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SOME RESULTS ON DIAGONAL MAPS

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Summary. In this paper we study some results about diagonal maps between sequence spaces. Our purpose is two-fold: In §1, we characterize, for a large class of scalar sequence spaces λ and μ, those diagonal maps $a:\lambda\to\mu$ wich transform bounded sets into relatively compact sets as those that can be approximated by finite sequences in the topology of uniform convergence on the bounded subsets of λ. Consequences within the frame of echelon spaces and examples are provided. In §2, we give a useful characterization of the space of compact diagonal maps between Cesáro sequence spaces. On the other hand, in §3, we deal with the space $\lambda\{E\}$ of absolutely λ-summable sequences from E (λ being a normal sequence space and E a Hausdorff lcs). Our main result establishes that, under certain conditions, the space $\ell^{\infty}\{L(E,F)(\tau_b)\}$ represents, both algebraic and topologically, the space of continuous diagonal maps between two such vector-valued sequence spaces.

Some results in §2 were presented to the VII Congress of the Group of Latin Expression Mathematicians held in Coimbra (Portugal) in 1985.

1. DIAGONAL MAPS ON T-BS SPACES.

Definitions. We use the notation and concepts of [10 and 17] for general theory and sequence space theory, respectively. We assume throughout that every sequence space to be considered contains the space ϕ.

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A locally convex sequence space λ is called a K-space if its topology is finer than $\sigma(\lambda,\phi)$. If λ and μ are K-spaces, $M(\lambda,\mu)$ stands for the space of those diagonal maps

$$a: x\in\lambda \longrightarrow ax:=(a_n x_n)_n \in \mu$$

wich are continuous. It is clear that $\phi\subset M(\lambda,\mu)$. Moreover, if τ_b stands for the topology of uniform convergence on the bounded subsets of λ, then $M(\lambda,\mu)$ is a closed subspace of $L(\lambda,\mu)(\tau_b)$; therefore we can define the space $S(\lambda,\mu):=\overline{\phi}^{\tau_b}\subset M(\lambda,\mu)$.

Let $T=(t_{nk})_{n,k}$ be an infinite, row-finite matrix such that $\lim_n t_{nk}=1$ for each k, i.e. a Sp_1-matrix. Let $t^n:=(t_{nk})_k\in\phi$ be the n-th row of T. If x is a sequence, $t^n x$ is called the n-th Toeplitz section of x with respect to the matrix T. A K-space λ is said to be a T-BS space if the set $\{t^n x: n=1,2,\dots\}$ is bounded for each x in λ. A sequence x in λ is said to have the property T-AK if $x=\lim_n t^n x$ in the topology of λ. If every x in λ has the property T-AK we say that λ is a T-AK space. T-BS and T-AK spaces have been widely studied by Buntinas [2] and Meyers [12] as a generalization of BS and AK spaces studied by Zeller [22], Sargent [18] and Garling [7,8] (BS and AK are T-BS and T-AK, respectively when T is the summability matrix: $t_{nk}=1$ if $n\geq k$ and 0 otherwise). We say that λ is a T-ES space if $\{t^n: n=1,2,\dots\}$ is an equicontinuous set in $M(\lambda,\lambda)$. Plainly T-AK implies T-BS and T-BS plus barrelledness implies T-ES.

Theorem 1. _Let_ λ _be a_ T-_BS space and_ μ _a quasi-complete_ U-_ES space with respect to_ Sp_1-_matrices_ T _and_ U, _respectively_. _Then_ $S(\lambda,\mu)$ _coincides with the space of all continuous diagonal maps which transform each bounded subset of_ λ _into a relatively compact set of_ μ. _In fact, if_ a $S(\lambda,\mu)$, _then_ a _has the property_ U-_AK_.

Proof. One part is well-known [11, §42.1.(3)]. Now let $a\in M(\lambda,\mu)$ be such that a(A) is relatively compact for every bounded subset A of λ. Firstly, let us see that $ax=\lim_m at^m x$ in μ for every x in λ. Indeed, $\{x,t^m x: m\in\mathbb{N}\}$ is a bounded subset of λ, thus $B:=\{ax,at^m x: m\in\mathbb{N}\}$ is relatively compact; but, since $\lim_m t_{mk}=1$ for all k, ax is the only possible limit point of every sequence from B. Now we have that $ax \in \overline{\phi}$ (closure in μ) so that, by a well-known property of the equicontinuous sets [11,§39.4.(1)] and the fact that $y=\lim_n u^n y$ in μ if $y \in \phi$, we have that $ax=\lim_n u^n ax$ in μ. Finally, let us see that $a=\tau_b-\lim_n u^n a$. Indeed, if A is a bounded subset of λ and V is a zero-neighborhood in μ, then we can find $x^{(1)},\dots, x^{(p)}$ in A such that $a(A) \subset \cup_{j=1}^{p}(ax^{(j)}+V/3)$. For each x in A take j such that $ax-ax^{(j)} \in V$ and write:

$$ax-u^n ax = (ax-ax^{(j)})+(ax^{(j)}-u^n ax^{(j)})+(u^n(ax^{(j)}-ax))$$

then, by the equicontinuity of $\{u^n: n=1,2\dots\}$ and the fact that $ax^{(j)}=\lim_n u^n ax^{(j)}$ $j=1,\dots,p$ it is easy to deduce that $ax-u^n ax$ is in V for each n greater than a suitable n_0 and every x in A Q.E.D.

Corollary 1. _Under the assumptions of the theorem, if λ is a normed space then $S(\lambda,\mu)$ is the space of all compact diagonal maps from λ into μ. The same conclusion holds (see [5]) if λ is a DF-space and μ is a Fréchet space._

This corollary generalizes results due to Crofts [3] and Florencio [6]

Corollary 2. _If λ is a Montel, T-BS space then λ has the approximation property._

Example 1. If λ is a normal (in the sense of Köthe) sequence space such that $\lambda(\beta(\lambda,\lambda^{\times}))$ is a Banach space, then $M(\lambda,\lambda)$ is ℓ^{∞} and, therefore, $S(\lambda,\lambda)$ equals c_0.

Example 2. The space Λ of all bounded linear maps from ℓ^2 into itself is a Banach space without the approximation property [19]. The elements of this space can be viewed as infinite matrices [17,4.1.6]. Reordering a matrix $A=(a_{pk})$ according to:

$$b_n = \begin{cases} a_{p,k+1} & \text{if } n=k+p \quad p=1,2,\dots,k+1 \\ a_{k+1,k+1+p} & \text{if } n=k+k+1+p \quad p=1,2,\dots,k \end{cases}$$

we can associate the sequence $b=(b_n)$ with A in a unique way. In this sense, Λ can be viewed as a Banach K-space. If we show that Λ is a T-BS space then, by using corollary 1, we obtain that every compact diagonal map from Λ into Λ can be approximated by finite rank although Λ does not have the approximation property.

Let us take the operators T_n that assign to each matrix its square box of order n: $(T_n(A))_{ij}=a_{ij}$ if $0\leq i, j\leq n$ and zero otherwise. Then $T_n \in L(\Lambda,\Lambda)$ when Λ is taken as a matrix space. To each T_n corresponds the diagonal map P_{n^2} ($P_n:=(1,1,\ldots,1,0,0,\ldots)$ the last 1 in the n-th place) defined on Λ when Λ is taken as a sequence space. If T is the matrix whose rows are P_{n^2}, we need only to show that the operators T_n are uniformly bounded in $L(\Lambda,\Lambda)$: Let $n\in N$, if x is in ℓ^2 then $||p_n x||_2 \leq ||x||_2$. Now if (a_{ij}) is in Λ we have

$$|| T_n(a_{ij}) ||_\Lambda = \sup \{|| (Tn(a_{ij}))x ||_2 : || x ||_2 \leq 1\}$$

but it is easy to see that $(T_n(a_{ij}))x = P_n((a_{ij})P_n x)$, therefore we obtain:

$$|| T_n(a_{ij}) ||_\Lambda \leq \sup\{|| (a_{ij})P_n x ||_2 : || x ||_2 \leq 1\} \leq$$

$$\leq \sup\{|| (a_{ij})y ||_2 : || y ||_2 \leq 1 \} = || (a_{ij}) ||_\Lambda$$

then $|| T_n ||_{L(\Lambda,\Lambda)} \leq 1$ for all n in N.

Echelon spaces. Let $\{\lambda_k, a^{(k)}: k=1,2,\ldots\}$ be an echelon system, i.e each λ_k is a Fréchet K-space and $a^{(k)}$ is a sequence of non-zero terms such that $a^{(k)}/a^{(k+1)}$ is in $M(\lambda_{k+1},\lambda_k)$ so that we can write the continuous embeddings:

$$(1/a^{(1)})\lambda_1 \hookrightarrow \ldots \hookrightarrow (1/a^{(k)})\lambda k \hookrightarrow \ldots$$

the space $\lambda:=\text{proj lim } (1/a^{(k)})\lambda_k$ is called the echelon space associated to $\{\lambda_k, a^{(k)}\}$. λ is a Fréchet K-space. The following result is a straightforward corollary of our preceding results:

Corollary 3. Let $\{\lambda_k, a^{(k)}\}$ be an echelon system such that each λ_k is a T_k-BS space for T_k a Sp_1-matrix. If for each k in $\mathbb{N}$ there *exists* $r>k$ *such that* $a^{(k)}/a^{(r)}$ *is in* $S(\lambda_r,\lambda_k)$ *then* λ *is a Montel space. Moreover, if each* λ_k *is a Banach space then the above condition is equivalent to* λ *be a Schwartz space.*

This corollary includes and reproves the characterizations of the "Schwartzness" of an echelon space given for the classical cases, see e.g. [3, 6, 14 or 20].

2. AN EXAMPLE WITH CESÀRO SUMMABILITY

Definitions. Let α be a nonnegative integer. The space C_α of those sequences which are summables in the sense of Cesàro of order α is the summability field associated to the Sp_1-matrix $T_\alpha=(t_{nk})_{n,k\geq 0}$ with

$$t_{nk}:=\binom{n-k+\alpha}{\alpha}\binom{n+\alpha}{\alpha}^{-1} \quad \text{if} \quad n\geq k, \text{ and } t_{nk}=0 \text{ otherwise}$$

in other words:

$$C_\alpha:=\{(x_n)_n: \quad (\sum_k t_{nk}x_k)_n \text{ converges}\}$$

Under the norm

$$\| x \|_\alpha := \sup\{|\sum_k t_{nk}x_k|: n=0,1,2,\ldots\}$$

C_α is a Banach, T_α-AK space [2, 17, 21]. Therefore $S(C_\alpha,C_\beta)$ is the space of compact diagonal maps from C_α into C_β. Our purpose in this section is to derive a useful characterization of this space. If $\alpha\leq\beta$ then, according to Bosanquet [1], we have:

$$\text{(B1)} \quad M(C_\alpha,C_\beta)=M(C_\alpha,C_\alpha)=\{u: \| u \| :=(\sum_{n\geq 0} n^\alpha|\Delta^{\alpha+1}u_n|+\| u\|_\infty)<\infty\}$$

Using this and [2, Prop.5] we can deduce readily:

Proposition 1. *Let* $\alpha\leq\beta$ *be nonnegative integers, then*

(1) *The norm* $\| \ \|$ *that appears in* (B1) *induces the topology* τ_b *on* $M(C_\alpha,C_\beta)$

(2) $S(C_\alpha,C_\beta)$ *coincides with* $M(C_\alpha,C_\alpha)\cap c_0=$
$=\{u: u \text{ has } T_\alpha\text{-AK in } M(C_\alpha,C_\alpha)(\tau_b)\}$

If $\alpha>\beta$ then, again by Bosanquet [1]:

(B2) $M(C_\alpha,C_\beta)=\{u: \| u \|^* := (\sum_{n\geq 0} n^\alpha |\Delta^{\alpha+1} u_n| + \|((n+1)^{\alpha-\beta} u_n)_n\|_\infty) < \infty\}$

Proposition 2. Let $\alpha>\beta$ be nonnegative integers. Then,

(1) $\| \ \|^*$ in (B2) is a norm on $M(C_\alpha,C_\beta)$ and induces the topology τ_b.

(2) $S(C_\alpha,C_\beta)$ is the space $E:=\{u\in M(C_\alpha,C_\beta) : ((n+1)^{\alpha-\beta}u_n)_n$ is in $c_0\}$

Proof. (1) By using the linearity of the difference operator Δ we obtain that $\| \ \|^*$ is a norm. Since the identity $M(C_\alpha,C_\beta)(\| \ \|^*) \longrightarrow M(C_\alpha,C_\beta)(\tau_b)$ has closed graph, we only need to show that $M(C_\alpha,C_\beta)(\| \ \|^*)$ is complete. Let $u^{(m)}=(u_n^{(m)})_n$ $m=1,2,\dots$ be a $\| \ \|^*$-Cauchy sequence in $M(C_\alpha,C_\beta)$, then $v^{(m)} := ((n+1)^{\alpha-\beta} u_n^{(m)})_n$ $m=1,2,\dots$ is a $\| \ \|_\infty$-Cauchy sequence. Set $v=\| \ \|_\infty\text{-}\lim_m v^{(m)}$, then $u:=((n+1)^{\beta-\alpha}v_n)_n$ is such that $((n+1)^{\alpha-\beta}u_n)_n \in \ell^\infty$ and $\|((n+1)^{\alpha-\beta}(u_n^{(m)}-u_n))_n\|_\infty \longrightarrow 0$. Since the inclusion $M(C_\alpha,C_\beta)(\| \ \|^*) \longrightarrow M(C_\alpha,C_\alpha)(\| \ \|)$ is continuous, $u^{(m)}$ is $\| \ \|$-Cauchy and, a fortiori, $\| \ \|$-convergent to some z in $M(C_\alpha,C_\alpha)$. Necessarily $z=u\in M(C_\alpha,C_\beta)$ and $u=\| \ \|^*\text{-}\lim_m u^{(m)}$.

(2) Let $u\in E\subset M(C_\alpha,C_\beta)\subset M(C_\alpha,C_\alpha)\cap c_0 = S(C_\alpha,C_\alpha)$. Using (2) from the above proposition, we have that

$$\text{(i)} \quad \lim_m \sum_{n\geq 0} n^\alpha |\Delta^{\alpha+1}(u_n-(t^m u)_n)|=0$$

where t^m is the m-th row of T_α. Let

$$z^{(m)} := u-t^m u = (0,(1-t_{m1})u_1,\dots,(1-t_{mm})u_m,u_{m+1},u_{m+2},\dots)$$

since $u\in E$ and $0\leq t_{mk}\leq 1$, given $\varepsilon>0$ there exists n_0 in $\mathbb{N}$ such that - for all m in $\mathbb{N}$ and $n\geq n_0$ we have

$$\text{(ii)} \quad |(n+1)^{\alpha-\beta} z_n^{(m)}| \leq |(n+1)^{\alpha-\beta}u_n| < \varepsilon$$

On the other hand, $\lim_m t_{mn}=1$, hence $m_0(\geq n_0)$ exists such that if $m\geq m_0$ and $0\leq n\leq n_0$ then

$$\text{(iii)} \quad |(n+1)^{\alpha-\beta} z_n^{(m)}| \leq (1-t_{mn}) |(n+1)^{\alpha-\beta}u_n| < \varepsilon$$

(i), (ii) and (iii) yield $\| u-t^m u\|^* \xrightarrow[m]{} 0$, i.e, $u\in S(C_\alpha,C_\beta)$.

Conversely, it is clear that E is $\| \ \|^*$-closed, therefore $\phi\subset E\subset S(C_\alpha,C_\beta)$ implies $E=S(C_\alpha,C_\beta)$ by using (1) and the definition of $S(C_\alpha,C_\beta)$

Q.E.D.

Example 3. Given k in $\mathbb{N}$, we take $a^{(k)}=((n+1)^k)_n$. Since $a^{(k)}/a^{(k+1)}=$ $=(1/(n+1))_n$ is in $M(C_{k+1},C_k)$, we can consider the echelon space λ associated to the system $\{C_k,\ a^{(k)}\}$. Now, if $r>k$, then $a^{(k)}/a^{(r)}=$ $=(1/(n+1)^{r-k})_n$ is not in $S(C_r,C_k)$; therefore, λ is not a Schwartz space although the quotients $a^{(k)}/a^{(r)}$ are absolutely summable if $r\geq k+2$.

3. DIAGONAL MAPS ON VECTOR SEQUENCE SPACES.

Definitions. Let λ be a normal sequence space. Let $E(\tau)$ be a Hausdorff lcs and $U(E)$ a zero-neighborhood basis for τ. A sequence $(x_n)_n$ from E is said to be absolutely λ-summable if $(q_U(x_n))_n$ is in λ for all U in $U(E)$. The space $\lambda\{E\}$ of all absolutely λ-summable sequences from E is a linear subspace of $E^{\mathbb{N}}$ that contains $E^{(\mathbb{N})}$. From τ and the strong topology $\beta(\lambda,\lambda^{\times})$ we can define a topology, which we call τB, given by the seminorms:

$$q_{M,U}(x):=\sup\{\Sigma|\beta_n|q_u(x_n):\ \beta\in M\}=q_{M^0}((q_U(x_n)_n),\quad x=(x_n)_n\in\lambda\{E\}$$

where U runs through $U(E)$ and M runs through the family $B(\lambda^{\times})$ of all normal, $\sigma(\lambda^{\times},\lambda)$-bounded sets of $\lambda^{\times}$ (the topology on λ of uniform convergence on $B(\lambda^{\times})$ is, precisely, $\beta(\lambda,\lambda^{\times})$).

The spaces $\lambda\{E\}$ were introduced by Pietsch [15] and studied by De - Grande-De Kimpe [4] and Rosier [16]. From these papers we recall that the maps:

$$I_k: x\in E \longrightarrow I_k(x):=xe_k\in\lambda\{E\}$$

$$\Pi_k: x=(x_n)_n\in\lambda\{E\}\longrightarrow \Pi_k(x):=x_k\in E$$

are continuous for all k=1,2,... and also that the projections $\{P_n: n\in\mathbb{N}\}$ (defined analogously to the scalar case) form an equicontinuous subset of $L(\lambda\{E\},\lambda\{E\})$. It is also clear (see [16]) that $\lambda\{E\}$ is an AK-space ($x=\tau B\text{-}\lim_n P_n(x)$ for all x in $\lambda\{E\}$) if and only if $\lambda(\beta(\lambda,\lambda^{\times}))$ is also an AK-space. Following Rosier [16], E is said to be fundamentally λ-bounded if the sets

$$[R,B]:=\{\alpha x=(\alpha_n x_n)_n:\ \alpha\in R,\ x_n\in B,\ n=1,2,\dots\}$$

form a fundamental system of bounded sets in $\lambda\{E\}(\tau B)$ when R runs through the (absolutely convex) normal bounded sets in λ and B runs through the (absolutely convex and closed) bounded sets in E. (Recall that all the $(\lambda,\lambda^{\times})$-polar topologies in λ have the same family of bounded sets because $\lambda^{\times}$ is weakly sequentially complete [20,Ch.2,4.(14)]). In particular, if E is normed then E is fundamentally λ-bounded for all normal spaces (see [16] for further examples).

If μ is another normal sequence space and F another Hausdorff lcs, we denote by $M(\lambda\{E\},\mu\{F\})$ the space of all continuous diagonal maps from $\lambda\{E\}$ into $\mu\{F\}$, i.e. of those sequences $A=(A_n)_n$ from $L(E,F)$ such that

$$A:x=(x_n)_n \in \lambda\{E\}(\tau B) \longrightarrow Ax:=(A_n x_n)_n \in \mu\{F\}(\tau B)$$

is continuous. On $M(\lambda\{E\},\mu\{F\})$ we consider the topology τ_b of uniform convergence on the family of all τB-bounded subsets of $\lambda\{E\}$. We shall need the following scalar-type lemma:

Lemma 1. Let λ be a normal sequence space, then

(1) If μ is a normal sequence space such that $\lambda\subset\mu$ then the injection $\lambda(\beta(\lambda,\lambda^{\times}))\to\mu(\beta(\mu,\mu^{\times}))$ is continuous.

(2) $\beta(\lambda^{\times\times},\lambda^{\times})$ induces on λ the strong topology $\beta(\lambda,\lambda^{\times})$

(3) If $\lambda^{\times}\subset\ell^{\infty}$ and M is a normal $\sigma(\lambda^{\times},\lambda)$-bounded subset of $\lambda^{\times}$, then M is $\|\ \|_{\infty}$-bounded.

Proof. (1) is straightforward. Now, using [20,Ch.2,4(14) and 5.(1)] and the Banach-Mackey theorem we obtain that $\sigma(\lambda^{\times},\lambda^{\times\times})$ and $\sigma(\lambda^{\times},\lambda)$ have the same family of bounded sets, hence (2). (3): as in (2) M is $\sigma(\lambda^{\times},\lambda^{\times\times})$-bounded so that M is also $\beta(\lambda^{\times},\lambda^{\times\times})$-bounded and finally, by (1), M is $\beta(\ell^{\infty},\ell^{1})$-bounded Q.E.D.

Theorem 2. Let λ and μ be normal sequence spaces such that

(i) $\lambda\subset\mu\subset\ell^{\infty}$ and (ii) $\lambda^{\times}\subset\ell^{\infty}$

Let E and F be Hausdorff lcs, then:

(1) $M(\lambda\{E\},\mu\{F\}) = \{A \in (L(E,F))^{\mathbb{N}}: \{A_n: n=1,2,\dots$ is equicontinuous$\}\}$

(2) If in addition E is quasi-$\aleph_0$-barrelled, then

$$M(\lambda\{E\},\mu\{F\} = \ell^\infty \{L(E,F)(\tau_b)\}$$

(3) If in addition to (2), E is fundamentally λ-bounded, then the following topological equality holds:

$$M(\lambda\{E\},\mu\{F\})(\tau_b) = \ell^\infty\{L(E,F)(\tau_b)\}(\tau B)$$

Proof. (1)($\subset$) Assume $A=(A_n)_n$ is in $M(\lambda\{E\},\mu\{F\})$, then for each $k=1,2,\dots$ we have $A_k=\Pi_k A I_k$. By the hypothesis A is continuous, so we will deduce the equicontinuity of $\{A_n: n=1,2,\dots\}$ by showing that $\{\Pi_k\}_k$ and $\{I_k\}_k$ are, respectively, equicontinuous subsets of $L(\mu\{F\},F)$ and $L(E,\lambda\{E\})$. On one hand, if M is in $B(\lambda^\times)$ we can find $r>0$, by (3) in the lemma above, such that $|\beta_n|\leq r$ for all n in $\mathbb{N}$ and β in M, hence $q_{M,U}(I_k(x))\leq rq_U(x)$ for all x in E. On the other hand, if $V\in U(F)$ and x is in $\mu\{F\}$, then $q_V(\Pi_k(x))=q_V(x_k)\leq\|(q_V(x_n)_n\|_\infty$; now apply (1) in the lemma above to μ and ℓ^∞.

(1)($\supset$) Given $V\in U(F)$, let $U\in U(E)$ be such that $q_V(A_kx)\leq q_U(x)$ for all x in E and $k=1,2,\dots$ Now, if $x=(x_n)_n$ is in $\lambda\{E\}$, then $q_V(A_kx_k)\leq q_U(x_k)$. By using the normality of λ and the fact that $\lambda\subset\mu$, we obtain that $Ax\in\mu\{F\}$. Finally, A is continuous because $q_{M,V}(Ax)\leq q_{M,U}(x)$ and $M\in B(\mu^\times)\subset B(\lambda^\times)$ (the last inclusion by (1) in the lemma above).

(2) follow from (1) and the definition of quasi-$\aleph_0$-barrelledness.

(3) (τB is finer that τ_b). Suppose $q_{(B^*,V^*)}(\cdot)$ is a τ_b-seminorm on $M(\lambda\{E\},\mu\{F\})$, i.e. $B^*=[R,B]$ and $q_{V^*}(\cdot)=q_{M,V}(\cdot)$ for certain R (normal and bounded in λ), B (bounded in E), $V\in U(F)$ and $M\in B(\mu^\times)\subset B(\lambda^\times)$. Now, if $A=(A_n)_n\in M(\lambda\{E\},\mu\{F\})$, then one can see that

$$q_{(B^*,V^*)}(A):=\sup\{q_{M,V}((A_nx_n)_n): x\in[R,B]\}\leq$$

$$\leq\sup\{q_V(A_ny_n):y_n\in B,\ n\in\mathbb{N}\}\cdot\sup\{\Sigma|\alpha_n\beta_n|:\beta\in M,\alpha\in R\}=$$

$$=r(R,M)\cdot\|(q_{(B.V)}(A_n))_n\|_\infty$$

being $r(R,M)$ a constant and $\|(q_{(B,v)}(A_n))_n\|_\infty$ a τB-seminorm on $\ell^\infty\{L(E,F)(\tau_b)\}$.

(3) (τ_b is finer than τB). Consider the injection

$$I: M(\lambda\{E\},\mu\{F\})(\tau_b) \longrightarrow \ell^\infty\{L(E,F)(\tau_b)\}(\tau B)$$

we can make the following descomposition: $I=I_2 \circ I_1$ where

$$I_1: M(\lambda\{E\},\mu\{F\})(\tau_b) \longrightarrow M(\lambda\{E\}, \ell^\infty\{F\})(\tau_b)$$

$$I_2: M(\lambda\{E\},\ell^\infty\{F\})(\tau_b) \longrightarrow \ell^\infty\{L(E,F)(\tau_b)\}(\tau B)$$

the three spaces being algebraically isomorphic. We are done if we show that I_1 and I_2 are continuous. On one hand, if $V^* \in U(\ell^\infty\{F\})$, then $V_1^* := V^* \cap \mu\{F\}$ is in $U(\mu\{F\})$ ($\mu\{F\} \to \ell^\infty\{F\}$ is continuous: use (1) in lemma above). Therefore, since $q_{(B^*,V^*)}(A) = q_{(B^*,V_1^*)}(A)$ for all A in $M(\lambda\{E\},\mu\{F\})$ and all B^* bounded in $\lambda\{E\}$, I_1 is continuous. On the other hand, suppose that $q_{\infty,W}(\cdot)$ is a τB-seminorm on $\ell^\infty\{L(E,F)(\tau_b)\}$, i.e.

$$q_{\infty,W}(A) = \|(q_W(A_n))_n\|_\infty = \|(q_{(B,V)}(A_n))_n\|_\infty$$

for certain B (bounded in E) and $V \in U(F)$. Now the set of unit sequences $\{e_n : n=1,2,\dots\}$ is $\beta(\lambda,\lambda^\times)$-bounded (use $\ell^1 \subset \lambda^{\times\times}$, (1) and (2) in the lemma above. Take R as the normal hull of this set, R is a normal bounded set in λ. Now take $B^* := [R,B]$ and V^* in $U(\ell^\infty\{F\})$ such that $q_{V^*}(\cdot) = q_{\infty,V}(\cdot)$. If A is in $M(\lambda\{E\},\ell^\infty\{F\})$, and bearing in mind that xe_n is in B^* for all x in B, then

$$q_{\infty,W}(A) = \sup\{q_{(B,V)}(A_n) : n=1,2,\dots\} = \sup\{q_V(A_n x) : x \in B,\ n=1,2,\dots\} \leq$$

$$\leq \sup\{q_V(A_n y_n) : y \in B^*,\ n=1,2,\dots\} = \sup\{q_{V^*}(Ay) : y \in B^*\} = q_{(B^*,V^*)}(A)$$

$q_{(B^*,V^*)}(\cdot)$ being a τ_b-seminorm on $M(\lambda\{E\},\ell^\infty\{F\})$ Q.E D.

Remark. Observe that (i) and (ii) in the theorem above are satisfied if λ and μ are normal sequence spaces such that $\ell^1 \subset \lambda \subset \mu \subset \ell^\infty$. So that the theorem holds if λ,μ are Dubinski's step spaces [3].

Proposition 3. *Let λ and μ be normal sequence spaces such that $\lambda^\times \subset \ell^\infty$, $\lambda \subset \mu \subset \ell^\infty$ and $\mu(\beta(\mu,\mu^\times))$ is an AK-space. Let E be a quasi-$\aleph_0$-barrelled Hausdorff lcs which is fundamentally λ-bounded. Let F be a Hausdorff lcs. Then a diagonal map $A \in M(\lambda\{E\},\mu\{F\})$ transform bounded sets into* *precompact sets if and only if the following two conditions hold:*

(1) *Each* A_n *transforms bounded sets into precompact sets*.

(2) $\{A_n: n=1,2,\dots\}$ *is a null sequence in* $L(E,F)(\tau_b)$.

Proof. ($\Longrightarrow$(1)) This follows from the continuity of the maps I_k and Π_k.

($\Longrightarrow$(2)) Note that $Ax=(A_n x_n)_n$ has the property AK in $\mu\{F\}$ and proceed, with the obvious changes, as in theorem 1.

((1),(2)$\Longrightarrow$) If $\tau_b\text{-}\lim_n A_n = 0$, then, by [16], $A=\tau B\text{-}\lim_n P_n(A)$ in $\ell^\infty\{F\}$. But, again because I_k and Π_k are continuous, each map $P_n(A)=(A_1,\dots,A_n,0,\dots)$ transforms bounded sets of $\lambda\{E\}$ into precompact sets of $\mu\{F\}$. Now use [11,§42.1.(3)] Q.E.D.

Corollary 1. *Let* λ *and* μ *be normal sequence spaces such that* $\lambda(\beta(\lambda,\lambda^\times))$ *is normed*, $\mu(\beta(\mu,\mu^\times))$ *is an AK-space*, $\lambda^\times\subset\ell^\infty$ *and* $\lambda\subset\mu\subset\ell^\infty$. *Let* E *be a normed space and* F *be a Hausdorff lcs. Then a diagonal map* $A\in M(\lambda\{E\},\mu\{F\})$ *is precompact if and only if each* A_n *is precompact and* $(A_n)_n$ *is a null sequence in* $L(E,F)(\tau_b)$.

Remark. If $\lambda(\beta(\lambda,\lambda^\times))$ is a normal, normed sequence space such that $\|e_n\|_\lambda=1$, for all n=1,2,... and λ is an AK-space, then it is easy to see that λ and $\lambda^\times$ are contained in ℓ^∞; so that this corollary includes and reproves a recent result by Gupta and Patterson [9,Prop.4.6]

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