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Köthe echelon spaces à la Dieudonné

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ABSTRACT

Let (g_n) be a sequence of locally integrable functions defined on a Radon measure space. The echelon space associated to (g_n) was defined by J. Dieudonné as the Köthe-dual of (g_n), i.e. the space Λ of all locally integrable functions f such that all the integrals $\int |f \cdot g_n|$ are finite. Denote by $\Lambda^\times$ the Köthe-dual of Λ. We prove that $\Lambda(\beta(\Lambda, \Lambda^\times))$ is a Fréchet space with dual $\Lambda^\times$. This result gives its correct sense to a wrong affirmation of J. Dieudonné and validates those instances where it has been used. As a tool to prove this result, we study the problem of when the strong dual of a perfect space coincides with its Köthe-dual and give some necessary and sufficient conditions.

1. INTRODUCTION

The Köthe-Toeplitz theory of perfect sequence spaces [7, §30] has been one of the most influential in the study of the structure of locally convex spaces. This theory, and in particular the duality between echelon and co-echelon sequence spaces (see e.g. [1] and [6]), has provided the specialists with plenty of hints, examples and counterexamples. Different extensions of this theory are obtained by replacing the $\sum$ in the definition of G. Köthe and O. Toeplitz with a suitable $\int$. One of the first moves in this direction was made by J. Dieudonné.

It is in the last section of his seminal paper [3, §16] that J. Dieudonné considers echelon spaces of functions. His definition is as follows: let X be a locally compact and σ-compact Hausdorff topological space with a Radon measure μ. The echelon space associated to a sequence (g_n) of locally integrable functions is defined as the Köthe-dual of the sequence:

$$\Lambda := \left\{ f : f \text{ is locally integrable and } p_{g_n}(f) = \int_X |f \cdot g_n| \, \mathrm{d}\mu < +\infty, \ n = 1, 2, \ldots \right\}.$$

After giving this definition, he affirms that Λ endowed with the topology defined by the seminorms (p_{g_n}) is a Fréchet space and that its topological dual coincides with its Köthe-dual $\Lambda^\times$. This is not correct, as was pointed out by J.A. López Molina [8, Ex. (p. 187)] with the following example: consider the unit interval with its Lebesgue measure and the echelon space Λ associated to the function $g(x) = \exp(-1/x)$. Then $h = \chi_{[0,1]}$ is a function in $\Lambda^\times$ because the functions in Λ are, by definition, locally integrable. However, h is not continuous for the seminorm p_g because the sequence $(k \cdot \chi_{[0,1/k]})$ from Λ satisfies

$$\lim_k p_g(k \cdot \chi_{[0,1/k]}) = \lim_k \int_0^{1/k} k \cdot \exp(-1/x) \, \mathrm{d}x \leq \lim_k \exp(-k) = 0,$$

but for all $k = 1, 2, \ldots$ we have

$$\int_0^1 k \cdot \chi_{[0,1/k]}(x) \cdot h(x) \, \mathrm{d}x = \int_0^{1/k} k \, \mathrm{d}x = 1.$$

This mistake led J.A. López Molina to consider an alternative definition of echelon space: require the functions in Λ to be only measurable instead of locally integrable. The theory of echelon spaces defined in this way has been developed and generalized by J.A. López Molina [8] and [9], J.C. Díaz [2] and K. Reiher [12].

Our purpose in this paper is to prove that J. Dieudonné's affirmation is essentially correct in the sense that an echelon space Λ is a Fréchet space when endowed with the strong topology $\beta(\Lambda, \Lambda^\times)$ and its topological dual equals its Köthe-dual $\Lambda^\times$. This result will, in turn, validate those instances in which J. Dieudonné's affirmation has been used precisely in its correct sense, as in [11, Cor. 1].

In §2 we recall J. Dieudonné's definition of perfect spaces and give some necessary and sufficient conditions for the topological dual of a perfect space to coincide with its Köthe-dual. These conditions, which are of independent interest, generalize and unify some previously known results and will be used in §3, where the result announced in the preceding paragraph appears.

We refer the reader to W. Rudin's book [13] for the results concerning measure theory and integration, and to G. Köthe's monograph [7] for the theory of locally convex spaces.

2. WHEN DOES THE TOPOLOGICAL DUAL COINCIDE WITH THE KÖTHE-DUAL?

Although some of our results in this paper can be given in a more abstract measure-theoretic frame, we shall stick to J. Dieudonné's original formulation [3, §§10–16]. In what follows, X stands for a locally compact, Hausdorff topological space that is σ-compact, so that we can write $X = \bigcup_m X_m$, where every X_m is compact and $X_m \subset \text{int}(X_{m+1})$ for all $m \in \mathbb{N}$. Let μ be a positive Radon measure on X and Ω be the space of all (equivalence classes of) locally integrable functions from X into the field $\mathbb{K}$ of real or complex numbers. For a subset $\Lambda \subset \Omega$ we denote by $\Lambda^\times$ the set of all $g \in \Omega$ such that $f \cdot g$ is integrable for each $f \in \Lambda$; $\Lambda^\times$ is called the Köthe-dual of Λ. A linear subspace Λ of Ω is said to be a perfect space (or Köthe space) if $(\Lambda^\times)^\times = \Lambda$. In particular, $\Lambda^\times$ is always perfect. For instance, $L^1(\mu)$ and $L^\infty(\mu)$ are perfect spaces, and each one is the Köthe-dual of the other. The space Ω is also perfect and $\Omega^\times$ is the space Φ of all (equivalence classes of) measurable essentially bounded functions with compact support. When Λ contains Φ, the spaces Λ and $\Lambda^\times$ are put into duality by means of the canonical bilinear form

$$\langle f, g\rangle = \int_X f(x) \cdot g(x)\, \mathrm{d}\mu(x) \quad (f \in \Lambda,\ g \in \Lambda^\times).$$

Let B stand for the unit ball of $L^\infty(\mu)$. A subset H of Ω is called normal if $f \cdot t \in H$ for all $f \in H$ and $t \in B$ or, equivalently, if $f \in H$ and g is a measurable function such that $|g(x)| \leq |f(x)|$ μ-a.e. on X, then g is also in H. The normal hull of a set $H \subset \Omega$ is defined by $\{t \cdot f\colon\ f \in H,\ t \in B\}$. One important fact is that the normal hull of a weakly bounded set is also bounded [3, Prop. 6]. This means that the strong topology $\beta(\Lambda, \Lambda^\times)$ is generated by the seminorms

$$p_H(f) := \sup\left\{\int_X |f \cdot g|\, \mathrm{d}\mu\colon\ g \in H\right\},$$

where H runs through the absolutely convex, normal and $\sigma(\Lambda^\times, \Lambda)$-bounded subsets of $\Lambda^\times$. If Λ is a perfect space, then $\Lambda(\beta(\Lambda, \Lambda^\times))$ is complete [3, Th. 5].

Denote by Λ' the topological dual of $\Lambda(\beta(\Lambda, \Lambda^\times))$. The problem of when Λ' equals $\Lambda^\times$ or, equivalently, when the strong topology $\beta(\Lambda, \Lambda^\times)$ coincides with the Mackey topology $\mu(\Lambda, \Lambda^\times)$, has been addressed by several authors (we shall give precise references in the notes following our Theorem 1), and different necessary or sufficient conditions have been considered. In our first result we unify and extend previously known results.

Theorem 1. *Let Λ be a perfect space and denote by Λ' the topological dual of Λ endowed with the strong topology $\beta(\Lambda, \Lambda^\times)$. Consider the following conditions:*

(i) *$\Lambda' = \Lambda^\times$ or, equivalently, $\mu(\Lambda, \Lambda^\times) = \beta(\Lambda, \Lambda^\times)$.*

(ii) *For every function $f \in \Lambda$ and every sequence (A_n) of measurable sets such that $\lim_n \mu(A_n) = 0$, the sequences $(f \cdot \chi_{A_n})$ and $(f - f \cdot \chi_{X_n})$ converge to zero for the strong topology $\beta(\Lambda, \Lambda^\times)$.*

(iii) *If (f_n) is a decreasing sequence in Λ that converges to zero μ-a.e., then (f_n) converges to zero for the strong topology $\beta(\Lambda, \Lambda^\times)$.*

(iv) *$\Lambda(\beta(\Lambda, \Lambda^\times))$ is separable.*

(v) *The space $C_c(X)$ of continuous functions with compact support is dense in Λ for the strong topology $\beta(\Lambda, \Lambda^\times)$.*

Then we have (iv) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (v). *Besides, if the measure space* (X,μ) *is separable, then* (iv) *is equivalent to* (i) – (iii). *Moreover, if X is a metrizable space, then all the conditions above are equivalent.*

Proof. We start by proving the equivalence of (i) – (iii).

(i) $\Rightarrow$ (ii): Take $f \in \Lambda$. To see that $(f - f \cdot \chi_{X_n})$ converges strongly to zero, fix a strong seminorm p_H, where H is an absolutely convex, normal and $\sigma(\Lambda^\times,\Lambda)$-bounded subset of $\Lambda^\times$. Condition (i) tells us that H is relatively compact for the weak topology $\sigma(\Lambda^\times,\Lambda)$. Consider the mapping

$$T : g \in \Lambda^\times \to T(g) = f \cdot g \in L^1(\mu).$$

Since Λ is normal, the function $f \cdot h$ is in Λ for every $h \in L^\infty(\mu)$, and we have

$$\langle T(g), h\rangle_{(L^1(\mu),L^\infty(\mu))} = \int_X f \cdot g \cdot h \,\mathrm{d}\mu = \langle g, f \cdot h\rangle_{(\Lambda^\times,\Lambda)}.$$

Therefore, T is $\sigma(\Lambda^\times,\Lambda) - \sigma(L^1(\mu), L^\infty(\mu))$ continuous and consequently $T(H)$ is relatively compact for the weak topology $\sigma(L^1(\mu), L^\infty(\mu))$. Then, by [3, Th. 4], there is a compact set $K \subset X$ such that

$$\sup\left\{\int_{X\setminus K} |T(g)| \,\mathrm{d}\mu\colon\ g \in H\right\} = \sup\left\{\int_{X\setminus K} |f \cdot g| \,\mathrm{d}\mu\colon\ g \in H\right\} < 1.$$

Taking $m \in \mathbb{N}$ such that $K \subset X_m$ we have $p_H(f - f \cdot \chi_{X_n}) < 1$ for $n \geq m$.

On the other hand, if $\mu(A_n) \to 0$ but $(f \cdot \chi_{A_n})$ does not tend to zero for the Mackey topology, then we can find a normal set $H \subset \Lambda^\times$, absolutely convex and $\sigma(\Lambda^\times,\Lambda)$-compact, and an increasing sequence of indices (n_k) such that

$$\sum_{k=1}^{\infty} \mu(A_{n_k}) < +\infty,$$

but

$$\sup\left\{\int_{A_{n_k}} |f \cdot g| \,\mathrm{d}\mu\colon\ g \in H\right\} > 1 \quad \text{for all } k = 1, 2, \ldots$$

Consider the decreasing sequence (F_k) of measurable sets defined by $F_k := \bigcup_{j\geq k} A_{n_j}$. Then $\mu(F_k) \leq \sum_{j\geq k} \mu(A_{n_j})$, hence $\lim_k \mu(F_k) = 0$. For each $k \in \mathbb{N}$ define the set

$$H_k := \left\{g \in H\colon\ \int_{F_k} |f \cdot g| \,\mathrm{d}\mu \geq 1\right\}.$$

These sets (H_k) are non-empty, $\sigma(\Lambda^\times,\Lambda)$-closed and they have the finite intersection property because for $k_1 < k_2$ we have

$$\int_{F_{k_2}} |f \cdot h| \,\mathrm{d}\mu \leq \int_{F_{k_1}} |f \cdot h| \,\mathrm{d}\mu$$

for every $h \in \Lambda^\times$ so that $H_{k_2} \subset H_{k_1}$. Since H is $\sigma(\Lambda^\times,\Lambda)$-compact, there is a function $g \in H$ such that $g \in H_k$ for all k or, equivalently,

$$\int_{F_k} |f \cdot g| \,\mathrm{d}\mu \geq 1 \quad \text{for all } k \in \mathbb{N}.$$

But $f \cdot g \in L^1(\mu)$. Therefore, there is some $\delta > 0$ such that if $\mu(A) < \delta$ then $\int_A |f \cdot g| \, \mathrm{d}\mu < 1$, and this is in contradiction with the inequality above because $\lim_k \mu(F_k) = 0$.

(ii) $\Rightarrow$ (iii): Let (f_n) be a decreasing sequence in Λ that converges to zero μ-a.e. and take a normal and $\sigma(\Lambda^\times, \Lambda)$-bounded set H. Applying (ii) to $f = f_1 \in \Lambda$ we can find an index $N \in \mathbb{N}$ such that

$$p_H(f_n - f_n \cdot \chi_{X_N}) \leq p_H(f - f \cdot \chi_{X_N}) < \tfrac{1}{2} \quad \text{for all } n \geq 2.$$

Now, let $\alpha = p_H(\chi_{X_N})$. We may assume that $\alpha \neq 0$. (Otherwise, every $g \in H$ would be zero μ-a.e. in X_N and the proof of this implication would be finished.) Consider the measurable sets defined for each $n \in \mathbb{N}$ by:

$$A_n := \{x \in X_N \colon f_n(x) > 1/(4\alpha)\}.$$

Since $\mu(X_N) < +\infty$ and (f_n) converges to zero μ-a.e., we have that $\lim_n \mu(A_n) = 0$. By condition (ii), there is $m \in \mathbb{N}$ such that $p_H(f_1 \cdot \chi_{A_m}) < \frac{1}{4}$. Now, for $n \geq m$ we have that $|f_n(x)| \leq 1/(4\alpha)$ on $X_N \backslash A_m$. Therefore

$$\begin{aligned} p_H(f_n \cdot \chi_{X_N}) &\leq p_H(f_n \cdot \chi_{A_m}) + p_H(f_n \cdot \chi_{X_N \backslash A_m}) \\ &\leq p_H(f_1 \cdot \chi_{A_m}) + (1/(4\alpha)) \cdot p_H(\chi_{X_N \backslash A_m}) \\ &\leq \tfrac{1}{4} + (1/(4\alpha)) \cdot \alpha = \tfrac{1}{2}. \end{aligned}$$

Hence, if $n \geq m$ we have

$$p_H(f_n) \leq p_H(f_n - f_n \cdot \chi_{X_N}) + p_H(f_n \cdot \chi_{X_n}) < \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

(iii) $\Rightarrow$ (i): We always have $\Lambda^\times \subset \Lambda'$. On the other hand, take $\phi \in \Lambda'$. Fix an index $N \in \mathbb{N}$. For each measurable set $A \subset X_N$, define $G(A) = \phi(\chi_A)$. Condition (iii) yields that G is a σ-additive measure. Indeed, if (A_n) is a sequence of disjoint measurable sets in X_N with union A, apply (iii) to the functions defined by

$$t_m := \chi_A - \sum_{n=1}^{m} \chi_{A_n} \in \Phi \subset \Lambda, \quad \text{for each } m = 1, 2, \ldots$$

Then (t_m) converges to zero for the strong topology. Hence $\phi(\chi_A) = \sum_{n=1}^\infty \phi(\chi_{A_n})$ so that $G(A) = \sum_{n=1}^\infty G(A_n)$. Let us see now that G is absolutely continuous with respect to μ: if we have a measurable set $A \subset X_N$ with $\mu(A) = 0$, then $\chi_A = 0$ μ-a.e., thus $G(A) = \phi(\chi_A) = 0$.

Apply the Radon-Nikodým Theorem to deduce the existence, on each X_N, of a function $g_N \in L^1(\mu, X_N)$ such that

$$G(A) = \int_A g_N \, \mathrm{d}\mu \quad \text{for } A \subset X_N.$$

It is clear that if $N \geq M$, then $g_N = g_M$ on X_M so that the function $g = \lim_N g_N$ is well-defined and locally integrable. Moreover, for each compact K and each measurable set $A \subset K$, we have

$$G(A) = \int_A g \, \mathrm{d}\mu.$$

We prove now that $g \in \Lambda^\times$ and $\phi(f) = \langle f, g\rangle$. Suppose, without loss of generality, that $g \geq 0$. We proceed in several steps.

(1) Since ϕ is linear, we have that $\phi(f) = \langle g, f\rangle$ for f a simple function with compact support.

(2) For $f \in \Phi$ (the space of measurable and essentially bounded functions with compact support), let (f_α) be a net of simple functions with compact support such that $\lim_\alpha \|(f - f_\alpha)\cdot \chi_A\|_\infty = 0$ where A is the support of f. If H is a normal and $\sigma(\Lambda^\times, \Lambda)$-bounded set, then we have

$$\lim_\alpha p_H(f - f_\alpha) = \lim_\alpha \sup\left\{\int_A |f - f_\alpha|\cdot|h|\,\mathrm{d}\mu\colon\ h \in H\right\}$$
$$\leq \lim_\alpha \|(f - f_\alpha)\cdot\chi_A\|_\infty \cdot p_H(\chi_A) = 0.$$

This proves that (f_α) converges to f for the strong topology $\beta(\Lambda, \Lambda^\times)$. Using this, (1) above and that $f = \lim_\alpha f_\alpha$ in $L^\infty(\mu, A)$, we obtain

$$\phi(f) = \lim_\alpha \phi(f_\alpha) = \lim_\alpha \int_A f_\alpha \cdot g\,\mathrm{d}\mu = \int_A f\cdot g\,\mathrm{d}\mu = \int_X f\cdot g\,\mathrm{d}\mu.$$

(3) Now take a positive function $f \in \Lambda$. For each $n \in \mathbb{N}$, let f_n be the function in Φ defined by:

$$f_n(x) := \begin{cases} f(x), & \text{if } f(x) \leq n \text{ and } x \in X_n \\ 0, & \text{otherwise.}\end{cases}$$

Then $(f - f_n)$ is a decreasing sequence that converges to zero μ-a.e. By condition (iii), f_n converges to f for the topology $\beta(\Lambda, \Lambda^\times)$. Using this and the Monotone Convergence Theorem in $L^1(\mu)$ we have

$$\phi(f) = \lim_n \phi(f_n) = \lim_n \int_X f_n \cdot g\,\mathrm{d}\mu = \int_X f\cdot g\,\mathrm{d}\mu.$$

(4) Finally, for arbitrary $f \in \Lambda$ the equality $\phi(f) = \langle f, g\rangle$ follows by linearity.

(iv) $\Rightarrow$ (i): We always have $\Lambda^\times \subset \Lambda'$. Now, take $\phi \in \Lambda'$. Then there is some absolutely convex, closed and $\sigma(\Lambda^\times, \Lambda)$-bounded subset H of $\Lambda^\times$ such that $\phi \in H^{\circ\circ}$, the bipolar of H in Λ'. Since $\Lambda(\beta(\Lambda, \Lambda^\times))$ is separable, $H^{\circ\circ}$ is metrizable [7, §21.3.(4)] and compact for the topology $\sigma(\Lambda', \Lambda)$. Therefore, H is sequentially dense in $H^{\circ\circ}$. But, on the other hand, H is $\sigma(\Lambda^\times, \Lambda)$-sequentially complete because it is closed in $\Lambda^\times(\sigma(\Lambda^\times, \Lambda))$ and this space is sequentially complete [3, Prop. 12]. Hence, $\phi \in H^{\circ\circ} = H \subset \Lambda^\times$. Consequently, $\Lambda' \subset \Lambda^\times$.

(i) $\Rightarrow$ (v): For every non-zero $g \in \Lambda^\times$ there is a function h continuous and having compact support such that $\langle h, g\rangle \neq 0$ [3, p. 98]. Then, by the Hahn-Banach separation theorem, $C_c(X)$ is dense in Λ for any topology such that the dual of Λ is $\Lambda^\times$.

(i) – (iii) $\Rightarrow$ (iv) when the measure space (X, μ) is separable: For every non-zero $g \in \Lambda^\times$ there is a simple function h having compact support such that $\langle h, g\rangle \neq 0$. Condition (i) and the Hahn-Banach separation theorem ensure that the space $S_c(X)$ of simple functions having compact support is dense in

$\Lambda(\beta(\Lambda, \Lambda^\times))$. Therefore, we have to prove that $S_c(X)$ is separable for the strong topology $\beta(\Lambda, \Lambda^\times)$.

Since the measure space (X, μ) is separable, there is a countable family $\mathcal{C}$ of measurable sets (we may, and do, assume that this family contains all the sets of the form $Y \cap X_n$ for $Y \in \mathcal{C}$ and $n \in \mathbb{N}$) such that for every measurable set A there is a sequence (Y_j) from $\mathcal{C}$ with $\lim_j \mu(A \triangle Y_j) = 0$, where $\triangle$ stands for the symmetric difference operator. If, in particular, A is contained in X_n, then both χ_A and $\chi_{X_n \setminus A}$ are in $S_c(X) \subset \Lambda$. Since $\lim_j \mu(A \triangle Y_j) = 0$, we can apply condition (ii) to χ_A and $\chi_{X_n \setminus A} = \chi_{X_n} - \chi_A$ (recall that $A \subset X_n$) to obtain, for the strong topology $\beta(\Lambda, \Lambda^\times)$,

$$0 = \lim_j \chi_A \chi_{A \triangle Y_j} = \lim_j \chi_A(\chi_A + \chi_{Y_j} - 2\chi_A \chi_{Y_j}) = \lim_j (\chi_A - \chi_A \chi_{Y_j})$$

and

$$0 = \lim_j (\chi_{X_n} - \chi_A)\chi_{A \triangle Y_j} = \lim_j (\chi_{X_n} - \chi_A)(\chi_A + \chi_{Y_j} - 2\chi_A \chi_{Y_j})$$
$$= \lim_j (\chi_{X_n}\chi_{Y_j} - \chi_A \chi_{Y_j}).$$

It follows that $\lim_j (\chi_A - \chi_{(Y_j \cap X_n)}) = 0$ for the strong topology $\beta(\Lambda, \Lambda^\times)$. This ensures that the countable set D of all simple functions of the form $\sum \alpha_Y \chi_Y$, where each Y in the sum is from $\mathcal{C}$ and each α_Y is rational, is dense in $S_c(X)$ for the strong topology $\beta(\Lambda, \Lambda^\times)$.

(v) $\Rightarrow$ (iv) when X is metrizable: According to condition (v), we have to prove that $C_c(X)$ endowed with the restriction of the topology $\beta(\Lambda, \Lambda^\times)$ is separable. Now, if X is metrizable, then each of the spaces $C(X_n)(\|\cdot\|_\infty)$ is separable. For every $n \in \mathbb{N}$, let D_n be a countable dense set in $C(X_n)(\|\cdot\|_\infty)$. Now, for $f \in C(X_n)$ take $g \in D_n$ such that

$$|f(x) - g(x)| \leq 1 \quad \text{for all } x \in X_n.$$

Let H be a normal and $\sigma(\Lambda^\times, \Lambda)$-bounded subset of $\Lambda^\times$. Since f and g are supported in X_n, we have

$$p_H(f-g) = \sup\left\{\int_{X_n} |f-g| \cdot |h| \, d\mu \colon h \in H\right\}$$
$$\leq \sup\left\{\int_{X_n} |h| \, d\mu \colon h \in H\right\} = p_H(\chi_{X_n}).$$

This shows that $D = \bigcup_n D_n$ is a countable set dense in $C_c(X)$ for the strong topology $\beta(\Lambda, \Lambda^\times)$. □

Notes. J. Dieudonné [3, §13 (p. 107)] claimed that (iv) $\Rightarrow$ (i). Y. Kōmura [5, Th. 1.3] showed the equivalence of (i) and (iv) for $X = \mathbb{R}^n$. G.G. Lorentz [10, Th. 3] proved that (iii) $\Rightarrow$ (i) for the case when X is a finite interval in the real line and Λ is a normed space. R. Welland [18, Th. 2] proved the equivalence of (i), (iii) and (iv) under the hypothesis that the measure space (X, μ) is separable; here we have shown that this hypothesis is not really necessary to prove (i) $\Leftrightarrow$ (iii). Condition (ii) may be easier to use that (iii) and the proof of (ii) $\Rightarrow$ (iii) above follows the ideas given by A.C. Zaanen in [19, §72]. Finally, condition (v) has been considered by G. Silverman for translation invariant perfect spaces [17, Th. 2 and Th. 3].

3. THE STRONG DUAL OF AN ECHELON SPACE

Let (g_n) be an increasing sequence of non-negative locally integrable functions. The Köthe-dual of (g_n) is called the echelon space associated to this sequence and we shall denote it by $\Lambda(g_n)$. Thus

$$\Lambda(g_n) = \left\{ f \in \Omega\colon\ p_{g_n}(f) := \int_X |f(x)| \cdot g_n(x)\, \mathrm{d}\mu(x) < +\infty \right.$$
$$\left. \text{for all } n = 1, 2, \ldots \right\}.$$

When (g_n) reduces to a simple function g, the space $\Lambda(g)$ is commonly denoted by L_g^1 and has many interesting properties (see [3, §§10–12], [4] and [15]). Note that we can write $\Lambda(g_n) = \bigcap_n L_{g_n}^1$. As we said in the introduction, the main purpose of this paper is to give the following result.

Theorem 2. *Let $\Lambda = \Lambda(g_n)$ be the echelon space associated to an increasing sequence (g_n) of non-negative locally integrable functions. Then Λ is a perfect space with Köthe-dual $\Lambda^\times = \bigcup_n (L_{g_n}^1)^\times$. Moreover, $\Lambda(\beta(\Lambda, \Lambda^\times))$ is a Fréchet space with dual $\Lambda^\times$ and the strong topology $\beta(\Lambda, \Lambda^\times)$ is generated by the family of seminorms*

$$p_{g_n}\colon f \in \Lambda \to p_{g_n}(f) := \int_X |f(x)| \cdot g_n(x)\, \mathrm{d}\mu(x), \quad n = 1, 2, \ldots$$

$$q_m\colon f \in \Lambda \to q_m(f) := \int_{X_m} |f(x)|\, \mathrm{d}\mu(x), \quad m = 1, 2, \ldots$$

Proof. First, we prove the theorem when (g_n) reduces to a single function $g \in \Omega$. According to [15, Prop. 1], the strong topology $\beta(L_g^1, (L_g^1)^\times)$ is given by the family of seminorms $\{p_g \text{ and } q_m,\ m = 1, 2, \ldots\}$ where $p_g(f) = \int_X |f| \cdot g\, \mathrm{d}\mu$. The space L_g^1 is perfect because it is the Köthe-dual of $\{g\}$ and therefore, it is complete when endowed with the strong topology [3, Thm. 5]. Hence $L_g^1(\beta(L_g^1, (L_g^1)^\times))$ is a Fréchet space. To prove that the topological dual of this space is $(L_g^1)^\times$, we shall apply condition (ii) of Theorem 1 above. Let $f \in L_g^1$. Obviously, $\lim_n q_m(f - f \cdot \chi_{X_n}) = 0$ for every $m \in \mathbb{N}$, and since $f \cdot g \in L^1(\mu)$, we also have

$$\lim_n p_g(f - f \cdot \chi_{X_n}) = \lim_n \int_{X \setminus X_n} |f \cdot g|\, \mathrm{d}\mu = 0.$$

Now let (A_n) be a sequence of measurable sets with $\lim_n \mu(A_n) = 0$. Since $f \cdot \chi_{X_m}$ is integrable for each m, we have

$$\lim_n q_m(f \cdot \chi_{A_n}) = \lim_n \int_{A_n} |f| \cdot \chi_{X_m}\, \mathrm{d}\mu = 0.$$

On the other hand, using again that $f \cdot g \in L^1(\mu)$, we have

$$\lim_n p_g(f \cdot \chi_{A_n}) = \lim_n \int_{A_n} |f| \cdot g \, d\mu = 0.$$

This finishes the proof for L_g^1.

We turn now to the general case. Let τ be the topology defined by both sets of seminorms $\{p_{g_n}: \ n = 1, 2, \ldots\}$ and $\{q_m: \ m = 1, 2, \ldots\}$. Since $\Lambda \subset L_{g_n}^1$ for every $n \in N$ we have that $(L_{g_n}^1)^\times \subset \Lambda^\times$ and therefore $\bigcup_n (L_{g_n}^1)^\times \subset \Lambda^\times$. Denote by τ_n the corresponding strong topology $\beta(L_{g_n}^1, (L_{g_n}^1)^\times)$ on $L_{g_n}^1$. Since $g_n \leq g_{n+1}$, the inclusion $L_{g_{n+1}}^1(\tau_{n+1}) \to L_{g_n}^1(\tau_n)$ is continuous, so that $\Lambda(\tau)$ is the reduced (because $C_c(X)$ is dense in each of these spaces by (i) $\Rightarrow$ (v) in Theorem 1) countable projective limit of a family of Fréchet spaces. Therefore, $\Lambda(\tau)$ is a Fréchet space. Denote by Λ' the dual of $\Lambda(\tau)$. Note that $\tau = \beta(\Lambda, \Lambda')$. On the other hand,

$$\Lambda' \subset \bigcup_n (L_{g_n}^1(\tau_n))' = \bigcup_n (L_{g_n}^1)^\times \subset \Lambda^\times.$$

The proof will be finished if we show that $\Lambda^\times \subset \Lambda'$. Take $h \in \Lambda^\times$ and assume, without loss of generality, that $h \geq 0$. Call T the linear form induced by h on Λ, $T(f) = \langle f, h \rangle$. For $k = 1, 2, \ldots$ define the functions

$$h_k(x) = \begin{cases} h(x), & \text{if } h(x) \leq k \text{ and } x \in X_k \\ 0, & \text{otherwise,} \end{cases}$$

then (h_k) is an increasing sequence that converges pointwise to h. We can apply the Monotone Convergence Theorem to deduce that

$$T(f) = \int_X f \cdot h \, d\mu = \lim_k \int_X f \cdot h_k \, d\mu = \lim_k \langle f, h_k \rangle.$$

Now, observe that every $h_k \in \Phi$ and that the seminorm $|\langle \cdot, h_k \rangle|$ is dominated by $k \cdot q_k$. Therefore, T is the pointwise limit of the sequence $(\langle \cdot, h_k \rangle)$ of τ-continuous linear forms. By the Banach-Steinhaus Theorem, T is also τ-continuous, i.e. $h \in \Lambda'$. $\square$

One can see now that the trouble in J. Dieudonné's affirmation was that he missed the family of seminorms $\{q_m: \ m \in \mathbb{N}\}$.

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