^{1-n\delta_n^{1/2}} \frac{\delta_n^p}{(1+\delta_n - r)^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr \right)^{1/p} \leq \frac{\delta_n}{(n\delta_n^{1/2}+\delta_n)^{1+\frac{1}{q}}}$$
$$\leq \frac{\delta_n}{(n\delta_n^{1/2})^{1+\frac{1}{q}}} \leq \frac{1}{n^2}.$$

In the next inequalities we use that, for $|\theta| \leq 1$ and $r \geq 1 - n\delta_n^{1/2}$

$$|1+\delta_n - re^{i\theta}|^2 = (1+\delta_n - r)^2 + 2r(1+\delta_n)(1-\cos(\theta)) \geq (1+\delta_n-r)^2 + \frac{r\theta^2}{2}$$
$$\geq (1+\delta_n-r)^2 + (1-n\delta_n^{1/2})\frac{\theta^2}{2} \geq (1+\delta_n-r)^2 + \frac{\theta^2}{4}.$$

Thus, it follows

$$\rho_{p,q}^q \left(u_{\delta_n}(ze^{-i\theta_n}) \chi_{\{re^{i\theta}\,:\,1-n\delta_n^{1/2}<r<1,\ |\theta-\theta_n|>n^2\delta_n\}}(z) \right)$$
$$= 2\int_{n^2\delta_n}^{\pi} \left(\int_{1-n\delta_n^{1/2}}^1 \frac{\delta_n^p}{|1+\delta_n - re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr \right)^{q/p} \frac{d\theta}{2\pi}$$
$$\leq 2\left(\frac{\pi - n^2\delta_n}{1-n^2\delta_n}\right) \int_{n^2\delta_n}^1 \left(\int_{1-n\delta_n^{1/2}}^1 \frac{\delta_n^p}{|1+\delta_n - re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr \right)^{q/p} \frac{d\theta}{2\pi},$$

since the inner integral is a decreasing function in θ. Therefore,

$$\rho_{p,q}^q \left(u_{\delta_n}(ze^{-i\theta_n}) \chi_{\{re^{i\theta}\,:\,1-n\delta_n^{1/2}<r<1,\ |\theta-\theta_n|>n^2\delta_n\}}(z) \right) \tag{3.3.2}$$
$$\leq 2^{3q+1} \int_{n^2\delta_n}^{n^2\delta_n+n\delta_n^{1/2}} \left(\int_{1-n\delta_n^{1/2}}^{1+\delta_n-\theta} \frac{\delta_n^p}{(1+\delta_n-r)^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr \right.$$
$$\left. + \int_{1+\delta_n-\theta}^1 \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr \right)^{q/p} d\theta$$
$$+ 2^{3q+1} \int_{n^2\delta_n+n\delta_n^{1/2}}^1 \left(\int_{1-n\delta_n^{1/2}}^1 \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr \right)^{q/p} d\theta$$
$$\leq 2^{3q+1} \int_{n^2\delta_n}^{n^2\delta_n+n\delta_n^{1/2}} \left(\frac{1}{p\left(1+\frac{1}{q}\right)} \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{q}\right)}} + \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{q}\right)}} \right)^{q/p} d\theta$$
$$+ 2^{3q+1} \int_{n^2\delta_n+n\delta_n^{1/2}}^1 \frac{\delta_n^{q+\frac{q}{2p}} n^{q/p}}{\theta^{q\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, d\theta$$

$$\leq 2^{3q+1} 2^{q/p} \delta_n^q \int_{n^2\delta_n}^{n^2\delta_n + n\delta_n^{1/2}} \frac{1}{\theta^{q\left(1+\frac{1}{q}\right)}} + 2^{3q+1}\delta_n^q \int_{n^2\delta_n+n\delta_n^{1/2}}^{1} \frac{1}{\theta^{q\left(1+\frac{1}{q}\right)}}\, d\theta$$
$$\leq 2^{3q+1+\frac{q}{p}} \delta_n^q \int_{n^2\delta_n}^{1} \frac{1}{\theta^{q\left(1+\frac{1}{q}\right)}}\, d\theta \leq 2^{3q+1+\frac{q}{p}} \frac{\delta_n^q}{(n^2\delta_n)^q} \leq 2^{3q+2}\frac{1}{n^{2q}}.$$

Similarly, we obtain

$$\rho_{p,q}^q \left(u_{\delta_n}(ze^{-i\theta_n}) \chi_{\{re^{i\theta}\,:\,1-\frac{\delta_n}{n^{2p}}<r<1,\ |\theta-\theta_n|<n^2\delta_n\}}(z)\right) \tag{3.3.3}$$
$$= 2\int_0^{n^2\delta_n} \left(\int_{1-\frac{\delta_n}{n^{2p}}}^{1} \frac{\delta_n^p}{|1+\delta_n - re^{i\theta}|^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr\right)^{q/p} \frac{d\theta}{2\pi}$$
$$\leq 2\int_0^{\delta_n} \left(\int_{1-\frac{\delta_n}{n^{2p}}}^{1} \frac{\delta_n^p}{(1+\delta_n - r)^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr\right)^{q/p} \frac{d\theta}{2\pi}$$
$$+ 2^{3q+1}\int_{\delta_n}^{n^2\delta_n} \left(\int_{1-\frac{\delta_n}{n^{2p}}}^{1} \frac{\delta_n^p}{\theta^{p\left(1+\frac{1}{p}+\frac{1}{q}\right)}}\, dr\right)^{q/p} \frac{d\theta}{2\pi}$$
$$\leq 2\delta_n^q \left(\frac{\delta_n}{n^{2p}}\right)^{q/p} \int_0^{\delta_n} \frac{1}{\delta_n^{q\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \frac{d\theta}{2\pi} + 2^{3q+1}\delta_n^q \left(\frac{\delta_n}{n^{2p}}\right)^{q/p} \int_{\delta_n}^{n^2\delta_n} \frac{1}{\theta^{q\left(1+\frac{1}{p}+\frac{1}{q}\right)}} \frac{d\theta}{2\pi}$$
$$\leq \frac{8^{q+1}}{2\pi}\delta_n^q \left(\frac{\delta_n}{n^{2p}}\right)^{q/p} \frac{1}{\delta_n^{q\left(1+\frac{1}{p}\right)}} = \frac{8^{q+1}}{2\pi}\frac{1}{n^{2q}}.$$

Using inequalities (3.3.1), (3.3.2), and (3.3.3) we deduce

$$\|u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)\|_{H^{q,p,1/p}} \leq \rho_{p,q}(u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)) \lesssim \frac{1}{n^2}.$$

Now, by the very definition of A_n, the sets $\{A_n\}$ are pairwise disjoint. Define the functions $g_n(z) = u_{\delta_n}(ze^{-i\theta_n})\chi_{\mathbb{D}\setminus A_n}(z)$ and $f_n(z) = u_{\delta_n}(ze^{-i\theta_n})\chi_{A_n}(z)$ such that $u_{\delta_n}(ze^{-i\theta_n}) = f_n(z) + g_n(z)$. As we have seen, we have that $\rho_{p,q}(g_n) \lesssim \frac{1}{n^2}$ and $\|g_n\|_{H^{q,p,1/p}} \lesssim \frac{1}{n^2}$ for $n \in \mathbb{N}$. In addition, one can see that $\rho_{p,q}(f_n) \asymp \|f_n\|_{H^{q,p,1/p}} \asymp 1$ because $\rho_{p,q}(u_{\delta_n}(ze^{-i\theta_n})) \asymp \|u_{\delta_n}(ze^{-i\theta_n})\|_{H^{q,p,1/p}} \asymp 1$.

Given a measure space $(\Omega, \mathcal{A}, \mu)$. We have that for any sequence of measurable functions $h_n : \Omega \to \mathbb{C}$ whose supports are pairwise disjoint, it holds that

$$\int_\Omega \left|\sum_n h_n(w)\right|^s d\mu(w) = \sum_n \int_\Omega |h_n(w)|^s\, d\mu(w)$$

for all $s > 0$. Then, using this fact twice (one in each variable, first with $s = q$ and then with $s = p/q$), we obtain

$$\begin{aligned}\left\|\sum \alpha_n f_n\right\|_{H^{q,p,1/p}} &= \left(\int_0^1 \sum |\alpha_n|^p \left(\int_0^{2\pi} \left|f_n(re^{i\theta})\right|^q \frac{d\theta}{2\pi}\right)^{p/q} dr\right)^{1/p} \\ &= \left(\sum |\alpha_n|^p \|f_n\|^p_{H^{q,p,1/p}}\right)^{1/p} \asymp \left(\sum |\alpha_n|^p\right)^{1/p}.\end{aligned}$$

By the same reason,

$$\begin{aligned}\rho_{p,q}\left(\sum \alpha_n f_n\right) &= \left(\int_0^{2\pi} \sum |\alpha_n|^q \left(\int_0^1 \left|f_n(re^{i\theta})\right|^p dr\right)^{q/p} \frac{d\theta}{2\pi}\right)^{1/q} \\ &= \left(\sum |\alpha_n|^q \rho^q_{p,q}(f_n)\right)^{1/q} \asymp \left(\sum |\alpha_n|^q\right)^{1/q}.\end{aligned}$$

Hence, if we consider the function $F_m(z) := \sum_{n=1}^m u_{\delta_n}(ze^{-i\theta_n})$ we obtain that

$$\begin{aligned}\rho_{p,q}(F_m) &\leq \rho_{p,q}\left(\sum_{n=1}^m f_n\right) + \rho_{p,q}\left(\sum_{n=1}^m g_n\right) \lesssim m^{1/q} + \sum_{n=1}^m \frac{1}{n^2} \\ &\leq m^{1/q} + \frac{\pi^2}{6} \leq \left(1 + \frac{\pi^2}{6}\right) m^{1/q}\end{aligned}$$

and

$$\rho_{p,q}(F_m) \geq \rho_{p,q}\left(\sum_{n=1}^m f_n\right) - \rho_{p,q}\left(\sum_{n=1}^m g_n\right) \gtrsim m^{1/q}$$

for m big enough. So that $\rho_{p,q}(F_m) \asymp m^{1/q}$. In the same way for the norm in $H^{q,p,1/p}$, it follows that $\|F_m\|_{H^{q,p,1/p}} \asymp m^{1/p}$ using the same argument.

Therefore, if it were true that $RM(p,q) = H^{q,p,1/p}$, then we would have $\rho_{p,q}(F_m) \asymp \|F_m\|_{H^{q,p,1/p}}$. But if $p > q$ this is imposible, because this implies that $m^{1/p} \asymp m^{1/q}$ for all $m \in \mathbb{N}$. Thus, we conclude *a)*.

b) If $q \geq p$ then $H^{q,p,1/p} \subset RM(p,q)$, using Minkowski's integral inequality we have

$$\left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr\right)^{q/p} \frac{d\theta}{2\pi}\right)^{1/q} \leq \left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi}\right)^{p/q} dr\right)^{1/p}.$$

Let us see that $H^{q,p,1/p} \neq RM(p,q)$ for $q > p$. Assume that $H^{q,p,1/p} = RM(p,q)$ then $(H^{q,p,1/p})^* = (RM(p,q))^*$. By Theorem 3.1.7 and Proposition 3.3.4, we have that $H^{q',p',1/p'} = RM(p',q')$ for $p' > q'$. But this contradicts *a)*. So *b)* holds and we are done. □

3.4 On the bidual of $RM(p,0)$

In this section, we identify in a natural way the bidual of $RM(p,0)$ with $RM(p,\infty)$. It is clear that $RM(p,\infty)$ is a subspace of the Bergman space $A^p = RM(p,p)$ (see Theorem 2.1.3). Throughout this section, we denote by I the inclusion map from $RM(p,0)$ into A^p. We follow the scheme of the proof of K.-M. Perfekt in [54], but it is worth mentioning we can not use his results directly.

Lemma 3.4.1. *Let $1 < p < +\infty$. Then $I^*((A^p)^*)$ is dense (in the norm topology) in $(RM(p,0))^*$.*

Proof. To simplify the notation in this proof we just write $\|\cdot\|$ to denote $\|\cdot\|_{(RM(p,0))^*}$. Given $\lambda \in (RM(p,0))^*$, consider the following family of bounded linear functionals $\lambda_r(f) := \lambda(f_r)$, $0 < r < 1$ (see page 35 for the definition of f_r). We will prove that $\lambda_r \in (A^p)^*$ and $\lim_{r\to 1} \|I^*(\lambda_r) - \lambda\| = 0$. It is clear that $I^*(\lambda_r)$ acts as λ_r over $RM(p,0)$. So that, as customary, with a slight abuse of notation, we will write λ_r instead of $I^*(\lambda_r)$.

Given $f \in A^p$, using Proposition 1.5.2, we have

$$|\lambda_r(f)| = |\lambda(f_r)| \leq \|\lambda\|\, \rho_{p,\infty}(f_r) \leq \|\lambda\| \sup_{D(0,r)} |f(w)| \lesssim \frac{\|\lambda\|}{(1-r)^{2/p}} \rho_{p,p}(f).$$

Therefore, $\lambda_r \in (A^p)^*$.

Assume that $\lim_{r\to 1} \|I^*(\lambda_r) - \lambda\|$ is not 0. Then there exists $\epsilon > 0$, a sequence $\{r_n\}$ in $(0,1)$, with $r_n \to 1$, and a sequence $\{h_n\}$ in the unit ball of $RM(p,0)$ such that

$$\epsilon < |\lambda(h_n) - \lambda_{r_n}(h_n)| = |\lambda(h_n - (h_n)_{r_n})| \leq \|\lambda\|\, \rho_{p,\infty}(h_n - (h_n)_{r_n})$$

for all n. By Proposition 1.5.11(2), $\rho_{p,\infty}((h_n)_{r_n}) \leq 2$, thus writing $g_n := h_n - (h_n)_{r_n}$, we have a bounded sequence $\{g_n\}$ in $RM(p,0)$ that goes to zero uniformly on compacta of $\mathbb{D}$, and by Corollary 1.5.3, such that $\frac{\varepsilon}{\|\lambda\|} < \rho_{p,\infty}(g_n) < 3$ for all n.

Fix a sequence of positive numbers $\{\varepsilon_k\} \in \ell^{p'}$ and $\rho_1 \in (0,1)$. There exists n_1 such that $\sup\{|g_{n_1}(z)| : |z| \leq \rho_1\} < \varepsilon_1$. Since $g_{n_1} \in RM(p,0)$, we can choose $\rho_2 > \rho_1$ so that

$$\sup_\theta \left(\int_{\rho_2}^1 |g_{n_1}(re^{i\theta})|^p \, dr \right)^{1/p} < \varepsilon_1.$$

With the same argument, we obtain n_2 and $\rho_3 > \rho_2$ such that $\sup\{|g_{n_2}(z)| : |z| \leq$

$\rho_2\} < \varepsilon_2$ and

$$\sup_\theta \left(\int_{\rho_3}^1 |g_{n_2}(re^{i\theta})|^p \, dr \right)^{1/p} < \varepsilon_2.$$

By induction, we build a sequence $\{g_{n_k}\}$, an increasing sequence of numbers $\{\rho_k\}$ that converges to 1, such that $\sup\{|g_{n_k}(z)| : |z| \le \rho_k\} < \varepsilon_k$ and

$$\sup_\theta \left(\int_{\rho_{k+1}}^1 |g_{n_k}(re^{i\theta})|^p \, dr \right)^{1/p} < \varepsilon_k.$$

Given $\{\alpha_k\} \in \ell^p$, we have that $\sum_{k=0}^\infty \alpha_k g_{n_k} \in RM(p,0)$. Indeed, the sequence $\{h_n\}_n := \{\sum_{k=0}^n \alpha_k g_{n_k}\}_n$ is a Cauchy sequence, since $\{\alpha_k\} \in \ell^p$ and $\rho_{p,\infty}(h_n - h_m) \le \left(3 + 2\|\{\varepsilon_k\}\|_{\ell^{p'}}\right) \left(\sum_{k=n}^m |\alpha_k|^p\right)^{1/p}$ due to the fact that

$$\begin{aligned}
&\rho_{p,\infty}\left(\sum_{k=1}^\infty \alpha_k g_{n_k} \right) \\
&= \sup_\theta \left(\int_0^1 \left| \sum_{k=1}^\infty \alpha_k g_{n_k}(re^{i\theta}) \left(\chi_{[0,\rho_k)}(r) + \chi_{[\rho_k,\rho_{k+1})}(r) + \chi_{[\rho_{k+1},1)}(r) \right) \right|^p dr \right)^{1/p} \\
&\le \sup_\theta \left(\int_0^1 \left(\sum_{k=1}^\infty |\alpha_k| \varepsilon_k \right)^p dr \right)^{1/p} \\
&\quad + \sup_\theta \left(\int_0^1 \left(\sum_{k=1}^\infty |\alpha_k| |g_{n_k}(re^{i\theta})| \chi_{[\rho_k,\rho_{k+1})}(r) \right)^p dr \right)^{1/p} \\
&\quad + \sup_\theta \left(\int_0^1 \left(\sum_{k=1}^\infty |\alpha_k| |g_{n_k}(re^{i\theta})| \chi_{[\rho_{k+1},1)}(r) \right)^p dr \right)^{1/p} \\
&\le \|\{\alpha_k\}\|_{\ell^p} \|\{\varepsilon_k\}\|_{\ell^{p'}} + \sup_\theta \left(\sum_{k=1}^\infty \int_{\rho_k}^{\rho_{k+1}} |\alpha_k|^p |g_{n_k}(re^{i\theta})|^p \, dr \right)^{1/p} \\
&\quad + \sum_{k=1}^\infty |\alpha_k| \sup_\theta \left(\int_{\rho_{k+1}}^1 |g_{n_k}(re^{i\theta})|^p \, dr \right)^{1/p} \\
&\le \|\{\alpha_k\}\|_{\ell^p} \|\{\varepsilon_k\}\|_{\ell^{p'}} + 3\|\{\alpha_k\}\|_{\ell^p} + \|\{\alpha_k\}\|_{\ell^p} \|\{\varepsilon_k\}\|_{\ell^{p'}} \\
&= \left(3 + 2\|\{\varepsilon_k\}\|_{\ell^{p'}}\right) \|\{\alpha_k\}\|_{\ell^p}.
\end{aligned}$$

Therefore, there exists a bounded linear operator $T : \ell^p \to RM(p,0)$ such that $T(e_k) = g_{n_k}$, for all k.

Now, if we consider the composition of the operators λ and T it follows that $\lambda \circ T : \ell^p \to \mathbb{C}$ is a bounded linear functional, that is, $\lambda \circ T \in (\ell^p)^* \cong \ell^{p'}$. So, it satisfies that $(\lambda \circ T)(e_k) \to 0$, $k \to \infty$, but this is impossible because $|(\lambda \circ T)(e_k)| > \varepsilon$

for all $k \in \mathbb{N}$. Therefore, $I^*((A^p)^*)$ is dense in $(RM(p,0))^*$. □

Lemma 3.4.2. *For all $f \in RM(p,\infty)$ there exists a sequence $\{f_n\} \subset RM(p,0)$ such that $f_n \to f$ in A^p and $\limsup_n \rho_{p,\infty}(f_n) \le \rho_{p,\infty}(f)$.*

Proof. Fix $f \in RM(p,\infty)$ and for each $r \in (0,1)$ consider $f_r(z) := f(rz)$, $z \in \mathbb{D}$. The function f_r belongs to $RM(p,0)$ for all $r < 1$ (see Proposition 1.5.11(2)).

Since $RM(p,\infty) \subset A^p$, by Proposition 1.5.11, $\rho_{p,p}(f_r - f) \to 0$ when $r \to 1$. Moreover, since $\int_0^1 |f_r(se^{i\theta})|^p\, ds < \frac{1}{r}\int_0^1 |f(se^{i\theta})|^p\, ds$, for all $\theta \in [0,2\pi]$, we have that $\limsup_{r\to 1} \rho_{p,\infty}(f_r) \le \rho_{p,\infty}(f)$. □

Theorem 3.4.3. *Let $1 < p < +\infty$ and the inclusion $I : RM(p,0) \to A^p$. Then $I^{**} : (RM(p,0))^{**} \to A^p$ is a continuous and injective inclusion. Moreover*

(1) $I^{**}((RM(p,0))^{**}) = RM(p,\infty)$,

(2) $I^{**} : (RM(p,0))^{**} \to RM(p,\infty)$ is an isometry.

If $\{x_n\}$ is a bounded sequence in $(RM(p,0))^{**}$ that converges to 0 in the weak-* topology, then $\{I^{**}(x_n)\}$ converges to 0 uniformly on compact subsets of the unit disc.

Proof. Since the set of all polynomials is dense both in $RM(p,0)$ and A^p, then $RM(p,0)$ is dense in A^p. Therefore, it follows that $I^* : (A^p)^* \to (RM(p,0))^*$ is continuous and injective.

Since $I^*((A^p)^*)$ is dense in $(RM(p,0))^*$ (see Lemma 3.4.1), we obtain that

$$I^{**} : (RM(p,0))^{**} \to (A^p)^{**} \cong A^p$$

is continuous and injective. Moreover, it is easy to see that I^{**} acts as the identity on $RM(p,0)$.

Let us show that $RM(p,\infty) \subset I^{**}((RM(p,0))^{**})$. Given $\psi \in RM(p,\infty)$, by Lemma 3.4.2, there exists a sequence $\{\psi_n\} \subset RM(p,0)$ such that $\psi_n \to \psi$ in A^p and $\limsup_n \rho_{p,\infty}(\psi_n) \le \rho_{p,\infty}(\psi)$.

Assume that $\{x^*(\psi_n)\}$ is not a convergent sequence. Then it has two convergent subsequences to α and β respectively, where $\alpha \neq \beta$. We take $\varepsilon < \frac{|\alpha-\beta|}{2}$ and $x^*_\varepsilon \in I^*((A^p)^*)$ such that $\|x^* - x^*_\varepsilon\|_{(RM(p,0))^*} < \varepsilon$ (remember that $I^*((A^p)^*)$ is dense in $(RM(p,0))^*$). Since $\psi_n \to \psi$ in A^p, we know that $x^*_\varepsilon(\psi_n) \to x^*_\varepsilon(\psi)$. Therefore, it follows that $|\alpha - x^*_\varepsilon(\psi)| < \varepsilon$ and $|\beta - x^*_\varepsilon(\psi)| < \varepsilon$. By means of the triangle inequality, we obtain the following contradiction: $|\alpha-\beta| < 2\varepsilon < |\alpha-\beta|$. Thus, we can define $\hat{\psi} \in (RM(p,0))^{**}$ of the following form

$$\hat{\psi}(x^*) := \lim_{n\to\infty} x^*(\psi_n)$$

for $x^* \in (RM(p,0))^*$. It is clear that $\psi_n \to \hat{\psi}$ in $\sigma((RM(p,0))^{**}, (RM(p,0))^*)$. Hence, $I^{**}(\psi_n) \to I^{**}(\hat{\psi})$ in (A^p, w^*).

Moreover, as $\psi_n \to \psi$ on A^p we have that $\psi_n = I^{**}(\psi_n) \to \psi$ in (A^p, w). From this and the reflexivity of A^p, we conclude that $I^{**}(\hat{\psi}) = \psi$. Moreover,

$$\|\hat{\psi}\|_{(RM(p,0))^{**}} \le \limsup_n \rho_{p,\infty}(\psi_n) \le \rho_{p,\infty}(\psi). \tag{3.4.1}$$

We turn our attention to show the inclusion $I^{**}((RM(p,0))^{**}) \subset RM(p,\infty)$. Let $m \in (RM(p,0))^{**}$ and $\psi = I^{**}(m) \in A^p$. Using [24, Proposition 4.1, Chapter V] we have that the unit ball of $RM(p,0)$ is w^*-dense in the unit ball of $(RM(p,0))^{**}$. Moreover, the weak*-topology of $(RM(p,0))^{**}$ is metrizable in the unit ball, because $(RM(p,0))^*$ is separable due to Lemma 3.4.1 and the fact that $(A^p)^*$ is separable. We choose a sequence $\{\psi_n\} \subset RM(p,0)$ with $\sup_n \rho_{p,\infty}(\psi_n) \le \|m\|$ such that $\psi_n \xrightarrow{w^*} m$. Therefore, it follows that $x^*(\psi_n) \to x^*(\psi)$ for all $x^* \in (A^p)^*$. Bearing in mind that $\delta_z \in (A^p)^*$ for all $z \in \mathbb{D}$, we have that $\psi_n(z) \to \psi(z)$ for all $z \in \mathbb{D}$. Using Fatou's lemma we obtain that

$$\rho_{p,\infty}(\psi) \le \liminf_n \rho_{p,\infty}(\psi_n) \le \|m\|_{(RM(p,0))^*}. \tag{3.4.2}$$

From (3.4.1) and (3.4.2) we obtain (2).

Moreover, if $\{x_n\} \in (RM(p,0))^{**} \subset (A^p)^{**} = A^p$ that converges to 0 in the weak-* topology, then $\{I^{**}(x_n)\} \subset A^p$ converges pointwise to 0, and this implies that $\{I^{**}(x_n)\}$ converges to 0 uniformly on compact subsets of the unit disc. □

Chapter 4

Littlewood-Paley type inequalities

In this chapter we provide the Littlewood-Paley type inequalities associated to $RM(p,q)$ and their converses for certain cases. Beyond its intrinsic interest, such results will be an important tool in the study of the integration operator in Chapter 5. These inequalities will have a great importance in the proof of the fact that the belonging of g to the Bloch space is a sufficient condition for the boundedness of the integration operator T_g acting on $RM(p,q)$.

4.1 Littlewood-Paley type inequalities

An important result to prove the Littlewood-Paley type inequalities is the Hardy's inequality whose proof has been added for sake of completeness.

Lemma 4.1.1. *[27, Hardy's inequality, p. 234] Let $1 < p < +\infty$. If $f \in L^p((0,\infty))$, then the function*

$$F(x) = \frac{1}{x}\int_0^x f(t)\ dt, \qquad 0 < x < +\infty$$

belongs to $L^p((0,\infty))$. Moreover, it holds that $\|F\|_p \le \frac{p}{p-1}\|f\|_p$.

Proof. Fix $a > 0$. Since

$$F(x) = \frac{1}{a}\int_0^a f\left(\frac{xt}{a}\right)\ dt,$$

using Minkowski's integral inequality we have

$$\left(\int_0^a |F(x)|^p\ dx\right)^{1/p} \le \frac{1}{a}\int_0^a \left(\int_0^a \left|f\left(\frac{xt}{a}\right)\right|^p\ dx\right)^{1/p}\ dt.$$

By means of the change of variable $u = \frac{xt}{a}$, it follows that

$$\begin{aligned}\left(\int_0^a |F(x)|^p\ dx\right)^{1/p} &\le a^{\frac{1}{p}-1}\int_0^a t^{-1/p}\left(\int_0^a |f(u)|^p\ du\right)^{1/p}\ dt\\ &= \frac{p}{p-1}\left(\int_0^a |f(u)|^p\ du\right)^{1/p} \le \frac{p}{p-1}\|f\|_p.\end{aligned}$$

Now, taking $a \to +\infty$ we obtain the Hardy's inequality. □

Lemma 4.1.2. *Let $1 \le p < +\infty$. If $f \in \mathcal{C}^1([0,1)) \cap L^p([0,1])$, then it satisfies that*

$$\left(\int_0^1 |f(x)|^p\ dx\right)^{1/p} \le p\left(\int_0^1 |f'(x)|^p(1-x)^p\ dx\right)^{1/p} + |f(0)|.$$

Proof. Without lost of generality we can assume that $f(0) = 0$. To prove this inequality we need study first the case $p = 1$. We have that

$$\int_0^1 |f(x)|\ dx = \int_0^1 \left| \int_0^x f'(t)\ dt \right|\ dx \leq \int_0^1 \int_0^x |f'(t)|\ dt\ dx$$

Applying Fubini's theorem it follows that

$$\int_0^1 |f(x)|\ dx \leq \int_0^1 |f'(t)| \left(\int_t^1\ dx \right)\ dt = \int_0^1 |f'(t)|(1-t)\, dt.$$

To prove such inequality for $p > 1$, we show that the operator

$$g \mapsto Rg(x) = \int_0^x \frac{g(t)}{1-t}\ dt$$

is bounded in $L^p([0,1])$. Let $Ch(t) := \frac{1}{1-t} \int_t^1 h(x)\ dx$ for an integrable function h. For any $g \in L^p([0,1])$ and $h \in L^1([0,1])$, we have

$$\langle Rg, h \rangle = \int_0^1 \left(\int_0^x \frac{g(t)}{1-t}\ dt \right) h(x)\ dx = \int_0^1 g(t) \left(\frac{1}{1-t} \int_t^1 h(x)\ dx \right)\ dt = \langle g, Ch \rangle.$$

Using Lemma 4.1.1 with the function $h(1-t)\chi_{[0,1]}(t)$, $t \in (0, \infty)$, we have that

$$\left(\int_0^1 \left| \frac{1}{x} \int_0^x h(1-t)\, dt \right|^{p'} dx \right)^{1/p'} \leq \left(\int_0^\infty \left| \frac{1}{x} \int_0^x h(1-t)\chi_{[0,1]}(t)\, dt \right|^{p'} dx \right)^{1/p'}$$
$$\leq p\|h\|_{L^{p'}([0,1])}.$$

Doing a change of variable, we have that $\|Ch\|_{L^{p'}([0,1])} \leq p\|h\|_{L^{p'}([0,1])}$ and therefore, by duality, $||Rg||_{L^p([0,1])} \leq p||g||_{L^p([0,1])}$. Taking $g(x) = f'(x)(1-x)$, $x \in [0,1]$, we conclude

$$\|f\|_{L^p([0,1])} \leq p\|f'(x)(1-x)\|_{L^p([0,1])}.$$

□

We are ready to state the first result of this section.

Proposition 4.1.3. *Let $1 \leq p < +\infty$, $1 \leq q \leq +\infty$. If $f \in RM(p,q)$ then we have*

$$\rho_{p,q}(f) \leq p\ \rho_{p,q}(f'(z)(1-|z|)) + |f(0)|. \tag{4.1.1}$$

Proof. The result follows using Lemma 4.1.2 and taking the $L^q([0,2\pi])$-norm. □

Remark 4.1.4. Notice that Littlewood-Paley type inequalities do not hold for the cases where $p = \infty$, that is, for Hardy spaces. This can be seen through lacunary series. Assume that $\rho_{\infty,q}(f) \lesssim \rho_{\infty,q}(f'(z)(1-|z|)) + |f(0)|$ for any $f \in RM(\infty, q) = H^q$. Taking the lacunary series

$$f(z) = \sum_{k=0}^{\infty} z^{2^k}$$

we would have

$$\|f\|_{H^q} \asymp \rho_{\infty,q}(f) \lesssim \rho_{\infty,q}(f'(z)(1-|z|)) \leq \|f\|_{\mathcal{B}}.$$

But, this is impossible since f belongs to the Bloch space $\mathcal{B}$ but not to Hardy space H^q (see [27, p. 241]).

Next, we will show the version of the Littlewood-Paley type inequalities for the spaces $RM(p, 0)$. We need a preliminary result that will be used to show a sufficient condition on the functions of $RM(p, \infty)$ to belong to the subspace $RM(p, 0)$.

Lemma 4.1.5. *Let $1 \leq p < +\infty$. If $f \in \mathcal{H}(\mathbb{D})$ we have*

$$\left(\int_{\rho}^{1} |f(re^{i\theta})|^p \, dr\right)^{1/p} \leq p \left(\int_{\rho}^{1} |f'(re^{i\theta})|^p (1-r)^p \, dr\right)^{1/p} + (1-\rho)^{1/p} |f(\rho e^{i\theta})|$$

for all $\rho \in [0, 1)$ and all $\theta \in [0, 2\pi]$.

Proof. Fix $\rho \in [0, 1)$ and $\theta \in [0, 2\pi]$. Consider the function $g(r) = f((r+(1-r)\rho)e^{i\theta})$ and apply Lemma 4.1.2,

$$\left(\int_{0}^{1} |f((r+(1-r)\rho)e^{i\theta})|^p \, dr\right)^{1/p}$$
$$\leq p \left(\int_{0}^{1} |f'((r+(1-r)\rho)e^{i\theta})|^p (1-\rho)^p (1-r)^p \, dr\right)^{1/p} + |f(\rho e^{i\theta})|.$$

Using the change of variable $u = r + (1-r)\rho$ we obtain

$$\left(\int_{\rho}^{1} |f(ue^{i\theta})|^p \, \frac{du}{1-\rho}\right)^{1/p} \leq p \left(\int_{\rho}^{1} |f'(ue^{i\theta})|^p (1-u)^p \, \frac{du}{1-\rho}\right)^{1/p} + |f(\rho e^{i\theta})|.$$

Therefore, we have concluded the proof. □

Proposition 4.1.6. *Let $1 \leq p < +\infty$. If $f \in \mathcal{H}(\mathbb{D})$ and satisfies*

$$\lim_{\rho \to 1^-} \sup_{\theta \in [0, 2\pi]} \left(\int_{\rho}^{1} |f'(re^{i\theta})|^p (1-r)^p \, dr\right)^{1/p} = 0 \tag{4.1.2}$$

then $\lim_{\rho\to 1^-} \sup_\theta\ (1-\rho)^{1/p}|f(\rho e^{i\theta})| = 0$ *and* $f \in RM(p,0)$.

Proof. If $f \in \mathcal{H}(\mathbb{D})$ and satisfies (4.1.2), then we clearly obtain that

$$\sup_{\theta\in[0,2\pi]} \left(\int_0^1 |f'(re^{i\theta})|^p(1-r)^p\ dr\right)^{1/p} < +\infty.$$

Using Proposition 4.1.3 it follows that $\rho_{p,\infty}(f) < +\infty$, that is, $f \in RM(p,\infty)$. So, without loss of generality we assume that $f(0) = 0$ and $\rho_{p,\infty}(f) = 1$. We observe that

$$(1-\rho)^{1/p}|f(\rho e^{i\theta})| = (1-\rho)^{1/p}\left|\int_0^\rho f'(te^{i\theta})e^{i\theta}\ dt\right| \le (1-\rho)^{1/p}\int_0^\rho |f'(te^{i\theta})|\ dt. \tag{4.1.3}$$

Let us see that $\lim_{\rho\to 1^-} \sup_\theta (1-\rho)^{1/p}\int_0^\rho |f'(te^{i\theta})|\, dt = 0$. Fix $0 < \rho_1 < \rho$. Using that $|f'(z)| \le \frac{C}{(1-|z|)^{1+\frac{1}{p}}}$ for all z and for some constant $C = C(p) > 0$ (see Proposition 1.5.6) we obtain that

$$\begin{aligned}\int_0^\rho |f'(te^{i\theta})|(1-\rho)^{1/p}\ dt &\le C\frac{(1-\rho)^{1/p}}{(1-\rho_1)^{1/p}}\int_0^{\rho_1} \frac{(1-\rho_1)^{1/p}}{(1-t)^{1+\frac{1}{p}}}\ dt + \int_{\rho_1}^\rho |f'(te^{i\theta})|(1-\rho)^{1/p}\ dt\\ &\le Cp\frac{(1-\rho)^{1/p}}{(1-\rho_1)^{1/p}} + \int_{\rho_1}^\rho |f'(te^{i\theta})|(1-t)\frac{(1-\rho)^{1/p}}{(1-t)}\ dt.\end{aligned}$$

On the one hand, if $p = 1$

$$\begin{aligned}\int_0^\rho |f'(te^{i\theta})|(1-\rho)\ dt &\le C\frac{1-\rho}{1-\rho_1} + \int_{\rho_1}^\rho |f'(te^{i\theta})|(1-t)\ dt\\ &\le C\frac{1-\rho}{1-\rho_1} + \int_{\rho_1}^1 |f'(te^{i\theta})|(1-t)\ dt.\end{aligned}$$

On the other hand, using Hölder's inequality for $1 < p < +\infty$, we obtain

$$\begin{aligned}&\int_0^\rho |f'(te^{i\theta})|(1-\rho)^{1/p}\ dt\\ &\le Cp\frac{(1-\rho)^{1/p}}{(1-\rho_1)^{1/p}} + \left(\int_{\rho_1}^\rho |f'(te^{i\theta})|^p(1-t)^p\ dt\right)^{1/p}\left(\int_{\rho_1}^\rho \frac{(1-\rho)^{p'/p}}{(1-t)^{p'}}\ dt\right)^{1/p'}\\ &\le Cp\frac{(1-\rho)^{1/p}}{(1-\rho_1)^{1/p}} + \left(\int_{\rho_1}^1 |f'(te^{i\theta})|^p(1-t)^p\ dt\right)^{1/p}\frac{1}{(p'-1)^{1/p'}}.\end{aligned}$$

Now, taking supremum with respect to θ in the previous inequalities and considering $\rho_1 = 1 - \sqrt{1-\rho}$, we obtain that

$$\lim_{\rho\to 1^-} \sup_\theta \int_0^\rho |f'(te^{i\theta})|(1-\rho)^{1/p}\, dt = 0.$$

Therefore, bearing in mind (4.1.3) it follows that $\lim_{\rho\to 1^-} \sup_\theta (1-\rho)^{1/p}|f(\rho e^{i\theta})| = 0$ and by means of Lemma 4.1.5 we conclude that $f \in RM(p,0)$. □

4.2 Converse Littlewood-Paley type inequalities

Now, we tackle the converse Littlewood-Paley inequality for the cases $1 < p, q < +\infty$, $(1,q)$ with $1 \le q < +\infty$, and (∞, q) with $1 \le q \le +\infty$. However, this result will be extended for all $(p,1)$ with $1 < p < +\infty$ in Chapter 6. Firstly, we recall the Luecking regions and their expanded regions.

Definition 4.2.1. Given a non-negative integer n, set

$$\Gamma_n = \left\{ z \in \mathbb{D} \ : \ 1 - \frac{1}{2^n} \le |z| < 1 - \frac{1}{2^{n+1}} \right\}.$$

We define the *Luecking regions* $R_{n,j}$ as follows

$$R_{n,j} = \left\{ z \in \Gamma_n \ : \ \arg(z) \in \left[\frac{2\pi j}{2^n}, \frac{2\pi(j+1)}{2^n} \right) \right\}, \quad j = 0, 1, \ldots, 2^n - 1.$$

In the next picture we show some Luecking regions.

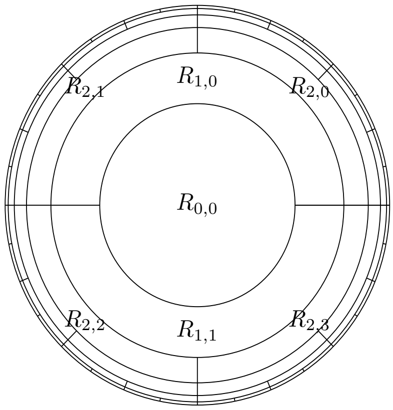


Figure 4.1

The *expanded Luecking region* $\tilde{R}_{n,j}$ is the union of $R_{n,j}$ and all Luecking regions contiguous with it. That is,

$$\tilde{R}_{n,j} = \bigcup_{\partial R_{n,j} \cap \partial R_{m,k} \neq \emptyset} R_{m,k}.$$

Now, we state some properties of these regions that will be useful for the estimation of the norm of certain maximal operators over these regions.

Lemma 4.2.2. *Let $z \in R_{n,j}$, then $D\left(z, \frac{1-|z|}{2}\right) \subset \tilde{R}_{n,j}$.*

Proof. Let $z \in R_{n,j}$. We will study the regions for $n \geq 1$ since this fact is easy to check for $\tilde{R}_{0,0} = D(0, 3/4)$. Indeed, for $0 \leq |z| < \frac{1}{2}$ we have that

$$0 \leq \left| z + \lambda \frac{1-|z|}{2} e^{i\theta} \right| \leq \frac{\lambda + (2-\lambda)|z|}{2} < \frac{3}{4}$$

for $0 \leq \lambda < 1$ and $\theta \in [0, 2\pi]$.

Let $R_{n,j}$ be a region as in Figure 4.1. Firstly, we see that for every $z \in R_{n,j}$, it holds that

$$D\left(z, \frac{1-|z|}{2}\right) \subset \left\{ \xi \in \mathbb{D} : 1 - \frac{1}{2^{n-1}} < |\xi| < 1 - \frac{1}{2^{n+2}} \right\}.$$

We have that every point of such disc is of the form $\xi = z + \lambda \frac{1-|z|}{2} e^{i\theta}$ with $\lambda \in [0,1)$ and $\theta \in [0, 2\pi]$. It follows that

$$|\xi| \leq |z| + \lambda \frac{1-|z|}{2} < |z| + \frac{1-|z|}{2} = \frac{1+|z|}{2} < \frac{1 + (1 - 2^{-(n+1)})}{2} = 1 - \frac{1}{2^{n+2}},$$
$$|\xi| \geq |z| - \lambda \frac{1-|z|}{2} \geq \frac{(2+\lambda)\left(1 - \frac{1}{2^n}\right) - \lambda}{2} = 1 - \frac{\lambda+2}{2^{n+1}} > 1 - \frac{3}{2^{n+1}} > 1 - \frac{1}{2^{n-1}}.$$

With this we see that the discs are contained in the union of the three *levels* $n-1$, n and $n+1$.

For $\frac{1}{2} \leq x < 1$, consider the disc $D\left(x, \frac{1-x}{2}\right)$. Take $w = u + iv$, the point of $\partial D\left(x, \frac{1-x}{2}\right)$ with the biggest argument. Such point satisfies that $(u,v) \perp (u-x, v)$.

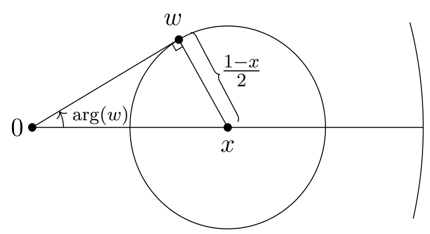


Figure 4.2

From Picture 4.2, one conclude that

$$\arg(w) = \arcsin\left(\frac{1-x}{2x}\right).$$

Consider the function $h(x) = \pi(1-x) - \arcsin\left(\frac{1-x}{2x}\right)$ for $x \in [1/2, 1)$. Then $h'(x) = \frac{1}{\sqrt{x^2(x+1)(3x-1)}} - \pi$ and $h''(x) = -\frac{x(6x^2+3x-1)}{(x^2(x+1)(3x-1))^{3/2}}$.

Since $h''(x) < 0$ for $x \in [1/2, 1)$, we have $h'(x) \le h'(1/2) = \frac{8\sqrt{3}-6\pi}{6} < 0$. Thus, h decreases in $[1/2, 1)$ so that $h(x) > \lim_{s\to 1^-} h(s) = 0$ for all $x \in [1/2, 1)$. This means that for any $z \in D\left(x, \frac{1-x}{2}\right)$, it holds that $\arg(z) \le \pi(1-|z|)$.

Now, take $z \in R_{n.j}$ and $w \in D\left(z, \frac{1-|z|}{2}\right)$. Then $|\arg(w) - \arg(z)| \le \pi(1-|z|)$. Therefore

$$\arg(w) \le \arg(z) + \pi(1-|z|) < \frac{2\pi(j+1)}{2^n} + \frac{2\pi}{2^{n+1}}$$

and

$$\arg(w) \ge \arg(z) - \pi(1-|z|) \ge \frac{2\pi j}{2^n} - \frac{2\pi}{2^{n+1}}.$$

Hence, $w \in \tilde{R}_{n,j}$. □

Lemma 4.2.3. *Denote by $NC(R_{n,j})$ the number of regions $R_{m,k}$ such that $\partial R_{n,j} \cap \partial R_{m,k} \neq \emptyset$. Then,*

- *$NC(R_{0,0}) = 3$;*
- *$NC(R_{1,j}) = 7$, $j = 0, 1$;*
- *If $n \ge 2$, then $NC(R_{n,j}) = 9$, $j = 0, \ldots, 2^n - 1$.*

In particular, $NC(R_{n,j}) \le 9$.

Proof. It is clear that $NC(R_{0,0}) = 3$ because $\tilde{R}_{0,0} = R_{0,0} \cup R_{1,0} \cup R_{1,1}$. The case $n = 1$, where $NC(R_{1,j}) = 7$ for $j \in \{0, 1\}$, follows from the fact that

$$\tilde{R}_{1,0} = \tilde{R}_{1,1} = R_{0,0} \cup \left(\bigcup_{j=0}^{1} R_{1,j}\right) \cup \left(\bigcup_{j=0}^{3} R_{2,j}\right), \quad \text{(see Figure 4.1).}$$

The remaining cases ($n \ge 2$) comes from

$$\tilde{R}_{n,0} = \left(R_{n-1,0} \cup R_{n-1,2^{n-1}-1}\right) \cup \left(R_{n,2^n-1} \cup \bigcup_{k=0}^{1} R_{n,k}\right)$$
$$\cup \left(R_{n+1,2^{n+1}-1} \cup \bigcup_{k=0}^{3} R_{n+1,k}\right),$$

$$\tilde{R}_{n,j} = \left(\bigcup_{k=\lfloor \frac{j-1}{2} \rfloor}^{\lfloor \frac{j+1}{2} \rfloor} R_{n-1,k} \right) \cup \left(\bigcup_{k=j-1}^{j+1} R_{n,k} \right)$$
$$\cup \left(\bigcup_{k=2j-1}^{2j+2} R_{n+1,k} \right) \quad \text{for} \quad j = 1, \dots, 2^n - 2,$$

$$\tilde{R}_{n,2^n-1} = \left(R_{n-1,0} \cup R_{n-1,2^{n-1}-1} \right) \cup \left(R_{n,0} \cup \bigcup_{k=2^n-2}^{2^n-1} R_{n,k} \right)$$
$$\cup \left(R_{n+1,0} \cup \bigcup_{k=2^{n+1}-3}^{2^{n+1}-1} R_{n+1,k} \right).$$

Hence, we have that $NC(R_{n,j}) = 9$ for $n \geq 2$ and $j = 0, \dots, 2^n - 1$. □

Lemma 4.2.4. *Let n be a non-negative integer. Then*

$$m_2(R_{n,j}) \asymp m_2(\tilde{R}_{n,j}) \asymp 4^{-n}$$

for $j = 0, 1, \dots, 2^n - 1$.

Proof. The case $n = 0$ is trivial, because

$$m_2(R_{0,0}) = m_2(D(0,1/2)) = \frac{\pi}{4} = \frac{4}{9} m_2(D(0,3/4)) = \frac{4}{9} m_2(\tilde{R}_{0,0}).$$

If $n \geq 1$, then it is easy to see that

$$m_2(R_{n,j}) = \frac{\pi}{2} \frac{1}{4^n} \left(2 - \frac{3}{2^{n+1}} \right) \asymp \frac{1}{4^n}$$

for $j = 0, \dots, 2^n - 1$. Using this estimate together with Lemma 4.2.3, it follows that $m_2(\tilde{R}_{n,j}) \asymp 4^{-n}$ for $j = 0, \dots, 2^n - 1$. □

From now on, given a measurable set $A \subset \mathbb{D}$ of positive measure we will denote the mean $\frac{1}{m_2(A)} \int_A f \, dm_2$ by $⨍_A f \, dm_2$ for $f \in L^1(A)$.

Definition 4.2.5. For every locally integrable function f on $\mathbb{D}$ we set:

1. $M_R f := \sum_{n,j} \left(⨍_{R_{n,j}} |f| \, dm_2 \right) \chi_{R_{n,j}}$;
2. $M_{\tilde{R}} f := \sum_{n,j} \left(⨍_{\tilde{R}_{n,j}} |f| \, dm_2 \right) \chi_{R_{n,j}}$;
3. $M_D f(z) := ⨍_{D\left(z, \frac{1-|z|}{2}\right)} |f| \, dm_2$, for all $z \in \mathbb{D}$.

Now, we estimate these sublinear operators in the following proposition.

Proposition 4.2.6. *Let f be a locally integrable function on $\mathbb{D}$.*

1. *If $1 \leq p \leq q < +\infty$, there is a constant $C(p,q) > 0$ such that $\rho_{p,q}(M_R f) \leq C(p,q)\rho_{p,q}(f)$.*

2. *If $1 \leq p \leq q < +\infty$ or $1 < q \leq p \leq +\infty$, there is a constant $C(p,q) > 0$ such that $\rho_{p,q}(M_{\tilde{R}} f) \leq C(p,q)\rho_{p,q}(f)$.*

3. *If $1 \leq p \leq q < +\infty$ or $1 < q \leq p \leq +\infty$, there is a constant $C(p,q) > 0$ such that $\rho_{p,q}(M_D f) \leq C(p,q)\rho_{p,q}(f)$.*

Proof. (1) Since $1 \leq p \leq q < \infty$ we have that $r = \frac{q}{p} \geq 1$. So

$$\begin{aligned}\rho^p_{p,q}(M_R f) &= \left(\int_0^{2\pi} \left(\int_0^1 |M_R f(re^{i\theta})|^p \, dr\right)^r \frac{d\theta}{2\pi}\right)^{1/r} \\ &= \sup_{\substack{\xi \in B_{L^{r'}(\mathbb{T})} \\ \xi \geq 0}} \int_0^{2\pi} \left(\int_0^1 |M_R f(re^{i\theta})|^p \, dr\right) \xi(e^{i\theta}) \frac{d\theta}{2\pi}.\end{aligned}$$

Fix $\xi \in B_{L^{r'}(\mathbb{T})}$ with $\xi \geq 0$. Bearing in mind that the regions $R_{n,j}$ are pairwise disjoint we have that

$$\int_0^{2\pi} \left(\int_0^1 |M_R f(re^{i\theta})|^p \, dr\right) \xi(e^{i\theta}) \frac{d\theta}{2\pi} \asymp \sum_{n,j} \int_{R_{n,j}} (M_R f(z))^p \, \xi\left(\frac{z}{|z|}\right) \, dm_2(z).$$

Now, using Jensen's inequality and the fact that $M_R f$ is constant in each region $R_{n,j}$, we obtain that

$$\begin{aligned}&\int_0^{2\pi} \left(\int_0^1 |M_R f(re^{i\theta})|^p \, dr\right) \xi(e^{i\theta}) \frac{d\theta}{2\pi} \\ &\lesssim \sum_{n,j} \left(\fint_{R_{n,j}} |f(z)|^p \, dm_2(z)\right) \int_{R_{n,j}} \xi\left(\frac{z}{|z|}\right) \, dm_2(z) \\ &\lesssim \frac{1}{4\pi} \sum_{n,j} \left(\fint_{R_{n,j}} |f(z)|^p \, dm_2(z)\right) m_1(I_{n,j}) \int_{I_{n,j}} \xi\left(e^{i\theta}\right) \, dm_1(\theta)\end{aligned}$$

where $I_{n,j} = \left\{e^{i\theta} : \, \theta \in \left[\frac{2\pi j}{2^n}, \frac{2\pi(j+1)}{2^n}\right)\right\}$. Taking into account Lemma 4.2.4, observe that $m_1(I_{n,j}) \asymp 2^{-n}$ and then $m_2(R_{n,j}) \asymp (m_1(I_{n,j}))^2$. Using the Hardy-Littlewood maximal operator $\mathcal{M}$ we have

$$\begin{aligned}\int_0^{2\pi}\left(\int_0^1 |M_R f(re^{i\theta})|^p\,dr\right)\xi(e^{i\theta})\frac{d\theta}{2\pi} &\lesssim \sum_{n,j}\int_{R_{n,j}}|f(z)|^p \inf_{e^{\theta i}\in I_{n,j}}\mathcal{M}\xi(e^{i\theta})\,dm_2(z)\\ &\le \sum_{n,j}\int_{R_{n,j}}|f(z)|^p\mathcal{M}\xi\left(\frac{z}{|z|}\right)\,dm_2(z) = \int_{\mathbb{D}}|f(z)|^p\mathcal{M}\xi\left(\frac{z}{|z|}\right)\,dm_2(z)\\ &\lesssim \int_0^{2\pi}\mathcal{M}\xi(e^{i\theta})\left(\int_0^1|f(re^{i\theta})|^p\,dr\right)\frac{d\theta}{2\pi}\le \|\mathcal{M}\xi\|_{L^{r'}(\mathbb{T})}\,\rho_{p,q}^p(f).\end{aligned}$$

Finally, since $\|\mathcal{M}\xi\|_{L^{r'}(\mathbb{T})}\le C_{r'}\|\xi\|_{L^{r'}(\mathbb{T})}$ we conclude the proof of statement (1).

(2) The proof of the case $1\le p\le q<+\infty$ follows the same argument we have used in statement (1), but using Lemma 4.2.3, Lemma 4.2.4 and the projection of the regions $\tilde{R}_{n,j}$, $n\in\mathbb{N}$, $j\in\{0,1,\dots,2^n-1\}$ over $\mathbb{T}$ instead of arcs $I_{n,j}$. Since $1\le p\le q<\infty$ we have that $r=\frac{q}{p}\ge 1$. So

$$\begin{aligned}\rho_{p,q}^p(M_{\tilde{R}}f) &= \left(\int_0^{2\pi}\left(\int_0^1|M_{\tilde{R}}f(re^{i\theta})|^p\,dr\right)^r\frac{d\theta}{2\pi}\right)^{1/r}\\ &= \sup_{\substack{\xi\in B_{L^{r'}(\mathbb{T})}\\ \xi\ge 0}}\int_0^{2\pi}\left(\int_0^1|M_{\tilde{R}}f(re^{i\theta})|^p\,dr\right)\xi(e^{i\theta})\frac{d\theta}{2\pi}.\end{aligned}$$

Fix $\xi\in B_{L^{r'}(\mathbb{T})}$ with $\xi\ge 0$. Bearing in mind that the regions $R_{n,j}$ are pairwise disjoint we have that

$$\int_0^{2\pi}\left(\int_0^1|M_{\tilde{R}}f(re^{i\theta})|^p\,dr\right)\xi(e^{i\theta})\frac{d\theta}{2\pi}\asymp\sum_{n,j}\int_{R_{n,j}}\left(M_{\tilde{R}}f(z)\right)^p\xi\left(\frac{z}{|z|}\right)\,dm_2(z).$$

Now, using Jensen's inequality and the fact that $M_{\tilde{R}}f$ is constant in each region $R_{n,j}$, we obtain that

$$\begin{aligned}&\int_0^{2\pi}\left(\int_0^1|M_{\tilde{R}}f(re^{i\theta})|^p\,dr\right)\xi(e^{i\theta})\frac{d\theta}{2\pi}\\ &\quad\lesssim\sum_{n,j}\left(\fint_{\tilde{R}_{n,j}}|f(z)|^p\,dm_2(z)\right)\int_{R_{n,j}}\xi\left(\frac{z}{|z|}\right)\,dm_2(z)\\ &\quad\lesssim\frac{1}{4\pi}\sum_{n,j}\left(\fint_{\tilde{R}_{n,j}}|f(z)|^p\,dm_2(z)\right)m_1(\tilde{I}_{n,j})\int_{\tilde{I}_{n,j}}\xi\left(e^{i\theta}\right)\,dm_1(\theta)\end{aligned}$$

where $\tilde{I}_{n,j}$ is the projection of the region $\tilde{R}_{n,j}$ over $\mathbb{T}$. Bearing in mind Lemma 4.2.4, observe that $m_1(\tilde{I}_{n,j})\asymp 2^{-n}$ and then $m_2(\tilde{R}_{n,j})\asymp(m_1(\tilde{I}_{n,j}))^2$. Using the Hardy-

Littlewood maximal operator $\mathcal{M}$ and Lemma 4.2.3, we have

$$\begin{aligned}\int_0^{2\pi}\left(\int_0^1 |M_{\tilde{R}}f(re^{i\theta})|^p\,dr\right)\xi(e^{i\theta})\frac{d\theta}{2\pi} &\lesssim \sum_{n,j}\int_{\tilde{R}_{n,j}}|f(z)|^p \inf_{e^{\theta i}\in\tilde{I}_{n,j}}\mathcal{M}\xi(e^{i\theta})\,dm_2(z)\\ &\le \sum_{n,j}\int_{\tilde{R}_{n,j}}|f(z)|^p\mathcal{M}\xi\left(\frac{z}{|z|}\right)\,dm_2(z) \le 9\int_{\mathbb{D}}|f(z)|^p\mathcal{M}\xi\left(\frac{z}{|z|}\right)\,dm_2(z)\\ &\lesssim \int_0^{2\pi}\mathcal{M}\xi(e^{i\theta})\left(\int_0^1 |f(re^{i\theta})|^p\,dr\right)\,\frac{d\theta}{2\pi} \le \|\mathcal{M}\xi\|_{L^{r'}(\mathbb{T})}\,\rho_{p,q}^p(f).\end{aligned}$$

Finally, since $\|\mathcal{M}\xi\|_{L^{r'}(\mathbb{T})} \le C_{r'}\|\xi\|_{L^{r'}(\mathbb{T})}$ we conclude that there is a constant $C(p,q)>0$ such that $\rho_{p,q}(M_{\tilde{R}}f)\le C(p,q)\,\rho_{p,q}(f)$ for $1\le p\le q<+\infty$. In fact, it can be shown that the linear operator

$$\tilde{M}f=\sum_{n,j}\left(⨍_{\tilde{R}_{n,j}} f(z)\,dm_2(z)\right)\chi_{R_{n,j}}$$

is bounded on the space of measurable functions on $\mathbb{D}$ where the $\rho_{p,q}$-norm is finite.

Now, we assume that $1<q\le p\le+\infty$. Let us describe the adjoint of the operator $\tilde{M}$. Given g, we have

$$\begin{aligned}\int_{\mathbb{D}}\left(\tilde{M}f(z)\right)g(z)\,dm_2(z) &= \sum_{n,j}\left(⨍_{\tilde{R}_{n,j}} f(w)\,dm_2(w)\right)\left(\int_{R_{n,j}} g(z)\,dm_2(z)\right)\\ &= \sum_{n,j}\left(\int_{\tilde{R}_{n,j}} f(w)\,dm_2(w)\right)\frac{m_2(R_{n,j})}{m_2(\tilde{R}_{n,j})}\left(⨍_{R_{n,j}} g(z)\,dm_2(z)\right)\\ &= \int_{\mathbb{D}} f(w)\left(\sum_{n,j}\beta_n\left(⨍_{R_{n,j}} g(z)\,dm_2(z)\right)\chi_{\tilde{R}_{n,j}}(w)\right)\,dm_2(w)\end{aligned}$$

where $\beta_n=\frac{m_2(R_{n,j})}{m_2(\tilde{R}_{n,j})}$ (notice that this quotient does not depend on j). Hence, the adjoint operator is

$$\tilde{M}^*f=\sum_{n,j}\beta_n\left(⨍_{R_{n,j}} f(z)\,dm_2(z)\right)\chi_{\tilde{R}_{n,j}}.$$

We know that there is a constant $C(p,q)>0$ such that $\rho_{p',q'}(\tilde{M}f)\le C(p,q)\rho_{p',q'}(f)$. Thus, $\rho_{p,q}(\tilde{M}^*f)\le C(p,q)\rho_{p,q}(f)$ (see Theorem 3.1.5). Moreover, we claim that $\tilde{M}f\asymp\tilde{M}^*f$ for positive functions. Therefore, it follows

$$\rho_{p,q}(M_{\tilde{R}}f)=\rho_{p,q}(\tilde{M}|f|)\asymp\rho_{p,q}(\tilde{M}^*|f|)\le C(p,q)\rho_{p,q}(f).$$

To proof the claim take a positive function f. Bearing in mind Lemma 4.2.4, we have

$$\begin{aligned}
\tilde{M}^* f &= \sum_{n,j} \beta_n \left(\fint_{R_{n,j}} f(z)\, dm_2(z) \right) \chi_{\tilde{R}_{n,j}} \\
&= \sum_{n,j} \beta_n \sum_{\partial R_{n,j} \cap \partial R_{m,k} \neq \emptyset} \left(\fint_{R_{n,j}} f(z)\, dm_2(z) \right) \chi_{R_{m,k}} \\
&= \sum_{m,k} \sum_{\partial R_{n,j} \cap \partial R_{m,k} \neq \emptyset} \frac{1}{m_2(\tilde{R}_{n,j})} \left(\int_{R_{n,j}} f(z)\, dm_2(z) \right) \chi_{R_{m,k}} \\
&\asymp \sum_{m,k} \left(\fint_{\tilde{R}_{m,k}} f(z)\, dm_2(z) \right) \chi_{R_{m,k}} = \tilde{M} f.
\end{aligned}$$

(3) Let $z \in \mathbb{D}$ and take $R_{n,j}$ such that $z \in R_{n,j}$. Hence, using Lemma 4.2.2, we have $D\left(z, \frac{1-|z|}{2}\right) \subset \tilde{R}_{n,j}$. Also, it can be proved that $m_2(\tilde{R}_{n,j}) \asymp m_2\left(D\left(z, \frac{1-|z|}{2}\right)\right)$. Therefore, the result follows because $M_D f(z) \lesssim M_{\tilde{R}} f(z)$ for every $z \in \mathbb{D}$. □

Using these results we obtain the converse Littlewood-Paley inequality for certain cases. These inequalities will be important in the subsequent study of the weak compactness of the operator $T_g : RM(1,q) \to RM(1,q)$ in the Chapter 5.

Proposition 4.2.7. *Assume (p,q) are in one of the following three cases: $1 < p, q < +\infty$, $(1,q)$ with $1 \leq q < +\infty$, or (∞, q) with $1 \leq q \leq +\infty$. Then, there is a constant $C = C(p,q) > 0$ such that*

$$\rho_{p,q}(f'(z)(1-|z|)) \leq C \rho_{p,q}(f), \qquad f \in RM(p,q).$$

Proof. By means of Cauchy's integral formula over $\partial D(z,r)$ for $0 \leq r \leq \frac{1-|z|}{2}$ we have that

$$2\pi r^2 |f'(z)| \leq r \int_0^{2\pi} |f(z + re^{i\theta})|\, d\theta.$$

Integrating with respect to r over $0 \leq r \leq \frac{1-|z|}{2}$

$$\frac{1}{3}(1-|z|)|f'(z)| \leq \fint_{D\left(z, \frac{1-|z|}{2}\right)} |f(\xi)|\, dm_2(\xi) = M_D f(z).$$

Assume that $1 \leq p \leq q < +\infty$ or $1 < q \leq p \leq +\infty$. Taking $RM(p,q)$-norm and using the statement *(3)* of Proposition 4.2.6 we conclude

$$\rho_{p,q}(f'(z)(1-|z|) \leq 3\rho_{p,q}(M_D f) \leq C_{p,q} \rho_{p,q}(f).$$

It remains to show the case $p = +\infty$, $q = 1$. Fix $e^{i\theta} \in \partial\mathbb{D}$ and $r \in (0,1)$. From $(1-r)|f'(re^{i\theta})| \leq 3M_D f(re^{i\theta})$ and taking a certain Stolz region $S_C\left(e^{i\theta}\right)$ such that $D\left(re^{i\theta}, \frac{1-r}{2}\right) \subset S_C\left(e^{i\theta}\right)$, it follows that

$$(1-r)|f'(re^{i\theta})| \leq 3 \sup_{w \in S_C\left(e^{i\theta}\right)} |f(w)| =: 3H_f(e^{i\theta}).$$

Moreover, if we take supremum with respect to r, it follows that $\sup_{r\in[0,1)}(1-r)|f'(re^{i\theta})| \leq 3H_f(e^{i\theta})$.

Therefore, taking $L^1(\mathbb{T})$-norm and using Theorem 1.2.5, we obtain

$$\rho_{\infty,q}((1-|z|)f'(z)) \leq 3\|H_f\|_{L^1(\mathbb{T})} \leq 3C\|f\|_{H^1} \leq 3C\rho_{\infty,1}(f). \tag{4.2.1}$$

□

To finish this section, we mention that the converse Littlewood-Paley type inequality holds for the cases $(p,1)$ with $1 < p < +\infty$ (see Chapter 6). We point out that we do not know if the converse Littlewood-Paley type inequality holds in the cases (p,∞), with $1 \leq p < +\infty$.

Chapter 5

Integration operators

In this chapter we study integration operators acting on the $RM(p,q)$ spaces. In particular, we obtain a characterization of the boundedness, compactness and weak compactness of such operators. Moreover, one may wonder about the integration operators acting between different $RM(p,q)$ spaces, but this will be studied in Chapter 6, where the boundedness will be characterized.

5.1 Boundedness and compactness of the integration operator T_g

In this section we begin to study the operators of the form

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\, d\zeta,$$

where $g \in \mathcal{H}(\mathbb{D})$, in the spaces $RM(p,q)$.

Lemma 5.1.1. *Let $1 \leq p < +\infty$ and $\mathcal{B}$ the Bloch space. Then $\mathcal{B} \subset RM(p,0)$.*

Proof. Let $g \in \mathcal{B}$. Then, by [28, Proposition 1, p. 43], there are $M, C > 0$ such that

$$|g(z)| \leq M \ln\left(\frac{1}{1-|z|}\right) + C, \quad z \in \mathbb{D}, \tag{5.1.1}$$

and we have

$$\begin{aligned}\sup_\theta \left(\int_\rho^1 |g(se^{i\theta})|^p\, ds\right)^{1/p} &\leq \left(\int_\rho^1 \left(M\ln\left(\frac{1}{1-s}\right) + C\right)^p ds\right)^{1/p} \\ &\leq M\left(\int_\rho^1 \ln^p\left(\frac{1}{1-s}\right) ds\right)^{1/p} + C(1-\rho)^{1/p} \to 0\end{aligned}$$

when $\rho \to 1$, since $\ln^p\left(\frac{1}{1-s}\right)$ is integrable. Therefore, $\mathcal{B} \subset RM(p,0)$. □

By Corollary 1.5.3, every bounded sequence in $RM(p,q)$ is uniformly bounded on each compact set of the unit disc and then it is a normal family. Thus a standard argument shows that:

Lemma 5.1.2. *Let $1 \leq p, q \leq +\infty$ and let $T : RM(p,q) \to X$ be a linear and bounded operator, where X is a Banach space.*

1. *If for every bounded sequence $\{f_n\} \subset RM(p,q)$ uniformly convergent on compact sets to 0 it holds that $||T(f_n)|| \to 0$, then the operator T is compact.*

2. *Assuming that $T = T_g$ for some holomorphic function g and $X = RM(p,q)$, then T is compact in $RM(p,q)$ if and only if for every bounded sequence*

$\{f_n\} \subset RM(p,q)$ uniformly convergent on compact sets to 0 it holds that $\rho_{p,q}(T_g(f_n)) \to 0$.

Proof. (1) To prove that T is compact in $RM(p,q)$ we need to see that for all sequence $\{h_n\}$ in $B_{RM(p,q)}$, the sequence $\{T(h_n)\}$ has a convergent subsequence in $RM(p,q)$. Take a sequence $\{h_n\}$ in $B_{RM(p,q)}$. $\{h_n\}$ is locally bounded since $\rho_{p,q}(h_n) \leq 1$ for every $n \in \mathbb{N}$ (see Corollary 1.5.3). Using Montel's theorem we obtain that $\{h_n\}$ is normal, that is, there exists a subsequence $\{h_{n_k}\}$ that converges uniformly on every compact set of $\mathbb{D}$. Therefore, there is a function $h \in \mathcal{H}(\mathbb{D})$ such that $h_n \to h$ uniformly on compact set of $\mathbb{D}$. By Fatou's lemma, one can see that $\rho_{p,q}(h) \leq 1$ (see Lemma 1.5.4). Taking the sequence $\{f_k\} := \left\{\frac{h_{n_k}-h}{2}\right\}$ that converges uniformly to 0 on compact sets of $\mathbb{D}$ and $\rho_{p,q}(f_k) \leq 1$, we obtain that

$$\rho_{p,q}\left(\frac{T(h_{n_k}) - T(h)}{2}\right) = \frac{1}{2}\rho_{p,q}(T(h_{n_k}) - T(h)) \to 0,$$

that is, $T(h_{n_k}) \to T(h)$ in $RM(p,q)$-norm. Thus, T is compact in $RM(p,q)$.

(2) One of the implications was proven in statement *(1)* in a more general setting. Let us continue with the proof of the other implication. Suppose that T_g is compact in $RM(p,q)$ and assume that there are a constant $\gamma > 0$ and a bounded sequence $\{f_n\}$ in $RM(p,q)$ that converges uniformly to zero on compact sets of $\mathbb{D}$ such that $\rho_{p,q}(T_g(f_n)) \geq \gamma > 0$. From the definition of T_g we have that $T_g(f_n)(z)$ converges uniformly to 0 on compact sets of $\mathbb{D}$. By the compactness of T_g, there are a subsequence $\{f_{n_k}\}$ and $h \in RM(p,q)$ such that $\rho_{p,q}(T_g(f_{n_k}) - h) \to 0$. Since $T_g(f_n)(z)$ converges uniformly to 0 on compact sets of $\mathbb{D}$, we have that $h = 0$. But, we obtain a contradiction because $\rho_{p,q}(T_g(f_{n_k})) \to 0$ when $k \to \infty$. □

It is well-known that, for $1 \leq p < +\infty$, the operator T_g is bounded (resp. compact) over the Hardy spaces H^p if and only if $g \in BMOA$ (resp. $VMOA$) [5,55], and the operator $T_g : A^p \to A^p$ is bounded (resp. compact) if and only if g belongs to the Bloch space $\mathcal{B}$ (resp. the little Bloch space $\mathcal{B}_0$) [6]. Next result completes these characterizations to $RM(p,q)$ whenever $(p,q) \neq (+\infty, +\infty)$.

Theorem 5.1.3. *Let $1 \leq p < +\infty$, $1 \leq q \leq +\infty$. Then*

1. *The operator $T_g : RM(p,q) \to RM(p,q)$ is bounded if and only if $g \in \mathcal{B}$.*
2. *The operator $T_g : RM(p,0) \to RM(p,0)$ is bounded if and only if $g \in \mathcal{B}$.*
3. *The operator $T_g : RM(p,q) \to RM(p,q)$ is compact if and only if $g \in \mathcal{B}_0$.*
4. *The operator $T_g : RM(p,0) \to RM(p,0)$ is compact if and only if $g \in \mathcal{B}_0$.*

Proof. *(1)* Assume that $g \in \mathcal{B}$. If $q < +\infty$, by Proposition 4.1.3, there is a constant C_p such that

$$\begin{aligned}\rho_{p,q}^q(T_g(f)) &\leq C_p^q \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})g'(re^{i\theta})|^p (1-r^2)^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} \\ &\leq C_p^q \|g\|_{\mathcal{B}}^q \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{q/p} \frac{d\theta}{2\pi} = C_p^q \|g\|_{\mathcal{B}}^q \, \rho_{p,q}(f)^q,\end{aligned}$$

getting the boundedness of T_g if $q < +\infty$. A similar argument works if $q = +\infty$.

Conversely, assume that T_g is bounded on $RM(p,q)$. Fix $z \in \mathbb{D}$. By Corollary 1.5.7, there is $C > 0$ such that if $f \in B_{RM(p,q)}$, then

$$\begin{aligned}|g'(z)||f(z)| = |T_g(f)'(z)| &\leq \rho_{p,q}(T_g(f)) \|\delta_z'\|_{(RM(p,q))^*} \\ &\leq \|T_g\| \, \rho_{p,q}(f) \|\delta_z'\|_{(RM(p,q))^*} \leq C \|T_g\| \, \rho_{p,q}(f) \|\delta_z\|_{(RM(p,q))^*} \frac{1}{1-|z|}.\end{aligned}$$

We can choose $f \in RM(p,q)$, with $\rho_{p,q}(f) \leq 1$, such that $\|\delta_z\|_{(RM(p,q))^*} \leq 2|f(z)|$. Therefore,

$$|g'(z)|(1-|z|^2) \leq C\|T_g\| \, \rho_{p,q}(f) \frac{\|\delta_z\|_{(RM(p,q))^*}}{|f(z)|} \leq 2C\|T_g\|,$$

so that $g \in \mathcal{B}$ and (1) holds.

(2) Assume that $g \in \mathcal{B}$. By (1) we know that T_g is bounded from $RM(p,0)$ into $RM(p,\infty)$. If f is a polynomial and $z \in \mathbb{D}$, then

$$(1-|z|^2)|T_g(f)'(z)| \leq \|g\|_{\mathcal{B}} \|f\|_\infty.$$

That is $T_g(f) \in \mathcal{B}$ and, by Lemma 5.1.1, $T_g(f) \in RM(p,0)$. The density of the polynomials in $RM(p,0)$ (see Corollary 1.5.16) and the boundedness of T_g from $RM(p,0)$ into $RM(p,\infty)$ (by *(1)*) implies that $T_g(RM(p,0)) \subset RM(p,0)$.

Conversely, if T_g is bounded on $RM(p,0)$ we can argue as in the proof of statement *(1)* using Corollary 1.5.17 instead of Corollary 1.5.7.

(3) and *(4)* We start by proving that if T_g is compact in $RM(p,q)$, with $q < +\infty$, then $g \in \mathcal{B}_0$. Take $f \in RM(p,q)$, then

$$\langle f, T_g^*(\delta_z') \rangle = \langle T_g(f), \delta_z' \rangle = g'(z) f(z) = g'(z) \langle f, \delta_z \rangle,$$

multiplying by $\frac{1-|z|}{\|\delta_z\|}$ we obtain

$$\langle f, T_g^* \left(\frac{\delta_z'(1-|z|)}{\|\delta_z\|} \right) \rangle = g'(z)(1-|z|) \langle f, \frac{\delta_z}{\|\delta_z\|} \rangle.$$

Hence, it follows that

$$|g'(z)|(1-|z|)\left|\langle f, \frac{\delta_z}{\|\delta_z\|}\rangle\right| \leq \rho_{p,q}(f)\left\|T_g^*\left(\frac{\delta_z'(1-|z|)}{\|\delta_z\|}\right)\right\|_{(RM(p,q))^*}.$$

Taking supremum in $f \in B_{RM(p,q)}$, we have

$$|g'(z)|(1-|z|) \leq \left\|T_g^*\left(\frac{\delta_z'(1-|z|)}{\|\delta_z\|}\right)\right\|_{(RM(p,q))^*}.$$

We claim that $\frac{\delta_z'(1-|z|)}{\|\delta_z\|} \to 0$ in the weak-$*$ topology, when $|z| \to 1$. Assuming the claim holds, the compactness of T_g implies that

$$\left\|T_g^*\left(\frac{\delta_z'(1-|z|)}{\|\delta_z\|}\right)\right\|_{(RM(p,q))^*} \to 0,$$

as $|z| \to 1$, so that $g \in \mathcal{B}_0$.

Let us see the claim. Take f a polynomial. Then

$$\frac{|\delta_z'(f)|(1-|z|)}{\|\delta_z\|} \asymp \frac{|f'(z)|(1-|z|)}{(1-|z|)^{-\frac{1}{p}-\frac{1}{q}}} \lesssim \|f'\|_\infty (1-|z|)^{1+\frac{1}{p}+\frac{1}{q}} \to 0$$

as $|z| \to 1$. The density of the polynomials in $RM(p,q)$ and $\frac{\|\delta_z'\|(1-|z|)}{\|\delta_z\|} \lesssim 1$ (see Corollary 1.5.7) show that $\frac{\delta_z'(1-|z|)}{\|\delta_z\|} \xrightarrow{w^*} 0$ as $|z| \to 1$. So that the claim holds.

The same argument shows that if $T_g : RM(p,0) \to RM(p,0)$ is compact, then $g \in \mathcal{B}_0$.

Assume now that $q = +\infty$. The compactness of $T_g : RM(p,\infty) \to RM(p,\infty)$ implies $g \in \mathcal{B}$ so that $T_g : RM(p,0) \to RM(p,0)$ is bounded and, clearly compact. Thus $g \in \mathcal{B}_0$.

Let us see that if $g \in \mathcal{B}_0$, then the operator T_g is compact. Assume for the moment that g is a polynomial. Take $\{f_k\}$ a sequence in the unit ball of $RM(p,q)$ uniformly convergent to 0 on compact sets. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $|f_k(z)| \leq \varepsilon$ for all $|z| \leq \rho := 1-\varepsilon$ and $k \geq N$. Fix $z = re^{i\theta}$. If $r \leq \rho$, then

$$\left|T_g f_k(re^{i\theta})\right| \leq \|g'\|_\infty \int_0^r |f_k(se^{i\theta})|\, ds \leq \|g'\|_\infty \varepsilon$$

while, if $r > \rho$, then

$$\left|T_g f_k(re^{i\theta})\right| \leq \|g'\|_\infty \int_0^r |f_k(se^{i\theta})|\, ds \leq \|g'\|_\infty \left(\int_0^1 |f_k(se^{i\theta})|^p\, ds\right)^{1/p}.$$

Therefore

$$\left(\int_0^1 |T_g(f_k)(re^{i\theta})|^p\, dr\right)^{1/p} \leq \|g'\|_\infty \left(\varepsilon\rho^{1/p} + (1-\rho)^{1/p}\left(\int_0^1 |f_k(se^{i\theta})|^p\, ds\right)^{1/p}\right).$$

Hence, for all $k \geq N$,

$$\begin{aligned}\rho_{p,q}(T_g(f_k)) &\leq \|g'\|_\infty \varepsilon\rho^{1/p} + \|g'\|_\infty(1-\rho)^{1/p}\rho_{p,q}(f_k)\\ &\leq \|g'\|_\infty(\varepsilon + (1-\rho)^{1/p}) = \|g'\|_\infty(\varepsilon + \varepsilon^{1/p}).\end{aligned}$$

Therefore $\lim_k \rho_{p,q}(T_g(f_k)) = 0$. By Proposition 5.1.2, T_g is compact on $RM(p,q)$.

If $g \in \mathcal{B}_0$, there is a sequence of polynomials $\{g_n\}$ such that $\lim_n \|g - g_n\|_{\mathcal{B}} = 0$ (see [28, p. 45-46]). Moreover, there is a constant $C = C(p)$ (see the proof of statement (1)) such that

$$\|T_g - T_{g_n}\| = \|T_{g-g_n}\| \leq C_p\|g - g_n\|_{\mathcal{B}} \to 0.$$

Since T_{g_n} is compact for all n, then so is T_g.

The same argument works in $RM(p,0)$ so that we are done. □

Remark 5.1.4. Let $1 \leq p < +\infty$. If $g \in \mathcal{B}_0$ then $T_g(RM(p,\infty)) \subset RM(p,0)$. Indeed, if $f \in RM(p,\infty)$ and h is a polynomial, then for all $r \in (0,1)$ we have that

$$|T_hf(re^{i\theta})| \leq \|h'\|_\infty\ \rho_{p,\infty}(f).$$

That is, $T_hf \in H^\infty \subset RM(p,0)$. Hence, using density of polynomials in $\mathcal{B}_0$ (see [28, p. 45-46]) and the estimate $\|T_g\| \leq C_p\|g\|_{\mathcal{B}}$ (see the proof of statement (1) in above theorem), we can prove that if $g \in \mathcal{B}_0$ then $T_g(RM(p,\infty)) \subset RM(p,0)$, because $RM(p,0)$ is closed in $RM(p,\infty)$ and

$$\|T_g - T_{h_n}\| = \|T_{g-h_n}\| \leq C_p\|g - h_n\|_{\mathcal{B}} \to 0,$$

where h_n are polynomials such that $\|g - h_n\|_{\mathcal{B}} \to 0$.

However, the reverse implication does not hold as the next example shows. That is, the compactness cannot be characterized by the property of sending the big-O space into the little-o space, despite what happens in other spaces of holomorphic functions (see, i.e., [10] for mixed norm spaces, [12] for weighted Banach spaces, and [16] for the Bloch space and BMOA).

Example 5.1.5. Let $1 \leq p < +\infty$ and $g(z) = -\log(1-z)$. Then $g \in \mathcal{B} \setminus \mathcal{B}_0$ and $T_g(RM(p,\infty)) \subset RM(p,0)$.

Proof. Since $|g'(z)| = |1-z|^{-1}$, it is easy to see that $g \in \mathcal{B} \setminus \mathcal{B}_0$. Fix $f \in RM(p,\infty)$ such that $\rho_{p,\infty}(f) \leq 1$. By Proposition 4.1.6, in order to prove that $T_g(f) \in RM(p,0)$, it is enough to show

$$\lim_{\rho \to 1^-} \sup_\theta \left(\int_\rho^1 |f(re^{i\theta}) g'(re^{i\theta})|^p (1-r)^p \, dr \right)^{1/p} = 0.$$

Suppose by contradiction that there are a constant $c > 0$ and sequences $\{\rho_k\} \to 1$ and $\{\theta_k\}$ in $(-\pi, \pi]$ such that

$$\left(\int_{\rho_k}^1 |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr \right)^{1/p} > c.$$

Notice that the sequence $\{\theta_k\}$ must converge to 0. Indeed, using that $|1-e^{i\theta_k}| \leq 2|1-re^{i\theta_k}|$ and that $\rho_k \to 1$ we have

$$\begin{aligned} c &< \left(\int_{\rho_k}^1 |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr \right)^{1/p} \\ &< 2 \left(\int_0^1 |f(re^{i\theta_k})|^p \frac{(1-\rho_k)^p}{|1-e^{i\theta_k}|^p} \, dr \right)^{1/p} < 2 \frac{(1-\rho_k)}{|1-e^{i\theta_k}|} \end{aligned}$$

so that it holds that $\theta_k \to 0$.
Claim 1. There is $\delta > 0$ such that if $\theta \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right] \setminus \{0\}$ and $1 > r > 1 - \delta|\theta|$, then $\left|\frac{1-r}{1-re^{i\theta}}\right| < c/4$.
Proof of Claim 1. Notice $\frac{1}{2} < \frac{1-\cos(\theta)}{\theta^2/2} < 1$ for $\theta \in [-\pi/4, \pi/4] \setminus \{0\}$. Therefore if $r > 1 - \delta|\theta|$, then

$$\left|\frac{1-r}{1-re^{i\theta}}\right|^2 = \frac{(1-r)^2}{(1-r)^2 + 2r(1-\cos(\theta))} < \frac{1}{1 + \frac{1-\delta|\theta|}{2\delta^2}} < \frac{1}{1 + \frac{1-\delta\frac{\pi}{4}}{2\delta^2}} < \frac{c^2}{16}$$

if δ is small enough and Claim 1 holds.

By Claim 1 and the fact that $\rho_{p,\infty}(f) \leq 1$ we have

$$\left(\int_{1-\delta|\theta_k|}^1 |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr \right)^{1/p} < \frac{c}{4}.$$

Therefore,

$$\int_{\rho_k}^1 |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr - \int_{1-\delta|\theta_k|}^1 |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr > c^p - \frac{c^p}{4^p} \geq \frac{3}{4} c^p > 0,$$

so that $1 - \delta|\theta_k| > \rho_k$.

Now, it is obtained that

$$\int_{\rho_k}^{1-\delta|\theta_k|} |f(re^{i\theta_k})|^p \, dr \geq \int_{\rho_k}^{1-\delta|\theta_k|} |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr \tag{5.1.2}$$
$$= \int_{\rho_k}^{1} |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr - \int_{1-\delta|\theta_k|}^{1} |f(re^{i\theta_k})|^p \frac{(1-r)^p}{|1-re^{i\theta_k}|^p} \, dr > \frac{3}{4}c^p.$$

Claim 2. There is $M > \delta$ and k_0 such that, if $\rho_k < 1 - M|\theta_k|$, then

$$\left(\int_{\rho_k}^{1-M|\theta_k|} |f(re^{i\theta_k})|^p dr \right)^{1/p} < \frac{c}{3}$$

for $k > k_0$.

Proof of Claim 2. By Proposition 1.5.6, there is a constant C_1 such that $|f'(w)| \leq C_1(1-|w|)^{-1-\frac{1}{p}}$, for all $w \in \mathbb{D}$. Take $M > \max\left\{\delta, \frac{4C_1}{c}\right\}$ and $k \in \mathbb{N}$ such that $M|\theta_k| < 1$. Then

$$\left(\int_0^{1-M|\theta_k|} |f(r) - f(re^{i\theta_k})|^p \, dr \right)^{1/p}$$
$$\leq \left(\int_0^{1-M|\theta_k|} r^p \, |1-e^{i\theta_k}|^p \sup_{w\in[r,re^{i\theta_k}]} |f'(w)|^p \, dr \right)^{1/p}$$
$$\leq C_1 |1-e^{i\theta_k}| \left(\int_0^{1-M|\theta_k|} \frac{dr}{(1-r)^{p+1}} \right)^{1/p} < C_1 \frac{|1-e^{i\theta_k}|}{|\theta_k|} \frac{1}{p^{1/p} M} \leq \frac{C_1}{M} < \frac{c}{4}.$$

By the integrability of $|f(r)|^p$ on the interval $[0,1)$ (see Remark 1.2.3) and the fact that $\rho_k \to 1^-$, there exists k_0 such that for all $k > k_0$ we have

$$\left(\int_{\rho_k}^{1-M|\theta_k|} |f(re^{i\theta_k})|^p \, dr \right)^{1/p}$$
$$\leq \left(\int_{\rho_k}^{1-M|\theta_k|} |f(re^{i\theta_k}) - f(r)|^p \, dr \right)^{1/p} + \left(\int_{\rho_k}^{1-M|\theta_k|} |f(r)|^p \, dr \right)^{1/p} \leq \frac{c}{4} + \frac{c}{13} < \frac{c}{3}$$

and Claim 2 holds.

If $\rho_k < 1 - M|\theta_k|$, by (5.1.2) and Claim 2, it follows

$$\int_{1-M|\theta_k|}^{1-\delta|\theta_k|} |f(re^{i\theta_k})|^p \, dr = \int_{\rho_k}^{1-\delta|\theta_k|} |f(re^{i\theta_k})|^p \, dr - \int_{\rho_k}^{1-M|\theta_k|} |f(re^{i\theta_k})|^p \, dr$$
$$> \frac{3}{4}c^p - \frac{c^p}{3^p} = c^p \left(\frac{3}{4} - \frac{1}{3^p} \right) \geq \frac{5c^p}{12}.$$

If $\rho_k > 1 - M|\theta_k|$, by (5.1.2), we obtain

$$\int_{1-M|\theta_k|}^{1-\delta|\theta_k|} |f(re^{i\theta_k})|^p \, dr > \int_{\rho_k}^{1-\delta|\theta_k|} |f(re^{i\theta_k})|^p \, dr > \frac{3c^p}{4} > \frac{5c^p}{12}.$$

Therefore, there exists $r_k \in (1 - M|\theta_k|, 1 - \delta|\theta_k|)$ such that

$$|f(r_k e^{i\theta_k})|^p (M - \delta)|\theta_k| \geq \int_{1-M|\theta_k|}^{1-\delta|\theta_k|} |f(re^{i\theta_k})|^p \, dr > \frac{5c^p}{12}.$$

Thus, as $r_k < 1 - \delta|\theta_k|$ it follows that

$$|f(r_k e^{i\theta_k})(1 - r_k e^{i\theta_k})^{1/p}| \geq |f(r_k e^{i\theta_k})|(1 - r_k)^{1/p} > \frac{5^{1/p}\delta^{1/p} c}{12^{1/p}(M - \delta)^{1/p}}$$

what contradicts Proposition 2.2.2, because $r_k e^{i\theta_k}$ tends nontangentially to 1 since $r_k \in (1 - M|\theta_k|, 1 - \delta|\theta_k|)$ and $|\theta_k| \to 0$. □

5.2 Weak compactness of the integration operator T_g

It is well-known that any weakly compact integration operator on the Hardy space $H^1 (= RM(\infty, 1))$ is compact (see [45]). Since the Bergman space $A^1 (= RM(1, 1))$ is isomorphic to ℓ_1 (see [65, p. 89]) and then it has the Schur property, it also holds that if T_g is weakly compact on A^1 then it is compact. In this section, we will show that this happens in other spaces of average radial integrability but not in all of them. When the weak compactness does not coincide with the compactness we will provide different characterizations.

Since $RM(p, q)$ is reflexive if either $1 < p, q < +\infty$ (see Theorem 3.1.7) or $p = +\infty$ and $1 < q < +\infty$ (see [27, Theorem 7.3, p. 113]), the problem we are dealing with in this section is only interesting in the next three cases:

- $1 \leq p \leq +\infty$ and $q = +\infty$;
- $p = 1$ and $1 \leq q \leq +\infty$;
- $1 \leq p \leq +\infty$ and $q = 1$.

There is a useful characterization of the weak compactness of T_g in terms of the norm convergence of certain convex combinations.

Lemma 5.2.1. *Let $1 \leq p, q, \tilde{p}, \tilde{q} \leq +\infty$ and X a Banach space.*

1. *Let $T : RM(p, q) \to X$ be a linear and bounded operator. Assume that every sequence $\{f_n\}$ in the unit ball of $RM(p, q)$ convergent to 0 uniformly on compact*

sets of $\mathbb{D}$ *satisfies that there exists a sequence* $\{g_k\}$*, with* $g_k \in \mathrm{co}\{f_k, f_{k+1}, \dots\}$ *for all* k*, such that* $\|Tg_k\| \to 0$ *when* $k \to \infty$*. Then* T *is weakly compact.*

2. *Assume that* $T_g : RM(p,q) \to RM(\tilde{p},\tilde{q})$ *is bounded. Then* $T_g : RM(p,q) \to RM(\tilde{p},\tilde{q})$ *is weakly compact if and only if every sequence* $\{f_n\}$ *in the unit ball of* $RM(p,q)$ *convergent to* 0 *uniformly on compact sets of* $\mathbb{D}$ *satisfies that there exists a sequence* $\{g_k\}$*, with* $g_k \in \mathrm{co}\{f_k, f_{k+1}, \dots\}$ *for all* k*, such that* $\rho_{\tilde{p},\tilde{q}}(T_g g_k) \to 0$ *when* $k \to \infty$*.*

Proof. Let us begin with (1). Assume by contradiction that T is not weakly compact. Then there is a bounded sequence $\{f_n\}$ such that $\{Tf_n\}$ does not have weakly convergent subsequences. Applying Montel's theorem, there are a holomorphic function f and a subsequence $\{f_{n_k}\}$ such that it converges uniformly to f on compact sets of $\mathbb{D}$. By Lemma 1.5.4, it holds that $f \in RM(p,q)$. Consider the bounded sequence $\{h_k\} := \{f_{n_k} - f\}$. Clearly it converges uniformly to 0 on compact sets of $\mathbb{D}$. Since $\{Th_k\}$ does not converge weakly to zero, there are $\lambda \in X^*$, $\delta > 0$, and a subsequence $\{Th_{k_j}\}$ such that

$$\mathsf{Re}\,\left(\lambda\left(Th_{k_j}\right)\right) \geq \delta > 0.$$

By our assumption, there exist $g_j \in \mathrm{co}\{h_{k_j}, h_{k_{j+1}}, \dots\}$ such that $\|Tg_j\|_X \to 0$. Then, $g_j = \sum_{l=j}^{\infty} \alpha_{l,j} h_{k_l}$ for certain coefficients $0 \leq \alpha_{k,j} \leq 1$, with $\sum_{j=k}^{\infty} \alpha_{k,j} = 1$, and, for each k, the set $\{j \geq k : \alpha_{k,j} \neq 0\}$ is finite. Therefore,

$$\mathsf{Re}\,\lambda(Tg_j) = \sum_{l=j}^{\infty} \alpha_{l,j} \mathsf{Re}\,\lambda\left(Th_{k_l}\right) \geq \delta > 0,$$

and we obtain a contradiction because

$$0 < \delta \leq \mathsf{Re}\,\lambda(Tg_j) \leq |\lambda(Tg_j)| \leq \|\lambda\|\,\|Tg_j\|_X.$$

Let us prove (2). By (1), we just have to check one implication. Assume that $T_g : RM(p,q) \to RM(\tilde{p},\tilde{q})$ is weakly compact. Let $\{f_n\} \subset B_{RM(p,q)}$ be a sequence that converges uniformly to 0 on compact sets of $\mathbb{D}$. By the very definition of integration operator, we also have that $T_g f_n$ converges to 0 uniformly on compact sets of the unit disc. By the weak compactness of T_g, there exists a subsequence $\{T_g f_{n_k}\}$ that converges weakly to some $h \in RM(\tilde{p},\tilde{q})$. Since the convergence in the weak topology implies pointwise convergence, we have that $h = 0$. Therefore $\{T_g f_{n_k}\}$ converges weakly to 0. By Theorem A.2.1, we obtain that there exist $g_k \in \mathrm{co}\{f_{n_k}, f_{n_{k+1}}, \dots\} \subset \mathrm{co}\{f_k, f_{k+1}, \dots\}$ such that $\rho_{\tilde{p},\tilde{q}}(T_g g_k) \to 0$. $\square$

5.2.1 The case $q = +\infty$

Unlike what happens in other spaces of holomorphic functions, Example 5.1.5 shows that the compactness cannot be characterized by the property of sending the big-O space into the little-o space. Nevertheless, this property characterizes the weak compactness in the spaces $RM(p,\infty)$ for $1 < p < +\infty$.

Theorem 5.2.2. *Let $1 < p < +\infty$ and $g \in \mathcal{B}$. The following are equivalent:*

(1) $T_g(RM(p,\infty)) \subset RM(p,0)$.

(2) $T_g : RM(p,0) \to RM(p,0)$ is weakly compact.

(3) $T_g : RM(p,\infty) \to RM(p,\infty)$ is weakly compact.

Proof. Let us recall that given a Banach space X and a bounded operator $T : X \to X$ it holds that T is weakly compact if and only if $T^{**} : X^{**} \to X$ is bounded (see Theorem A.2.2) if and only if $T^{**} : X^{**} \to X^{**}$ is weakly compact.

Let us consider the bounded operator $T_g : RM(p,0) \to RM(p,0)$. A standard argument using Theorem 3.4.3 gives that the next diagram is commutative:

$$\begin{array}{ccc} (RM(p,0))^{**} & \xrightarrow{(T_g)^{**}} & (RM(p,0))^{**} \\ {\scriptstyle I^{**}}\downarrow & & \downarrow{\scriptstyle I^{**}} \\ RM(p,\infty) & \xrightarrow{\quad T_g \quad} & RM(p,\infty) \end{array}$$

Since I^{**} is an isomorphism, above general results give the theorem. □

A similar result to above theorem for $p = +\infty$ was obtained in [23]. Namely, they proved that T_g is weakly compact on H^∞ if and only if it is weakly compact on the disc algebra and if and only if T_g sends H^∞ into the disc algebra. Let us recall that the disc algebra is the closure of the polynomials in H^∞ in an analogous way to the couple $RM(p,0)$ and $RM(p,\infty)$.

Next example shows that Theorem 5.2.2 does not hold for $p = 1$.

Example 5.2.3. Let $g(z) = -\log(1-z) \in \mathcal{B}\backslash\mathcal{B}_0$. Then $T_g : RM(1,\infty) \to RM(1,0)$ fixes a copy of ℓ^1 (see Definition A.3.1). In particular, $T_g : RM(1,\infty) \to RM(1,0)$ is not weakly compact.

Proof. By Example 5.1.5, $T_g : RM(1,\infty) \to RM(1,0)$ is a bounded operator. Take $\beta \geq 2$ a natural number and write $\delta := \frac{2}{3}\frac{\beta}{1+\beta}$. Notice that the sequence $\{T_g(\beta^n z^{\beta^n})\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Consider the sequence of functions $f_n : [0,1) \to \mathbb{C}$ given by $f_n(r) := T_g(\beta^n z^{\beta^n})(r)$ for $r \in [0,1)$. Notice that

$$\int_0^1 \left| T_g(\beta^n z^{\beta^n})(r) \right| dr = \left| \int_0^1 \int_0^r \beta^n \frac{u^{\beta^n}}{1-u}\, du\, dr \right|$$
$$= \left| \int_0^1 \int_u^1 \beta^n \frac{u^{\beta^n}}{1-u}\, dr\, du \right| = \frac{\beta^n}{1+\beta^n} \geq \frac{3}{2}\delta$$

for all $n \in \mathbb{N}$. That is, $1 \geq \|f_n\|_1 \geq 3\delta/2$ for $n \in \mathbb{N}$. Since all the functions f_n are integrable in $[0,1)$ and goes to zero uniformly on compacta of such interval we can choose $r_1 \in [0,1)$ and $n_2 > n_1 := 1$ such that

$$\int_{r_1}^1 |f_{n_1}|\, dr < \frac{\delta}{2} \quad \text{and} \quad \int_0^{r_1} |f_{n_2}|\, dr < \frac{\delta}{4}.$$

Repeating the argument we choose $r_2 \in (r_1, 1)$ and $n_3 > n_2$ such that such that

$$\int_{r_2}^1 |f_{n_2}|\, dr < \frac{\delta}{4} \quad \text{and} \quad \int_0^{r_2} |f_{n_3}|\, dr < \frac{\delta}{4}.$$

Continuing inductively we obtain a subsequence $\{f_{n_k}\}$ and a sequence of disjoint intervals $\{I_k\} = \{(r_{k-1}, r_k)\}$, setting $r_0 = 0$, such that

$$\int_{I_k} |f_{n_k}|\, dr > \delta \quad \text{and} \quad \int_{\cup_{j\neq k} I_j} |f_{n_k}|\, dr < \frac{\delta}{2}.$$

Now, we consider the operator $\Theta : \ell^1 \to RM(1,\infty)$ given by $\Theta(\{\alpha_k\}) = \sum_{k=1}^\infty \alpha_k \beta^{n_k} z^{\beta^{n_k}}$. By Proposition 1.4.1,

$$\rho_{1,\infty}(\Theta(\{\alpha_k\})) = \rho_{1,\infty}\left(\sum_{k=1}^\infty \alpha_k \beta^{n_k} z^{\beta^{n_k}} \right) \asymp \left(\sum_{k=0}^\infty \frac{|\alpha_k| \beta^{n_k}}{\beta^{n_k}+1} \right) \leq \|\{\alpha_k\}\|_{\ell^1}.$$

Therefore, the boundedness of T_g implies that $T_g \circ \Theta : \ell^1 \to RM(1,0)$ is continuous. On the other hand,

$$\rho_{1,\infty}\left(\sum_{k=1}^\infty \alpha_k T_g(\beta^{n_k} z^{\beta^{n_k}}) \right) \geq \int_0^1 \left| \sum_{k=1}^\infty \alpha_k f_{n_k}(r) \right| dr$$
$$\geq \sum_{k=1}^\infty \int_{I_k} \left(|\alpha_k||f_{n_k}(r)| - \sum_{j\neq k} |\alpha_j||f_{n_j}(r)| \right) dr$$
$$\geq \delta \sum_{k=1}^\infty |\alpha_k| - \sum_{j=1}^\infty |\alpha_j| \sum_{k\neq j} \int_{I_k} |f_{n_j}(r)|\, dr \geq \delta \sum_{k=1}^\infty |\alpha_k| - \frac{\delta}{2} \sum_{k=1}^\infty |\alpha_k| = \frac{\delta}{2} \|\{\alpha_k\}\|_{\ell^1}.$$

Hence, we have that $T_g : RM(1,\infty) \to RM(1,0)$ fixes a copy of ℓ^1 and we are done. □

5.2.2 The case $p = 1$

We start with a characterization of the weak compactness of the operator $T_g : RM(1,q) \to RM(1,q)$ in terms of non-fixing copies of ℓ^1. We need the following lemma which probably is well-known by specialist but we could not find any reference so that we include the proof for the sake of completeness.

Lemma 5.2.4. *Let X be a Banach space and μ a positive and finite measure on Ω. If $T : X \to L^1(\mu)$ is bounded and not weakly compact, then it fixes a copy of ℓ^1.*

Proof. Since $T(B_X)$ is not relatively weakly compact, by Theorem A.2.5, there exists a sequence $\{f_n\}$ in $T(B_X)$ which is equivalent to the basis of ℓ^1. That is, there is a positive constant δ such that

$$\|\sum_n \alpha_n f_n\| \geq \delta \sum_n |\alpha_n|$$

for all sequences $\{\alpha_n\}$ of complex numbers. Take $x_n \in B_X$ such that $T(x_n) = f_n$. Then

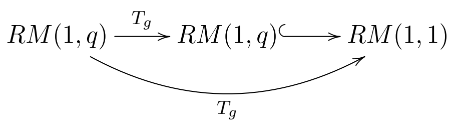
$$\sum_n |\alpha_n| \geq \|\sum_n \alpha_n x_n\| \geq \frac{1}{||T||}\|\sum_n \alpha_n f_n\| \geq \frac{\delta}{||T||}\sum_n |\alpha_n|$$

for all sequences $\{\alpha_n\}$ of complex numbers and we are done. □

Proposition 5.2.5. *Let $1 < q < +\infty$ and $g \in \mathcal{B}$. The following assertions are equivalent:*

1. *$T_g : RM(1,q) \to RM(1,q)$ is weakly compact.*
2. *$T_g : RM(1,q) \to RM(1,1)$ is compact.*
3. *$T_g : RM(1,q) \to RM(1,1)$ is weakly compact.*
4. *$T_g : RM(1,q) \to RM(1,1)$ does not fix a copy of ℓ^1.*
5. *$T_g : RM(1,q) \to RM(1,q)$ does not fix a copy of ℓ^1.*

Proof. It is obvious that (1) implies (5). Bearing in mind the following commutative diagram



it is clear that (5) implies (4) (see Theorem A.3.2). Notice that $RM(1,1) = A^1 \subset L^1(\mathbb{D})$. By Lemma 5.2.4, if $T_g : RM(1,q) \to RM(1,1)$ does not fix a copy of ℓ^1, it is weakly compact. In addition, since $RM(1,1)$ is isomorphic to ℓ^1 (see [65, Theorem

11, p. 89]), it has the Schur property and it must be compact. Thus, (4) implies (3) and (3) implies (2). Therefore, it remains to show that (2) implies (1).

Assume that $T_g : RM(1,q) \to RM(1,1)$ is compact. Let $\{f_n\} \subset B_{RM(1,q)}$ be a sequence that converges uniformly to 0 on compact sets of $\mathbb{D}$. Then, by Lemma 5.1.2(2), the compactness implies that $\rho_{1,1}(T_g f_n) \to 0$.

The value $H_n(\theta) := \int_0^1 |T_g f_n(re^{i\theta})|\, dr$ is finite for almost every θ. Since $g \in \mathcal{B}$, by Theorem 5.1.3, there is a constant $C > 0$ such that $\|H_n\|_{L^q(\mathbb{T})} = \rho_{1,q}(T_g(f)) \leq C$. Moreover, $\lim_n \|H_n\|_{L^1(\mathbb{T})} = 0$. Therefore, we obtain a subsequence $\{H_{n_k}\}$ such that $H_{n_k} \to 0$ weakly in $L^q(\mathbb{T})$. Hence, there is $F_k \in \text{co}\{H_{n_k}, H_{n_{k+1}}, \dots\}$ such that $\|F_k\|_{L^q(\mathbb{T})} \to 0$ (see Theorem A.2.1). Write

$$F_k = \sum_{j=k}^{\infty} \alpha_{k,j} H_{n_j},$$

where $\alpha_{k,j} \geq 0$, $\sum_{j=k}^{\infty} \alpha_{k,j} = 1$, and, for each k, the set $\{j \geq k : \alpha_{k,j} \neq 0\}$ is finite. The functions

$$g_k := \sum_{j=k}^{\infty} \alpha_{k,j} f_{n_j},$$

belong to $RM(1,q)$ and

$$\int_0^1 |T_g g_k(re^{i\theta})|\, dr \leq \sum_{j=k}^{\infty} \alpha_{k,j} H_{n_j} = F_k(\theta).$$

It follows that $\rho_{1,q}(T_g g_k) \to 0$, as $k \to \infty$. Using Lemma 5.2.1 we conclude that $T_g : RM(1,q) \to RM(1,q)$ is weakly compact. □

The main result of this section provides a characterization of the weak compactness of the operator $T_g : RM(1,q) \to RM(1,q)$ in terms of the symbol g. For this purpose, we introduce a pointwise version of the little Bloch space.

Definition 5.2.6. The weakly little Bloch space, denoted by $\mathcal{B}_{0,w}$, is the subspace of $\mathcal{B}$ consisting of analytic functions $f \in \mathcal{B}$ with

$$\lim_{r\to 1} (1 - r^2)|f'(re^{i\theta})| = 0,$$

for almost every $e^{i\theta} \in \mathbb{T}$.

Remark 5.2.7. Let us see that the weakly little Bloch space $\mathcal{B}_{0,w}$ is a closed subspace of the Bloch space $\mathcal{B}$. Let $\{f_n\}$ be a sequence in $\mathcal{B}_{0,w}$ that converges in $\mathcal{B}$, that is, there is $f \in \mathcal{B}$ such that $\|f_n - f\|_{\mathcal{B}} \to 0$ when $n \to \infty$.

Considering the sequence of subsets $\{A_n\}$ of $\mathbb{T}$ such that each subset A_n satisfies that $m_1(A_n) = 0$ and

$$\lim_{r\to 1^-} |g_n'(re^{i\theta})|(1-r) = 0$$

for every $\theta \in \mathbb{T}\setminus A_n$, define the set $A := \cup_{n\geq 1} A_n$. Clearly, A is a set of measure zero.

Fix $\varepsilon > 0$. There is N such that $\|f - f_N\|_{\mathcal{B}} < \varepsilon$. For each $r \in [0,1)$ and $\theta \in \mathbb{T}\setminus A$,

$$\begin{aligned}|f'(re^{\theta i})|(1-r) &\leq |f'(re^{\theta i}) - f_N'(re^{\theta i})|(1-r) + |f_N'(re^{\theta i})|(1-r)\\ &\leq \|f_N - f\|_{\mathcal{B}} + |f_N'(re^{\theta i})|(1-r) < \varepsilon + |f_N'(re^{\theta i})|(1-r).\end{aligned}$$

Therefore, $\limsup_{r\to 1^-} |f'(re^{i\theta})|(1-r) \leq \varepsilon$ for all $\theta \in \mathbb{T}\setminus A$, because $f_N \in \mathcal{B}_{0,w}$. The arbitrariness of ε gives that $f \in \mathcal{B}_{0,w}$.

Remark 5.2.8. By [56, Proposition 4.8], if $g \in \mathcal{H}(\mathbb{D})$ and $\mathsf{Im}\, g$ has a finite angular limit at e^{it}, then $(z - e^{it})g'(z)$ has angular limit 0 at e^{it}. This result implies that if $g \in \mathcal{B}$ and $\mathsf{Im}\, g$ has a finite angular limit at e^{it} for almost every $e^{it} \in \mathbb{T}$, then $g \in \mathcal{B}_{0,w}$. In particular, $H^p \cap \mathcal{B} \subset \mathcal{B}_{0,w}$. This last inclusion was firstly noticed by Pavlović [51, Corollary, 2.1]. As far as we know this is the first paper where the space $\mathcal{B}_{0,w}$ was considered.

We are going to prove that $T_g\colon RM(1,q) \to RM(1,q)$ is weakly compact if and only $g \in \mathcal{B}_{0,w}$. Next theorem provides one of the implications. A preliminary lemma is needed.

Lemma 5.2.9. *Given B, $c > 0$ there exists $\delta \in (0, 1/2)$ such that for $g \in \mathcal{B}$ satisfying*

$$|g'(z)|(1-|z|) \leq B, \qquad \text{for all } z \in \mathbb{D}, \tag{5.2.1}$$

$\eta \in (0, 1/2)$ and $e^{ia} \in \mathbb{T}$ satisfying

$$|g'((1-\eta)e^{ia})|\eta > 2c, \tag{5.2.2}$$

we have

$$|g'(re^{i\theta})| > \frac{c}{\eta},$$

whenever $|r - (1-\eta)| < \delta\eta$ and $|\theta - a| < \delta\eta$.

Proof. It is not difficult to show (see for instance the proof of [27, Theorem 5.5]) that if $g \in \mathcal{B}$ satisfies (5.2.1) for $B > 0$, then it also satisfies

$$|g''(z)| \leq \frac{4B}{(1-|z|)^2}, \qquad \text{for all } z \in \mathbb{D}. \tag{5.2.3}$$

Assume now that $\eta \in (0, 1/2)$ and $e^{ia} \in \mathbb{T}$ satisfy (5.2.2), and pick any $\delta \in (0, 1/2)$. If $|r - (1-\eta)| < \delta\eta$ and $|\theta - a| < \delta\eta$, then $|re^{i\theta} - (1-\eta)e^{ia}| < 2\eta\delta$ and every point w in the segment joining $re^{i\theta}$ and $(1-\eta)e^{ia}$ has module $|w| < (1-\eta) + \delta\eta$. Hence, for all these w, we have

$$|g''(w)| < \frac{4B}{(1-\delta)^2\eta^2}$$

and, by the mean value inequality,

$$|g'((1-\eta)e^{ia}) - g'(re^{i\theta})| < \frac{8B\eta\delta}{(1-\delta)^2\eta^2} \leq \frac{32B\delta}{\eta} \leq \frac{c}{\eta},$$

if $\delta \leq c/32B$. The lemma follows since, by (5.2.2),

$$|g'(re^{i\theta})| \geq |g'((1-\eta)e^{ia})| - \frac{c}{\eta} > \frac{2c - c}{\eta} = \frac{c}{\eta}.$$

□

Theorem 5.2.10. *Let $1 < q < +\infty$ and $g \in \mathcal{B} \setminus \mathcal{B}_{0,w}$. Then the operator*

$$R_g : RM(1, q) \to L^1([0, 1) \times \mathbb{T})$$

defined by

$$R_g(f)(r, e^{i\theta}) := f(re^{i\theta})g'(re^{i\theta})(1 - r), \qquad r \in [0, 1), \quad e^{i\theta} \in \mathbb{T},$$

is not weakly compact.

Proof. We will denote by m_1 both the Lebesgue measure on $[0, 1)$ and the arc length measure on $\mathbb{T}$, and by m_2 the product measure $m_2 = m_1 \times m_1$ on $[0, 1) \times \mathbb{T}$. In order to prove that R_g is not weakly compact, and using the Dunford-Pettis Theorem (see Theorem A.2.4), we need to show that the image by R_g of the unit ball of $RM(1, q)$ is not uniformly integrable (see Definition A.2.3). This will be done if we show the existence of two constants C, $\alpha > 0$ such that, for every $\varepsilon > 0$, there exits $f \in RM(1, q)$ and a measurable set $D \subset [0, 1) \times \mathbb{T}$ such that

$$\text{(a)} \;\; m_2(D) < \varepsilon, \qquad \text{(b)} \;\; \rho_{1,q}(f) \leq C, \qquad \text{and} \qquad \text{(c)} \;\; \int_D |R_g(f)|\, dm_2 > \alpha. \tag{5.2.4}$$

The condition $g \in \mathcal{B} \setminus \mathcal{B}_{0,w}$ yields the existence of two constants B, $c > 0$ and a measurable set $A \subset \mathbb{T}$ of positive measure such that

$$|g'(z)|(1 - |z|) \leq B, \qquad \text{for all } z \in \mathbb{D},$$

and

$$\limsup_{r\to 1^-} |g'(re^{i\theta})|(1-r) > 2c, \qquad \text{for every } e^{i\theta} \in A. \tag{5.2.5}$$

Fix $m_1(A) > \beta > 0$. Finally take $M > 1$ big enough (to be determined later).

In order to get the conditions in (5.2.4), fix $\varepsilon \in (0,1)$. For every $e^{ia} \in A$, there exists $\varepsilon_a \in (0, \varepsilon/4\pi)$ such that

$$|g'((1-\varepsilon_a)e^{ia})|\varepsilon_a > 2c. \tag{5.2.6}$$

Recall that $M > 1$ is big enough and consider, for every $e^{ia} \in A$, the open arc

$$J_a := \{e^{it} : t \in (a - M\varepsilon_a, a + M\varepsilon_a)\}.$$

The family of all these arcs is a covering of A. So passing first through a compact set $K \subset A$ with $m_1(K) > \beta$ in order to get a finite covering and then using Hardy-Littlewood covering lemma (see for instance [61, Lemma 7.3]) there exist $N \in \mathbb{N}$ and $a_1, a_2, \ldots, a_N$ such that $\{J_{a_k} : k = 1, 2, \ldots, N\}$ is a family of pairwise disjoint arcs with

$$\sum_{k=1}^{N} m(J_{a_k}) > \beta/3. \tag{5.2.7}$$

To simplify the notation we put ε_k and J_k instead of ε_{a_k} and J_{a_k} respectively. From (5.2.7) we get

$$\sum_{k=1}^{N} \varepsilon_k > \frac{\beta}{6M}. \tag{5.2.8}$$

We will also consider the arcs

$$L_k := \{e^{it} : t \in [a_k - \delta\varepsilon_k, a_k + \delta\varepsilon_k]\},$$

where δ is the one in Lemma 5.2.9, and the subsets of $[0,1) \times \mathbb{T}$,

$$D_k := [1 - \varepsilon_k - \delta\varepsilon_k, 1 - \varepsilon_k + \delta\varepsilon_k] \times L_k, \qquad D = \bigcup_{k=1}^{N} D_k.$$

Observe that D is a compact subset of $(1 - \varepsilon/2\pi, 1) \times \mathbb{T}$ and therefore

$$m_2(D) < \frac{\varepsilon}{2\pi} 2\pi = \varepsilon.$$

This yields (5.2.4)(a).

Let us define the function f. Consider, for $z \in \mathbb{D}$,

$$u_k(z) = \frac{\varepsilon_k^2}{\big(z-(1+\varepsilon_k)e^{ia_k}\big)^3}, \qquad \text{and} \qquad f(z) = \sum_{k=1}^N u_k(z). \tag{5.2.9}$$

For every $e^{i\theta} \in \mathbb{T}$ and $1 \le k \le N$, define

$$\varphi_k(e^{i\theta}) = \int_0^1 |u_k(re^{i\theta})|\, dr.$$

We will use the following estimate about φ_m to be proved later.
Claim. Let $\theta \in \mathbb{R}$ such that $|\theta - a_k| \le \pi$. Then

$$\varphi_k(e^{i\theta}) \le \min\Big\{1, \frac{8\varepsilon_k^2}{|\theta-a_k|^2}\Big\}.$$

We prove this claim at the end of the proof.

Observe that, if $(r, e^{i\theta}) \in D_k$, then

$$\begin{aligned} |re^{i\theta} - (1+\varepsilon_k)e^{ia_k}| &\le r|e^{i\theta} - e^{ia_k}| + (1+\varepsilon_k) - r \\ &\le r\delta\varepsilon_k + (1+\varepsilon_k) - (1-\varepsilon_k - \delta\varepsilon_k) \le 2(1+\delta)\varepsilon_k < 3\varepsilon_k, \end{aligned}$$

consequently

$$|u_k(re^{i\theta})| \ge \frac{\varepsilon_k^2}{(3\varepsilon_k)^3} = \frac{1}{27\varepsilon_k},$$

and, by Lemma 5.2.9 with ε_k in the place of η,

$$|u_k(re^{i\theta})||g'(re^{i\theta})|(1-r) \ge \frac{1}{27\varepsilon_k}\frac{c}{\varepsilon_k}(1-\delta)\varepsilon_k \ge \frac{c(1-\delta)}{27\varepsilon_k},$$

and

$$\int_{D_k} |R_g u_k|\, dm_2 \ge m_2(D_k)\frac{c(1-\delta)}{27\varepsilon_k} = \frac{(2\delta\varepsilon_k)^2 c(1-\delta)}{27\varepsilon_k} \ge \frac{c\delta^2\varepsilon_k}{14}. \tag{5.2.10}$$

Therefore, for every k, we have, since $|R_g h(r, e^{i\theta})| \le B|h(re^{i\theta})|$,

$$\int_{D_k} |R_g f|\, dm_2 \ge \int_{D_k} |R_g u_k| - \sum_{j\ne k}\int_{D_k} |R_g u_j|\, dm_2 \ge \frac{c\delta^2}{14}\varepsilon_k - B\sum_{j\ne k}\int_{L_k} \varphi_j(e^{i\theta})\, dm(e^{i\theta}).$$

As $L_k \subset J_k$ and the J_k's are pairwise disjoint, so are the L_k's and the D_k's. Hence, adding up these inequalities from $k=1$ to $k=N$, using (5.2.7) and taking into account that $L_k \subset \mathbb{T} \setminus J_j$, for $k \neq j$, we get

$$\int_D |R_g f|\, dm_2 = \sum_{k=1}^N \int_{D_k} |R_g f|\, dm_2 \geq \frac{c\delta^2}{14} \sum_{k=1}^N \varepsilon_k - B \sum_{j=1}^N \sum_{k=1, k\neq j}^N \int_{L_k} \varphi_j(e^{i\theta})\, dm(e^{i\theta})$$
$$\geq \frac{c\delta^2}{14} \sum_{k=1}^N \varepsilon_k - B \sum_{j=1}^N \int_{\mathbb{T}\setminus J_j} \varphi_j(e^{i\theta})\, dm(e^{i\theta}).$$

By the Claim, we have

$$\int_{\mathbb{T}\setminus J_j} \varphi_j(e^{i\theta})\, dm(e^{i\theta}) = \int_{M\varepsilon_j < |\theta - a_j| < \pi} \varphi_j(e^{i\theta})\, d\theta$$
$$\leq 2 \int_{M\varepsilon_j}^{\pi} \frac{8\varepsilon_j^2}{t^2}\, dt \leq 16\varepsilon_j^2 \int_{M\varepsilon_j}^{+\infty} \frac{dt}{t^2} = \frac{16\varepsilon_j^2}{M\varepsilon_j}.$$

Putting together the last two estimates, if $M > \frac{2\times 16\times 14}{c\delta^2} B$, we obtain

$$\int_D |R_g f|\, dm_2 \geq \Big(\frac{c\delta^2}{14} - \frac{16B}{M}\Big) \sum_{k=1}^N \varepsilon_k \geq \frac{c\delta^2}{28} \sum_{k=1}^N \varepsilon_k,$$

and, by (5.2.8),

$$\int_D |R_g f|\, dm_2 > \frac{c\delta^2}{28} \frac{\beta}{6M} = \frac{c\delta^2 \beta}{168M} := \alpha.$$

We have established (5.2.4) (c).

Now we prove the bound for $\rho_{1,q}(f)$. Naturally we have

$$\rho_{1,q}(f) \leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{k=1}^N \varphi_k(e^{i\theta})\Big)^q d\theta\right)^{1/q}.$$

In order to apply the estimate in the Claim, the condition $|\theta - a_k| \leq \pi$ has to be satisfied. Let us assume that all the a_k's belong to the interval $[0, 2\pi)$, then, for all $\theta \in (-\pi, \pi]$, either $|\theta - a_k| \leq \pi$ or $|\theta - (a_k - 2\pi)| \leq \pi$. Define

$$I_k^+ := [a_k - \varepsilon_k, a_k + \varepsilon_k] \qquad \text{and} \qquad I_k^- := [a_k - 2\pi - \varepsilon_k, a_k - 2\pi + \varepsilon_k].$$

Observe that, if g is the characteristic function of the interval $[a - \varepsilon, a + \varepsilon]$ and $\mathcal{M}g$ is its Hardy-Littlewood maximal function, we have $\mathcal{M}g(t) = 1$, if $t \in (a - \varepsilon, a + \varepsilon)$, and, for $t \notin (a - \varepsilon, a + \varepsilon)$,

$$\mathcal{M}g(t) \geq \frac{1}{2|t-a|} \int_{t-|t-a|}^{t+|t-a|} g(u)\, du = \frac{\varepsilon}{2|t-a|}\,.$$

Now if g_k^+ is the characteristic function of I_k^+ and g_k^- is the characteristic function of I_k^-, using the Claim, we have, for every $\theta \in (-\pi, \pi]$,

$$\varphi_k(e^{i\theta}) \leq 32\big[\big(\mathcal{M}g_k^+\big)^2(\theta) + \big(\mathcal{M}g_k^-\big)^2(\theta)\big].$$

Therefore,

$$\rho_{1,q}(f) \leq 32\bigg(\int_{\mathbb{R}} \Big(\sum_{k=1}^{N} \big[\big(\mathcal{M}g_k^+\big)^2(t) + \big(\mathcal{M}g_k^-\big)^2(t)\big]\Big)^q \, dt\bigg)^{1/q} = 32\|H\|^2_{L^{2q}(\mathbb{R})},$$

where

$$H = \Big(\sum_{k=1}^{N} \big[\big(\mathcal{M}g_k^+\big)^2 + \big(\mathcal{M}g_k^-\big)^2\big]\Big)^{1/2}.$$

Applying Theorem 3.2.7, there exists a constant $A_{2,2q} > 0$ such that

$$\|H\|_{L^{2q}(\mathbb{R})} \leq A_{2,2q}\|h\|_{L^{2q}(\mathbb{R})}, \qquad \text{for } \ h = \big(\sum_k [(g_k^+)^2 + (g_k^-)^2]\big)^{1/2}.$$

Since all the intervals I_k^+'s and I_k^-'s are pairwise disjoint, we see easily that h is the characteristic of the union of all these intervals and so $\|h\|_{L^{2q}(\mathbb{R})} \leq (4\pi)^{1/2q}$. Finally we get

$$\rho_{1,q}(f) \leq 32\|H\|^2_{L^{2q}(\mathbb{R})} \leq 32A^2_{2,2q}\|h\|^2_{L^{2q}(\mathbb{R})} \leq 32A^2_{2,2q}(4\pi)^{1/q} =: C,$$

and we finish because we have proved (5.2.4) (b).

Proof of the Claim. By rotation invariance, we can assume $a_k = 0$. Then, for all $\theta \in [-\pi, \pi]$ and all $r \in [0, 1]$, we have $|re^{i\theta} - (1 + \varepsilon_k)| \geq |(1 + \varepsilon_k) - r|$. This yields $|u_k(re^{i\theta})| \leq |u_k(r)|$, and

$$\varphi_k(e^{i\theta}) \leq \varphi_k(e^{i0}) = \int_0^1 \frac{\varepsilon_k^2}{(1 + \varepsilon_k - r)^3}\, dr = \Big(\frac{\varepsilon_k^2}{2(1 + \varepsilon_k - r)^2}\Big]_{r=0}^{r=1} < \frac{1}{2} \leq 1. \quad (5.2.11)$$

On the other side, for all $z = re^{i\theta} \in \mathbb{D}$, we have

$$|1 + \varepsilon_k - z| \geq |1 - z| \geq \begin{cases} |\sin\theta| \geq 2|\theta|/\pi, & \text{if } \ 0 < |\theta| < \pi/2, \\ 1, & \text{if } \ \pi/2 \leq |\theta| \leq \pi. \end{cases} \quad (5.2.12)$$

Therefore, if $\pi/2 \leq |\theta| \leq \pi$, we have

$$|1 - re^{i\theta}|^3 \geq |1 - re^{i\theta}|^2 = 1 + r^2 - 2r\cos\theta \geq 1 + r^2 \qquad \text{and} \qquad |u_k(re^{i\theta})| \leq \frac{\varepsilon_k^2}{1 + r^2}.$$

Integrating

$$\varphi_k(e^{i\theta}) \le \varepsilon_k^2 \int_0^1 \frac{dr}{1+r^2} = \frac{\pi\varepsilon_k^2}{4} \le \frac{\pi^3}{32}\frac{8\varepsilon_k^2}{|\theta|^2} < \frac{8\varepsilon_k^2}{|\theta|^2}, \qquad \text{if } \frac{\pi}{2} \le |\theta| \le \pi. \tag{5.2.13}$$

For $1 \le |\theta| \le \pi/2$, we use the first case in (5.2.12). We have

$$\varphi_k(e^{i\theta}) \le \varepsilon_k^2 \Big(\frac{\pi}{2|\theta|}\Big)^3 \le \frac{\pi^3\varepsilon_k^2}{8|\theta|^2} = \frac{\pi^3}{64}\frac{8\varepsilon_k^2}{|\theta|^2} < \frac{8\varepsilon_k^2}{|\theta|^2}, \qquad \text{if } 1 \le |\theta| \le \frac{\pi}{2}. \tag{5.2.14}$$

Finally, for $|\theta| < 1$, we have

$$\varphi_k(e^{i\theta}) \le \varepsilon_k^2 \int_0^1 \frac{dr}{|1-re^{i\theta}|^3} \le \varepsilon_k^2 \int_0^{1-|\theta|} \frac{dr}{(1-r)^3} + \varepsilon_k^2 \int_{1-|\theta|}^1 \frac{\pi^3}{8|\theta|^3}\, dr,$$

and we get

$$\varphi_k(e^{i\theta}) \le \varepsilon_k^2 \Big(\frac{1}{2|\theta|^2} + \frac{\pi^3}{8|\theta|^2}\Big) \le \Big(\frac{1}{2} + \frac{\pi^3}{8}\Big)\frac{\varepsilon_k^2}{|\theta|^2} < \frac{8\varepsilon_k^2}{|\theta|^2}, \qquad \text{if } 0 \le |\theta| < 1. \tag{5.2.15}$$

Putting together (5.2.11), (5.2.13), (5.2.14), and (5.2.15), the claim follows. □

Theorem 5.2.11. *Let $1 < q < +\infty$ and $g \in \mathcal{B}$. Then $T_g : RM(1,q) \to RM(1,q)$ is weakly compact if and only if $g \in \mathcal{B}_{0,w}$.*

Proof. Assume that $g \in \mathcal{B}_{0,w}$. For each $\varepsilon, \delta > 0$, we set

$$A(\delta,\varepsilon) := \Big\{\theta \in \mathbb{T} :\ (1-r)|g'(re^{i\theta})| < \varepsilon, \quad \text{for all } r \in (1-\delta, 1)\Big\}.$$

Fixed $m \in \mathbb{N}$. By hypothesis, we have that

$$m_1\left(A\left(\frac{1}{n}, \frac{1}{2^m}\right)\right) \to 2\pi \quad \text{as} \quad n \to \infty,$$

since $A\left(\frac{1}{n}, \frac{1}{2^m}\right) \subset A\left(\frac{1}{n+1}, \frac{1}{2^m}\right)$. Hence, for each $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ such that

$$m_1\left(A\left(\frac{1}{n_m}, \frac{1}{2^m}\right)\right) > 2\pi - \frac{1}{m^2}.$$

Since $\left(\bigcap_{m\ge k} A\left(\frac{1}{n_m}, \frac{1}{2^m}\right)\right)^c = \bigcup_{m\ge k} A\left(\frac{1}{n_m}, \frac{1}{2^m}\right)^c$, we have

$$\lim_{k\to\infty} m_1\left(\bigcap_{m\ge k} A\left(\frac{1}{n_m}, \frac{1}{2^m}\right)\right) = 2\pi. \tag{5.2.16}$$

Fix $\varepsilon > 0$, by (5.2.16), there is $k = k(\varepsilon)$ such that $m_1(A_\varepsilon) > 2\pi - \varepsilon$ where

$$A_\varepsilon := \bigcap_{m \geq k} A\left(\frac{1}{n_m}, \frac{1}{2^m}\right). \tag{5.2.17}$$

This means that given $\theta \in A_\varepsilon$, for each $m \geq k$,

$$(1-r)|g'(re^{i\theta})| < 1/2^m$$

whenever $1 - 1/n_m < r < 1$.

To obtain the weak compactness, we apply Lemma 5.2.1. Let $\{f_n\} \in B_{RM(1,q)}$ be a sequence uniformly convergent to 0 on compact sets. Define the functions

$$H_n(\theta) := \int_0^1 |f_n(re^{i\theta})g'(re^{i\theta})|(1-r)\,dr \quad \text{and} \quad F_n(\theta) := \int_0^1 |f_n(re^{i\theta})|\,dr.$$

Using that T_g is bounded on $RM(1,q)$ and Proposition 4.2.7, the sequence $\{H_n\}$ is bounded on $L^q(\mathbb{T})$. Then, by the reflexivity of this space, we can find a subsequence $\{H_{n_k}\}$ convergent in the weak topology to a function $h \in L^q(\mathbb{T})$. Therefore, there is $G_k \in \mathrm{co}\{H_{n_k}, H_{n_{k+1}}, \dots\}$ such that $\|G_k - h\|_{L^q(\mathbb{T})} \to 0$ (see Theorem A.2.1). We claim that $h = 0$ almost everywhere. To settle this fact, fix $\varepsilon > 0$. By (5.2.17), there are $N = N(\varepsilon) \in \mathbb{N}$ and a measurable set A_ε with $m_1(A_\varepsilon) > 2\pi - \varepsilon$ and for every $\theta \in A_\varepsilon$ and $m \geq N$,

$$(1-r)|g'(re^{i\theta})| < 1/2^m$$

whenever $1 - 1/n_m < r < 1$. We may assume that $1/2^N < \varepsilon$ and that for $m \geq M_0$ and $r < 1 - 1/n_N$,

$$|f_m(re^{i\theta})g'(re^{i\theta})|(1-r) \leq \varepsilon$$

(remember that the sequence $\{f_n\}$ converges uniformly to 0 on the disc centered at 0 and radius $1 - 1/n_N$). Thus, for $n \geq M_0$ and $\theta \in A_\varepsilon$,

$$\begin{aligned} H_n(\theta) &= \int_0^{1-1/n_N} |f_n(re^{i\theta})g'(re^{i\theta})|(1-r)\,dr + \int_{1-1/n_N}^1 |f_n(re^{i\theta})g'(re^{i\theta})|(1-r)\,dr \\ &\leq \varepsilon + \varepsilon F_n(\theta). \end{aligned} \tag{5.2.18}$$

So, it follows that $\|H_n \chi_{A_\varepsilon}\|_{L^q(\mathbb{T})} \leq 2\varepsilon$ for all $n > N$. Hence, $\|G_n \chi_{A_\varepsilon}\|_{L^q(\mathbb{T})} < 3\varepsilon$ for n large enough. This implies that $\|h\chi_{A_\varepsilon}\|_{L^q(\mathbb{T})} \leq 3\varepsilon$. The arbitrariness of ε and the fact that $m_1(A_\varepsilon) > 2\pi - \varepsilon$ implies that $h = 0$ almost everywhere and $\|G_k\|_{L^q(\mathbb{T})} \to 0$.

Notice that we can express G_k in the following way

$$G_k = \sum_{j=k}^{\infty} \alpha_{k,j} H_{n_j}$$

where $\alpha_{k,j} \geq 0$, $\sum_{j=k}^{\infty} \alpha_{k,j} = 1$ and, for each k, the set $\{j \geq k : \alpha_{k,j} \neq 0\}$ is finite. Thus the functions

$$g_k := \sum_{j=n}^{\infty} \alpha_{k,j} f_{n_j},$$

are well-defined and it follows, using Lemma 4.1.2, that

$$\int_0^1 |T_g g_k(re^{i\theta})| \, dr \leq \int_0^1 |g_k(re^{i\theta})||g(re^{i\theta})|(1-r) \, dr \leq \sum_{j=k}^{\infty} \alpha_{k,j} H_{n_j}(\theta) = G_k(\theta).$$

Hence $\rho_{1,q}(T_g g_k) \to 0$ when $k \to \infty$. Therefore, using Lemma 5.2.1 we conclude that $T_g : RM(1,q) \to RM(1,q)$ is weakly compact.

For the converse implication assume that $g \in \mathcal{B} \setminus \mathcal{B}_{0,w}$. By Propositions 4.1.3 and 4.2.7, given $f \in RM(1,q)$, it holds that $\rho_{1,q}(T_g f) \asymp \|R_g f\|_{Y_{1,q}}$ where R_g is the operator introduced in Theorem 5.2.10 with the identification $Y_{1,q} \subset Y_{1,1} = L^1([0,1) \times \mathbb{T})$. Therefore, $R_g : RM(1,q) \to L^1([0,1) \times \mathbb{T})$ is bounded and not weakly compact. By Lemma 5.2.1, there exists a sequence $\{f_n\}$ in the unit ball of $RM(1,q)$ convergent to 0 uniformly on compact sets of $\mathbb{D}$ such that no convex combination $g_k \in \text{co}\{f_k, f_{k+1}, \dots\}$ satisfies that $\|R_g g_k\|_{L^1([0,1)\times\mathbb{T})} \to 0$ when $k \to \infty$. Applying again Propositions 4.2.7, no convex combination $g_k \in \text{co}\{f_k, f_{k+1}, \dots\}$ satisfies that $\rho_{1,1}(T_g g_k) \to 0$ when $k \to \infty$. By Lemma 5.2.1(2), $T_g : RM(1,q) \to RM(1,1)$ is not weakly compact and, by Proposition 5.2.5, $T_g : RM(1,q) \to RM(1,q)$ is not weakly compact. □

We point out that beyond what Proposition 5.2.5 might suggest, the weak compactness $T_g : RM(1,q) \to RM(1,q)$ does not depend on q when it runs the interval $(1, +\infty)$.

Remark 5.2.12. Using [57, Proposition 5.4, p. 601], there are $g_1, g_2 \in \mathcal{B}$ such that

$$|g_1'(z)| + |g_2'(z)| \geq \frac{1}{1-|z|}, \quad \text{for all } z \in \mathbb{D}.$$

Therefore either g_1 or g_2 does not belong to $\mathcal{B}_{0,w}$, so that $\mathcal{B} \setminus \mathcal{B}_{0,w}$ is not empty. Moreover the function $g(z) = \log(1-z)$, $z \in \mathbb{D}$, belongs to $\mathcal{B}_{0,w} \setminus \mathcal{B}_0$. In fact, writing $g_\theta(z) = \log(1 - ze^{i\theta})$ for $\theta \in [0,\pi)$ and $z \in \mathbb{D}$, one can see that $\|g_\theta - g_{\tilde{\theta}}\|_{\mathcal{B}} \geq 1$ if

$\theta \neq \tilde{\theta}$, because

$$\|g_\theta - g_{\tilde{\theta}}\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} \frac{|1-e^{i(\tilde{\theta}-\theta)}|(1-|z|)}{|1-ze^{i\theta}||1-ze^{i\tilde{\theta}}|} \geq \sup_{r\in[0,1)} \frac{|1-e^{i(\tilde{\theta}-\theta)}|(1-r)}{|1-re^{-i\theta}e^{i\theta}||1-re^{-i\theta}e^{i\tilde{\theta}}|}$$
$$= \sup_{r\in[0,1)} \frac{|1-e^{i(\tilde{\theta}-\theta)}|}{|1-re^{i(\tilde{\theta}-\theta)}|} \geq \lim_{r\to 1^-} \frac{|1-e^{i(\tilde{\theta}-\theta)}|}{|1-re^{i(\tilde{\theta}-\theta)}|} = 1.$$

Then $\mathcal{B}_{0,w}$ is a non-separable closed subspace of $\mathcal{B}$. The separability of $\mathcal{B}_0$ and the non-separability of $\mathcal{B}_{0,w}$ show that the second one is much bigger than the first one. Therefore there are integral operators T_g bounded and not weekly compact on $RM(1,q)$ and integral operators T_g weakly compact and not compact.

Remark 5.2.13. For $q = 1$, Theorem 5.2.11 is not valid. Since $RM(1,1) \cong \ell^1$, the operator $T_g : RM(1,1) \to RM(1,1)$ is weakly compact if and only if $T_g : RM(1,1) \to RM(1,1)$ is compact if and only if $g \in \mathcal{B}_0$.

Remark 5.2.14. We also have that Theorem 5.2.11 is not valid for $q = +\infty$. Example 5.2.3 shows that for $g(z) = -\log(1-z)$ the operator $T_g : RM(1,\infty) \to RM(1,\infty)$ is not weakly compact, but $g \in \mathcal{B}_{0,w}$. We point out that we have no characterization of the weak compactness of T_g on $RM(1,\infty)$.

5.2.3 The case $q = 1$

To finish, we turn our attention to the weak compactness of $T_g : RM(p,1) \to RM(p,1)$. In fact, we are going to prove that the weak compactness and the compactness are equivalent. The following four lemmas will be necessary to give a characterization of the weak compactness of T_g by means of sequences in $(RM(p,1))^*$ which are equivalent to the basis of c_0.

Lemma 5.2.15. *Let $1 \leq p \leq +\infty$ and let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that there are constants $C_1, C_2, C_3 > 0$ satisfying:*

1. $\sum_{n=1}^{\infty} |f(z_n)|(1-|z_n|)^{1+\frac{1}{p}} \leq C_1\ \rho_{p,1}(f)$ *for all* $f \in RM(p,1)$.
2. *For all $m \in \mathbb{N}$, there is $f_m \in RM(p,1)$ with $\rho_{p,1}(f_m) \leq C_2$ such that*

$$|f_m(z_m)|(1-|z_m|)^{1+\frac{1}{p}} \geq \frac{1}{C_3}, \qquad \sum_{n\neq m} |f_m(z_n)|(1-|z_n|)^{1+\frac{1}{p}} \leq \frac{1}{2C_3}.$$

Then $\{(1-|z_n|)^{1+\frac{1}{p}}\ \delta_{z_n}\}$ is equivalent to the basis of c_0 in $(RM(p,1))^*$.

Proof. We will present the proof for p finite, being the other case similar. It is sufficient to prove that there are constants $A > 0$ and $B > 0$ such that

$$A \max_{1\le k\le N}\{|\alpha_k|\} \le \left\|\sum_{k=1}^{N} \alpha_k (1-|z_k|)^{1+\frac{1}{p}} \delta_{z_k}\right\|_{(RM(p,1))^*} \le B \max_{1\le k\le N}\{|\alpha_k|\}$$

for every N and for every sequence $\{\alpha_k\}$.

First, using assertion (1), we have that

$$\left\|\sum_{k=1}^{N} \alpha_k (1-|z_k|)^{1+\frac{1}{p}} \delta_{z_k}\right\|_{(RM(p,1))^*} = \sup_{f\in B_{RM(p,1)}} \left|\sum_{k=1}^{N} \alpha_k (1-|z_k|)^{1+\frac{1}{p}} f(z_k)\right|$$
$$\le \max_{1\le k\le N}\{|\alpha_k|\} \sup_{f\in B_{RM(p,1)}} \sum_{k=1}^{\infty} |f(z_k)|(1-|z_k|)^{1+\frac{1}{p}} \le C_1 \max_{1\le k\le N}\{|\alpha_k|\}.$$

The remaining inequality proceeds as follows employing this time assertion (2). We choose m such that $\max_{1\le k\le N}\{|\alpha_k|\} = |\alpha_m|$. Then, we obtain that

$$\left\|\sum_{k=1}^{N} \alpha_k (1-|z_k|)^{1+\frac{1}{p}} \delta_{z_k}\right\|_{(RM(p,1))^*} \ge \frac{1}{C_2}\left|\sum_{k=1}^{N} \alpha_k (1-|z_k|)^{1+\frac{1}{p}} f_m(z_k)\right|$$
$$\ge \frac{|\alpha_m|}{C_2}(1-|z_m|)^{1+\frac{1}{p}}|f_m(z_m)| - \frac{1}{C_2}\sum_{k=1,\,k\ne m}^{N} |\alpha_k|(1-|z_k|)^{1+\frac{1}{p}}|f_m(z_k)|$$
$$\ge \frac{1}{2C_2C_3} \max_{1\le k\le N}\{|\alpha_k|\}.$$

□

The proof for the following lemma follows in a similar way.

Lemma 5.2.16. *Let $1 \le p \le +\infty$ and let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that there are constants $C_1, C_2, C_3 > 0$ satisfying:*

1. $\sum_{n=1}^{\infty} |f'(z_n)|(1-|z_n|)^{2+\frac{1}{p}} \le C_1\ \rho_{p,1}(f)$ *for all $f \in RM(p,1)$.*
2. *For all $m \in \mathbb{N}$, there is $f_m \in RM(p,1)$ with $\rho_{p,1}(f_m) \le C_2$ such that*

$$|f_m'(z_m)|(1-|z_m|)^{2+\frac{1}{p}} \ge \frac{1}{C_3}, \qquad \sum_{n\ne m} |f_m'(z_n)|(1-|z_n|)^{2+\frac{1}{p}} \le \frac{1}{2C_3}.$$

Then $\{(1-|z_n|)^{2+\frac{1}{p}}\ \delta'_{z_n}\}$ is equivalent to the basis of c_0 in $(RM(p,1))^*$.

Lemma 5.2.17. *Let $c \in (0,1/2)$, then there are two constants μ_1, μ_2, depending* *only on c, such that for all $f \in \mathcal{H}(\mathbb{D})$*

$$|f(z)| \leq \frac{\mu_1}{(1-|z|)^2} \int_{\theta-c(1-r)}^{\theta+c(1-r)} \left(\int_{r-c(1-r)}^{r+c(1-r)} |f(\rho e^{it})| \, d\rho \right) dt,$$

$$(1-|z|)|f'(z)| \leq \frac{\mu_2}{(1-|z|)^2} \int_{\theta-c(1-r)}^{\theta+c(1-r)} \left(\int_{r-c(1-r)}^{r+c(1-r)} |f(\rho e^{it})| \, d\rho \right) dt,$$

where $z = re^{i\theta}$ with $1 > r \geq \frac{1}{2}$.

Proof. Let $c \in (0, 1/2)$, $z = re^{i\theta} \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$. It can be proved that for $\lambda \in \left(0, \frac{1}{\sqrt{4+c^2}}\right)$ we have that

$$D(z, \lambda c(1-|z|)) \subset \left\{\rho e^{it} \in \mathbb{D} : \; \rho \in I, t \in J\right\}$$

where $I = [r - c(1-r), r + c(1-r)]$ and $J = [\theta - c(1-r), \theta + c(1-r)]$.

Applying the mean value inequality over $D(z, \lambda c(1-|z|))$ we have that

$$\begin{aligned} |f(z)| &\leq \frac{1}{\pi\lambda^2 c^2 (1-r)^2} \int_{D(z,\lambda c(1-|z|))} |f(w)| \, dw \\ &\leq \frac{1}{\pi\lambda^2 c^2 (1-r)^2} \int_{\theta-c(1-r)}^{\theta+c(1-r)} \left(\int_{r-c(1-r)}^{r+c(1-r)} |f(\rho e^{it}| \, d\rho \right) dt. \end{aligned}$$

The estimate of the derivative follows in analogous way, but using Cauchy's integral formula as we did in the proof of Proposition 1.5.6. □

Lemma 5.2.18. *Let $1 \leq p < +\infty$ and $\{z_k\} \subset \mathbb{D} \setminus \frac{1}{2}\mathbb{D}$ such that $\frac{1-|z_{n+1}|}{1-|z_n|} \leq \frac{\beta}{n^2}$ with $\beta \in \left(0, \frac{1}{1+2^{4+\frac{1}{p}}}\right)$. Then both $\{(1-|z_n|)^{1+\frac{1}{p}} \; \delta_{z_n}\}$ and $\{(1-|z_n|)^{2+\frac{1}{p}} \; \delta'_{z_n}\}$ are equivalent to the basis of c_0 in $(RM(p,1))^*$.*

Proof. We will prove the result just for $\{(1-|z_n|)^{1+\frac{1}{p}} \; \delta_{z_n}\}$ using Lemma 5.2.15 and omit the proof of the other case because it can be obtained following a similar argument. Set $z_n = r_n e^{i\theta_n}$ and $\varepsilon_n = 1 - r_n$. For a certain constant $c \in (0, 1/2)$ we define the sets

$$I_n := [\theta_n - c(1-r_n), \theta_n + c(1-r_n)]$$

and

$$J_n := [r_n - c(1-r_n), r_n + c(1-r_n)].$$

Now, we denote by $A_n := \cup_{k>n} I_k$ where $m(A_n) \leq \sum_{k=n+1}^{\infty} 2c\varepsilon_k \leq \frac{1}{n^2}\varepsilon_n$.

Let $f \in B_{RM(p,1)}$. Applying Lemma 5.2.17 to each element of the sequence $\{z_n\}$, we obtain that

$$(1-|z_n|)^{1+\frac{1}{p}}|f(z_n)| \leq \frac{\mu_1}{\varepsilon_n^{1/p'}} \int_{I_n}\int_{J_n} |f(\rho e^{it})|\,d\rho\,dt.$$

We split the integrals, apply Hölder's inequality in the first integral and Proposition 1.5.2 in the second integral:

$$\begin{aligned}(1-|z_n|)^{1+\frac{1}{p}}|f(z_n)| &\leq \frac{\mu_1}{\varepsilon_n^{1/p'}} \int_{I_n\setminus A_n}\int_{J_n} |f(\rho e^{it})|\,d\rho\,dt + \frac{\mu_1}{\varepsilon_n^{1/p'}} \int_{A_n}\int_{J_n} |f(\rho e^{it})|\,d\rho\,dt \\ &\leq \frac{\mu_1(2c\varepsilon_n)^{1/p'}}{\varepsilon_n^{1/p'}} \int_{I_n\setminus A_n} \left(\int_{J_n} |f(\rho e^{it})|^p\,d\rho\right)^{1/p} dt + \frac{C\mu_1 m(A_n)m(J_n)}{\varepsilon_n^{1/p'}(1-(r_n+c(1-r_n)))^{1+\frac{1}{p}}} \\ &\leq \mu_1(2c)^{1/p'} \int_{I_n\setminus A_n} \left(\int_{J_n} |f(\rho e^{it})|^p\,d\rho\right)^{1/p} dt + \frac{2Cc\mu_1}{n^2(1-c)^{1+\frac{1}{p}}}.\end{aligned}$$

Since $\{I_n \setminus A_n\}$ are disjoint sets, we have that

$$\sum_{n=1}^{\infty}(1-|z_n|)^{1+\frac{1}{p}}|f(z_n)| \leq \mu_1(2c)^{1/p'} + \frac{\pi^2 Cc\mu_1}{3(1-c)^{1+\frac{1}{p}}} = C_1.$$

Hence, we have proved that the sequence $\{z_n\}$ satisfies statement (1) of Lemma 5.2.15. To prove the remaining condition we consider the family of holomorphic functions $f_n(z) := \frac{1-r_n}{(1-\overline{z_n}z)^{2+\frac{1}{p}}}$, $z \in \mathbb{D}$. We have to show that $\{f_n\}$ satisfies statement (2) of Lemma 5.2.15.

Let us see that $\rho_{p,1}(f_n) \lesssim 1$. First of all, we observe that

$$\begin{aligned}\rho_{p,1}(f_n) &= \int_0^{2\pi}\left(\int_0^1 \frac{(1-r_n)^p}{|1-\overline{z_n}re^{i\theta}|^{2p+1}}\,dr\right)^{1/p} \frac{d\theta}{2\pi} \\ &\leq 8(1-r_n)\int_0^{\pi/4}\left(\int_0^1 \frac{1}{|1-rr_ne^{i\theta}|^{2p+1}}\,dr\right)^{1/p} \frac{d\theta}{2\pi}.\end{aligned}$$

Since

$$\begin{aligned}|1-rr_ne^{i\theta}|^2 &= (1-rr_n)^2 + 2rr_n(1-\cos(\theta)) \geq 2rr_n(1-\cos(\theta)) \\ &\geq 2(1-\theta)(1-\cos(\theta)) \geq 2\left(1-\frac{\pi}{4}\right)(1-\cos(\theta)) \geq \frac{1}{16}\theta^2\end{aligned}$$

whenever $0 \leq \theta \leq \pi/4$ and $1 \geq r \geq \frac{1-\theta}{r_n}$, we have

$$\begin{aligned}\int_0^{\pi/4}\left(\int_0^1 \frac{1}{|1-rr_ne^{i\theta}|^{2p+1}}\,dr\right)^{1/p} \frac{d\theta}{2\pi} &\leq \int_0^{1-r_n}\left(\int_0^1 \frac{1}{|1-rr_ne^{i\theta}|^{2p+1}}\,dr\right)^{1/p} \frac{d\theta}{2\pi} \\ &+ \int_{1-r_n}^{\pi/4}\left(\int_0^{\frac{1-\theta}{r_n}} \frac{1}{|1-rr_ne^{i\theta}|^{2p+1}}\,dr + \int_{\frac{1-\theta}{r_n}}^1 \frac{1}{|1-rr_ne^{i\theta}|^{2p+1}}\,dr\right)^{1/p} \frac{d\theta}{2\pi}\end{aligned}$$

$$\begin{aligned}
&\le \int_0^{1-r_n} \left(\int_0^1 \frac{1}{(1-rr_n)^{2p+1}}\, dr \right)^{1/p} \frac{d\theta}{2\pi} \\
&\quad + \int_{1-r_n}^{\pi/4} \left(\int_0^{\frac{1-\theta}{r_n}} \frac{1}{(1-rr_n)^{2p+1}}\, dr + 4^{2p+1} \int_{\frac{1-\theta}{r_n}}^1 \frac{1}{\theta^{2p+1}}\, dr \right)^{1/p} \frac{d\theta}{2\pi} \\
&\le \frac{1-r_n}{2\pi(2pr_n)^{1/p}} \left(\frac{1}{(1-r_n)^{2p}} \right)^{1/p} \\
&\quad + 4^3 \int_{1-r_n}^{\pi/4} \left(\frac{1}{2pr_n} \left(\frac{1}{\theta^{2p}} \right) + \frac{1}{\theta^{2p+1}} \left(1 - \frac{1-\theta}{r_n} \right) \right)^{1/p} \frac{d\theta}{2\pi} \\
&\le \frac{1}{2\pi(2pr_n)^{1/p}(1-r_n)} + 4^3 \frac{(2p+1)^{1/p}}{2\pi(2pr_n)^{1/p}} \int_{1-r_n}^{\pi/4} \frac{1}{\theta^2}\, d\theta \\
&\le \frac{1}{2\pi(2pr_n)^{1/p}(1-r_n)} + 4^3 \frac{(2p+1)^{1/p}}{2\pi(2pr_n)^{1/p}(1-r_n)} \le 4^3 \frac{(2p+1)^{1/p}+1}{2\pi(2pr_n)^{1/p}(1-r_n)}.
\end{aligned}$$

Therefore, we conclude that

$$\rho_{p,1}(f_n) \le 4^3 \frac{(2p+1)^{1/p}+1}{p^{1/p}} \frac{8}{2\pi} = C_2.$$

To finish the proof, we have to show that for a certain constant $C_3 > 0$ it holds

$$|f_m(z_m)|(1-|z_m|)^{1+\frac{1}{p}} \ge \frac{1}{C_3}, \qquad \sum_{n \ne m} |f_m(z_n)|(1-|z_n|)^{1+\frac{1}{p}} \le \frac{1}{2C_3}.$$

It is easy to see that

$$\begin{aligned}
|f_m(z_m)|(1-|z_m|)^{1+\frac{1}{p}} &= \frac{1-r_m}{(1-r_m^2)^{2+\frac{1}{p}}} (1-r_m)^{1+\frac{1}{p}} \\
&= \frac{1}{(1+r_m)^{2+\frac{1}{p}}} \ge \frac{1}{2^{2+\frac{1}{p}}} =: \frac{1}{C_3} > 0.
\end{aligned}$$

If $k > m$ then

$$|f_m(z_k)|(1-|z_k|)^{1+\frac{1}{p}} \le \frac{1-r_m}{(1-r_m)^{2+\frac{1}{p}}} (1-r_k)^{1+\frac{1}{p}} = \left(\frac{\varepsilon_k}{\varepsilon_m} \right)^{1+\frac{1}{p}}.$$

And if $k < m$, then

$$|f_m(z_k)|(1-|z_k|)^{1+\frac{1}{p}} \le \frac{1-r_m}{(1-r_k)^{2+\frac{1}{p}}} (1-r_k)^{1+\frac{1}{p}} = \frac{\varepsilon_m}{\varepsilon_k}.$$

Now, bearing in mind that $0 < \beta < \frac{1}{1+2^{4+\frac{1}{p}}}$ and $\frac{\varepsilon_{n+1}}{\varepsilon_n} < \beta$ for all n, we obtain

$$\sum_{k\neq m} |f_m(z_k)|(1-|z_k|)^{1+\frac{1}{p}} < \sum_{k=m+1}^{\infty} \beta^{k+\frac{k}{p}} + \sum_{k=1}^{m-1} \beta^k$$
$$\leq \frac{\beta^{1+\frac{1}{p}}}{1-\beta^{1+\frac{1}{p}}} + \frac{\beta-\beta^m}{1-\beta} < \frac{2\beta}{1-\beta} < \frac{1}{2C_3}.$$

Therefore, applying Lemma 5.2.15 we have proved that $\{(1-|z_n|)^{1+\frac{1}{p}}\ \delta_{z_n}\}$ is equivalent to the basis of c_0 in $(RM(p,1))^*$. □

As a consequence of these lemmas we obtain in the following three results new characterizations of the compactness of $T_g : RM(p,1) \to RM(p,1)$ and its equivalence with the weak compactness.

Theorem 5.2.19. *Let $g \in \mathcal{B}$ and $1 \leq p < +\infty$. If $T_g : RM(p,1) \to RM(p,1)$ is not compact, then T_g^* fixes a copy of c_0.*

Proof. Since $g \in \mathcal{B}$ and $T_g : RM(p,1) \to RM(p,1)$ is not compact, by Theorem 5.1.3, there are constants $C, \delta > 0$ and a sequence $\{z_n\}$ where $|z_n| \to 1$ such that

$$\delta \leq |g'(z_n)|(1-|z_n|) \leq C.$$

Using that

$$T_g^*((1-|z_n|)^{2+\frac{1}{p}}\delta'_{z_n}) = g'(z_n)(1-|z_n|)^{2+\frac{1}{p}}\delta_{z_n}, \quad |g'(z_n)|(1-|z_n|)^{2+\frac{1}{p}} \asymp (1-|z_n|)^{1+\frac{1}{p}},$$

and extracting a subsequence such that $\frac{1-|z_{n_{k+1}}|}{1-|z_{n_k}|} \leq \frac{\beta}{k^2}$ with $\beta \in \left(0, \frac{1}{1+2^{4+\frac{1}{p}}}\right)$, by Lemma 5.2.18, we conclude that T_g^* fixes a copy of c_0. □

Corollary 5.2.20. *Let $g \in \mathcal{B}$ and $1 \leq p < +\infty$. Then the following are equivalent:*

1. *$T_g : RM(p,1) \to RM(p,1)$ is not compact;*
2. *T_g^* fixes a copy of c_0;*
3. *T_g fixes a copy of ℓ^1.*

Proof. Being clear that (3) implies (1) and, using Theorem 5.2.19, that (1) implies (2), we just have to justify that (2) implies (3). But this is a consequence of the fact that if the adjoint of a bounded operator between two Banach spaces fixes a copy of c_0, then the operator fixes a copy of ℓ^1. This result is probably well-known by specialist (and essentially due to C. Bessaga and A. Pełczyński), but we could not find any reference so that we schedule its proof for the sake of completeness.

Assume that $T : X \to Y$ is bounded and T^* is fixes a copy of c_0. Then T^* is not unconditionally converging (see Theorem A.3.4), so that T is not an ℓ^1-cosingular operator (see Theorem A.3.6). But a standard argument shows that in this case T fixes a copy of ℓ^1. □

It is worth pointing out that if an operator fixes a copy of ℓ^1 then, in general, its adjoint does not fix a copy of c_0 [41, Example 1.2].

Corollary 5.2.21. *Let $g \in \mathcal{B}$ and $1 \leq p < +\infty$. Then $T_g : RM(p,1) \to RM(p,1)$ is weakly compact if and only if it is compact (and if and only if $g \in \mathcal{B}_0$).*

Chapter 6

Tent spaces

This chapter is dedicated to present part of the work about tent spaces carried out in collaboration with Prof. P. Galanopoulos (see [4]). This collaboration began during my research stay at the Aristotle University of Thessaloniki at the beginning of 2020. We introduce the tent spaces and some properties that will be of great importance in the study of our spaces of average radial integrability. In particular, we show that, for $1 \leq p, q < +\infty$, we can express the $RM(p,q)$ space as a tent space for certain measure. With this fact we can extend Proposition 4.2.7 of Chapter 4 for the cases (p, ∞), with $1 \leq p < +\infty$.

In addition, in the last sections we study a problem of Carleson measure, posed by Luecking in [48], which allows us to obtain a characterization of the boundedness of the integration operators $T_g : RM(p,q) \to H^s$ and $T_g : RM(p,q) \to RM(t,s)$ for $1 \leq p, q, t, s < +\infty$.

6.1 Tent spaces

Let $\xi \in \mathbb{T}$. We define the non-tangential regions $\Gamma(\xi)$ and $S_C(\xi)$ as follows

$$\Gamma(\xi) := \left\{ r\xi e^{i\theta} : |\theta| < 1 - r,\ 0 < r < 1 \right\},$$
$$S_C(\xi) := \{ z \in \mathbb{D} : |z - \xi| < C(1 - |z|) \},$$

where $C > 1$ and $|\theta| := \min \{|\theta + 2k\pi| : k \in \mathbb{Z}\}$. Let us remember that Stolz regions $S_C(\xi)$ were already introduced in Chapter 1.

The boundary of $\Gamma(\xi)$ is a curve that goes non-tangentially to ξ. See Figure 6.1a. In fact, for C large enough one can see that $\Gamma(\xi) \subset S_C(\xi)$ (see Figure 6.1b) and for C close to 1, if $z \in S_C(\xi)$ and z is close to ξ, it holds that $z \in \Gamma(\xi)$ (see Figure 6.1c).

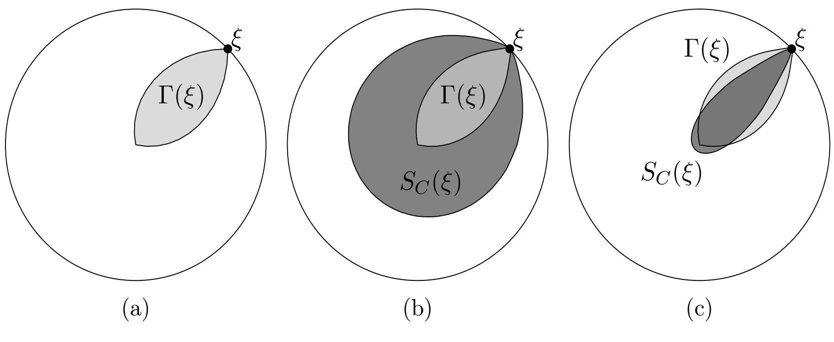


Figure 6.1

Tent spaces T_p^q were introduced by Coifman, Meyer, and Stein in [22] to provide a natural setting for the study of things like maximal functions, square functions, and to simplify some results about Cauchy integrals on Lipschitz curves. Since then, these spaces have been studied widely by many authors (e.g. [47], [48], [42], [11], [21], [53], [50]). Let us recall their definitions.

Definition 6.1.1. The tent space T_p^q, when $0 < p, q < +\infty$, consists of those measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{T_p^q} = \left(\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f(z)|^p \frac{dm_2(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{1/q} < +\infty. \tag{6.1.1}$$

Analogously, the space T_∞^q consists of those measurable functions f on $\mathbb{D}$ with

$$\|f\|_{T_\infty^q} = \left(\int_{\mathbb{T}} \operatorname*{ess\,sup}_{z \in \Gamma(\xi)} |f(z)|^q \, |d\xi| \right)^{1/q} < +\infty, \quad \text{if } q < +\infty, \tag{6.1.2}$$

where the essential supremum is taken with respect to the Lebesgue measure m_2. For the case $q = +\infty$ and $p < +\infty$, the space T_p^∞ consists of those measurable functions f on $\mathbb{D}$ with

$$\|f\|_{T_p^\infty} = \sup_{\substack{\xi \in \mathbb{T} \\ 0<h<1}} \left(\frac{1}{h} \int_{D(\xi,h)\cap\mathbb{D}} |f(z)|^p \, dm_2(z) \right)^{1/p} < +\infty. \tag{6.1.3}$$

Notice that $f \in T_p^\infty$ if and only if $d\mu_f(z) = |f(z)|^p \, dm_2(z)$ is a Carleson measure on $\mathbb{D}$ (see Section 6.4 of this chapter, especially condition (6.4.1)).

A first non-elementary result we will use is the following lemma.

Lemma 6.1.2. *[11, Lemma 4, p. 66] Let $0 < p, q < +\infty$, $C > 1$, and $\lambda > \max\{1, p/q\}$. Then there are positive constants $C_1 = C_1(p, q, \lambda, C)$ and $C_2 = C_2(p, q, \lambda, C)$ such that*

$$C_1 \int_{\mathbb{T}} \mu(S_C(\xi))^{q/p} |d\xi| \le \int_{\mathbb{T}} \left(\int_{\mathbb{D}} \left(\frac{1-|z|}{|1-z\bar{\xi}|} \right)^{\lambda} d\mu(z) \right)^{q/p} |d\xi| \le C_2 \int_{\mathbb{T}} \mu(S_C(\xi))^{q/p} |d\xi|$$

for every positive measure μ on $\mathbb{D}$.

Remark 6.1.3. Using Lemma 6.1.2 for the measure $d\mu_f(z) = |f(z)|^p \frac{dm_2(z)}{1-|z|}$, it is clear that in (6.1.1) one can replace the set $\Gamma(\xi)$ for any Stolz region $S_C(\xi)$ getting an equivalent norm.

Definition 6.1.4. If we take holomorphic functions instead of measurable functions, we define the tent space of holomorphic functions as $AT_p^q := T_p^q \cap \mathcal{H}(\mathbb{D})$.

In [53] Perälä proved the Littlewood-Paley inequalities for tent spaces AT_p^q.

Theorem 6.1.5. *[53, Theorem 2, p. 9] Let $1 \leq p, q < +\infty$. Then, we have that*

$$\|f'(z)(1-|z|)\|_{T_p^q} \asymp \|f\|_{T_p^q}$$

for $f \in \mathcal{H}(\mathbb{D})$.

The following remark is a technical issue that we will use from time to time through the chapter.

Remark 6.1.6. Let $C > 1$ and $z \in \mathbb{D}$, then

$$\int_{\mathbb{T}} \chi_{\Gamma(\xi)}(z)\ |d\xi| \asymp \int_{\mathbb{T}} \chi_{S_C(\xi)}(z)\ |d\xi| \asymp (1-|z|).$$

Firstly, we can assume without loss of generality that $z \in [0,1)$. The estimate

$$\int_{\mathbb{T}} \chi_{\Gamma(\xi)}(z)\ |d\xi| \asymp (1-z)$$

follows immediately, because $z \in \Gamma(e^{i\theta})$ if and only if $|\theta| < 1-z$.

In order to prove the estimate

$$\int_{\mathbb{T}} \chi_{S_C(\xi)}(z)\ |d\xi| \asymp (1-z),$$

we will study the following two cases. If $C > \frac{1+z}{1-z}$, we have that

$$|z-\xi| \leq 1+z < C(1-z)$$

for all $\xi \in \mathbb{T}$. So that, $\int_{\mathbb{T}} \chi_{S_C(\xi)}(z)\ |d\xi| = 2\pi$. Now, assume that $C \leq \frac{1+z}{1-z}$. One can check easily that

$$\int_{\mathbb{T}} \chi_{S_C(\xi)}(z)\ |d\xi| = 2\theta_0,$$

where $\theta_0 \in [0,\pi]$ is such that $|z - e^{i\theta_0}| = C(1-z)$. So that, we have that

$$1-\cos(\theta_0) = \frac{(C^2-1)(1-|z|)^2}{2|z|}.$$

Using that $\frac{2}{\pi^2} \leq 2\frac{1-\cos(\theta)}{\theta^2} \leq 1$, for $0 \leq \theta \leq \pi/2$, and $C \leq \frac{1+z}{1-z}$, we obtain that

$$\theta_0 \asymp \sqrt{1-\cos(\theta_0)} \asymp (1-z).$$

Therefore, we conclude that

$$\int_{\mathbb{T}} \chi_{S_C(\xi)}(z)\ |d\xi| \asymp (1-|z|).$$

The following lemma will be useful in order to obtain equivalent expressions when we consider different non-tangential regions.

Lemma 6.1.7. *Let $M_1 > M > 1$. There is a constant $C > 0$ such that*

$$\int_{\mathbb{T}} \left(\sup_{z_k \in S_{M_1}(\xi)} |\lambda_k|^q \right) |d\xi| \leq C \int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} |\lambda_k|^q \right) |d\xi|$$

for all $0 < q < +\infty$, and for all sequences $\{z_k\} \subset \mathbb{D}$ and $\{\lambda_k\} \subset \mathbb{C}$.

Proof. If $z \in \mathbb{D}$, we define the open arcs $L_M(z) = \{\xi \in \mathbb{T} \ : z \in S_M(\xi)\}$. Take $f_M(\xi) = \sup_{z_k \in S_M(\xi)} |\lambda_k|^q$. Since

$$\int_{\mathbb{T}} |f(\xi)|^q \, |d\xi| = q \int_0^\infty \lambda^{q-1} \, m_1(\{\xi \in \mathbb{T} \ : \ |f(\xi)| > \lambda\}) \, d\lambda,$$

for all measurable functions, to prove this lemma, it is enough to show that there is a constant $C > 0$ such that

$$m_1 \left(\{\xi \in \mathbb{T} \ : f_{M_1}(\xi) > \alpha\} \right) \leq C m_1 \left(\{\xi \in \mathbb{T} \ : f_M(\xi) > \alpha\} \right)$$

for all $\alpha > 0$.

Fix $\alpha > 0$ and let A be the set

$$A = \{k \in \mathbb{N} \ : \ |\lambda_k|^q > \alpha\}.$$

One can check that

$$\{\xi \in \mathbb{T} \ : f_{M_1}(\xi) > \alpha\} = \bigcup_{k \in A} L_{M_1}(z_k)$$

and analogously for M.

Since there is a compact set $F \subset \bigcup_{k \in A} L_{M_1}(z_k)$ such that

$$m_1(F) \geq \frac{1}{2} m_1 \left(\{\xi \in \mathbb{T} \ : f_{M_1}(\xi) > \alpha\} \right),$$

we can obtain a finite cover $\{L_{M_1}(z_k) \ : \ k \in A_0\}$ of F, where $A_0 \subset A$ is a finite set. Applying the covering lemma (see [61, Lemma 7.3, p. 137]), there is a sub-collection of these arcs $\{L_{M_1}(z_k) \ : \ k \in A_1\}$, with $A_1 \subset A_0$, which are pairwise disjoint and

$$m_1(F) \leq 3 \sum_{k \in A_1} m_1(L_{M_1}(z_k)).$$

Hence, it follows that

$$m_1(\{\xi\in\mathbb{T}\ : f_{M_1}(\xi)>\alpha\}) \le 6\sum_{k\in A_1} m_1(L_M(z_k)).$$

Now, by Remark 6.1.6, there exists $C_1 = C_1(M, M_1)$ such that $m_1(L_{M_1}(z)) \le C_1\ m_1(L_M(z))$ for all $z\in\mathbb{D}$. So that, it follows

$$m_1(\{\xi\in\mathbb{T}\ : f_{M_1}(\xi)>\alpha\}) \le 6\sum_{k\in A_1} m_1(L_{M_1}(z_k)) \le 6C_1\sum_{k\in A_1} m_1(L_M(z_k)).$$

Moreover, the sets $\{L_M(z_k)\ :\ k\in A_1\}$ are pairwise disjoint because $L_M(z_k)\subset L_{M_1}(z_k)$ where $M_1>M>1$. Therefore, we conclude that

$$\begin{aligned} m_1(\{\xi\in\mathbb{T}\ : f_{M_1}(\xi)>\alpha\}) &\le 6C_1\sum_{k\in A_1} m_1(L_M(z_k)) \le 6C_1\ m_1\left(\bigcup_{k\in A} L_M(z_k)\right) \\ &= 6C_1\ m_1(\{\xi\in\mathbb{T}\ : f_M(\xi)>\alpha\}). \end{aligned}$$

□

6.2 Tent spaces and $RM(p,q)$

In this section, we provide an identification of the spaces $RM(p,q)$ as tent spaces T_p^q for holomorphic functions. This relationship between Triebel spaces, a more general family of spaces that includes our $RM(p,q)$ spaces, and tent spaces is commented in some works, such as [64], [32] and [21]. Nevertheless, we could not find any explicit reference with a proof of this fact, so that we include it.

We start with the case $p=+\infty$.

Theorem 6.2.1. *Let $1\le q<+\infty$. Then $H^q = RM(\infty,q) = AT_\infty^q$.*

Proof. First, we take $C>0$ such that $\Gamma(\xi)\subset S_C(\xi)$ for every $\xi\in\mathbb{T}$. Fix $f\in\mathcal{H}(\mathbb{D})$. By Theorem 1.2.5, we have that there is a constant $\gamma=\gamma(C)$ such that

$$\begin{aligned} \rho_{\infty,q}^q(f) &= \int_{\mathbb{T}}\left(\sup_{r\in[0,1)}|f(r\xi)|^q\right)\ |d\xi| \le \int_{\mathbb{T}}\left(\sup_{z\in\Gamma(\xi)}|f(z)|^q\right)\ |d\xi| \\ &= \|f\|_{T_\infty^q}^q \le \int_{\mathbb{T}}\left(\sup_{z\in S_C(\xi)}|f(z)|^q\right)\ |d\xi| \le \gamma\|f\|_{H^q}^q. \end{aligned}$$

Since $RM(\infty,q)=H^q$ (see page 23), we are done. □

Remark 6.2.2. Notice that above proof shows that one can obtain an equivalent norm in AT_∞^q replacing in (6.1.2) the set $\Gamma(\xi)$ by any Stolz region $S_C(\xi)$. This fact complements Remark 6.1.3 for the case $p = +\infty$. This can also be obtained applying Lemma 6.1.7 to $\{z_k\}$ a dense sequence in $\mathbb{D}$ and with $\lambda_k = f(z_k)$ for all k.

In the next result we use a similar argument to one used by Pavlović in [51] in order to prove Littlewood-Paley type inequalities for Hardy spaces.

Theorem 6.2.3. *Let $1 \le p, q < +\infty$. Then for $f \in \mathcal{H}(\mathbb{D})$ we have that*

$$\rho_{p,q}(f) \asymp \|f\|_{T_p^q},$$
$$\rho_{p,q}(f'(z)(1-|z|)) \asymp \|f'(z)(1-|z|)\|_{T_p^q}.$$

In particular, $RM(p,q) = AT_p^q$.

Proof. Fix $f \in \mathcal{H}(\mathbb{D})$. First of all, we will show, for $C = 7$,

$$\rho_{p,q}(f) \lesssim \left(\int_0^{2\pi} \left(\int_{S_C(e^{i\theta})} |f(w)|^p \, \frac{dm_2(w)}{1-|w|} \right)^{q/p} \frac{d\theta}{2\pi} \right)^{1/q}. \qquad (6.2.1)$$

Fixed $\theta \in [0, 2\pi]$, we have

$$\begin{aligned} \int_0^1 |f(re^{i\theta})|^p \, dr &= \sum_{n=0}^{\infty} \int_{1-2^{-n}}^{1-2^{-(n+1)}} |f(re^{i\theta})|^p \, dr \\ &\le \sum_{n=0}^{\infty} \sup_{1-2^{-n}<r<1-2^{-(n+1)}} |f(re^{i\theta})|^p (2^{-n} - 2^{-(n+1)}) \\ &= \sum_{n=0}^{\infty} 2^{-(n+1)} \sup_{2^{-(n+1)}<1-r<2^{-n}} |f(re^{i\theta})|^p. \end{aligned}$$

Applying the mean value property for subharmonic functions, given $1 - 2^{-n} < r < 1 - 2^{-(n+1)}$, we obtain

$$\begin{aligned} \frac{2^{-2(n+2)}}{2} |f(re^{i\theta})|^p &\le \frac{1}{2\pi} \int_0^{2^{-(n+2)}} \int_0^{2\pi} |f(re^{i\theta} + \rho e^{i\varphi})|^p \rho \, d\varphi \, d\rho \\ &\le \frac{1}{2\pi} \int_{E_n(\theta)} |f(w)|^p \, dm_2(w) \end{aligned}$$

where $E_n(\theta) = \{z \in \mathbb{D} : |z - (1-2^{-n})e^{i\theta}| < \frac{3}{2^{n+2}}\}$. Thus,

$$\int_0^1 |f(re^{i\theta})|^p \, dr \le \frac{16}{\pi} \sum_{n=0}^{\infty} \frac{2^{-(n+1)}}{2^{-2n}} \int_{E_n(\theta)} |f(w)|^p \, dm_2(w).$$

Moreover, it is easy to see that $E_j(\theta) \cap E_l(\theta) = \emptyset$, if $|j - l| \ge 3$. For $z \in E_n(\theta)$ we have

$$|z| < \frac{3}{2^{n+2}} + 1 - \frac{1}{2^n} = 1 - \frac{1}{2^{(n+2)}},$$

and we deduce

$$|z-e^{i\theta}| \le |z-(1-2^{-n})e^{i\theta}+(1-2^{-n})e^{i\theta}-e^{i\theta}| < \frac{3}{2^{n+2}}+\frac{1}{2^n}$$
$$= \frac{7}{2^{(n+2)}} < 7(1-|z|).$$

That is, $E_n(\theta)\subset S_7(e^{i\theta})$ for all $n\in\mathbb{N}$. Therefore, it follows that

$$\int_0^1 |f(re^{i\theta})|^p\, dr \le C\sum_{n=0}^{\infty} 2^n \int_{E_n(\theta)} |f(w)|^p \frac{(1-|w|)}{(1-|w|)}\, dm_2(w)$$
$$\le 2C\sum_{n=0}^{\infty}\int_{E_n(\theta)} |f(w)|^p\frac{dm_2(w)}{1-|w|} \le 6\ C\int_{S_7(e^{i\theta})}|f(w)|^p\frac{dm_2(w)}{1-|w|}.$$

From this, we clearly have (6.2.1).

Now, we prove that

$$\left(\int_0^{2\pi}\left(\int_{\Gamma(e^{i\theta})}|f(w)|^p\ \frac{dm_2(w)}{1-|w|}\right)^{q/p}\frac{d\theta}{2\pi}\right)^{1/q} \lesssim \rho_{p,q}(f).$$

Notice that for $\xi\in\mathbb{T}$

$$\int_{\Gamma(\xi)}|f(w)|^p\ \frac{dm_2(w)}{1-|w|} = \int_0^1\int_{|\theta|<1-r}|f(re^{i\theta}\xi)|^p(1-r)^{-1}\ r\ d\theta\ dr$$
$$\le \sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}}\int_{|\theta|<2^{-n}}|f(re^{i\theta}\xi)|^p\ (1-r)^{-1}\ d\theta dr$$
$$= \sum_{n=0}^{\infty}\int_{|\theta|<2^{-n}}\int_{1-2^{-n}}^{1-2^{-(n+1)}}|f(re^{i\theta}\xi)|^p(1-r)^{-1}\ dr\ d\theta$$
$$\le \sum_{n=0}^{\infty}2^{n+1}\int_{|\theta|<2^{-n}}h_n((1-2^{-n-2})\xi e^{i\theta})\ d\theta$$

where

$$h_n(z)=\int_{1-2^{-n}}^{1-2^{-(n+1)}}\left|f\left(\frac{rz}{1-2^{-n-2}}\right)\right|^p\ dr.$$

For each n, the function h_n is logarithmically-subharmonic (see Theorem A.4.1). Bearing in mind that $|(1-2^{-n-2})\xi e^{i\theta}-\xi|\le C2^{-n-2}$ for $|\theta|<2^{-n}$, where C is an absolute constant, it follows that $(1-2^{-(n+2)})\xi e^{i\theta}\in S_C(\xi)$ and

$$\int_0^1\int_{|\theta|<1-r}|f(re^{i\theta}\xi)|^p(1-r)^{-1}\ d\theta\ dr \le 4\sum_{n=0}^{\infty}M_*h_n(\xi)$$

where, as usual, M_* denotes the non-tangential maximal operator, that is, $M_*u(\xi) = \sup_{z\in S_C(\xi)} u(z)$ for every positive function $u : \mathbb{D} \to \mathbb{R}$.

Applying Theorem A.4.2, there is a constant $C(p,q)$ such that

$$\begin{aligned}
&\left(\int_{\mathbb{T}} \left(\int_0^1 \int_{|\theta|<1-r} |f(re^{i\theta}\xi)|^p (1-r)^{-1}\, d\theta\, dr\right)^{q/p} |d\xi|\right)^{1/q}\\
&\le \left(\int_{\mathbb{T}} \left(4\sum_{n=0}^{\infty} M_* h_n(\xi)\right)^{q/p} |d\xi|\right)^{1/q}\\
&\le C(p,q) \sup_{0\le s<1} \left(\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} h_n(s\xi)\right)^{q/p} |d\xi|\right)^{1/q}\\
&= C(p,q) \sup_{0\le s<1} \left(\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} \int_{1-2^{-n}}^{1-2^{-(n+1)}} \left|f\left(\frac{rs\xi}{1-2^{-n-2}}\right)\right|^p dr\right)^{q/p} |d\xi|\right)^{1/q}.
\end{aligned}$$

Doing the change of variable $u = \frac{r}{1-2^{-n-2}}$, it follows

$$\begin{aligned}
\int_{1-2^{-n}}^{1-2^{-(n+1)}} \left|f\left(\frac{rs\xi}{1-2^{-n-2}}\right)\right|^p dr &= \int_{\frac{1-2^{-n}}{1-2^{-(n+2)}}}^{\frac{1-2^{-(n+1)}}{1-2^{-(n+2)}}} |f(rs\xi)|^p \, (1-2^{-(n+2)})\, dr\\
&\le \int_{1-2^{-n}}^{1-2^{-(n+2)}} |f(rs\xi)|^p \, dr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f(z\xi)|^p \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right)^{1/q}\\
&\le C(p,q) \sup_{0\le s<1} \left(\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} \int_{1-2^{-n}}^{1-2^{-(n+2)}} |f(rs\xi)|^p \, dr\right)^{q/p} |d\xi|\right)^{1/q}\\
&\le 2C(p,q) \sup_{0\le s<1} \rho_{p,q}(f_s).
\end{aligned}$$

By Proposition 1.5.11(1), we conclude that

$$\left(\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f(z\xi)|^p \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right)^{1/q} \lesssim \rho_{p,q}(f).$$

Following the same argument, we can obtain the equivalent result for the derivative. □

On the contrary to the case when q is finite, $RM(p,\infty)$ does not coincide with T_p^∞ as we show in the next result.

Theorem 6.2.4. *Let $1 \le p < +\infty$. Then $RM(p,\infty) \subsetneq AT_p^\infty \subsetneq H_{v_p}^\infty$, where $H_{v_p}^\infty$ is the space of the holomorphic functions f such that $(1-|z|)^{1/p}|f(z)|$ is bounded in $\mathbb{D}$.*

Proof. Let f be a function that belongs to $RM(p,\infty)$. Fixed $w = re^{i\theta} \in \mathbb{D} \setminus \{0\}$, one can easily check that

$$D(e^{i\theta}, 1-r) \cap \mathbb{D} \subset \left\{\rho e^{\varphi i} \;:\; \rho \in [0,1),\; |\theta - \varphi| \le 2(1-r)\right\} =: A(re^{i\theta}).$$

It follows that

$$\begin{aligned}\int_{D(e^{i\theta},1-r)\cap\mathbb{D}} |f(z)|^p \, dm_2(z) &\le \int_{A(re^{i\theta})} |f(z)|^p \, dm_2(z)\\ &\le \int_{\{|\theta-\varphi|\le 2(1-r)\}} \int_0^1 |f(\rho e^{i\varphi})|^p \, d\rho \, d\varphi\\ &\le 4(1-r)\rho_{p,\infty}^p(f).\end{aligned}$$

Therefore, dividing by $1-r$ and taking supremum we obtain that $RM(p,\infty) \subset AT_p^\infty$.

Moreover, $RM(p,\infty) \ne AT_p^\infty$, because the function $f(z) = (1-z)^{-1/p}$ does not belong to $RM(p,\infty)$ (see Example 1.3.1), but does to AT_p^∞. To check the belonging of f to AT_p^∞ it is enough to prove that there is a constant $C > 0$ such that

$$\frac{1}{h} \int_{D(1,h)\cap\mathbb{D}} \frac{1}{|1-z|} \, dm_2(z) \le C$$

for all $h \in (0,1)$. Using a change of variable to polar coordinates, we obtain

$$\begin{aligned}\int_{D(1,h)\cap\mathbb{D}} \frac{1}{|1-z|} \, dm_2(z) &\le \int_{D(1,h)} \frac{1}{|1-z|} \, dm_2(z)\\ &= \int_0^{2\pi} \int_0^h \frac{\rho}{\rho} \, d\rho \, d\theta = 2\pi h\end{aligned}$$

for all $h \in (0,1)$. Therefore, $f \in AT_p^\infty$.

Let us show the inclusion $AT_p^\infty \subsetneq H_{v_p}^\infty$. Let $f \in AT_p^\infty$. Since f is a holomorphic function in $\mathbb{D}$, in order to show that $f \in H_{v_p}^\infty$, it is enough to consider the supremum in the set $|z| > 1/3$. Applying the mean value property, for $z = r\xi \in \mathbb{D} \setminus D(0,1/3)$, $r = |z|$, it follows

$$(1-|z|)^{1/p}|f(z)| \le \left(\frac{4}{\pi}\right)^{1/p} \left(\frac{1}{1-r} \int_{D\left(r\xi, \frac{1-r}{2}\right)} |f(w)|^p \, dm_2(w)\right)^{1/p}.$$

Since $D\left(z, \frac{1-|z|}{2}\right) \subset D\left(\xi, \frac{3}{2}(1-r)\right) \cap \mathbb{D}$, we have

$$(1-|z|)^{1/p}|f(z)| \leq \left(\frac{6}{\pi}\right)^{1/p} \left(\frac{2}{3(1-r)} \int_{D\left(\xi, \frac{3(1-r)}{2}\right)\cap\mathbb{D}} |f(w)|^p \, dm_2(w)\right)^{1/p}$$
$$\leq \left(\frac{6}{\pi}\right)^{1/p} \|f\|_{T_p^\infty}.$$

Thus, we have shown that $AT_p^\infty \subset H_{v_p}^\infty$. To check that $AT_p^\infty \neq H_{v_p}^\infty$, we use the fact that there are $f, g \in H_{v_p}^\infty$ such that

$$|f(z)| + |g(z)| \geq (1-|z|)^{-1/p}$$

for all $z \in \mathbb{D}$ (see [1, Lemma 1, p. 401]). Since $(1-|z|)^{-1/p}$ is not $L^p(\mathbb{D})$-integrable, then either f or g is not in A^p. So that, $H_{v_p}^\infty \not\subseteq A^p$. Moreover, one have that $AT_p^\infty \subset A^p$, because $f \in AT_p^\infty$ if and only if the measure $d\mu_f(z) = |f|^p \, dm_2(z)$ is a Carleson measure. Therefore, one can conclude that there is a function in $H_{v_p}^\infty$ that does not belong to AT_p^∞. So that we are done. □

In the case of $RM(p,q)$ spaces, the Littlewood-Paley type inequalities for certain cases have been proved in Proposition 4.1.3 and Proposition 4.2.7. Another novelty of this section is that Proposition 4.2.7 is extended to $RM(p,1)$, for $1 < p < +\infty$. It is enough to combine Theorem 6.1.5, due to Perälä, and Theorem 6.2.3.

Corollary 6.2.5. *Let $1 \leq p, q < +\infty$. Then for $f \in \mathcal{H}(\mathbb{D})$ we have that*

$$\rho_{p,q}(f) \asymp \rho_{p,q}(f'(z)(1-|z|)).$$

6.3 Tent space of sequences $T_p^q(Z)$

The discrete version of tent spaces will be an important tool to obtain the main results of this chapter. Most of the issues we present in this section are well-known for specialist so that many times we only need state them and provide a reference.

Let us recall that the hyperbolic metric (also called the Bergman metric or the Poincaré metric) is given by $\left(1-|z|^2\right)^{-1} |dz|$ (see [18, p. 10]) and whose induced distance on $\mathbb{D}$ is given by

$$\beta(z,w) = \frac{1}{2}\log\left(\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}\right),$$

where $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$, is the hyperbolic distance (also known as Bergman or Poincaré distance) on $\mathbb{D}$ and let $\Delta(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$ be the hyperbolic disc of radius $r > 0$ centered at $z \in \mathbb{D}$. It is well-known (see for instance [28, p. 40]) that $\Delta(z,r)$ is an euclidean disc where its euclidean center is

$$c = \frac{z(1-\tanh^2(r))}{1-\tanh^2(r)|z|^2}$$

and its euclidean radius is

$$R = \frac{\tanh(r)(1-|z|^2)}{1-\tanh^2(r)|z|^2}.$$

So that, it follows that

$$\pi \tanh^2(r)(1-|z|)^2 \le m_2(\Delta(z,r)) \le \frac{4\pi\tanh^2(r)}{(1-\tanh^2(r))^2}(1-|z|)^2. \tag{6.3.1}$$

Definition 6.3.1. The sequence $Z = \{z_n\}$ is a separated sequence if there is a constant $\delta > 0$ such that $\beta(z_j, z_k) \ge \delta$ for $j \ne k$. Moreover, a sequence $Z = \{z_n\}$ is said to be an (r,κ)-lattice (in the hyperbolic distance), for $r > \kappa > 0$, if

1. $\mathbb{D} = \bigcup_k \Delta(z_k, r)$,
2. the sets $\Delta(z_k, \kappa)$ are pairwise disjoint.

Notice that any (r,κ)-lattice is a separated sequence with constant $\delta = 2\kappa$.

Remark 6.3.2. Let us see that if $z, w \in \mathbb{D}$ such that $\beta(z,w) < r$, then there is a constant $C = C(r) > 0$ such that $\frac{1}{C}(1-|z|) \le 1-|w| \le C(1-|z|)$. It is enough to show that $\frac{1-|z|}{1-|w|} \le C$. Using that $1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2}$, it follows

$$\frac{1-|z|^2}{1-|w|^2} = \frac{(1-|z|^2)^2}{(1-|\varphi_z(w)|^2)|1-\overline{w}z|^2}.$$

Thus, we have

$$\frac{1-|z|^2}{1-|w|^2} \le \frac{(1-|z|^2)^2}{(1-\tanh^2(r))|1-\overline{w}z|^2} \le \frac{(1-|z|^2)^2}{(1-\tanh^2(r))(1-|z|)^2} \le \frac{4}{1-\tanh^2(r)},$$

because $|\varphi_z(w)| = \tanh(\beta(z,w)) \le \tanh(r)$.

Proposition 6.3.3. *Let $K \ge 1$, $R > 0$, and an (r,κ)-lattice $Z = \{z_k\}$. There is a positive integer $N = N(K,R,Z)$ such that for each point $z \in \mathbb{D}$ there are at most N hyperbolic discs $\Delta(z_k, Kr)$ satisfying $\Delta(z,R) \cap \Delta(z_k, Kr) \ne \emptyset$.*

Proof. Fix $z \in \mathbb{D}$ and $J_z = \{z_k \in Z\ : \Delta(z_k, Kr) \cap \Delta(z,R) \ne \emptyset\}$. Triangle inequality shows

$$\bigcup_{z_k \in J_z} \Delta(z_k, Kr) \subset \Delta(z, 2Kr + R).$$

Moreover, since $\Delta(z_k, \kappa) \subset \Delta(z_k, Kr)$, it follows

$$\bigcup_{z_k \in J_z} \Delta(z_k, \kappa) \subset \Delta(z, 2Kr + R).$$

Using (6.3.1) and the fact that the sets $\Delta(z_k, \kappa)$ are pairwise disjoint, we have that there are $C_1 = C_1(r, \kappa)$ and $C_2 = C_2(r, K, R)$ such that

$$C_1 \sum_{z_k \in J_z} (1 - |z_k|)^2 \leq \sum_{z_k \in J_z} m_2(\Delta(z_k, \kappa)) \leq m_2(\Delta(z, 2Kr + R)) \leq C_2(1 - |z|)^2.$$

By Remark 6.3.2 there is $C = C(r, K, R) > 0$ such that

$$(\# J_z)\ C_1\ C\ (1 - |z|^2) \leq C_2(1 - |z|^2),$$

because $\beta(z, z_k) < Kr + R$. So that, $\# J_z \leq \frac{C_2}{C_1\ C}$ for all $z \in \mathbb{D}$.

□

Next lemma is due to Wu and we include its proof for the sake of completeness.

Lemma 6.3.4. *[66, Lemma 2.3, p. 992] Let $M > 1$, $r \geq 0$ and $\xi \in \mathbb{T}$. If $M_* = (M + 1)e^{2r} - 1$, then*

$$\Delta(z, r) \subset S_{M_*}(\xi)$$

for all $z \in S_M(\xi)$.

Proof. Let $z \in S_M(\xi)$ and $w \in \Delta(z, r)$. We have to show that $w \in S_{M_*}(\xi)$. Using the triangle inequality, we obtain

$$\begin{aligned} |\xi - w| &\leq |\xi - z| + |z - w| < M(1 - |z|) + |z - w| \\ &\leq M(1 - |w| + |z - w|) + |z - w| \\ &= M(1 - |w|) + (M + 1)|z - w|. \end{aligned}$$

Recall that the euclidean center and radius of the disc $\Delta(w, r)$ are $c = \frac{w(1-\tanh^2(r))}{1-\tanh^2(r)|w|^2}$ and $R = \frac{\tanh(r)(1-|w|^2)}{1-\tanh^2(r)|w|^2}$, respectively. Since $w \in \Delta(z, r)$ is equivalent to $z \in \Delta(w, r)$, we have

$$\begin{aligned}|z-w| &\le |z-c|+|c-w| \\ &< \frac{\tanh(r)(1-|w|^2)}{1-\tanh^2(r)|w|^2}+|w|\left(1-\frac{1-\tanh^2(r)}{1-\tanh^2(r)|w|^2}\right) \\ &= \frac{\tanh(r)(1-|w|^2)}{1-\tanh(r)|w|}.\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}|\xi-w| &< M(1-|w|)+(M+1)\frac{\tanh(r)(1-|w|^2)}{1-\tanh(r)|w|} \\ &\le \left(M+(M+1)\frac{2\tanh(r)}{1-\tanh(r)}\right)(1-|w|) \\ &= \left(M\frac{1+\tanh(r)}{1-\tanh(r)}+\frac{2\tanh(r)}{1-\tanh(r)}\right)(1-|w|) \\ &= \left((M+1)e^{2r}-1\right)(1-|w|).\end{aligned}$$

□

Lemma 6.3.5. *Let $C>1$ and an (r,κ)-lattice $Z=\{z_k\}$. There is N such that for all $n\in\mathbb{N}$ and $\xi\in\mathbb{T}$ there are N hyperbolic discs $\Delta(z_k,r)$ that cover the set*

$$S_C(\xi)\cap\left\{z\in\mathbb{D}\;:\;\frac{1}{2^{n+1}}\le 1-|z|\le\frac{1}{2^n}\right\}.$$

Proof. Assume without loss of generality that $\xi=1$. We fix $\alpha_n=1-\frac{1}{2^n}$ for all $n\in\mathbb{N}$.

Using that $1-|\varphi_{\alpha_n}(z)|^2=\frac{(1-|z|^2)(1-\alpha_n^2)}{|1-\alpha_n z|^2}$, we have that

$$1-|\varphi_{\alpha_n}(z)|^2\ge\frac{1}{4^{n+1}}\frac{1}{|1-\alpha_n z|^2}$$

for $z\in A_C^n:=S_C(1)\cap\left\{z\in\mathbb{D}\;:\;\frac{1}{2^{n+1}}\le 1-|z|\le\frac{1}{2^n}\right\}$. Triangle inequality shows

$$|1-\alpha_n z|\le|1-z|+|z||1-\alpha_n|<C(1-|z|)+C(1-\alpha_n)\le\frac{2C}{2^n}$$

for $z\in A_C^n$. So that, it follows

$$1-|\varphi_{\alpha_n}(z)|^2>\frac{1}{16C^2}.$$

Therefore, for all $n\in\mathbb{N}$ we have $A_C^n\subset\Delta\left(\alpha_n,\tanh^{-1}\left(\sqrt{1-\frac{1}{16C^2}}\right)\right)$. Applying

Proposition 6.3.3, there are at most N hyperbolic discs such that

$$\Delta\left(\alpha_n, \tanh^{-1}\left(\sqrt{1-\frac{1}{16C^2}}\right)\right) \cap \Delta(z,r) \neq \emptyset.$$

Therefore, A_C^n is included in the union of such discs and we are done. □

Example 6.3.6. Let Z be the sequence formed by the centers of the Luecking regions (see Definition 4.2.1). Then there exist $r > \kappa > 0$ such that Z is an (r,κ)-lattice.

Proof. Observe that the centers of the Luecking regions are $z_{n,j} = \left(1 - \frac{3}{2^{n+2}}\right) e^{i\frac{2\pi(2j+1)}{2^{n+1}}}$ for all $n \in \mathbb{N}\cup\{0\}$ and $j \in \{0,1,\dots,2^n-1\}$. First, let us check that there is $r > 0$ such that $R_{n,j} \subset \Delta(z_{n,j}, r)$ for all $n \in \mathbb{N}\cup\{0\}$ and $j \in \{0,1,\dots,2^n-1\}$. Let $z \in R_{n,j}$. Considering the family of all path γ in $R_{n,j}$ joining z with $z_{n,j}$, we have

$$\beta(z, z_{n,j}) \leq \inf_{\gamma} \int_{\gamma} \frac{|dw|}{1-|w|^2} \leq 2^{n+1} \inf_{\gamma} \operatorname{length}(\gamma),$$

where length(γ) denotes the euclidean length of the curve γ. It is clear that we can always find such a path whose length is smaller than the length of the path γ_1 represented in the Figure 6.2. Consequently,

$$\begin{aligned}\beta(z, z_{n,j}) \leq 2^{n+1}\operatorname{length}(\gamma_1) &= 2^{n+1}\left(\frac{1}{2^{n+2}} + \frac{2\pi}{2^{n+1}}\left(1-\frac{1}{2^{n+1}}\right)\right)\\ &= \frac{1}{2} + 2\pi\left(1-\frac{1}{2^{n+1}}\right) < 2\pi + \frac{1}{2}.\end{aligned}$$

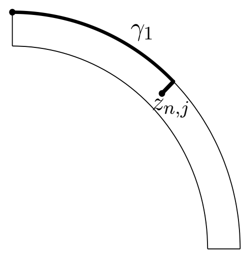


Figure 6.2

So that we can take $r = 2\pi + \frac{1}{2}$.

Now, we show that there is $\kappa > 0$ such that $\Delta(z_{n,j}, \kappa) \subset R_{n,j}$ for all $n \in \mathbb{N}\cup\{0\}$ and $j \in \{0,1,\dots,2^n-1\}$. If we consider the Luecking region $R_{n,j}$, we can take $\kappa > 0$

such that

$$\kappa < \min\left\{\beta\left(z_{n,j}, \left(1-\frac{1}{2^n}\right)e^{i\frac{2\pi(2j+1)}{2^{n+1}}}\right), \beta\left(z_{n,j}, \left(1-\frac{1}{2^{n+1}}\right)e^{i\frac{2\pi(2j+1)}{2^{n+1}}}\right)\right\}.$$

Indeed

$$\beta\left(z_{n,j}, \left(1-\frac{1}{2^n}\right)e^{i\frac{2\pi(2j+1)}{2^{n+1}}}\right) = \frac{1}{2}\log\left(\frac{2^{n+3}-3}{3(2^{n+1}-1)}\right) > \frac{1}{2}\log\left(\frac{4}{3}\right)$$

and

$$\beta\left(z_{n,j}, \left(1-\frac{1}{2^{n+1}}\right)e^{i\frac{2\pi(2j+1)}{2^{n+1}}}\right) = \frac{1}{2}\log\left(\frac{3(2^{n+2}-1)}{2^{n+3}-3}\right) > \frac{1}{2}\log\left(\frac{3}{2}\right).$$

So that, taking $\kappa = \frac{1}{2}\log\left(\min\left\{\frac{3}{2}, \frac{4}{3}\right\}\right) = \frac{1}{2}\log\left(\frac{4}{3}\right)$, we obtain that $\Delta(z_{n,j},\kappa) \subset R_{n,j}$. Since the Luecking regions are pairwise disjoint, then so are the hyperbolic discs $\Delta(z_{n,j},\kappa)$. □

We are aware that the constants r and κ that we have obtained in above example are far from optimal, but for our purposes they are enough.

Definition 6.3.7. Let $Z = \{z_n\}$ be an (r,κ)-lattice and $0 < p, q < +\infty$. We say that $\{\lambda_n\} \in T_p^q(Z)$ if

$$\|\{\lambda_n\}\|_{T_p^q(Z)} := \left(\int_{\mathbb{T}}\left(\sum_{z_n\in\Gamma(\xi)}|\lambda_n|^p\right)^{q/p}|d\xi|\right)^{1/q} < +\infty, \tag{6.3.2}$$

the sequence $\{\lambda_n\} \in T_\infty^q(Z)$ if

$$\|\{\lambda_n\}\|_{T_\infty^q(Z)} := \left(\int_{\mathbb{T}}\left(\sup_{z_n\in\Gamma(\xi)}|\lambda_n|\right)^{q}|d\xi|\right)^{1/q} < +\infty, \tag{6.3.3}$$

and $\{\lambda_n\} \in T_p^\infty(Z)$ if

$$\|\{\lambda_n\}\|_{T_p^\infty(Z)} = \sup_{\substack{\xi\in\mathbb{T}\\ 0<h<1}}\left(\frac{1}{h}\sum_{z_n\in D(\xi,h)\cap\mathbb{D}}|\lambda_n|^p(1-|z_n|^2)\right)^{1/p} < +\infty. \tag{6.3.4}$$

Remark 6.3.8. Using Lemma 6.1.2 for the measure $\mu = \sum_n |\lambda_n|^p\delta_{z_n}$, one can also replace the set $\Gamma(\xi)$ in (6.3.2) for any Stolz region $S_C(\xi)$ getting an equivalent norm.

The next version of the sub-mean value property will be useful several times in what remains of this chapter. As usual we write $f^{(0)} = f$.

Lemma 6.3.9. *[48, p. 338](See also [28, Corollary 1, p. 68] for the case $n = 0$)*

If $0 < p < +\infty$, $r > 0$, $n \in \mathbb{N} \cup \{0\}$, *and* f *is analytic in* $\mathbb{D}$, *then*

$$|f^{(n)}(\alpha)|^p \lesssim \frac{1}{(1-|\alpha|)^{2+np}} \int_{\Delta(\alpha,r)} |f(w)|^p \quad dm_2(w),$$

for each point $\alpha \in \mathbb{D}$.

The relationship between the discrete version of tent spaces and the continuous version is given in the next proposition.

Proposition 6.3.10. *Let* $0 < p, q < \infty$ *and* $Z = \{z_k\}$ *be an* (r, κ)*-lattice. Given* $f \in AT_p^q$ *and* $\lambda_k = \lambda_k(f) = \sup_{w \in \overline{\Delta(z_k,r)}} |f(w)|(1-|z_k|)^{1/p}$. *Then*

$$\|f\|_{T_p^q} \asymp \|\lambda_k\|_{T_p^q(Z)}.$$

Proof. Take $\tilde{z}_k \in \overline{\Delta(z_k, r)}$ such that $|f(\tilde{z}_k)| = \sup_{w \in \overline{\Delta(z_k,r)}} |f(w)|$. First, let us check that $\||f(\tilde{z}_k)|(1-|z_k|)^{1/p}\|_{T_p^q(Z)} \lesssim \|f\|_{T_p^q}$. Applying the mean value property over the hyperbolic disc $\Delta(\tilde{z}_k, s)$ (see Lemma 6.3.9) with $s < 3r$, we obtain

$$\||f(\tilde{z_k})|(1-|z_k|)^{1/p}\|_{T_p^q(Z)} \asymp \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} |f(\tilde{z_k})|^p (1-|z_k|) \right)^{q/p} |d\xi| \right)^{1/q}$$

$$\lesssim \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} \left(\int_{\Delta(\tilde{z}_k,s)} |f(z)|^p \frac{dm_2(z)}{(1-|\tilde{z}_k|)^2} \right) (1-|z_k|) \right)^{q/p} |d\xi| \right)^{1/q}.$$

Since $\Delta(\tilde{z}_k, s) \subset \Delta(z_k, 4r)$, applying Remark 6.3.2 we have

$$\||f(\tilde{z_k})|(1-|z_k|)^{1/p}\|_{T_p^q(Z)}$$

$$\lesssim \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} \left(\int_{\Delta(z_k,4r)} |f(z)|^p \frac{dm_2(z)}{1-|z_k|} \right) \right)^{q/p} |d\xi| \right)^{1/q}.$$

By Lemma 6.3.4, we can take $M_+ > M > 1$ such that

$$\bigcup_{\Delta(z_k,4r) \cap S_M(\xi) \neq \emptyset} \Delta(z_k, 4r) \subset S_{M_+}(\xi)$$

and applying Remark 6.3.2, it follows

$$\begin{aligned}
&\||f(\tilde{z_k})|(1-|z_k|)^{1/p}\|_{T_p^q(Z)}\\
&\lesssim \left(\int_{\mathbb{T}}\left(\sum_{z_k\in S_M(\xi)}\left(\int_{\Delta(z_k,4r)}|f(z)|^p\frac{dm_2(z)}{1-|z|}\right)\right)^{q/p}|d\xi|\right)^{1/q}\\
&\le \left(\int_{\mathbb{T}}\left(\int_{S_{M_+}(\xi)}\left(\sum_{z_k\in S_M(\xi)}\chi_{\Delta(z_k,4r)}(z)\right)|f(z)|^p\frac{dm_2(z)}{1-|z|}\right)^{q/p}|d\xi|\right)^{1/q}.
\end{aligned}$$

Due to the fact that to $Z=\{z_k\}$ is an (r,κ)-lattice, by Proposition 6.3.3 there is N such that

$$\sum_k \chi_{\Delta(z_k,4r)}(z)\le N$$

for all $z\in\mathbb{D}$. Therefore, we obtain

$$\begin{aligned}
&\||f(\tilde{z_k})|(1-|z_k|)^{1/p}\|_{T_p^q(Z)}\\
&\lesssim N^{1/p}\left(\int_{\mathbb{T}}\left(\int_{S_{M_+}(\xi)}|f(z)|^p\frac{dm_2(z)}{1-|z|}\right)^{q/p}|d\xi|\right)^{1/q}\lesssim N^{1/p}\|f\|_{T_p^q}.
\end{aligned}$$

Now, we proceed with the converse inequality. Using the pointwise estimate $|f(z)|\le\sum_k|f(z)|\chi_{\Delta(z_k,r)}(z)$ and the fact (given by Proposition 6.3.4) that we can take $M_+>M>1$ such that

$$\bigcup_{\Delta(z_k,r)\cap S_M(\xi)\ne\emptyset}\Delta(z_k,r)\subset S_{M_+}(\xi),$$

it follows

$$\begin{aligned}
\|f\|_{T_p^q}&\asymp\left(\int_{\mathbb{T}}\left(\int_{S_M(\xi)}|f(z)|^p\frac{dm_2(z)}{1-|z|}\right)^{q/p}|d\xi|\right)^{1/q}\\
&\le\left(\int_{\mathbb{T}}\left(\sum_k\int_{S_M(\xi)\cap\Delta(z_k,r)}|f(z)|^p\frac{dm_2(z)}{1-|z|}\right)^{q/p}|d\xi|\right)^{1/q}\\
&\le\left(\int_{\mathbb{T}}\left(\sum_{z_k\in S_{M_+}(\xi)}\int_{\Delta(z_k,r)}|f(z)|^p\frac{dm_2(z)}{1-|z|}\right)^{q/p}|d\xi|\right)^{1/q}.
\end{aligned}$$

Taking $\tilde{z}_k\in\overline{\Delta(z_k,r)}$ such that $|f(\tilde{z}_k)|:=\sup_{w\in\overline{\Delta(z_k,r)}}|f(w)|$ and applying (6.3.1) and Remark 6.3.2, we have

$$\|f\|_{T_p^q} \lesssim \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_{M_+}(\xi)} |f(\tilde{z}_k)|^p (1-|z_k|) \right)^{q/p} |d\xi| \right)^{1/q}$$
$$\asymp \| |f(\tilde{z_k})| (1-|z_k|)^{1/p} \|_{T_p^q(Z)}.$$

Therefore, we are done. □

In [21] Cohn and Verbitsky proved a result about factorizations of tent spaces of functions over the upper half-space. Throughout this chapter, we will use a result concerning factorizations of tent spaces of sequences due to Miihkinen, Pau, Perälä, and Wang. Before stating their result we recall that given three quasinormed spaces of sequences X, Y and W, we say that $W = X \cdot Y$, if for $x = \{x_k\} \in X$ and $y = \{y_n\} \in Y$, we have $\|\{x_k y_k\}\|_W \lesssim \|x\|_X \|y\|_Y$, and every $w = \{w_k\} \in W$ can be expressed as $\{w_k\} = \{x_k y_k\}$ with $\|w\|_W \gtrsim \inf \|x\|_X \|y\|_Y$, where the infimum is taken over all possible factorizations of w.

The announced result of Miihkinen, Pau, Perälä, and Wang is the following.

Proposition 6.3.11. *[50, Proposition 6, p. 19] Let $0 < p, q < +\infty$ and $Z = \{z_k\}$ be an (r, κ)-lattice. If $p \le p_1, p_2 \le +\infty$, $q \le q_1, q_2 \le +\infty$, with $\min\{p_1, q_1\} < +\infty$ and $\min\{p_2, q_2\} < +\infty$, and they satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$T_p^q(Z) = T_{p_1}^{q_1}(Z) \cdot T_{p_2}^{q_2}(Z).$$

Remark 6.3.12. In [50, Proposition 6, p. 19] this result is stated for the cases $p < p_1, p_2 < +\infty$ and $q < q_1, q_2 < +\infty$. However, their argument clearly works in the extreme cases.

The following propositions give us the duality for tent spaces of sequences. They will be a cornerstone in the proof of the main result of this chapter.

Proposition 6.3.13. *[11, Lemma 6, p. 68] Let $Z = \{z_k\}$ be an (r, κ)-lattice and $1 \le p < +\infty$, $1 < q < +\infty$. Then $(T_p^q(Z))^* \cong T_{p'}^{q'}(Z)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. The isomorphism between $T_{p'}^{q'}(Z)$ and $(T_p^q(Z))^*$ is given by the operator*

$$\{b_k\} \mapsto \langle \cdot, \{b_k\} \rangle$$

where $\langle \cdot, \{b_k\} \rangle$ is defined by

$$\langle \{a_k\}, \{b_k\} \rangle = \sum_k a_k \, b_k (1-|z_k|), \quad \{a_k\} \in T_p^q(Z).$$

In fact, $\|\{b_k\}\|_{T^{q'}_{p'}(Z)} \asymp \sup\left\{|\sum_k a_k b_k(1-|z_k|)| \ : \ \|\{a_k\}\|_{T_p^q(Z)} = 1\right\}$.

Proposition 6.3.14. *[11, Proposition 2, p. 72] Let $Z = \{z_k\}$ be an (r,κ)-lattice and $0 < p < 1 < q < +\infty$. Then $(T_p^q(Z))^* \cong T_\infty^{q'}(Z)$ where $\frac{1}{q} + \frac{1}{q'} = 1$. The isomorphism between $T_\infty^{q'}(Z)$ and $(T_p^q(Z))^*$ is given by the operator*

$$\{b_k\} \mapsto \langle \cdot, \{b_k\}\rangle$$

where $\langle \cdot, \{b_k\}\rangle$ is defined by

$$\langle \{a_k\}, \{b_k\}\rangle = \sum_k a_k \ b_k(1-|z_k|), \quad \{a_k\} \in T_p^q(Z).$$

In fact, $\|\{b_k\}\|_{T^{q'}_{\infty}(Z)} \asymp \sup\left\{|\sum_k a_k b_k(1-|z_k|)| \ : \ \|\{a_k\}\|_{T_p^q(Z)} = 1\right\}$.

Proposition 6.3.15. *[42, Lemma 3.4, p. 184] Let $Z = \{z_k\}$ be an (r,κ)-lattice and $1 < p < +\infty$. Then $(T_p^1(Z))^* \cong T_{p'}^{\infty}(Z)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. The isomorphism between $T_{p'}^{\infty}(Z)$ and $(T_p^1(Z))^*$ is given by the operator*

$$\{b_k\} \mapsto \langle \cdot, \{b_k\}\rangle$$

where $\langle \cdot, \{b_k\}\rangle$ is defined by

$$\langle \{a_k\}, \{b_k\}\rangle = \sum_k a_k \ b_k(1-|z_k|), \quad \{a_k\} \in T_p^1(Z).$$

In fact, $\|\{b_k\}\|_{T^{\infty}_{p'}(Z)} \asymp \sup\left\{|\sum_k a_k b_k(1-|z_k|)| \ : \ \|\{a_k\}\|_{T_p^1(Z)} = 1\right\}$.

A particular version of the next proposition can be seen in [48, Proposition 2, p. 352]. In this paper, Luecking proved it for the case when the (r,κ)-lattice is the one given by the centers of the Luecking regions (see Example 6.3.6). Our proof is a modification of the Luecking's one.

Proposition 6.3.16. *Let $Z = \{z_k\}$ be an (r,κ)-lattice. If either $0 < p < +\infty$, $0 < q < 1$ or $0 < p \leq 1$, $q = 1$, then*

$$\sup\left\{\left|\sum_k a_k b_k(1-|z_k|)\right| \ : \ \|\{b_k\}\|_{T_p^q(Z)} = 1\right\} \asymp \sup_k |a_k|(1-|z_k|)^{1-1/q}$$

for any sequence $\{a_k\}$.

Proof. If $z \in \mathbb{D}$, we define the set $L(z) = \{\xi \in \mathbb{T} \ : z \in \Gamma(\xi)\}$. For an interval I on $\mathbb{T}$, we denote by $T(I)$ the complement of the union of every nontangential region $\Gamma(\xi)$

such that $\xi \notin I$. First, we will check that there is a constant A such that

$$\sup_k |a_k|(1-|z_k|)^{1-1/q} \leq A \sup \left\{ \left| \sum_k a_k b_k (1-|z_k|) \right| \ : \ \|\{b_k\}\|_{T_p^q(Z)} = 1 \right\}. \quad (6.3.5)$$

Fix k_0. Consider the sequence $\{b_k\}$ such that $b_k = 0$ when $k \neq k_0$ and $b_{k_0} = (1-|z_{k_0}|)^{-1/q}$. By Remark 6.1.6, it is clear that

$$\|\{b_k\}\|_{T_p^q(Z)} = \left(\int_{\mathbb{T}} (1-|z_{k_0}|)^{-1} \chi_{\Gamma(\xi)}(z_{k_0}) \ |d\xi| \right)^{1/q} \asymp 1.$$

Hence

$$\begin{aligned} |a_{k_0}|(1-|z_{k_0}|)^{1-\frac{1}{q}} &= |a_{k_0}||b_{k_0}|(1-|z_{k_0}|) \\ &\lesssim \sup \left\{ \left| \sum_k a_k b_k (1-|z_k|) \right| \ : \ \|\{b_k\}\|_{T_p^q(Z)} = 1 \right\} \end{aligned}$$

and we have (6.3.5).

Now, we proceed with the proof of the converse inequality.

Claim: If I is an interval on $\mathbb{T}$ and $\xi \in I$, then there is a constant $C > 1$ and an endpoint α of I such that $\Gamma(\xi) \setminus T(I) \subset S_C(\alpha)$.

Proof of the claim: We can assume without loss of generality that

$$I = \{e^{i\theta} \in \mathbb{T} \ : \theta \in (-\theta_0, \theta_0)\}$$

for $\theta_0 \in (0, \pi]$. Fix $\gamma \in [0, \theta_0)$.

One can easily check that

$$\Gamma(e^{i\gamma}) \setminus T(I) \subset \Gamma(e^{i\gamma}) \cap \left\{ |z| < 1 - \frac{|\gamma - \theta_0|}{2} \right\} =: A(\gamma).$$

Therefore, it follows that for $re^{i\beta} \in A(\gamma)$

$$\begin{aligned} |re^{i\beta} - e^{i\theta_0}| &\leq |re^{i\beta} - e^{i\gamma}| + |e^{i\gamma} - e^{i\theta_0}| \\ &= \sqrt{(1-r)^2 + 2r(1-\cos(\beta-\gamma))} + \sqrt{2(1-\cos(\gamma-\theta_0))} \\ &\leq \sqrt{(1-r)^2 + |\beta-\gamma|^2} + |\gamma-\theta_0| < (2+\sqrt{2})(1-r). \end{aligned}$$

Setting $C = 2 + \sqrt{2}$, we have that $\Gamma(e^{i\gamma}) \setminus T(I) \subset S_C(e^{i\theta_0})$ and the claim holds.

Let $\{a_k\}_k$ be a sequence such that $|a_k| \leq (1-|z_k|)^{\frac{1}{q}-1}$ and let $\{b_k\}$ be an arbitrary sequence in $T_p^q(Z)$. Without loss of generality we can assume that both sequences are positive and $\{b_k\}$ is eventually zero. We need to show that there is a positive constant C_0 such that

$$\sum_{k=0}^{\infty} a_k b_k (1-|z_k|) \leq C_0 \|\{b_k\}\|_{T_p^q(Z)}. \tag{6.3.6}$$

For each $m \in \mathbb{Z}$, let

$$E_m = \left\{ \xi \in \mathbb{T} \ : \ \left(\sum_{z_k \in S_C(\xi)} |b_k|^p \right)^{1/p} > 2^m \right\}.$$

The set E_m is open. Let $I_{1,m}$, $I_{2,m}$, ... denote the decomposition of E_m into disjoint open intervals. Let $G_m := \bigcup_j T(I_{j,m})$. Since the norms of $T_p^q(Z)$ with different nontangential regions are equivalent, the norm of $\{b_k\}_k$ is equivalent to

$$\left(\sum_m 2^{mq} |E_m| \right)^{1/q}.$$

Now we continue with the estimation of the sum in (6.3.6). Because $\{z_k : b_k \neq 0\} \subset \bigcup_m G_m \setminus G_{m+1}$, we have

$$\begin{aligned}
\sum_k a_k b_k (1-|z_k|) &= \sum_m \sum_{z_k \in G_m \setminus G_{m+1}} a_k b_k (1-|z_k|) \\
&\asymp \sum_m \sum_k \sum_{z_k \in G_m \setminus G_{m+1}} a_k b_k \int_{\mathbb{T}} \chi_{T(z_k)}(\xi) \ |d\xi|.
\end{aligned}$$

Since $\xi \in L(z_k)$ if and only if $z_k \in \Gamma(\xi)$ and $\Gamma(\xi) \cap T(I_{j,m}) = \emptyset$ unless $\xi \in I_{j,m}$, it follows that

$$\begin{aligned}
\sum_k a_k b_k (1-|z_k|) &= \sum_m \sum_{z_k \in G_m \setminus G_{m+1}} a_k b_k (1-|z_k|) \\
&\asymp \sum_m \sum_j \int_{I_{j,m}} \sum_{z_k \in \Gamma(\xi) \cap T(I_{j,m}) \setminus G_{m+1}} a_k b_k \ |d\xi|.
\end{aligned} \tag{6.3.7}$$

Assume that $p > 1$ and $q < 1$. Then

$$\sum_{z_k \in \Gamma(\xi) \cap T(I_{j,m}) \setminus G_{m+1}} a_k b_k \leq \left(\sum_{z_k \in \Gamma(\xi) \setminus G_{m+1}} b_k^p \right)^{1/p} \left(\sum_{z_k \in \Gamma(\xi) \cap T(I_{j,m})} a_k^{p'} \right)^{1/p'}.$$

On the one hand, if $\xi \notin E_{m+1}$, then the first factor of the right is less than 2^{m+1} because the set $\Gamma(\xi) \setminus G_{m+1} \subset S_C(\xi)$. On the other hand, if $\xi \in E_{m+1}$, then it belongs to some $I_{j,m+1}$ for some j. So that, using the previous claim, $\Gamma(\xi) \setminus G_{m+1} \subset$

$\Gamma(\xi) \setminus T(I_{j,m+1}) \subset S_C(\alpha)$, for an endpoint α of $I_{j,m+1}$. Also, the first factor is less than 2^{m+1} because $\alpha \notin E_{m+1}$. Therefore, it follows that

$$\sum_{z_k \in \Gamma(\xi)\cap T(I_{j,m})\setminus G_{m+1}} a_k b_k \le 2^{m+1} \left(\sum_{z_k \in \Gamma(\xi)\cap T(I_{j,m})} a_k^{p'} \right)^{1/p'}.$$

Let us continue with the estimation of the second factor. Using Lemma 6.3.5, in $\Gamma(\xi)$ there are at most N points z_k such that $1-\frac{1}{2^n} \le |z_k| < 1-\frac{1}{2^{n+1}}$, $n \in \mathbb{N}$. Notice that N does not depend on n. Thus

$$\begin{aligned}\left(\sum_{z_k \in \Gamma(\xi)\cap T(I_{j,m})} a_k^{p'} \right)^{1/p'} &\le \left(\sum_{z_k \in \Gamma(\xi)\cap T(I_{j,m})} (1-|z_k|)^{\left(\frac{1}{q}-1\right)p'} \right)^{1/p'} \\ &\lesssim \left(\sum_{2^{-n}<|I_{j,m}|} 2^{-n\left(\frac{1}{q}-1\right)p'} \right)^{1/p'} \lesssim |I_{j,m}|^{\frac{1}{q}-1}.\end{aligned}$$

Therefore, we have

$$\sum_{z_k \in \Gamma(\xi)\cap T(I_{j,m})\setminus G_{m+1}} a_k b_k \lesssim 2^m |I_{j,m}|^{\frac{1}{q}-1}.$$

Using (6.3.7) and the fact that $q<1$, it follows

$$\begin{aligned}\sum_k a_k b_k (1-|z_k|) &\lesssim \sum_m \sum_j \int_{I_{j,m}} 2^m |I_{j,m}|^{\frac{1}{q}-1} \, |d\xi| = \sum_m \sum_j 2^m |I_{j,m}|^{\frac{1}{q}} \\ &\le \left(\sum_m \sum_j 2^{mq} |I_{j,m}| \right)^{1/q} = \left(\sum_m 2^{mq} |E_m| \right)^{1/q} \lesssim \|\{b_k\}\|_{T_p^q(Z)}.\end{aligned}$$

The case $q, p \le 1$ follows similarly, using the estimation

$$\begin{aligned}\sum_{z_k \in \Gamma(\xi)\cap T(I_{j,m})\setminus G_{m+1}} a_k b_k &\le \left(\sum_{z_k \in \Gamma(\xi)\setminus G_{m+1}} b_k^p \right)^{1/p} \sup_{z_k \in T(I_{j,m})} a_k \\ &\lesssim 2^m \sup_{z_k \in T(I_{j,m})} (1-|z_k|)^{\frac{1}{q}-1} \lesssim 2^m |I_{j,m}|^{\frac{1}{q}-1}.\end{aligned}$$

□

Remark 6.3.17. We point out that in [48, Proposition 2, p. 352] above result for the (r,κ)-lattice given in Example 6.3.6 is stated for $0<p<+\infty$ and $0<q\le 1$, but if $p>1$ and $q=1$, its proof does not work.

The following result, due to Perälä, will be an important tool in the characterization of the Carleson measures in the next section.

Proposition 6.3.18. *[53, Lemma 14, p. 24] Let* $Z = \{z_n\}$ *be an* (r,κ)*-lattice,* $0 < p, q < +\infty$*, and* $M > \max\{1, p/q, 1/q, 1/p\} + 1/p$*. Then the operator* $S : T_p^q(Z) \to AT_p^q$*, where*

$$S(\{\lambda_k\})(z) := \sum_{k=0}^{\infty} \lambda_k \frac{(1-|z_k|)^{M-\frac{1}{p}}}{(1-\overline{z_k}z)^M}$$

is bounded.

6.4 Carleson measures

Assume that X is a quasi-Banach space of analytic functions in $\mathbb{D}$. For $s > 0$, a positive Borel measure μ on the unit disc is called (s, X)-Carleson measure if there is a positive constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(w)|^s \, d\mu(w) \le C\|f\|_X^s$$

for all $f \in X$.

The notion of (s, X)-Carleson measures appeared in the work [20] of Carleson on the theory of interpolating sequences for Hardy spaces. He proved that the (p, H^p)-Carleson measures are exactly described by the geometric condition

$$\sup_{I \subseteq \mathbb{T}} \frac{\mu(S(I))}{|I|} < +\infty, \tag{6.4.1}$$

where the supremum is taken over all I intervals of $\mathbb{T}$,

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \le |z| < 1 \text{ and } \frac{z}{|z|} \in I \right\},$$

and $|I|$ is the arc length of I. From now on, if μ is a positive Borel measure on $\mathbb{D}$ that satisfies (6.4.1) then we will refer to it simply as a Carleson measure.

This section is devoted to the study of Carleson measure type problems for tent spaces. Namely, given $s, p, q \in (0, \infty)$, find necessary and sufficient conditions for a positive Borel measure μ on $\mathbb{D}$ in order to exist a positive constant C such that

$$\left(\int_{\mathbb{D}} |f(z)|^s \, d\mu(z) \right)^{1/s} \le C\|f\|_{T_p^q}, \text{ for all } f \in AT_p^q. \tag{6.4.2}$$

Moreover, Luecking in [48, p. 356] pursued these ideas even further by posing the following:

Problem: Let $p, q, s, t > 0$. Characterize the positive Borel measures μ on $\mathbb{D}$ for which there is a positive constant C such that

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} |f(z)|^t \, d\mu(z) \right)^{s/t} |d\xi| \leq C \|f\|_{T_p^q}^s, \text{ for all } f \in AT_p^q. \tag{6.4.3}$$

Setting $p = q$, that is the Bergman case, this problem was solved by Wu [66, Theorem 1, p. 996; Theorem 2, p. 998; Theorem 3, p. 1002; Theorem 4, p. 1006].

In the next result we solve this problem for all p and q.

Theorem 6.4.1. *Let $0 < p, q, s, t < +\infty$, $M > 1$, $Z = \{z_k\}$ an (r, κ)-lattice, and let μ be a positive measure on $\mathbb{D}$. Then the following are equivalent:*

1. *There is a constant $C > 0$ such that*

$$\left(\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |f(w)|^t \, d\mu(w) \right)^{s/t} |d\xi| \right)^{1/s} \leq C \|f\|_{T_p^q}$$

   for all $f \in AT_p^q$.

2. *The measure μ satisfies the following:*

   (a) If $0 < s < q < +\infty$, $0 < t < p < +\infty$, then

$$\int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} \left(\frac{\mu^{1/t}(\Delta(z_k, r))}{(1 - |z_k|)^{1/p}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| < +\infty.$$

   (b) If $0 < s < q < +\infty$, $0 < p \leq t < +\infty$, then

$$\int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} \frac{\mu^{1/t}(\Delta(z_k, r))}{(1 - |z_k|)^{1/p}} \right)^{\frac{qs}{q-s}} |d\xi| < +\infty.$$

   (c) If $0 < q < s < +\infty$, $0 < p, t < \infty$ or $0 < q = s < +\infty$, $0 < p \leq t < +\infty$, then

$$\sup_k \frac{\mu^{1/t}(\Delta(z_k, r))}{(1 - |z_k|)^{\frac{1}{p} + \frac{1}{q} - \frac{1}{s}}} < +\infty.$$

(d) If $0 < q = s < +\infty$, $0 < t < p < +\infty$, *then*

$$\sup_{\xi \in \mathbb{T}} \left(\sup_{\xi \in I} \frac{1}{|I|} \sum_{z_k \in S(I)} \left(\frac{\mu^{1/t}(\Delta(z_k, r))}{(1 - |z_k|)^{1/p}} \right)^{\frac{pt}{p-t}} (1 - |z_k|) \right)^{\frac{p-t}{pt}} < +\infty,$$

where I *runs the intervals in* $\mathbb{T}$,

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1 \text{ and } \frac{z}{|z|} \in I \right\},$$

and $|I|$ *is the arc length of* I.

Proof. First, we will prove that *(2)* implies *(1)*. Let $b > \max\{\frac{1}{t}, \frac{1}{s}\}$. By Lemma 6.3.4, one can take a constant $M_+ > M > 1$ such that, for all $\xi \in \mathbb{T}$,

$$\bigcup_{\Delta(z_k, r) \cap S_M(\xi) \neq \emptyset} \Delta(z_k, r) \subset S_{M_+}(\xi). \tag{6.4.4}$$

Fix $f \in AT^q_p$. Using the pointwise estimate $|f(z)| \leq \sum_k |f(z)| \chi_{\Delta(z_k, r)}(z)$, it follows

$$\begin{aligned}
&\left(\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |f(w)|^t \, d\mu(w) \right)^{s/t} |d\xi| \right)^{\frac{1}{s}} \\
&\leq \left(\int_{\mathbb{T}} \left(\sum_k \int_{\Delta(z_k, r) \cap S_M(\xi)} |f(w)|^t \, d\mu(w) \right)^{s/t} |d\xi| \right)^{\frac{1}{s}} \\
&\leq \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_{M_+}(\xi)} \sup_{w \in \overline{\Delta(z_k, r)}} |f(w)|^t \mu(\Delta(z_k, r)) \right)^{s/t} |d\xi| \right)^{\frac{1}{s}}.
\end{aligned}$$

For the simplicity of the presentation we set $|f_k| := \sup_{w \in \overline{\Delta(z_k, r)}} |f(w)|$. Bearing in mind Lemma 6.1.2 and $b > \max\{\frac{1}{t}, \frac{1}{s}\}$, we have

$$\left(\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |f(w)|^t \, d\mu(w) \right)^{s/t} |d\xi| \right)^{\frac{1}{s}} \tag{6.4.5}$$
$$\leq \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_{M_+}(\xi)} |f_k|^t \mu(\Delta(z_k, r)) \right)^{s/t} |d\xi| \right)^{\frac{1}{s}}$$
$$= \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_{M_+}(\xi)} \left(|f_k|^{1/b} \, \mu^{1/bt}(\Delta(z_k, r)) \right)^{bt} \right)^{bs/bt} |d\xi| \right)^{\frac{b}{bs}}$$
$$\asymp \|\{|f_k|^{1/b} \, \mu^{1/bt}(\Delta(z_k, r))\}\|^b_{T^{bs}_{bt}(Z)} \, .$$

Using the duality relation for the $T^{bs}_{bt}(Z)$ as stated in Proposition 6.3.13, we get

$$\|\{|f_k|^{1/b} \, \mu^{1/bt}(\Delta(z_k, r))\}\|_{T^{bs}_{bt}(Z)} \tag{6.4.6}$$
$$= \sup_{\substack{\lambda_k \geq 0 \\ \|\{\lambda_k\}\|_{T^{(bs)'}_{(bt)'}(Z)} \leq 1}} \sum_k \lambda_k |f_k|^{1/b} \, \mu^{1/bt}(\Delta(z_k, r)) \, (1 - |z_k|)$$
$$= \sup_{\substack{\lambda_k \geq 0 \\ \|\{\lambda_k\}\|_{T^{(bs)'}_{(bt)'}(Z)} \leq 1}} \sum_k \lambda_k |f_k|^{1/b} (1 - |z_k|)^{1/bp} \frac{\mu^{1/bt}(\Delta(z_k, r))}{(1 - |z_k|)^{1/bp}} (1 - |z_k|).$$

By Proposition 6.3.10, we have

$$\|f\|_{T^q_p} \asymp \|\{|f_k|(1 - |z_k|)^{1/p}\}\|_{T^q_p(Z)} = \|\{|f_k|^{1/b}(1 - |z_k|)^{1/bp}\}\|^b_{T^{bq}_{bp}(Z)} \, .$$

Therefore, for any sequence of positive numbers $\{\lambda_k\}_k \in T^{(bs)'}_{(bt)'}(Z)$, Proposition 6.3.11 implies that

$$\{\lambda_k |f_k|^{1/b} (1 - |z_k|)^{1/bp}\} \in T^{(bs)'}_{(bt)'}(Z) \cdot T^{bq}_{bp}(Z) = T^{\frac{qbs}{s+q(bs-1)}}_{\frac{pbt}{t+p(bt-1)}}(Z).$$

Taking into account the conditions *(2)(a)-(d)* together with Proposition 6.3.13, Proposition 6.3.14, Proposition 6.3.15, and Proposition 6.3.16, respectively, we obtain that there is a constant $C(\mu)>0$ (see the estimated value of $C(\mu)$ below) such that

$$\begin{aligned}&\left(\sum_k \lambda_k |f_k|^{1/b}(1-|z_k|)^{1/bp}\frac{\mu^{1/bt}(\Delta(z_k,r))}{(1-|z_k|)^{1/bp}}(1-|z_k|)\right)^b \qquad (6.4.7)\\ &\le C(\mu)^b \|\{\lambda_k|f_k|^{1/b}(1-|z_k|)^{1/bp}\}\|^b_{T^{\frac{qbs}{s+q(bs-1)}}_{\frac{pbt}{t+p(bt-1)}}(Z)}\\ &\lesssim C(\mu)^b\,\|\{\lambda_k\}\|^b_{T^{(bs)'}_{(bt)'}(Z)}\,\|\{|f_k|^{1/b}(1-|z_k|)^{1/bp}\}\|^b_{T^{bq}_{bp}(Z)}\\ &= C(\mu)^b\,\|\{\lambda_k\}\|^b_{T^{(bs)'}_{(bt)'}(Z)}\,\|\{|f_k|(1-|z_k|)^{1/p}\}\|_{T^q_p(Z)}\\ &\lesssim C(\mu)^b\,\|\{\lambda_k\}\|^b_{T^{(bs)'}_{(bt)'}(Z)}\,\|f\|_{AT^q_p}.\end{aligned}$$

Now, we proceed to clarify which is the constant $C(\mu)$ in each of the cases of the statement *(2)*.

(a) By Proposition 6.3.13, it follows that

$$C(\mu)\asymp\left\|\left\{\frac{\mu^{1/bt}(\Delta(z_k,r))}{(1-|z_k|)^{1/bp}}\right\}\right\|_{T^{\frac{bqs}{q-s}}_{\frac{bpt}{p-t}}(Z)}=\left\|\left\{\frac{\mu^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{1/p}}\right\}\right\|^{1/b}_{T^{\frac{qs}{q-s}}_{\frac{pt}{p-t}}(Z)}.$$

(b) Using Proposition 6.3.13 for the cases $s<q$, $p=t$ and Proposition 6.3.14 for the case $s<q$, $p<t$ we have that

$$C(\mu)\asymp\left\|\left\{\frac{\mu^{1/bt}(\Delta(z_k,r))}{(1-|z_k|)^{1/bp}}\right\}\right\|_{T^{\frac{bqs}{q-s}}_{\infty}(Z)}=\left\|\left\{\frac{\mu^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{1/p}}\right\}\right\|^{1/b}_{T^{\frac{qs}{q-s}}_{\infty}(Z)}.$$

(c) By means of Proposition 6.3.16, the constant $C(\mu)$ in these cases is determined by

$$C(\mu)\asymp\sup_k\frac{\mu^{1/bt}(\Delta(z_k,r))}{(1-|z_k|)^{1/bp}}(1-|z_k|)^{\frac{q-s}{qbs}}=\left(\sup_k\frac{\mu^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac1p+\frac1q-\frac1s}}\right)^{1/b}.$$

(d) In the remaining cases, we apply Proposition 6.3.15 to obtain

$$C(\mu)\asymp\left\|\left\{\frac{\mu^{1/bt}(\Delta(z_k,r))}{(1-|z_k|)^{1/bp}}\right\}\right\|_{T^{\infty}_{\frac{bpt}{p-t}}(Z)}=\left\|\left\{\frac{\mu^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{1/p}}\right\}\right\|^{1/b}_{T^{\infty}_{\frac{pt}{p-t}}(Z)}.$$

Therefore, combining (6.4.5), (6.4.6), and (6.4.7), we conclude that

$$\left(\int_{\mathbb{T}}\left(\int_{S_M(\xi)}|f(w)|^t\,d\mu(w)\right)^{s/t}|d\xi|\right)^{1/s}\lesssim\|f\|_{T_p^q}.$$

Now, we continue with the proof of *(1)* implies *(2)*. By Lemma 6.3.4, $\Delta(z,r)\subset S_{M_*}(\xi)$ for all $z\in S_M(\xi)$ where $M_*=(M+1)e^{2r}-1$. By hypothesis and Lemma 6.1.2 there is $C>1$ such that

$$\left(\int_{\mathbb{T}}\left(\int_{S_{M_*}(\xi)}|f(z)|^t\,d\mu(z)\right)^{s/t}|d\xi|\right)^{1/s}\leq C\,\|f\|_{T_p^q}\tag{6.4.8}$$

for all $f\in AT_p^q$. Take $\lambda=\{\lambda_k\}$ a sequence of $T_p^q(Z)$ and let $r_k:[0,1]\to\{-1,1\}$ be the Rademacher functions (see Subsection A.6). Moreover, for $K>\max\{1,p/q,1/q,1/p\}+1/p$, we consider the function

$$F_x(z):=\sum_k\lambda_k r_k(x)\frac{(1-|z_k|)^{K-\frac{1}{p}}}{(1-\overline{z_k}z)^K}\ ,\quad z\in\mathbb{D}.$$

From (6.4.8) and Proposition 6.3.18, it follows that

$$\int_{\mathbb{T}}\left(\int_{S_{M_*}(\xi)}|F_x(z)|^t\,d\mu(z)\right)^{s/t}|d\xi|\lesssim\|\lambda_k\|^s_{T_p^q(Z)}.$$

Integrating both sides with respect to x, we obtain

$$\int_0^1\int_{\mathbb{T}}\left(\int_{S_{M_*}(\xi)}|F_x(z)|^t\,d\mu(z)\right)^{s/t}|d\xi|\,dx\lesssim\|\lambda_k\|^s_{T_p^q(Z)}.$$

Applying Fubini's theorem and Khinchine-Kahane-Kalton inequality (see Theorem A.6.2), it follows

$$\int_{\mathbb{T}}\left(\int_0^1\int_{S_{M_*}(\xi)}|F_x(z)|^t\,d\mu(z)\,dx\right)^{s/t}|d\xi|\lesssim\|\lambda_k\|^s_{T_p^q(Z)}.$$

Now, by means of Khinchine's inequality (see Theorem A.6.1) we have

$$\int_{\mathbb{T}}\left(\int_{S_{M_*}(\xi)}\left(\sum_k|\lambda_k|^2\frac{(1-|z_k|)^{2K-\frac{2}{p}}}{|1-\overline{z_k}z|^{2K}}\right)^{t/2}d\mu(z)\right)^{s/t}|d\xi|\lesssim\|\lambda_k\|^s_{T_p^q(Z)}.$$

It holds that

$$\chi_{\Delta(z_k,r)}(z) \lesssim \frac{1-|z_k|}{|1-\overline{z_k}z|}, \quad z \in \mathbb{D},$$

because, using Remark 6.3.2, it follows that

$$\frac{(1-|z_k|^2)^2}{|1-\overline{z_k}z|^2} = \frac{(1-|\varphi_{z_k}(z)|^2)(1-|z_k|^2)}{1-|z|^2} \geq \frac{(1-\tanh^2(r))^2}{2}$$

for $z \in \Delta(z_k, r)$.

In addition, each $z \in \mathbb{D}$ belongs to no more than $N = N(r,\kappa)$ discs $\Delta(z_k, r)$ (see Proposition 6.3.3). Using the fact that

$$\|w\|_t \leq \max\{1, N^{1/t-1/2}\}\|w\|_2$$

for $w \in \mathbb{C}^N$ and, combining it with the estimation above, one can see that

$$\begin{aligned}
&\int_{\mathbb{T}} \left(\int_{S_{M_*}(\xi)} \sum_k |\lambda_k|^t \frac{\chi_{\Delta(z_k,r)}(z)}{(1-|z_k|)^{t/p}} \, d\mu(z) \right)^{s/t} |d\xi| \\
&\leq \max\{1, N^{s/t-s/2}\} \int_{\mathbb{T}} \left(\int_{S_{M_*}(\xi)} \left(\sum_k |\lambda_k|^2 \frac{\chi_{\Delta(z_k,r)}(z)}{(1-|z_k|)^{2/p}} \right)^{t/2} d\mu(z) \right)^{s/t} |d\xi| \\
&\lesssim \max\{1, N^{s/t-s/2}\} \int_{\mathbb{T}} \left(\int_{S_{M_*}(\xi)} \left(\sum_k |\lambda_k|^2 \frac{(1-|z_k|)^{2K-\frac{2}{p}}}{|1-\overline{z_k}z|^{2K}} \right)^{t/2} d\mu(z) \right)^{s/t} |d\xi| \\
&\lesssim \max\{1, N^{s/t-s/2}\} \, \|\lambda_k\|^s_{T^q_p(Z)}.
\end{aligned}$$

Hence, it follows that

$$\int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} |\lambda_k|^t \frac{\mu(\Delta(z_k,r))}{(1-|z_k|)^{t/p}} \right)^{s/t} |d\xi| \lesssim \max\{1, N^{s/t-s/2}\}\|\lambda_k\|^s_{T^q_p(Z)}.$$

On the other hand, take b such that $tb > 1$ and $sb > 1$. For any $\{\tau_k\}_k \in T^{\frac{bs}{bs-1}}_{\frac{bt}{bt-1}}(Z)$, after applying Remark 6.1.6, Fubini's theorem, and Hölder's inequality twice, one gets that

$$\begin{aligned}
&\sum_k \tau_k \lambda_k^{1/b} \left(\frac{\mu^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{1/p}} \right)^{1/b} (1-|z_k|) \qquad (6.4.9)\\
&\asymp \sum_k \tau_k \lambda_k^{1/b} \left(\frac{\mu^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{1/p}} \right)^{1/b} \int_{\mathbb{T}} \chi_{S_M(\xi)}(z_k)\ |d\xi|\\
&= \int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \tau_k \lambda_k^{1/b} \left(\frac{\mu^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{1/p}} \right)^{1/b} \right) |d\xi|\\
&\lesssim \int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} |\lambda_k|^t \frac{\mu(\Delta(z_k,r))}{(1-|z_k|)^{t/p}} \right)^{1/tb} \left(\sum_{z_k\in S_M(\xi)} \tau_k^{\frac{bt}{bt-1}} \right)^{\frac{bt-1}{bt}} |d\xi|\\
&\leq \left(\int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} |\lambda_k|^t \frac{\mu(\Delta(z_k,r))}{(1-|z_k|)^{t/p}} \right)^{s/t} |d\xi| \right)^{1/bs} \|\tau_k\|_{T^{\frac{bs}{bs-1}}_{\frac{bt}{bt-1}}(Z)}\\
&\lesssim \max\{1, N^{\frac{s}{tb}-\frac{t}{2b}}\} \|\lambda_k\|^{1/b}_{T^q_p(Z)} \|\tau_k\|_{T^{\frac{bs}{bs-1}}_{\frac{bt}{bt-1}}(Z)}.
\end{aligned}$$

By Proposition 6.3.11 we have that

$$\{\tau_k \lambda_k^{1/b}\}_k \in T^{\frac{bqs}{s+q(bs-1)}}_{\frac{bpt}{t+p(bt-1)}}(Z) = T^{\frac{bs}{bs-1}}_{\frac{bt}{bt-1}}(Z) \cdot T^{bq}_{bp}(Z).$$

Then, using (6.4.9) together with Proposition 6.3.13, Proposition 6.3.14, Proposition 6.3.15, and Proposition 6.3.16 respectively, we obtain that *(2)* holds. Therefore, we are done. □

In [48] Luecking proved the following result in the setting of tent spaces of analytic functions defined on the upper half-plane. His argument can be adapted for tent spaces of holomorphic functions on the unit disc. Thus we omit the proof of the following theorem.

Theorem 6.4.2. *[48, Theorem 3, p. 354] Let $0 < s, p, q < +\infty$ such that if $s = q$ then $p \leq q$, $Z = \{z_k\}$ the (r,κ)-lattice consisting of the centers of the Luecking regions, and $n \in \mathbb{N}\cup\{0\}$. Let μ be a finite positive measure on $\mathbb{D}$. Then the following assertions are equivalent.*

1. *There is a constant $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f^{(n)}(w)|^s \ d\mu(w) \right)^{1/s} \leq C \|f\|_{T^q_p}$$

for all $f \in AT^q_p$.

2. The sequence

$$\mu_k := \mu(R_k)(1-|z_k|)^{-\frac{s}{p}-sn-1},$$

where R_k is a Luecking region and z_k is the center of this region, satisfies one of the following:

(a) If $s<p,q$, then $\{\mu_k\} \in T^{\frac{q}{q-s}}_{\frac{p}{p-s}}(Z)$.

(b) If $p \le s < q$, then $\{\mu_k\} \in T^{\frac{q}{q-s}}_{\infty}(Z)$.

(c) If $q<s$ or $p \le s = q$, then $\{\mu_k(1-|z_k|)^{1-\frac{s}{q}}\}$ is a bounded sequence.

In fact, the cases $0<s=q<p<+\infty$ in above theorem appeared in [48, Theorem 3, p. 354], but its proof does not work. We finish the subsection tackling this cases.

Theorem 6.4.3. *Let $0<s<p<+\infty$, $Z=\{z_k\}$ the (r,κ)-lattice consisting of the centers of the Luecking regions, and $n \in \mathbb{N}\cup\{0\}$. Let μ be a finite positive measure on $\mathbb{D}$. Then, there is a constant $C>0$ such that*

$$\left(\int_{\mathbb{D}} |f^{(n)}(w)|^s \, d\mu(w)\right)^{1/s} \le C\|f\|_{T^s_p} \tag{6.4.10}$$

for all $f \in AT^s_p$ if and only if $\left\{\frac{\mu(R_k)}{(1-|z_k|)^{\frac{s}{p}+sn+1}}\right\} \in T^{\infty}_{\frac{p}{p-s}}(Z)$.

Proof. First, we prove that the belonging of $\left\{\frac{\mu(R_k)}{(1-|z_k|)^{\frac{s}{p}+sn+1}}\right\}$ to $T^{\infty}_{\frac{p}{p-s}}(Z)$ implies (6.4.10). Let $f \in AT^s_p$, then

$$|f^{(n)}(z)| \le \sum_k |f^{(n)}(\tilde{z_k})|\chi_{R_k}(z), \quad z \in \mathbb{D},$$

where R_k is a Luecking region and $\tilde{z_k} \in \overline{R_k}$ is such that $|f^{(n)}(\tilde{z_k})| := \sup_{w\in\overline{R_k}} |f^{(n)}(w)|$. So that

$$\begin{aligned}\int_{\mathbb{D}} |f^{(n)}(w)|^s \, d\mu(w) &\le \sum_k |f^{(n)}(\tilde{z_k})|^s \, \mu(R_k) \\ &= \sum_k |f^{(n)}(\tilde{z_k})|^s \, (1-|z_k|)^{\frac{s}{p}+sn} \frac{\mu(R_k)}{(1-|z_k|)^{\frac{s}{p}+ns+1}}(1-|z_k|).\end{aligned}$$

Hence, by hypothesis and Proposition 6.3.15 it follows

$$\begin{aligned}\int_{\mathbb{D}} |f^{(n)}(w)|^s \, d\mu(w) &\leq \sum_k |f^{(n)}(\tilde{z_k})|^s \, (1-|z_k|)^{\frac{s}{p}+sn} \frac{\mu(R_k)}{(1-|z_k|)^{\frac{s}{p}+ns+1}}(1-|z_k|) \\ &\lesssim \left\| \left\{ \frac{\mu(R_k)}{(1-|z_k|)^{\frac{s}{p}+ns+1}} \right\} \right\|_{T^{\infty}_{\frac{p}{p-s}}(Z)} \|\{|f^{(n)}(\tilde{z_k})|^s \, (1-|z_k|)^{\frac{s}{p}+sn}\}\|_{T^1_{p/s}(Z)}.\end{aligned}$$

Let us check that

$$\|\{|f^{(n)}(\tilde{z_k})|^s \, (1-|z_k|)^{\frac{s}{p}+n}\}\|_{T^1_{p/s}(Z)} \lesssim \|f\|^s_{T^s_p}.$$

Using Example 6.3.6, we obtain that there is $\eta > 0$ such that $\overline{R_k} \subset \Delta(z_k, \eta)$. Applying the mean value property over the hyperbolic disc $\Delta(\tilde{z}_k, \eta)$ (see Lemma 6.3.9), it follows

$$\begin{aligned}&\|\{|f^{(n)}(\tilde{z}_k)|^s \, (1-|z_k|)^{\frac{s}{p}+sn}\}\|_{T^1_{p/s}(Z)} \\ &\asymp \int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} |f^{(n)}(\tilde{z}_k)|^p \, (1-|z_k|)^{1+pn} \right)^{s/p} |d\xi| \\ &\lesssim \int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} \frac{(1-|z_k|)^{np+1}}{(1-|\tilde{z}_k|)^{np+2}} \int_{\Delta(\tilde{z}_k,\eta)} |f(w)|^p \, dm_2(w) \right)^{s/p} |d\xi|.\end{aligned}$$

Since $\Delta(\tilde{z_k}, \eta) \subset \Delta(z_k, 2\eta)$, by Remark 6.3.2 we have

$$\begin{aligned}&\|\{|f^{(n)}(\tilde{z}_k)|^s \, (1-|z_k|)^{\frac{s}{p}+sn}\}\|_{T^1_{p/s}(Z)} \\ &\lesssim \int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} \int_{\Delta(z_k,2\eta)} |f(w)|^p \, \frac{dm_2(w)}{1-|w|} \right)^{s/p} |d\xi|.\end{aligned}$$

By Lemma 6.3.4 we can take $M_+ > M > 1$ such that

$$\bigcup_{z_k \in S_M(\xi)} \Delta(z_k, 2\eta) \subset S_{M_+}(\xi).$$

So that, we have

$$\begin{aligned}&\|\{|f^{(n)}(\tilde{z}_k)|^s \, (1-|z_k|)^{\frac{s}{p}+sn}\}\|_{T^1_{p/s}(Z)} \\ &\lesssim \int_{\mathbb{T}} \left(\int_{S_{M_+}(\xi)} \left(\sum_{z_k \in S_M(\xi)} \chi_{\Delta(z_k,2\eta)}(w) \right) |f(w)|^p \, \frac{dm_2(w)}{1-|w|} \right)^{s/p} |d\xi|.\end{aligned}$$

Since each $w \in \mathbb{D}$ belongs at most to N hyperbolic discs $\Delta(z_k, 2\eta)$ (see Proposition 6.3.3), we conclude that

$$\|\{|f^{(n)}(\tilde{z_k})|^s\ (1-|z_k|)^{\frac{s}{p}+n}\}\|_{T^1_{p/s}(Z)} \lesssim \|f\|^s_{T^s_p}.$$

Conversely, assume now that there is a constant $C > 0$ such that

$$\left(\int_{\mathbb{D}} |f^{(n)}(w)|^s\ d\mu(w)\right)^{1/s} \leq C\|f\|_{T^s_p}$$

for all $f \in AT^s_p$. Let us see that $\left\{\frac{\mu(R_k)}{(1-|z_k|)^{\frac{s}{p}+sn+1}}\right\} \in T^{\infty}_{\frac{p}{p-s}}(Z)$. Let $\lambda = \{\lambda_k\}$ be a sequence of $T^s_p(Z)$ and $r_k : [0,1] \to \{-1,1\}$ the Rademacher functions. Moreover, for $M > \max\{1, p/s, 1/s, 1/p\} + 1/p$ we consider the function

$$F_x(z) := \sum_k \lambda_k r_k(x) \frac{(1-|z_k|)^{M-\frac{1}{p}}}{(1-\overline{z_k}z)^M}.$$

Using Proposition 6.3.18, we have that

$$\left(\int_{\mathbb{D}} |F^{(n)}_x(w)|^s\ d\mu(w)\right)^{1/s} \lesssim \|F_x\|_{T^s_p} \lesssim \|\lambda\|_{T^s_p(Z)}.$$

Integrating with respect to x, we obtain

$$\int_{\mathbb{D}}\int_0^1 |F^{(n)}_x(w)|^s\ dx\ d\mu(w) \lesssim \|\lambda\|^s_{T^s_p(Z)}.$$

By means of Khinchine's inequality (see Theorem A.6.1)

$$\int_0^1 |F^{(n)}_x(w)|^s\ dx \asymp \left(\sum_k |\lambda_k|^2\ \frac{(1-|z_k|)^{2M-2/p}}{|1-\overline{z_k}z|^{2M+n}}\right)^{s/2}.$$

Thus

$$\int_{\mathbb{D}} \left(\sum_k |\lambda_k|^2 \frac{(1-|z_k|)^{2M-\frac{2}{p}}}{|1-z\overline{z_k}|^{2M+2n}}\right)^{s/2} d\mu(z) \lesssim \|\lambda\|^s_{T^s_p(Z)}.$$

Since $\chi_{R_k}(z) \lesssim \frac{1-|z_k|}{|1-\overline{z_k}z|}$, we have that

$$\int_{\mathbb{D}} \left(\sum_k |\lambda_k|^2 \frac{\chi_{R_k}(z)}{(1-|z_k|)^{2n+\frac{2}{p}}}\right)^{s/2} d\mu(z) \lesssim \|\lambda\|^s_{T^s_p(Z)}.$$

Therefore, it follows that

$$\sum_k |\lambda_k|^s \frac{\mu(R_k)}{(1-|z_k|)^{ns+\frac{s}{p}+1}} (1-|z_k|) \lesssim \|\lambda\|^s_{T^s_p(Z)} = \|\{|\lambda_k|^s\}_k\|_{T^1_{p/s}(Z)}.$$

The arbitrariness of $\{|\lambda_k|^s\}_k$ in $T^1_{p/s}(Z)$ and Proposition 6.3.15 show that $\left\{\frac{\mu(R_k)}{(1-|z_k|)^{\frac{s}{p}+sn+1}}\right\} \in T^\infty_{\frac{p}{p-s}}(Z)$. □

6.5 Integration operator

In Chapter 5, we proved that the operator

$$T_g : RM(p,q) \to RM(p,q)$$

is bounded, for $1 \le p < +\infty$, $1 \le q \le +\infty$, if and only if $g \in \mathcal{B}$ (see Theorem 5.1.3). Now we are going to apply the main result of this chapter to study integration operators acting between different $RM(p,q)$ spaces.

Many authors have considered the action of T_g between distinct Bergman and Hardy spaces (see, i.e., [67], [50]). In this section, we consider the more general question of characterizing the symbols g such that

$$T_g : RM(p,q) \to RM(t,s)$$

is bounded for $1 \le p,q,t,s \le +\infty$. It turns out that Theorem 6.4.1 is the key to this study.

First we present the answer to the question under consideration when the parameters are finite.

Theorem 6.5.1. *Let $1 \le p,q,s,t < +\infty$ and $g \in \mathcal{H}(\mathbb{D})$. The following statements are equivalent:*

1. *The operator $T_g : RM(p,q) \to RM(t,s)$ is bounded.*
2. *If $Z = \{z_k\}$ is an (r,κ)-lattice and denoting $d\mu_g(z) := |g'(z)|^t \, dm_2(z)$ it holds*
   (a) *If $1 \le s < q < +\infty$, $1 \le t < p < +\infty$, then*

$$\int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| < +\infty.$$

(b) *If* $1 \leq s < q < +\infty$, $1 \leq p \leq t < +\infty$, *then*

$$\int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} \frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{qs}{q-s}} |d\xi| < +\infty.$$

(c) *If* $1 \leq q < s < +\infty$, $1 \leq p, t < \infty$ *or* $1 \leq q = s < +\infty$, $1 \leq p \leq t < +\infty$, *then*

$$\sup_k \frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{q}+\frac{1}{t}-\frac{1}{s}-1}} < +\infty.$$

(d) *If* $1 \leq q = s < +\infty$, $1 \leq t < p < +\infty$, *then*

$$\sup_{\xi \in \mathbb{T}} \left(\sup_{\xi \in I} \frac{1}{|I|} \sum_{z_k \in S(I)} \left(\frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} (1-|z_k|) \right)^{\frac{p-t}{pt}} < +\infty,$$

where I *is an interval of* $\mathbb{T}$, $S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1 \text{ and } \frac{z}{|z|} \in I \right\}$ *and* $|I|$ *is the arc length of* I.

3. *The function* $g \in \mathcal{H}(\mathbb{D})$ *satisfies that*

(a) *If* $1 \leq s < q < +\infty$, $1 \leq t < p < +\infty$,

$$g \in RM\left(\frac{pt}{p-t}, \frac{qs}{q-s} \right).$$

(b) *If* $1 \leq s < q < +\infty$, $1 \leq p \leq t < +\infty$,

$$g'(z)(1-|z|)^{1+\frac{1}{t}-\frac{1}{p}} \in T_{\infty}^{\frac{qs}{q-s}}.$$

(c) *If* $1 \leq q < s < +\infty$, $1 \leq p, t < \infty$ *or* $1 \leq q = s < +\infty$, $1 \leq p \leq t < +\infty$,

$$\sup_{z \in \mathbb{D}} |g'(z)|(1-|z|)^{1+\frac{1}{t}+\frac{1}{s}-\frac{1}{p}-\frac{1}{q}} < +\infty.$$

(d) *If* $1 \leq q = s < +\infty$, $1 \leq t < p < +\infty$, *the measure*

$$d\mu(z) = |g'(z)|^{\frac{pt}{p-t}} (1-|z|)^{\frac{pt}{p-t}} \, dm_2(z)$$

is a Carleson measure.

Proof. Fix $M > 1$. In the begining of the proof we remind that, given $f \in \mathcal{H}(\mathbb{D})$,

$$\begin{aligned}\rho_{t,s}(T_g(f)) &\asymp \left(\int_{\mathbb{T}}\left(\int_{S_M(\xi)} |T_g(f)'(w)|^t(1-|w|)^{t-1}\, dm_2(w)\right)^{s/t} |d\xi|\right)^{1/s}\\ &= \left(\int_{\mathbb{T}}\left(\int_{S_M(\xi)} |f(w)|^t|g'(w)|^t(1-|w|)^{t-1}\, dm_2(w)\right)^{s/t} |d\xi|\right)^{1/s}\end{aligned}$$

by Theorem 6.2.3 and Theorem 6.1.5. Now, consider μ the measure

$$d\mu(w) = |g'(w)|^t(1-|w|)^{t-1}\, dm_2(w) = (1-|w|)^{t-1}\, d\mu_g(w)\,.$$

In addition, taking into account that given the (r,κ)-lattice $\{z_k\}$,

$$1-|w| \asymp 1-|z_k|\,,$$

for any w in the hyperbolic disc $\Delta(z_k,r)$ (see Remark 6.3.2), and we have

$$\mu(\Delta(z_k,r)) \asymp (1-|z_k|)^{t-1}\mu_g(\Delta(z_k,r)).$$

Bearing in mind these facts, the equivalences between *(1)* and *(2)* follow immediately by using Theorem 6.4.1.

From now on, we prove the equivalence between *(2)* and *(3)*. We split the proof in different cases.

Let us prove that *(2)(a)* is equivalent to *(3)(a)*. The idea of the proof is to show that

$$\begin{aligned}&\int_{\mathbb{T}}\left(\sum_{z_k\in S_M(\xi)}\left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}}\right)^{\frac{pt}{p-t}}\right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \qquad (6.5.1)\\ &\asymp \int_{\mathbb{T}}\left(\int_{S_M(\xi)} |g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\, dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|.\end{aligned}$$

First, let us check that

$$\begin{aligned}&\int_{\mathbb{T}}\left(\sum_{z_k\in S_M(\xi)}\left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}}\right)^{\frac{pt}{p-t}}\right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \qquad (6.5.2)\\ &\lesssim \int_{\mathbb{T}}\left(\int_{S_M(\xi)} |g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\, dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|.\end{aligned}$$

Setting $|g'(\tilde{z}_k)| := \sup_{w\in\overline{\Delta(z_k,r)}} |g'(w)|$ we have

$$
\begin{aligned}
&\int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \\
&\lesssim \int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \left(\sup_{w\in\overline{\Delta(z_k,r)}} |g'(w)|(1-|z_k|)^{\frac{1}{t}-\frac{1}{p}+1} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \\
&= \int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \left(|g'(\tilde{z}_k)|(1-|z_k|)^{\frac{1}{t}-\frac{1}{p}+1} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \,.
\end{aligned}
$$

At this point we apply again the mean value property over the hyperbolic disc $\Delta(\tilde{z}_k, s)$ with $s < 3r$:

$$
\begin{aligned}
&\int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \\
&\lesssim \int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \frac{(1-|z_k|)^{1+\frac{pt}{p-t}}}{(1-|\tilde{z}_k|)^2} \left(\int_{\Delta(\tilde{z}_k,s)} |g'(w)|^{\frac{pt}{p-t}} \, dm_2(w) \right) \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|.
\end{aligned}
$$

Since $\Delta(\tilde{z}_k, s) \subset \Delta(z_k, 4r)$ and by Remark 6.3.2, we have

$$
\begin{aligned}
&\int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \\
&\lesssim \int_{\mathbb{T}} \left(\sum_{z_k\in S_M(\xi)} \left(\int_{\Delta(z_k,4r)} |g'(w)|^{\frac{pt}{p-t}} (1-|w|)^{\frac{pt}{p-t}-1} \, dm_2(w) \right) \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|.
\end{aligned}
$$

Taking $M_+ > M > 1$ such that

$$
\bigcup_{\Delta(z_k,r)\cap S_M(\xi)\neq\emptyset} \Delta(z_k, 4r) \subset S_{M_+}(\xi),
$$

we get that

$$\int_{\mathbb{T}}\left(\sum_{z_k\in S_M(\xi)}\left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}}\right)^{\frac{pt}{p-t}}\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim \int_{\mathbb{T}}\left(\int_{S_{M_+}(\xi)}\left(\sum_{z_k\in S_M(\xi)}\chi_{\Delta(z_k,4r)}(w)\right)\frac{|g'(w)|^{\frac{pt}{p-t}}}{(1-|w|)^{1-\frac{pt}{p-t}}}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|.$$

Since each $z\in\mathbb{D}$ belongs to no more than N hyperbolic discs $\Delta(z_k,4r)$ (with N only depending on the lattice, see Proposition 6.3.3), we conclude that

$$\int_{\mathbb{T}}\left(\sum_{z_k\in S_M(\xi)}\left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}}\right)^{\frac{pt}{p-t}}\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim \int_{\mathbb{T}}\left(\int_{S_{M_+}(\xi)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim \int_{\mathbb{T}}\left(\int_{S_M(\xi)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|.$$

Last inequality follows by Lemma 6.1.2. So that, (6.5.2) holds.

Now, we proceed with the converse inequality. Using the pointwise estimate $|g'(z)|\le\sum_k|g'(z)|\chi_{\Delta(z_k,r)}(z)$ and the fact (given by Proposition 6.3.4) that we can take $M_+>M>1$ such that

$$\bigcup_{\Delta(z_k,r)\cap S_M(\xi)\neq\emptyset}\Delta(z_k,r)\subset S_{M_+}(\xi),$$

it follows

$$\int_{\mathbb{T}}\left(\int_{S_M(\xi)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\le \int_{\mathbb{T}}\left(\sum_k\int_{S_M(\xi)\cap\Delta(z_k,r)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\le \int_{\mathbb{T}}\left(\sum_{z_k\in S_{M_+}(\xi)}\int_{\Delta(z_k,r)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|.$$

Taking $\tilde{z_k}\in\overline{\Delta(z_k,r)}$ such that $|g'(\tilde{z_k})|:=\sup_{w\in\overline{\Delta(z_k,r)}}|g'(w)|$, and applying (6.3.1) and Remark 6.3.2, we have

$$\int_{\mathbb{T}}\left(\int_{S_M(\xi)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sum_{z_k\in S_{M_+}(\xi)}|g'(\tilde{z_k})|^{\frac{pt}{p-t}}\,(1-|z_k|)^{\frac{pt}{p-t}+1}\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|.$$

By the mean value property over the hyperbolic disc $\Delta(\tilde{z_k},s)$ with $s<3r$ (see Lemma 6.3.9), we obtain

$$\int_{\mathbb{T}}\left(\int_{S_M(\xi)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sum_{z_k\in S_{M_+}(\xi)}\left(\int_{\Delta(\tilde{z_k},s)}|g'(w)|^t\,\frac{dm_2(w)}{(1-|\tilde{z_k}|)^2}\right)^{\frac{p}{p-t}}(1-|z_k|)^{\frac{pt}{p-t}+1}\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|.$$

Since $\Delta(\tilde{z_k},s)\subset\Delta(z_k,4r)$ and Remark 6.3.2, we have

$$\int_{\mathbb{T}}\left(\int_{S_M(\xi)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sum_{z_k\in S_{M_+}(\xi)}\left(\int_{\Delta(z_k,4r)}|g'(w)|^t\,\,dm_2(w)\right)^{\frac{p}{p-t}}(1-|z_k|)^{\frac{pt-p-t}{p-t}}\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sum_{z_k\in S_{M_+}(\xi)}\left(\sum_j\frac{\int_{\Delta(z_j,r)}\chi_{\Delta(z_k,4r)}(w)|g'(w)|^t\,\,dm_2(w)}{(1-|z_k|)^{1+\frac{t}{p}-t}}\right)^{\frac{p}{p-t}}\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|.$$

Applying Remark 6.3.2, it follows that

$$\int_{\mathbb{T}}\left(\int_{S_M(\xi)}|g'(w)|^{\frac{pt}{p-t}}(1-|w|)^{\frac{pt}{p-t}-1}\,dm_2(w)\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sum_{z_k\in S_{M_+}(\xi)}\left(\sum_{z_j\in\Delta(z_k,5r)}\frac{\mu_g^{1/t}(\Delta(z_j,r))}{(1-|z_j|)^{\frac{1}{t}+\frac{1}{p}-1}}\right)^{\frac{pt}{p-t}}\right)^{\frac{(p-t)qs}{(q-s)pt}}|d\xi|.$$

Because each $z\in\mathbb{D}$ belongs to no more than $N=N(r,\kappa)$ discs (see Proposition 6.3.3) and using the fact that

$$\|w\|_{\ell^1}\le N^{t/p}\|w\|_{\ell^{\frac{p}{p-t}}}$$

for all $w \in \mathbb{C}^N$, we obtain

$$\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |g'(w)|^{\frac{pt}{p-t}} (1-|w|)^{\frac{pt}{p-t}-1} \, dm_2(w) \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|$$
$$\lesssim \int_{\mathbb{T}} \left(\sum_{z_k \in S_{M_+}(\xi)} \sum_{z_j \in \Delta(z_k, 5r)} N^{t/p} \left(\frac{\mu_g^{1/t}(\Delta(z_j, r))}{(1-|z_j|)^{\frac{1}{t}+\frac{1}{p}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|.$$

By Proposition 6.3.4, one can take $M_* > M_+ > 1$ such that

$$\bigcup_{z_k \in S_{M_+}(\xi)} \Delta(z_k, 5r) \subset S_{M_*}(\xi).$$

Hence, we have

$$\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |g'(w)|^{\frac{pt}{p-t}} (1-|w|)^{\frac{pt}{p-t}-1} \, dm_2(w) \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|$$
$$\lesssim N^{\frac{(p-t)qs}{(q-s)p^2}} \int_{\mathbb{T}} \left(\sum_j \sum_{z_k \in S_{M_+}(\xi)} \chi_{\Delta(z_j,5r)}(z_k) \left(\frac{\mu_g^{1/t}(\Delta(z_j, r))}{(1-|z_j|)^{\frac{1}{t}+\frac{1}{p}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|$$
$$\lesssim N^{\frac{(p-t)qs}{(q-s)p^2}} \int_{\mathbb{T}} \left(\sum_{z_j \in S_{M_*}(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_j, r))}{(1-|z_j|)^{\frac{1}{t}+\frac{1}{p}-1}} \right)^{\frac{pt}{p-t}} \sum_{z_k \in S_{M_+}(\xi)} \chi_{\Delta(z_j,5r)}(z_k) \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|$$

Since there are at most $N_1 = N_1(r, \kappa)$ points z_k of the (r, κ)-lattice (see Proposition 6.3.3) such that $z_k \in \Delta(z_j, 5r)$ for all $j \in \mathbb{N}$, we have

$$\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |g'(w)|^{\frac{pt}{p-t}} (1-|w|)^{\frac{pt}{p-t}-1} \, dm_2(w) \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|$$
$$\lesssim N^{\frac{(p-t)qs}{(q-s)p^2}} N_1^{\frac{(p-t)qs}{(q-s)pt}} \int_{\mathbb{T}} \left(\sum_{z_j \in S_{M_*}(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_j, r))}{(1-|z_j|)^{\frac{1}{t}+\frac{1}{p}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|$$
$$\lesssim \int_{\mathbb{T}} \left(\sum_{z_j \in S_M(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_j, r))}{(1-|z_j|)^{\frac{1}{t}+\frac{1}{p}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi|.$$

Last inequality follows by Lemma 6.1.2. Therefore (6.5.1) holds.

Combining Theorem 6.1.5 and Theorem 6.2.3, it follows

$$\rho_{\frac{pt}{p-t},\frac{qs}{q-s}}(g) \asymp \|g\|_{T^{\frac{qs}{q-s}}_{\frac{pt}{p-t}}} \asymp \|g'(z)(1-|z|)\|_{T^{\frac{qs}{q-s}}_{\frac{pt}{p-t}}}.$$

By (6.5.1) and Remark 6.1.3, we conclude

$$\rho_{\frac{pt}{p-t},\frac{qs}{q-s}}(g) \asymp \left(\int_{\mathbb{T}} \left(\sum_{z_k \in S_M(\xi)} \left(\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} \right)^{\frac{(p-t)qs}{(q-s)pt}} |d\xi| \right)^{\frac{q-s}{qs}}.$$

So that, we have proved that *(2)(a)* and *(3)(a)* are equivalent.

Let us show the equivalence between *(2)(b)* and *(3)(b)*. We have to show that

$$\int_{\mathbb{T}} \left(\sup_{w \in S_M(\xi)} |g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}} \right)^{\frac{qs}{q-s}} |d\xi| \tag{6.5.3}$$
$$\asymp \int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} \frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{qs}{q-s}} |d\xi|.$$

First, we will show that

$$\int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} \frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{qs}{q-s}} |d\xi| \tag{6.5.4}$$
$$\lesssim \int_{\mathbb{T}} \left(\sup_{w \in S_M(\xi)} |g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}} \right)^{\frac{qs}{q-s}} |d\xi|.$$

Taking supremum over each hyperbolic disc $\Delta(z_k, r)$, we have

$$\int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} \frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{qs}{q-s}} |d\xi|$$
$$\lesssim \int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} \left(\sup_{w \in \overline{\Delta(z_k,r)}} (1-|w|)^{1+\frac{1}{t}-\frac{1}{p}} |g'(w)| \right) \right)^{\frac{qs}{q-s}} |d\xi|.$$

Since one can take $M_+ > M > 1$ such that

$$\bigcup_{\Delta(z_k,r) \cap S_M(\xi) \neq \emptyset} \Delta(z_k, r) \subset S_{M_+}(\xi),$$

it follows that

$$\int_{\mathbb{T}}\left(\sup_{z_k\in S_M(\xi)}\frac{\mu_g^{1/t}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$\leq\int_{\mathbb{T}}\left(\sup_{w\in S_{M_+}(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$\leq\int_{\mathbb{T}}\left(\sup_{w\in S_M(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|.$$

Last inequality follows by Lemma 6.1.7 applied to $\{z_k\}$ a dense sequence in $\mathbb{D}$. So that, (6.5.4) holds.

Now, we continue with the proof of the converse inequality. Using the fact (given by Proposition 6.3.4) that we can take $M_+ > M > 1$ such that

$$\bigcup_{\Delta(z_k,r)\cap S_M(\xi)\neq\emptyset}\Delta(z_k,r)\subset S_{M_+}(\xi),$$

we have

$$\int_{\mathbb{T}}\left(\sup_{w\in S_M(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$=\int_{\mathbb{T}}\left(\sup_{z_k\in S_{M_+}(\xi)}\sup_{w\in\Delta(z_k,r)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|.$$

Taking $\tilde{z_k}\in\overline{\Delta(z_k,r)}$ such that $|g'(\tilde{z_k})|:=\sup_{w\in\overline{\Delta(z_k,r)}}|g'(w)|$ and applying Remark 6.3.2, we have

$$\int_{\mathbb{T}}\left(\sup_{w\in S_M(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sup_{z_k\in S_{M_+}(\xi)}|g'(\tilde{z_k})|(1-|z_k|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|.$$

By the mean value property over the hyperbolic disc $\Delta(\tilde{z_k},s)$ with $s<3r$ (see

Lemma 6.3.9), we obtain

$$\int_{\mathbb{T}}\left(\sup_{w\in S_M(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sup_{z_k\in S_{M_+}(\xi)}\left(\int_{\Delta(\tilde{z_k},s)}|g'(w)|^t\,\frac{dm_2(w)}{(1-|\tilde{z_k}|)^2}\right)^{1/t}(1-|z_k|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$

Since $\Delta(\tilde{z_k},s)\subset\Delta(z_k,4r)$ and by Remark 6.3.2, it follows

$$\int_{\mathbb{T}}\left(\sup_{w\in S_M(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$\lesssim\int_{\mathbb{T}}\left(\sup_{z_k\in S_{M_+}(\xi)}\left(\int_{\Delta(z_k,4r)}|g'(w)|^t\,dm_2(w)\right)^{1/t}(1-|z_k|)^{1-\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$=\int_{\mathbb{T}}\left(\sup_{z_k\in S_{M_+}(\xi)}\frac{\mu_g^{1/t}(\Delta(z_k,4r))}{(1-|z_k|)^{\frac{1}{t}+\frac{1}{p}-1}}\right)^{\frac{qs}{q-s}}|d\xi|$$

Because each $z\in\mathbb{D}$ belongs to no more than $N=N(r,\kappa)$ discs (see Proposition 6.3.3) and using the fact that

$$\|w\|_{\ell^1}\leq N^{1-t}\|w\|_{\ell^{1/t}}$$

for all $w\in\mathbb{C}^N$, we obtain

$$\int_{\mathbb{T}}\left(\sup_{w\in S_M(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$\lesssim N^{\frac{qs(1-t)}{q-s}}\int_{\mathbb{T}}\left(\sup_{z_k\in S_{M_+}(\xi)}\sum_{z_j\in\Delta(z_k,5r)}\frac{\mu_g^{1/t}(\Delta(z_j,r))}{(1-|z_k|)^{\frac{1}{t}+\frac{1}{p}-1}}\right)^{\frac{qs}{q-s}}|d\xi|.$$

Since there are at most $N_1=N_1(r,\kappa)$ points z_k of the (r,κ)-lattice (see Proposition 6.3.3) such that $z_k\in\Delta(z_j,5r)$ for all $j\in\mathbb{N}$, we have

$$\int_{\mathbb{T}}\left(\sup_{w\in S_M(\xi)}|g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}}\right)^{\frac{qs}{q-s}}|d\xi|$$
$$\lesssim N^{\frac{qs(1-t)}{q-s}}\int_{\mathbb{T}}\left(\sup_{z_k\in S_{M_+}(\xi)}N_1\left(\sup_{z_j\in\Delta(z_k,5r)}\frac{\mu_g^{1/t}(\Delta(z_j,r))}{(1-|z_k|)^{\frac{1}{t}+\frac{1}{p}-1}}\right)\right)^{\frac{qs}{q-s}}|d\xi|.$$

By Proposition 6.3.4, one can take $M_* > M_+ > 1$ such that

$$\bigcup_{z_k \in S_{M_+}(\xi)} \Delta(z_k, 5r) \subset S_{M_*}(\xi).$$

So that, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}} \left(\sup_{w \in S_M(\xi)} |g'(w)|(1-|w|)^{1+\frac{1}{t}-\frac{1}{p}} \right)^{\frac{qs}{q-s}} |d\xi| \\
&\lesssim N^{\frac{qs(1-t)}{q-s}} N_1^{\frac{qs}{q-s}} \int_{\mathbb{T}} \left(\sup_{z_k \in S_{M_+}(\xi)} \left(\sup_{z_j \in \Delta(z_k, 5r)} \frac{\mu_g^{1/t}(\Delta(z_j, r))}{(1-|z_k|)^{\frac{1}{t}+\frac{1}{p}-1}} \right) \right)^{\frac{qs}{q-s}} |d\xi| \\
&\lesssim \int_{\mathbb{T}} \left(\sup_{z_k \in S_{M_*}(\xi)} \frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{t}+\frac{1}{p}-1}} \right)^{\frac{qs}{q-s}} |d\xi| \\
&\lesssim \int_{\mathbb{T}} \left(\sup_{z_k \in S_M(\xi)} \frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{t}+\frac{1}{p}-1}} \right)^{\frac{qs}{q-s}} |d\xi|.
\end{aligned}$$

Last inequality follows by Lemma 6.1.7. Therefore, we have proved that *(2)(b)* and *(3)(b)* are equivalent.

The equivalence between *(2)(c)* and *(3)(c)* follows immediately. Observe that for each $\Delta(z_k, r)$

$$\frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{q}+\frac{1}{t}-\frac{1}{s}-1}} \lesssim \sup_{z \in \mathbb{D}} |g'(z)|(1-|z|)^{1+\frac{1}{t}+\frac{1}{s}-\frac{1}{p}-\frac{1}{q}}.$$

So, one implication holds. Now, set $z \in \mathbb{D}$. Applying the mean value property it occurs that

$$|g'(z)|(1-|z|)^{1+\frac{1}{t}+\frac{1}{s}-\frac{1}{p}-\frac{1}{q}} \lesssim \left(\int_{\Delta(z,r)} |g'(w)|^t (1-|w|)^{t+\frac{t}{s}-\frac{t}{p}-\frac{t}{q}-1} \, dm(w) \right)^{1/t}.$$

Since the number of the hyperbolic discs $\Delta(z_k, r)$ such that $\Delta(z_k, r) \cap \Delta(z, r) \neq \emptyset$ can be at most a fixed number (see Proposition 6.3.3), we have

$$|g'(z)|(1-|z|)^{1+\frac{1}{t}+\frac{1}{s}-\frac{1}{p}-\frac{1}{q}} \lesssim \sup_k \frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{q}+\frac{1}{t}-\frac{1}{s}-1}}.$$

Now, we show that *(3)(d)* implies *(2)(d)*. Fix an arc $I \subset \mathbb{T}$. Then

$$\sum_{z_k \in S(I)} \left(\frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} (1-|z_k|) \lesssim \sum_{z_k \in S(I)} |g'(\tilde{z}_k)|^{\frac{pt}{p-t}} (1-|z_k|)^{2+\frac{pt}{p-t}},$$

where $|g'(\tilde{z}_k)| := \sup_{w \in \overline{\Delta(z_k,r)}} |g'(w)|$. Using the mean value property on $\Delta(\tilde{z}_k, s)$ (see Lemma 6.3.9) with $s < 3r$, it follows

$$\sum_{z_k \in S(I)} \left(\frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} (1-|z_k|) \lesssim \sum_{z_k \in S(I)} \frac{(1-|z_k|)^{2+\frac{pt}{p-t}}}{(1-|\tilde{z}_k|)^2} \left(\int_{\Delta(\tilde{z}_k,s)} |g'(w)|^{\frac{pt}{p-t}} \, dm_2(w) \right)$$

Since $\Delta(\tilde{z}_k, s) \subset \Delta(z_k, 4r)$, we obtain

$$\begin{aligned} \sum_{z_k \in S(I)} \left(\frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} (1-|z_k|) &\lesssim \sum_{z_k \in S(I)} (1-|z_k|)^{\frac{pt}{p-t}} \left(\int_{\Delta(z_k,4r)} |g'(w)|^{\frac{pt}{p-t}} \, dm_2(w) \right) \\ &\asymp \sum_{z_k \in S(I)} \int_{\Delta(z_k,4r)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}} \, dm_2(w) \, . \end{aligned}$$

Taking an arc $I_+ \subset \mathbb{T}$ such that $\bigcup_{z_k \in S(I)} \Delta(z_k, 4r) \subset S(I_+)$ with $|I| \asymp |I_+|$ and employing the finite covering property of the lattice we have

$$\sum_{z_k \in S(I)} \left(\frac{\mu_g^{1/t}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} (1-|z_k|) \lesssim \int_{S(I_+)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}} \, dm_2(w)$$

which is enough in order to claim that we have accomplished our aim.

Let us continue proving that *(2)(d)* implies *(3)(d)*. Fix an arc $I \subset \mathbb{T}$. Then

$$\int_{S(I)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}} \, dm_2(w) \lesssim \sum_k \int_{S(I) \cap \Delta(z_k,r)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}} \, dm_2(w)$$

Bearing in mind that we can take I_+ such that $\bigcup_{\Delta(z_k,r) \cap S(I) \neq \emptyset} \Delta(z_k, r) \subset S(I_+)$ with

$|I| \asymp |I_+|$, we have

$$\int_{S(I)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}}\, dm_2(w) \leq \sum_{z_k \in S(I_+)} \int_{\Delta(z_k,r)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}}\, dm_2(w)$$
$$\asymp \sum_{z_k \in S(I_+)} (1-|z_k|)^{\frac{pt}{p-t}} \int_{\Delta(z_k,r)} |g'(w)|^{\frac{pt}{p-t}}\, dm_2(w).$$

The application of the submean value property leads to

$$\int_{S(I)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}}\, dm_2(w)$$
$$\lesssim \sum_{z_k \in S(I_+)} (1-|z_k|)^{\frac{pt}{p-t}} \int_{\Delta(z_k,r)} \left(\frac{1}{(1-|z_k|)^2} \int_{\Delta(z_k,2r)} |g'(u)|^t \, dm_2(u) \right)^{\frac{p}{p-t}} dm_2(w)$$
$$\asymp \sum_{z_k \in S(I_+)} (1-|z_k|)^{\frac{pt}{p-t} - 2\frac{p}{p-t} + 2} \left(\mu_g^{\frac{1}{t}}(\Delta(z_k,2r)) \right)^{\frac{pt}{p-t}}$$
$$= \sum_{z_k \in S(I_+)} (1-|z_k|)^{\frac{pt}{p-t} - 2\frac{p}{p-t} + 1} \left(\mu_g^{\frac{1}{t}}(\Delta(z_k,2r)) \right)^{\frac{pt}{p-t}} (1-|z_k|)$$
$$= \sum_{z_k \in S(I_+)} \left(\frac{\mu_g^{\frac{1}{t}}(\Delta(z_k,2r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} (1-|z_k|)\,.$$

Since the hyperbolic discs $\Delta(z_k,2r)$ can be covered at most by a fixed number of hyperbolic discs $\Delta(z_j,r)$ and by the fact that one can take $I_* \subset \mathbb{T}$ such that $\bigcup_{z_k \in S(I_+)} \Delta(z_k,3r) \subset S(I_*)$ with $|I_+| \asymp |I_*|$, then

$$\int_{S(I)} (1-|w|)^{\frac{pt}{p-t}} |g'(w)|^{\frac{pt}{p-t}}\, dm_2(w) \lesssim \sum_{z_k \in S(I_*)} \left(\frac{\mu_g^{\frac{1}{t}}(\Delta(z_k,r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{t}-1}} \right)^{\frac{pt}{p-t}} (1-|z_k|)\,.$$

Therefore, we are done. □

Notice that, a priori, statement *(2)* depends on the (r,κ)-lattice we are using. But the equivalence with *(1)* shows that it is independent of the lattice.

Now it remains to study the integration operator T_g when one of the parameters is not finite. In particular, if $t = +\infty$ we have the case $T_g : RM(p,q) \to H^s$, because $RM(\infty,s) = H^s$ (see Chapter 1). For cases $0 < p = q, s \leq +\infty$, that is, the integration operator $T_g : A^p \to H^s$ has been studied by Wu in [67] and subsequently by Miihkinen, Pau, Perälä, and Wang in [50] for the multivariable case. Remember that $RM(p,p) = A^p$ (see Chapter 1). The following result provides a characterization of the integration operator $T_g : RM(p,q) \to H^s$ for the cases $1 \leq p,q,s < +\infty$.

Hardy spaces H^p can also be seen as unweighted tent spaces. Following an idea of Calderon, Pavlovic proved that

Theorem 6.5.2. *[51, Theorem 1.3, p. 172] Let $0 < p, q < +\infty$ and $M > 1$. If f is an analytic function with $f(0) = 0$, then*

$$\|f\|_{H^p} \asymp \left(\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |f(w)|^{q-2} |f'(w)|^2 \, dm_2(w) \right)^{p/q} |d\xi| \right)^{1/p}.$$

Theorem 6.5.3. *Let $1 \leq p, q, s < +\infty$ and $g \in \mathcal{H}(\mathbb{D})$. The following statements are equivalent:*

1. *The operator $T_g : RM(p,q) \to H^s$ is bounded.*
2. *If $Z = \{z_k\}$ is an (r,κ)-lattice and denoting $d\mu_g(z) = |g'(z)|^2 \, dm(z)$ it holds*
   (a) *If $1 \leq s < q < +\infty$, $2 < p < +\infty$,*
   $$\int_{\mathbb{T}} \left(\sum_{z_k \in \Gamma(\xi)} \left(\frac{\mu_g^{1/2}(\Delta(z_k, r))}{(1-|z_k|)^{1/p}} \right)^{\frac{2p}{p-2}} \right)^{\frac{(p-2)qs}{(q-s)2p}} |d\xi| < +\infty.$$
   (b) *If $1 \leq s < q < +\infty$, $1 \leq p \leq 2$,*
   $$\int_{\mathbb{T}} \left(\sup_{z_k \in \Gamma(\xi)} \frac{\mu_g^{1/2}(\Delta(z_k, r))}{(1-|z_k|)^{1/p}} \right)^{\frac{qs}{q-s}} |d\xi| < +\infty.$$
   (c) *If $1 \leq q < s < +\infty$, $1 \leq p < \infty$ or $1 \leq q = s < +\infty$, $1 \leq p \leq 2$,*
   $$\sup_k \frac{\mu_g^{1/2}(\Delta(z_k, r))}{(1-|z_k|)^{\frac{1}{p}+\frac{1}{q}-\frac{1}{s}}} < +\infty.$$
   (d) *If $1 \leq q = s < +\infty$, $2 < p < +\infty$,*
   $$\sup_{\xi \in \mathbb{T}} \left(\sup_{\xi \in I} \frac{1}{|I|} \sum_{z_k \in S(I)} \left(\frac{\mu_g^{1/2}(\Delta(z_k, r))}{(1-|z_k|)^{1/p}} \right)^{\frac{2p}{p-2}} (1-|z_k|) \right)^{\frac{p-2}{2p}} < +\infty,$$
   where I is an interval of $\mathbb{T}$, $S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1 \text{ and } \frac{z}{|z|} \in I \right\}$ and $|I|$ is the arc length of I.
3. *The function $g \in \mathcal{H}(\mathbb{D})$ satisfies that*

(a) If $1 \le s < q < +\infty$, $2 < p < +\infty$,

$$|g'(z)|(1-|z|)^{\frac{1}{2}} \in T^{\frac{qs}{q-s}}_{\frac{2p}{p-2}}.$$

(b) If $1 \le s < q < +\infty$, $1 \le p \le 2$,

$$|g'(z)|(1-|z|)^{1-\frac{1}{p}} \in T^{\frac{qs}{q-s}}_{\infty}.$$

(c) If $1 \le q < s < +\infty$, $1 \le p < \infty$ *or* $1 \le q = s < +\infty$, $1 \le p \le 2$,

$$\sup_{z\in\mathbb{D}} |g'(z)|(1-|z|)^{1+\frac{1}{s}-\frac{1}{p}-\frac{1}{q}} < +\infty.$$

(d) If $1 \le q = s < +\infty$, $2 < p < +\infty$, *the measure* $|g'(z)|^{\frac{2p}{p-2}}(1-|z|)^{\frac{p}{p-2}}\, dm_2(z)$ *is a Carleson measure.*

Proof. The proof of the equivalences between *(1)* and *(2)* follows as a consequence of Theorem 6.4.1. This is due to the equivalent description of the Hardy norm as

$$\|T_g(f)\|_{H^s} \asymp \left(\int_{\mathbb{T}} \left(\int_{S_M(\xi)} |f(w)|^2|g'(w)|^2\, dm_2(z)\right)^{s/2} |d\xi|\right)^{1/s}$$

(see Theorem 6.5.2) and to the consideration $d\mu(z) = |g'(z)|^2\, dm_2(z)$.

The remaining equivalence results using the same arguments that were used in the proof of Theorem 6.5.1. □

Appendix A

Selection of some used –and probably little-known– results

In this appendix we state (and reference) some widely known results we have used in the previous chapters. We are confident the reader is acquainted with most of them, but we have decided to gather them here for the sake of being self-contained.

A.1 On L^p with $p < 1$

L^p spaces are a key topic for any mathematician. Nevertheless, when $p < 1$, the properties of L^p are less familiar. Thus, we briefly recall two results about them.

Definition A.1.1. The real-valued map $\|\cdot\|$ on the vector space X is a quasinorm if it satisfies the following conditions for all $x, y \in X$ and all scalar α:

1. $\|x\| \geq 0$,
2. $\|x\| = 0$ if and only if $x = 0$,
3. $\|\alpha x\| = |\alpha|\|x\|$,
4. there is a constant $C \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$.

If the vector space X is equipped with a quasinorm $\|\cdot\|$, it is said to be a quasi-normed vector space. Such quasinorm induces a vector topology on X, where a neighbourhood basis at the point $x_0 \in X$ is given by the sets

$$V_n(x_0) = \{x \in X \ : \|x - x_0\| < 1/n\}, \quad n \in \mathbb{N}.$$

Moreover, if it is also a complete space, then it is called a quasi-Banach space.

Notice that, for many purposes, statement 4 in above definition is a fine substitute for the triangle inequality.

Theorem A.1.2. *Let X, Y be quasi-Banach spaces. The linear operator $T : X \to Y$ is continuous if and only if there is a constant $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$.*

Given $p < 1$ and a measure μ on Ω, the space $L^p(\mu)$ is a quasi-Banach space. Indeed, given $f, g \in L^p(\mu)$ we have that

$$\|f + g\|_{L^p} \leq 2^{-1+1/p}(\|f\|_{L^p} + \|g\|_{L^p}).$$

In fact,

$$d(f, g) := \int_\Omega |f - g|^p d\mu$$

is a complete metric on $L^p(\mu)$. Notice that this metric is invariant under translation, that is,

$$d(f + h, g + h) = d(f, g), \qquad f, g, h \in L^p(\mu).$$

Therefore, for $0 < p < 1$, $L^p(\mu)$ is a topological vector space complete with respect to a translation invariant metric (so that, it is an F-space), and we have notions of continuity, completeness, and boundedness here. Thus several consequences of the Baire category theorem as applied to Banach spaces carry over without change to $L^p(\mu)$: the open mapping theorem, the closed graph theorem, and the principle of uniform boundedness. In particular, we have used

Theorem A.1.3. *[62, p. 50-51] Let X, Y be F-spaces. The linear operator $T : X \to Y$ is continuous if and only if the graph $G = \{(x, Tx) : x \in X\}$ of T is closed in $X \times Y$.*

It is easy to see that, in general, $L^p(\mu)$ is not locally convex if $0 < p < 1$.

A.2 On the weak topology

A relevant notion in our work has been the weak and weak-$*$ topologies on Banach spaces. They play an important role, both in the general theory and in applications. These topologies allow easy use of compactness arguments, and so they are helpful in existence proofs.

We used several properties of the weak-$*$ topology in Chapter 3. As a matter of fact, we used Alaoglu theorem (B_{X^*} is weak-$*$ compact), Goldstine theorem (B_X is weak-$*$ dense in $B_{X^{**}}$), and the metrizability of B_{X^*} with the weak-$*$ topology whenever X is a separable Banach space. We refer the reader to the book [65] for these properties.

With respect to the weak topology, the following result helps us to introduce the norm in the study of the weak compactness:

Theorem A.2.1. *[65, Corollary on p. 28] Let X be a Banach space. If the sequence $\{x_n\} \subset X$ converges to $x \in X$ in the weak topology, then there exists a sequence $\{y_j\}$, with $y_j \in co\{x_j, x_{j+1}, \dots\}$ for all j, such that $\|y_j - x\| \to 0$, when $j \to \infty$.*

Two fundamental properties when one manages weakly compact operators are:

Theorem A.2.2. *[65, Theorem 6, p. 52] Let X and Y be Banach spaces.*

- *An operator $T : X \to Y$ is weakly compact if and only if $T^* : Y^* \to X^*$ is weakly compact.*
- *An operator $T : X \to Y$ is weakly compact if and only if $T^{**}(X^{**}) \subset i(Y)$, where i is the canonical inclusion map from Y into Y^{**}.*

We end this section by showing different characterizations of the compactness in the weak topology in L^1:

Definition A.2.3. A subset H of $L^1(\mu)$ is uniformly integrable if

$$\lim_{\mu(E)\to 0}\int_E |f|\, d\mu = 0$$

uniformly in $f \in H$.

Theorem A.2.4 (Dunford-Pettis Theorem). *(see [26, Theorem 15, p. 76] or [65, p. 137]) A subset H of $L^1(\mu)$ is relatively weakly compact if and only if it is bounded and uniformly integrable.*

Theorem A.2.5 (Kadec-Pełczyński). *[25, p. 93, Corollary] Let H be a nonweakly compact bounded set of $L^1(\mu)$, where μ is a positive finite measure. Then H contains a sequence $\{f_n\}$ which is equivalent to the canonical basis of ℓ^1.*

A.3 Fixing copies of a given Banach space

Definition A.3.1. Let X, Y, and E be Banach spaces. An operator $T : X \to Y$ is said to fix a copy of E if there is a subspace $A \subset X$ such that A is isomorphic to E and $T : A \to T(A)$ is an isomorphism.

Typical Banach spaces E considered in the literature are the spaces of sequences c_0, ℓ^1, and ℓ^∞.

An elementary argument shows the following result:

Theorem A.3.2. *Let X, Y, Z, and E be Banach spaces and let $T : X \to Y$ and $S : Y \to Z$ be linear operators. If the operator $S \circ T$ fixes a copy of E, then the operators T and S fix a copy of E.*

A notion closely related to fixing copies of c_0 is the unconditional convergence:

Definition A.3.3. Let X and Y be Banach spaces. The bounded linear operator $T : X \to Y$ is unconditionally converging if $\sum_{n\geq 1} Tx_n$ is unconditionally convergent whenever $\sum_{n\geq 1} x_n$ is weakly unconditionally Cauchy (that is, whenever $\sum_{n\geq 1} |x^*(x_n)| < +\infty$ for all $x^* \in X^*$).

Based on ideas from Bessaga and Pełczyński characterizing the existence of copies of c_0 in Banach space, (see [15, Theorem 5]), it can be obtained:

Theorem A.3.4. *[25, Exercise 8, page 54]. Let X and Y be Banach spaces. The bounded linear operator $T : X \to Y$ is unconditionally converging if and only if T does not fix a copy of c_0.*

Definition A.3.5. Let X and Y be Banach spaces. The linear operator $T : X \to Y$ is ℓ^1-cosingular if there are no epimorphisms $h_1 : X \to \ell^1$ and $h_2 : Y \to \ell^1$ such that $h_1 = h_2 \circ T$.

Theorem A.3.6. *(see [52], [41, page 273].) Let $T : X \to Y$ be a bounded linear operator, where X and Y be Banach spaces. T^* is unconditionally converging if and only if T is ℓ^1-cosingular.*

Remark A.3.7. Let us see that if $T : X \to Y$ is not ℓ^1-cosingular, then T fixes a copy of ℓ^1. Since $T : X \to Y$ is not ℓ^1-cosingular, then there are epimorphisms $h_1 : X \to \ell^1$ and $h_2 : Y \to \ell^1$ such that $h_1 = h_2 \circ T$. By the open mapping theorem, there is bounded sequence $\{x_j\} \subset X$ such that

$$h_1(x_j) = h_2 \circ T(x_j) = e_j = (0, \dots, 0, \overset{(j)}{1}, 0, \dots).$$

Now, we consider the operator $\Theta : \ell^1 \to X$ given by $\Theta(\{\alpha_j\}) = \sum_{j=0}^{\infty} \alpha_j x_j$. Bearing in mind that $h_1 = h_2 \circ T$, we have

$$\|\{\alpha_j\}\|_{\ell^1} = \left\| h_1 \left(\sum_{j=0}^{\infty} \alpha_j x_j \right) \right\|_{\ell^1} \leq \|h_2\| \left\| \sum_{j=0}^{\infty} \alpha_j T(x_j) \right\|_Y .$$

Moreover, by the boundedness of the operator $T : X \to Y$ and the fact that $\{x_j\}$ is a bounded sequence, it follows

$$\frac{1}{\|T\|} \left\| \sum_{j=0}^{\infty} \alpha_j T(x_j) \right\|_Y \leq \left\| \sum_{j=0}^{\infty} \alpha_j x_j \right\|_X \leq \sum_{j=0}^{\infty} |\alpha_j| \|x_j\|_X \leq \left(\sup_j \|x_j\| \right) \|\{\alpha_j\}\|_{\ell^1}.$$

Hence, we obtain that $T : X \to Y$ fixes a copy of ℓ^1 and we are done.

A.4 Logarithmically-subharmonic functions

Let us recall that a positive function $u : \Omega \to \mathbb{R}$, being Ω a domain in $\mathbb{R}^n$, is said to be a logarithmically-subharmonic function if $\log \circ u$ is subharmonic. Every logarithmically-subharmonic function is subharmonic, but the converse is not true.

The interest of this subclass of subharmonic functions is that it has more stability properties than the whole family of subharmonic functions.

Theorem A.4.1. *[59, p. 36] Let u be a continuous function on $\{(x, y) : x \in G \subset \mathbb{R}^m,\ y \in \overline{D} \subset \mathbb{R}^n\}$, where G and D are domains. If for each $y \in \overline{D}$ the function $u(x, y)$ is a logarithmically-subharmonic in G as a function of x, then the function*

$$F(x) := \int_D u(x, y)\ d\mu(y),$$

where μ is the volume measure in $\mathbb{R}^n$, is also logarithmically-subharmonic in G.

A deep result due to Pavlović (and based on previous results of Fefferman and Stein) shows:

Theorem A.4.2. *[51, Theorem 1.8, p. 173] Let $0 < p, q < +\infty$. Let $\{f_n\}$ be a sequence of logarithmically-subharmonic functions on $\mathbb{D}$. Then there exists a constant $C_{p,q}$ such that*

$$\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} (M_* f_n(\xi))^q \right)^{p/q} |d\xi| \leq C_{p,q} \sup_{0<r<1} \int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} (f_n(r\xi))^q \right)^{p/q} |d\xi|,$$

where M_* denotes the non-tangential maximal operator, $M_* f(\xi) = \sup_{z \in S_C(\xi)} |f(z)|$.

A.5 Lindelöf's theorem

In studying boundary behavior of holomorphic maps, it is useful to define different approaching regions to boundary points. The most common approaching ways to the boundary are the radial limits, the non-tangential limits and the unrestricted limits. Concerning the non-tangential limits, one of the basic results about the boundary behavior of holomorphic functions is Lindelöf's Theorem.

Theorem A.5.1 (Lindelöf's theorem)**.** *[18, Theorem 1.5.7, p. 26] Let $f : \mathbb{D} \to \mathbb{C}$ be a holomoprhic function. Assume that $f(\mathbb{D})$ is contained in a half-plane. Let $\gamma : [0, 1) \to \mathbb{D}$ be a continuous curve such that $\lim_{t\to 1} \gamma(t) = \xi \in \mathbb{T}$. Suppose that $\lim_{t\to 1} f(\gamma(t)) = L \in \mathbb{C}_\infty$ exists. Then $\angle \lim_{z\to\xi} f(z) = L$.*

A.6 Khinchine-Kahane-Kalton inequalities

Let $\{r_k\}$ be the sequence of Rademacher functions on $[0, 1]$, which is defined in the following way:

$$r_k(t) = \operatorname{sgn}(\sin(2^k \pi t)), \quad k > 0,$$

where given $x \in \mathbb{R}$, $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = 0$ if $x = 0$, and $\operatorname{sgn}(x) = -1$ if $x < 0$. First, we set out the classical Khinchine's inequality.

Theorem A.6.1. *[27, Theorem A.2, p. 224] Let $0 < p < +\infty$. Then we have*

$$\left(\sum_k |\lambda_k|^2 \right)^{1/2} \asymp \left(\int_0^1 \left| \sum_k \lambda_k r_k(t) \right|^p dt \right)^{1/p},$$

for any sequence $\{\lambda_k\}$ of complex numbers.

The result we will state below is an extension of the Khinchine-Kahane inequality to quasi-Banach spaces.

Theorem A.6.2. *[44, Theorem 2.1, p. 251] Let $0 < p, q < +\infty$ and let X be a quasi-Banach space. Then there are $A = A(X, p, q) > 0$ and $B = B(X, p, q)$ such that*

$$A\left(\int_0^1 \left\|\sum_k r_k(t) f_k\right\|_X^p dt\right)^{1/p} \leq \left(\int_0^1 \left\|\sum_k r_k(t) f_k\right\|_X^q dt\right)^{1/q} \leq B\left(\int_0^1 \left\|\sum_k r_k(t) f_k\right\|_X^p dt\right)^{1/p}$$

for any sequence $\{f_k\} \subset X$.

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