A_∞–coalgebra structures on the $\mathbf{Z_p}$-homology of Eilenberg-Mac Lane spaces

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Abstract

We study here the $A(\infty)$-coalgebra structure of the homology $H_*(K(\pi, n); \mathbf{Z_p})$ of an Eilenberg-Mac Lane space $K(\pi, n)$, where π is a finitely generated abelian group and n is a positive integer. Using diverse techniques of homological perturbation, we get that the components $\Delta_{i(p-2)+2}$ of degree $i(p-2)$ (with $i \geq 0$) are the only (possibly) non-null morphisms of said structure.

1 Extended Abstract

An Eilenberg-Mac Lane space $K(\pi, n)$, depending on a group π and a non-negative integer n is a simplicial set having only two non-null integer homotopy groups $\pi_0(K(\pi, n)) = \mathbf{Z}$ and $\pi_n(K(\pi, n)) = \pi$ (except the case $n = 0$, which only have one non-null homotopy group in degree 0 $\pi_0(K(\pi, 0)) = \pi$). We only consider here $K(\pi, n)$ spaces being π an finitely generated abelian group. In this case, the chain complex $C_*(K(\pi, n))$ canonically associated to an Eilenberg-Mac Lane space have a Hopf-algebra structure, that is, $C_*(K(\pi, n))$ is endowed with an algebra and a coalgebra structures that are compatible in some strong sense. Henri Cartan determines in his complete study [1] of the homology of Eilenberg-Mac Lane spaces four types of commutative differential graded augmented algebras, called *elementary complexes*, which can be made by using exterior and divided power algebras. The first is the exterior algebra $E(u, 2n-1)$ with one generator u of degree $2n-1$; the second is the divided power algebra $\Gamma(v, 2n)$ with one "generator" v of degree $2n$; the third is the twisted tensor product $\Gamma(u, 2n)\tilde{\otimes}^{\rho_{\pm p^r}} E(v, 2n+1)$ with differential operator defined by $\rho_{\pm p^r}(v) = \pm\, p^r\, u$; and the fourth is the twisted tensor product $E(u, 2n-1)\tilde{\otimes}^{\rho_{\pm p^r}} \Gamma(v, 2n)$ with differential operator defined by $\rho_{\pm p^r}(v) = \pm\, p^r\, u$. Tensor products of these elementary complexes describe the integer homology algebra $H_*(K(\pi, n))$ for any π and $n \geq 1$. Alain Proute in his brilliant Ph. D. Thesis "Algébres differentielles fortement homotopiquement associatives" [8] studies the $\mathbf{Z_p}$-homology algebra (the field $\mathbf{Z_p}$ being the ground ring) of $K(\pi, n)$ and starts the work of determining its $A(\infty)$-coalgebra structure. In the case $H_*(K(\pi, n); \mathbf{Z_2})$, this comultiplicative structure is reduced to a simple coalgebra. Consequently, this homology has a Hopf-algebra structure. If $p \neq 2$ (p prime), the situation changes dramatically. To the difficult problems to control as the $A(\infty)$-structures by tensor product evolve, the fact that the $A(\infty)$-coalgebra structure explodes in the case $p \neq 2$ joins up. Prouté only gives partial results for elementary algebras embodied in this context, that they permit however to conjecture aspects of nature of this complicated structure. In this article, we intend to extend Prouté's work and to determine the total group of component morphisms possibly non zero of the $A(\infty)$-coalgebra structure of $H_*(K(\pi, n); \mathbf{Z_p})$.

Only tensor products of two elementary complexes appear in $H_*(K(\pi, n); \mathbf{Z_p})$. They are the exterior and divided power algebras. The Real's work [10] is our starting point, where, using homological perturbation, specify a contraction $c_{(\pi,n);p}$ (particular homotopy equivalence) among $C_*(K(\pi, n); \mathbf{Z_p})$ and $H_*(K(\pi, n); \mathbf{Z_p})$. This strong compatibility is possible basically thanks to the following elementary contractions:

- A contraction C_{WB} "connecting" the normalized chain complex $C_*(\overline{W}(G))$ of the geometric classifying space of an abelian group G, and the reduced bar construction $\overline{B}(C_*(G))$ of the normalized chain complex of G [9].
- An isomorphism from the reduced bar construction of an exterior algebra to a divided power algebra.
- A contraction from the reduced bar construction $\overline{B}(\Gamma(u, 2n))$ to an infinite tensor product of algebras of the kind $E(v, 2np^i + 1) \otimes \Gamma(w, 2np^{(i+1)} + 2)$.

Attending to the terminology delivered in [10], all these three contractions are semi-full algebra contractions and, in particular, that means that the algebra structure is transferred correctly and of simple manner to the homology $H_*(K(\pi, n); \mathbf{Z_p})$. Regarding coalgebra's structures, two first contractions respect to this structure, being the last the one and only contraction that does not show a property of compatibility. The explosion of coalgebra's structure in the homology is precisely provoked by this last contraction. As said contraction is generated starting from elementary contractions c_{BQ} among $\overline{B}(Q_p(u, 2n))$ (being $Q_p(u, 2n) = P(u, 2n)/(u^p)$) and the tensor product $E(v, 2n + 1) \otimes \Gamma(w, 2np + 2)$, a first step to examine the strongly homotopy associativity of the homology $H_*(K(\pi, n); \mathbf{Z_p})$ is to deal first with this question in elementary contractions.

Prouté's study was limited, for the previously reasons expounded, to the determination of the $A(\infty)$-coalgebra's structure of the 1-homology of $Q_p(u, 2n)$. Such structure $(\Delta_2, \Delta_3, \Delta_4, ...)$ only shows two non-null morphisms: Δ_2 and Δ_p. Our strategy is to discuss this work from the perspective of Homological Perturbation Theory and to try to extend this technique to more complex algebras.

First, given a contraction $c = \{N, M, f, g, \phi\}$ among a differential graded coalgebra (N, d_N, Δ) and a chain complex (M, d_M), the component morphisms of the $A(\infty)$-coalgebra structure on M follow (up to signs) the following formulae (from now on, we use Real's notation [10]):

$$\Delta_i \;=\; f^{\otimes i}(\Delta^{[(i-1)]}\phi^{[c,i-1]} \cdots (\Delta^{[2]}\phi^{[c,2]})\Delta g, \quad \forall i \geq 1. \tag{1}$$

The development of this formula in terms of the coproduct $\Delta : N \to N \otimes N$ and the homotopy operator $\phi : N_* \to N_{*+1}$, gives us a sum of compositions

$$f^{\otimes i}(\Delta^{[i-1,j_{i-1}]}\phi^{[c,i-1,k_{i-1}]})\cdots(\Delta^{[2,j_2]}\phi^{[c,2,k_2]})\Delta g,\ \ 1\leq j_r,k_r\leq r. \qquad (2)$$

An elementary work is to prove (we would be able to do it using banally Inversion's Theory [10, 2]), the following rule:

"The morphism (2) will be zero if $j_r\neq k_r$, for some r."

Now, let us focus in the contraction $c_{BQ}=\{\overline{B}(Q_p(u,2n)),E(v,2n+1)\otimes\Gamma(w,2np+2),f_{BQ},g_{BQ},\phi_{BQ}\}$.

We denote an element of $\overline{B}(Q_{(p)}(u,2n))$ of the form $[u^{r_1}|\ldots|u^{r_m}]$ by $[r_1|\ldots|r_m]$, where $0\leq r_i<p$. From now on, $E(v,2n+1)\otimes\Gamma(w,2np+2)$ will be denoted by H for short.

The explicit morphisms of c_{BQ} are the following:

$$f_{BQ}[r_1|t_1|\ldots|r_m|t_m]=\{\textstyle\prod_{k=1}^n\delta_{p,r_k+t_k}\}\gamma_m(w),$$

$$f_{BQ}[r_1|t_1|\ldots|r_m|t_m|l]=\delta_{1,l}\{\textstyle\prod_{k=1}^n\delta_{p,r_k+t_k}\}v\cdot\gamma_m(w),$$

where the symbols $\delta_{i,j}$ are defined by:

$$\delta_{i,j}=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$$

The morphism $g_{BQ}:H\to\bar{B}(Q_{(p)}(u,2n))$ is defined over the generators as follows:

$$g_{BQ}(v)=[1],$$

$$g_{BQ}(\gamma_k(w))=[1|p-1|\overset{\text{k times}}{\cdots}|1|p-1].$$

The homotopy operator ϕ_{BQ} is defined by:

$$\phi_{BQ}1=0;\qquad \phi_{BQ}[1]=0;$$

$$\phi_{BQ}[x]=-[1|x-1]\quad 1<x<p;$$

$$\phi_{BQ}[x|y]=-[1|x-1|y];$$

$$\phi_{BQ}[x|y|z]=-[1|x-1|y|z]-\delta_{p,x+y}[1|p-1|\phi(z)]$$

where $z\in\bar{B}(Q_{(p)}(u,2n))$.

The coassociative coproduct on $\overline{B}(Q_{(p)}(u,2n))$ is

$$\Delta([a_1|\cdots|a_r])=\sum_{i=0}^{r}[a_1|\cdots|a_i]\otimes[a_{i+1}|\cdots|a_r].$$

The following properties can be easily verified:

If $\Delta(x)=\sum\Delta^1(x)\otimes\Delta^2(x)$, being x an element of $\overline{B}(Q_{(p)}(u,2n))$,

$$\phi_{BQ}(\Delta^1)^i g=0,\quad i\geq 1$$

$$\phi_{BQ}\Delta^1\phi_{BQ} = 0, \quad i \geq 1.$$

Let us consider the submodule S of $\overline{B}(Q_{(p)}(u,2n))$ generated by the elements $[a_1|a_2|\cdots|a_r]$ such that or else $a_{2i+1} = 1$ for all i or else $a_{2i} = 1$ for all i.

Again, it is easy to prove the following properties:

$$\phi_{BQ}(S) \subset S,$$

$$\Delta(S) \subset S \otimes S,$$

$$\mathrm{Im}\ g_{BQ} \subset S.$$

Now, we will use Inversion Theory to get an economical formulation (in terms of number of summands embodied) of the morphisms $\Delta_i : H \to H \otimes H$. *We will say that an element* $[a_1|a_2|\cdots|a_r] \in \overline{B}(Q_{(p)}(u,2n))$ *has t inversions if there exists t different couples* (a_{2i-1}, a_{2i}) *of the form* $(1,a)$*, where* $a \in \{1,2,\ldots,p-3,p-2\}$.

With this definition in hand, we can verify the following properties:

$$\phi_{BQ}(\text{element with } i\text{-inversions}) = \sum \text{elements with at least (i+1)-inversions},$$

$$\Delta(\text{element with } i\text{-inversions}) = \sum \text{elements with at least } (i-1)\text{-inversions},$$

$$f_{BQ}(\text{element with } i\text{-inversions}) = 0, \quad \forall i \geq 1.$$

Therefore, in the economical formula of Δ_i (see (1)) derived from this technique, the coproduct Δ of $\overline{B}(Q_{(p)}(u,2n))$ will be only applied to elements with 1-inversions and the homotopy operator ϕ_{BQ} will be applied to elements with 0-inversions only. Using this argument, it is possible to deduce that only the morphisms Δ_2 and Δ_p are non-zero in the $A(\infty)$-coalgebra structure over H.

This work can be extended without too many problems to reduced bar constructions of tensor products of $Q_{(p)}(u,2m)$. In this way, we show the main result of the paper:

Theorem. *Let π be a finitely generated abelian group, $n \geq 1$ be a positive integer and $p \neq 2$ be a prime. The (possibly) non-null morphisms of the $A(\infty)$-coalgebra structure of* $H_*(K(\pi,n);\mathbf{Z_p})$ *are* $\Delta_{i(p-2)+2}$, $\forall i \geq 0$.

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