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ON THE GRAPHS WHICH ARE THE EDGE OF A PLANE TILING

R. AYALA, E. DOMÍNGUEZ, A. MÁRQUEZ and A. QUINTERO

0. Introduction.

In [2] it is proved that an infinite graph is planar if and only if each finite subgraph is planar. Thus, the Kuratowski Theorem on planarity of graphs holds for infinite graphs. In addition, R. Halin ([7]) has characterized the connected graphs which admit a plane representation without vertex accumulation points (VAP-free plane graphs). Later, C. Thomassen ([11; Cor. 4.1]) shows that all connected VAP-free plane graphs admit locally finite plane representations (EAP-free plane graphs in the sense of [11]).

Edge graphs of plane tilings provide a large class of locally finite plane representations of graphs.

A natural question asks for all graphs whose locally finite plane representations always are edge graphs of plane tilings. In this paper we characterize that family of graphs in (2.11). This result is a consequence of the characterization of all graphs whose locally finite plane representations only have bounded components (see (2.9) below). Previously we have also characterized the graphs whose locally finite plane representations always have a unbounded component (see (2.6) below).

An important consequence of (2.9) is the characterization of those graphs which admit a locally finite representation (see (2.13) below) in terms of forbidden graphs.

1. Properly planar graphs.

In this paper a *graph* is a locally finite 1-dimensional connected CW-complex. Given a subgraph $H \subseteq G$, we denote by $G - H'$ the subgraph of G defined by all the edges of $G - H$ and their vertices. By a tree we shall always mean an infinite tree.

In a graph G, a vertex $v \in G$ is called a *cutpoint* of G if $G - \{v\}$ is not connected.

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When G has no cutpoint, G is said to be 2-conneccted. See [4] for notations and basic results of graph theory.

A graph is said to be *planar* if it can embedded in the plane, and *properly planar* (*p-planar*) if it can be embedded in a locally finite way. A locally finite embedding $\varphi: G \to R^2$ means in topological terms that φ is a proper map. We recall that a *proper map* (*p-map*) is a continuous map $f: X \to Y$ such that $f^{-1}(K)$ is compact for each compact $K \subseteq Y$.

It is easy to check that the planar graphs pictured in Figure 1 are not properly planar. Those are the graphs such that their one-point compactification are homeomorphic to either K_5 or $K_{3,3}$ (the complete graph with 5 vertices and the complete bipartite graph with 3 + 3 vertices respectively).

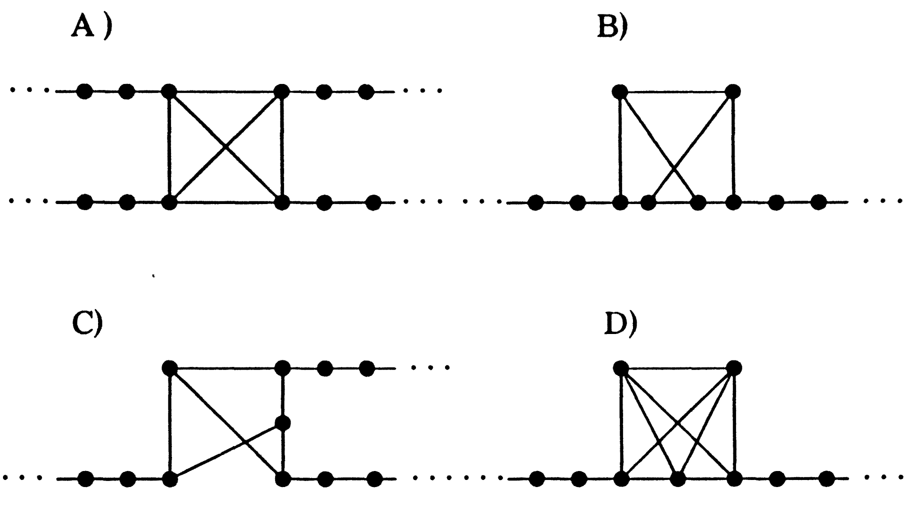

Figure 1: The graphs $A = K_5^\infty$, $B = L_{3,3}^\infty$, $C = K_{3,3}^\infty$, $D = L_5^\infty$.

An important notion for a non-compact space X is the set of ends of X. More explicitly, the *ends* of X are the elements of the inverse limit $\mathscr{F}(X) = \lim_{\leftarrow} \pi_0(X - K)$ where K ranges the family of compact sets of X and $\pi_0(X - K)$ stands for the set of connected components. The cardinal number of $\mathscr{F}(X)$ is denoted by $e(X)$. When G is a graph we can use a countable sequence $G_1 \subseteq G_2 \subseteq \ldots$ of finite subgraphs to define $\mathscr{F}(G)$. See [3] for details.

An end of a graph G is defined by a tree $F \subseteq G$ homeomorphic to the positive half-line R_+ (see [1]). Two trees F_1 and F_2 homeomorphic to R_+ represent the same end if given any compact subset $K \subseteq G$, there exists a path in $G - K$ joining F_1 to F_2. An end defined by F is said *stable* if for some K compact subset of G, the connected component of $G - K$ defined by F is a tree, otherwise the end is said to be *unstable* (see [1]).

1.1 REMARK. a) Notice that $e(K_5^\infty) = 4$, $e(L_5^\infty) = e(L_{3,3}^\infty) = 2$, and $e(K_{3,3}^\infty) = 3$. Furthermore, all the ends are stable.

b) If we join two vertices of different infinite edges in K_5^∞ we get a copy of $K_{3,3}^\infty$ embedded in the new graph. When we proceed in the same way with $K_{3,3}^\infty$ we get an embedding of $L_{3,3}^\infty$. And, if we pairwise join three vertices of three different infinite edges of K_5^∞ we get an embedding of $L_{3,3}^\infty$ in the new graph.

In [7] R. Halin characterizes VAP-free graphs in terms of the family of forbidden subgraphs consisting of the Kuratowski graphs and the four graphs in Figure 1. Moreover, C. Thomassen ([11; Cor. 4.1]) shows that any connected VAP-free plane graph admits a locally finite representation. That is, the following result holds.

1.2 THEOREM. *A connected planar graph G is p-planar if and only if G contains no subgraph homeomorphic to* K_5^∞, $K_{3,3}^\infty$, L_5^∞, *or* $L_{3,3}^\infty$. *These graphs will be called minimal non-p-planar graphs.*

1.3. COROLLARY. *Let G be a connected planar graph with* $e(G) = k$, *then G is p-planar if and only if it contains no subgraph homeomorphic to a minimal non-p-planar graph H with* $e(H) \leqq k$.

PROOF. If $H \subseteq G$ is a minimal non-p-planar graph with $e(H) \geqq k + 1$, we can use (1.1.b)) to get a new minimal non-p-planar graph with $H' \subseteq G$ with $e(H') \leqq k$.

1.4 COROLLARY. *Let G be a graph with* $e(G) = 1$, *then G is p-planar if and only if it is planar.*

Later, we shall also use the following lemma, whose first statement is actually used in Halin's proof of Theorem (1.2).

1.5 LEMMA. *Let G be a graph. There exists an increasing sequence* $\{G_i\}_{i \geqq 1}$ *of connected finite subgraphs such that* $G = \cup \{G_i; i \geqq 1\}$, *all the components of* $G - G_i$ *are unbounded, and* $H_i = G_i \cap (G - G_i)'$ *is a finite set of vertices.*

Moreover, if for any tree $T \subseteq G$, $e(T) = 1$ *we can choose the above sequence in such a way that each* H_i *is just a cutpoint of G.*

PROOF. We start with any incressing sequence $\{C_i\}_{i \geqq 1}$ of connected finite subgraphs of G with $G = \cup \{C_i; i \geqq 1\}$.

Since G is locally finite there are only finitely many components in $G - C_i$. If $\{K_\alpha; \alpha \in I_i\}$ is the family of bounded components of $G - C_i$, we take $C_i' = C_i \cup \{K_\alpha; \alpha \in I_i\}$. Finally we define G_i to be the union of C_i' with all the edges whose vertices are in C_i'.

Assume now that $e(T) = 1$ for any tree $T \subseteq G$, and let S be the set of cutpoints of G. The set S is infinite since, otherwise, by the Menger Theorem for infinite graphs ([6; Th. 3]), if from a vertex v there are not two infinite paths, vertex-disjoint apart from v, then there is a cutpoint w such that the component of $G - w$ containing v is finite. It follows that, if G contains only finitely many cutpoints, choosing v suitably we could find a tree T with $e(T) \geqq 2$.

The sequence $\{G_i\}_{i \geqq 1}$ is now constructed as follows. We take a vertex $v_1 \in S$. Then $G - \{v_1\}$ has a finite set $\{H_\beta; \beta \in I_1\}$ of bounded components and only one unbounded component since $e(G) = 1$. Let H_1 denote the union $\cup \{H_\beta; \beta \in I_1\}$, and $G_1 = H_1' = H_1 \cup \{v_1\}$.

As S is infinite, we can choose a vertex $v_2 \in S \cap U_1$. Again, we know that $G - \{v_2\}$ has a finite set $\{H_\beta; \beta \in I_2\}$ of bounded components and only one unbounded component U_2. If $H_2 = \cup \{H_\beta; \beta \in I_2\}$, it is clear that $G_1 \subseteq H_2$. Let $G_2 = H_2' = H_2 \cup \{v_2\}$.

We proceed inductively to define the increasing sequence $\{G_i\}_{i \geqq 1}$.

2. Spanning graphs.

Given a graph G, a locally finite (proper) embedding $\varphi: G \to R^2$ is said to be a *spanning embedding* if all the components of $R^2 - \varphi(G)$ are bounded. There exist obvious examples of infinite graphs which admit both spanning and non-spanning embeddings (see Example 2.3):

A graph G is a *spanning graph* if all of its properly planar embeddings are spanning embeddings.

2.1 LEMMA. *If a graph G admits a spanning embedding then $e(G) = 1$ and the end is unstable.*

PROOF. Let $F_1, F_2 \subseteq \varphi(G)$ be disjoint subgraphs homeomorphic to R_+. Given a disk $B(n)$ of center $(0, 0)$, and radius n, we take $k = k(n)$ such that the subgraph $G_{k-1} \subseteq G$ given by (1.5) satisfies $\varphi^{-1}(B(n)) \subseteq G_{k-1}$. Then the subgraph $C(k)$ which defines the outer face of G_k contains a cycle in $R^2 - B(n)$ which meets F_1 and F_2, and so $C(k)$ provides paths joining F_1 and F_2 outside $B(n)$. Therefore F_1 and F_2 define the same end and $e(G) = 1$. It is clear that this end must be unstable since otherwise an unbounded component would appear in $R^2 - \varphi(G)$.

2.2 LEMMA. *Assume $e(G) = 1$ and the proper embedding $\varphi: G \to R^2$ is not spanning. Then $R^2 - \varphi(G)$ has only one unbounded component U. Moreover, if $e(\mathrm{Fr}\, U) = 1$ then any tree $T \subseteq G$, $e(T) = 1$. Here $\mathrm{Fr}\, U$ denotes the subgraph consisting of all the edges which define the unbounded component U.*

PROOF. Obviously $R^2 - \varphi(G)$ has unbounded components. Assume that there are two unbounded components $U_1, U_2 \subseteq R^2 - \varphi(G)$. By joining two points $x_i \in U_i$ $(i = 1, 2)$ by an arc γ, we can construct an embedding $\varepsilon: R \to R^2$ with $\varepsilon(R) \cap \varphi(G) \subseteq \gamma$, and such that U_1, and U_2 meet the two components $W_1, W_2 \subseteq R^2 - \varepsilon(R)$. Therefore the graph $\varphi(G)$ must also meet both components.

In addition, $\varphi(G) \cap W_i$ ($i = 1, 2$) is infinite. Otherwise, if $W_1 \cap \varphi(G)$ is finite we can replace γ by a new are γ' such that all the graph $\varphi(G)$ is contained in W_2.

Finally, as $\varepsilon(R) \cap \varphi(G)$ is compact, we can find at least two ends in $\varphi(G)$, and this leads to contradiction.

Assume now that $e(\mathrm{Fr}\, U) = 1$. If $T \subseteq G$ is a tree with $e(T) = 2$, by connectedness U is contained in a component $\Omega \subseteq R^2 - \phi(T)$. If $F \subseteq U$ is an infinite ray, we can find two unbounded sequences, $\{l_n\}$, $\{r_n\}$, in $\mathrm{Fr}\, U \subseteq \Omega \cup \phi(T)$, one on each side of F. Since $e(\mathrm{Fr}\, U) = 1$ both sequences define the same end and there exists a locally finite family of arcs $\gamma_n \subseteq \mathrm{Fr}\, U$ joining l_n to r_n. But this can occur only if $\gamma_n \cap F \neq \emptyset$ for finitely many γ_n's. This fact leads to a contradiction.

2.3 EXAMPLE. Figure 2 shows that we can find spanning embeddings and non spanning embeddings of the same graph. More generally, we can prove

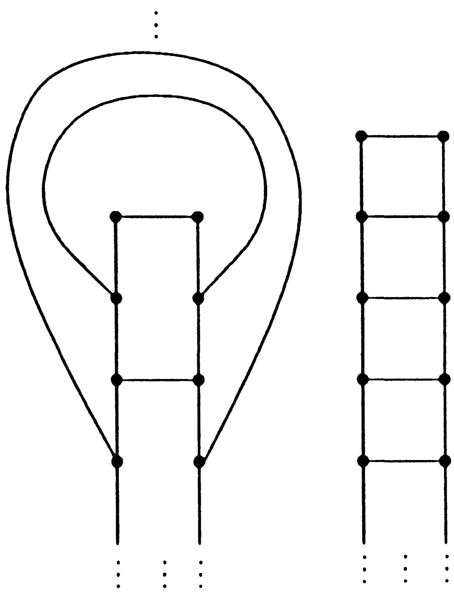

Figure 2: Two different embeddings of the same graph.

2.4 PROPOSITION. *Let G be an unstable one-ended planar graph. If for any tree $T \subseteq G$ we have $e(T) \leqq 2$ then G always admits a spanning embedding.*

PROOF. Let $\varphi: G \to R^2$ be a proper embedding. If φ is not spanning, $R^2 - \varphi(G)$ has only one unbounded component U according to (2.2).

a) Assume that there exists $T \subseteq \mathrm{Fr}\, U$ with $e(T) = 2$. We can also assume that T has no endpoints. Moreover, we can identify $\varphi(T)$ with the OX-axis $R \subseteq R^2$ since there exist homeomorphisms h: $R^2 \to R^2$ carrying $\varphi(T)$ to R.

As G is one-ended, we can find a sequence of pairwise disjoint arcs $\{\gamma_i\}_{j \geqq 1}$ which join the two branches of T. In addition, since $R = \varphi(T) \subseteq \operatorname{Fr} U$, we have that all the arcs $\varphi(\gamma_j)$ must be contained in the same component of $R^2 - R$. Assume $\varphi(\gamma_j) \subseteq R^2_+ = \{(x, y); y \geqq 0\}$.

If R^2_- denotes the lower half-plane, the subgraph $\varphi(G) \cap R^2_-$ is a union of a sequence $\{K_n\}$ of pairwise disjoint finite subgraphs such that each K_n meets $\varphi(T)$ in a vertex. Indeed, for each vertex $v \in \varphi(T)$ let K_v be the subgraph of $\varphi(G) \cap R^2_-$ generated by all the vertices which are joined to v outside $\varphi(T)$. Since $R = \varphi(T) \subseteq \operatorname{Fr} U$, it is easy to check that $K_v \cap K_w = \emptyset$ id $v \neq w$.

Therefore, it will suffice to find a spanning embedding of $L = \varphi(G) \cap R^2_+$ since it is easy to extend any proper embedding of L to a proper embedding of the whole graph $\varphi(G)$.

We now proceed to construct a spanning embedding $\psi: L \to R^2$. We consider the sequence of arcs $\{\gamma_j\}$ given above, and we start by defining $L_1 \subseteq L$ as the subgraph generated by $\varphi(\gamma_1\}$ and all the vertices which can be joined to $\varphi(\gamma_1)$ outside $\varphi(T)$. The subcomplex L_1 is finite. In fact, if L_1 is infinite we can apply the König Lemma ([8; VI.2.6]) to get an infinite ray in L_1 which together with T defines a tree T' with $e(T') = 3$.

As L_1 is finite, we take a disk $B(n_1)$ with $L_1 \subseteq B(n_1)$, and we choose a new arc $\varphi(\gamma_{j_2})$ outside $B(n_1)$. We use this arc to define $L_2 \subseteq L$ in the same way as L_1 above. It is clear that $L_1 \cap L_2 = \emptyset$, and we can inductively proceed to define a sequence of arcs $\{\gamma_{j_k}\}$ and the corresponding sequence of subgraphs $\{L_k\}$ with $\varphi(\gamma_{j_k}) \subseteq L_k$.

Notice that for any vertex v outside $L_k \cup \varphi(T)$ there is no arc in $\varphi(G - T)$ joining v to L_k. This remark allows us to define $\psi: L \to R^2$ as the identity in $\varphi(G) - \cup\{L_{2k-1}; k \geqq 1\}$, and $\psi | L_{2k-1}$ is the reflection with respect to the OX-axis. In this way we have defined a spanning embedding of L.

b) Assume that $e(\operatorname{Fr} U) = 1$. Then by (2.2) we have $e(T) = 1$ for any tree $T \subseteq G$. We fix a tree T. By using (1.5) it is easy to find a sequence $\{G_n\}$ of finite subgraphs with $G = \cup\{G_n; n \geqq 1\}$, $G_n \cap G_m = \emptyset$ if $|n - m| \geqq 2$, and $G_n \cap G_{n+1} = \{x_n\}$ with x_n a cutpoint of G.

As G is planar, we find embeddings $\varphi_n: G_n \to [n-1, n] \times (-1, 1)$ with $\varphi_n^{-1}(\{i\} \times (-1, 1)) = x_i (i = n-1, n; n \geqq 2)$. By gathering together all the φ_n''s we get a proper embedding $\varphi: G \to R_+ \times (-1, 1) \subseteq R^2$.

We shall now change φ into a spanning embedding as follows. We pick an edge e_1 of G_1 ending at x_1, and such that $\varphi(e_1)$ is an edge of $\operatorname{Fr} U$ with U the unbounded component of $R^2 - \varphi(G)$. Then, we reembed e_1 by carrying it to an arc $\xi_1 \subseteq U$ such that $\xi_1 \cup \varphi(e_1)$ define a non-trivial cycle in $R^2 - \{(0,0)\}$. We inductively follow this procedure by taking an edge e_n in G_n with $\varphi(e_n)$ in $\operatorname{Fr} U$, and reembed e_n by choosing an arc $\xi_n \subseteq U - \cup\{\xi_i; i \leqq n-1\}$ in such a way that $\xi_n \cup \varphi(e_n)$ is a non-trivial cycle of $R^2 - \{(0,0)\}$.

It is now clear that we have a spanning embedding $\varphi'\colon G \to R^2$ given by $\varphi' = \varphi$ on $G - \{e_i; i \geqq 1\}$, and $\varphi'(e_n) = \xi_n$.

2.5 Remark. It is not possible to give a characterization of graphs which admit an spanning embedding in terms of minimal graphs with such a property, since it is easy to check that the graph described in Figure 3 not have a spanning embedding, but, it contains a subgraph homeomorphic to that one of (2.3).

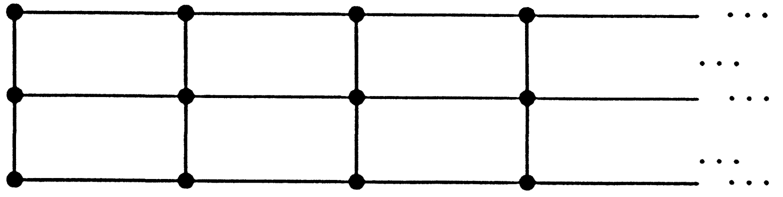

Figure 3: The graph of Remark 2.5.

2.6 Theorem. *Let G be an unstable one-ended planar graph. Then G does not admit a spanning embedding if and only if there exists a proper embedding* $\alpha\colon R_+ \to G$ *such that* $e(G - N(\alpha)) \geqq 2$, *for some open neighbourhood* $N(\alpha)$ *of* $\alpha(R_+)$.

Proof. Let $\varphi\colon G \to R^2$ be a spanning embedding, and assume that we can find a tree $T \subseteq G - N(\alpha)$ homeomorphic to R which defines two different ends of $G - N(\alpha)$. Then $N(\alpha)$ is contained in one of the two unbounded components, U_1, U_2, of $R^2 - \varphi(T)$. Assume $\alpha(R_+) \subseteq U_1$. As in the proof of (2.1), let $C(k) \subseteq G$ be the subgraph which defines the outer face of $G_k \subseteq G$ with $\varphi^{-1}(B(n)) \subseteq G_{k-1}$. Then $C(k)$ meets the two branches of T and provides a path $\eta_n \subseteq G \cap U_2$ outside $B(n)$ for each $n \geqq 1$. Thus the two branches of T define the same end of G. This is a contradiction.

Conversely, if G does not admit a spanning embedding we can find a tree $T \subseteq G$ with $e(T) = 3$ according to (2.4).

If $\varphi\colon G \to R^2$ is a proper embedding, we can assume without loss of generality that $\varphi(T) = OX \cup OY_+$ with $OY_+ = \{(0, y); y \geqq 0\}$. In fact, we can always find a homeomorphism $h\colon R^2 \to R^2$ such that $h \circ \varphi$ satisfies the required condition.

Since φ is not spanning, we know that all (except possibly finitely many) of the arcs $\{\gamma_i\}$ joining OX_+ to OX_- must be contained in one of the half-planes defined by OX.

If the arcs $\{\gamma_i\}$ are contained in the upper half-plane then $e(G - N(OY_+)) \geqq 2$.

Assumes that $\{\gamma_i\}$ is contained in the lower half-plane R^2_+. We can argue with $OX_+ \cup OY_+$ as we have already done with OX and all the arcs $\{\sigma_i\}$ joining OY_+ to OX_+ must be contained in the region $\{(x, y); x, y \geqq 0\}$. Otherwise $e(G - N(OX_-)) \geqq 2$.

Therefore, as φ is not spanning, all the arcs joining OY_+ to OX_- must meet OX_+ and so $e(G - N(OX_+)) \geqq 2$.

As a consequence of (2.6) we can state

2.7 COROLLARY. *Let G be as in* (2.6), *and assume that we can find a tree* $T \subseteq G$ *with* $e(T) \geqq 3$. *Then G admits a spanning embedding if and only if G is a spanning graph.*

PROOF. Assume that G admits a spanning embedding and let $\varphi: G \to R^2$ be a proper embedding. We choose a tree $T \subseteq G$ with $e(T) = 3$. As in the proof of (2.6), we can assume that $\varphi(T) = OX \cup OY_+$.

By (2.6) we have $e(G - N(L)) = 1$ for $L = OX_+, OX_-, OY_+$, and so there must exist locally finite families of disjoint arcs in the three regions of R^2 defined by $\varphi(T)$. This shows that φ is a spanning embedding.

2.8 REMARK. The proof of (2.7) also shows the existence of an embedding of the graph pictured in Figure 4 in any spanning graph G for which there exists a tree $T \subseteq G$ with $e(T) \geqq 3$.

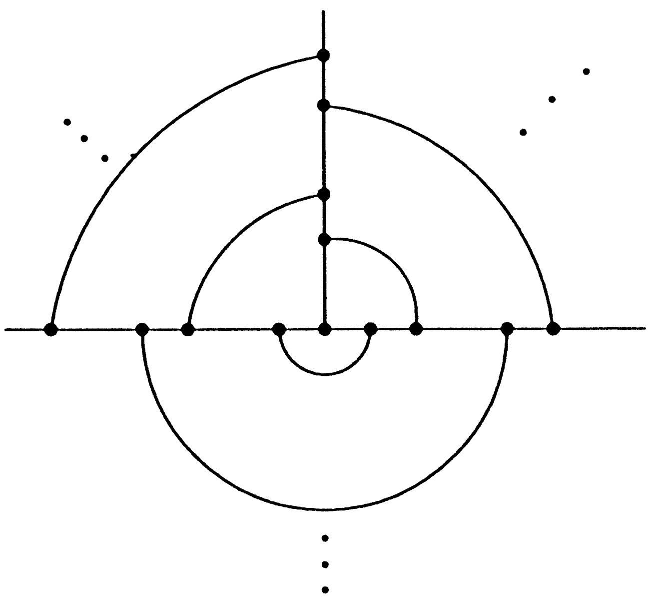

Figure 4.

Finally we shall give a characterization of spanning graphs in terms of the two minimal tiling graphs described in Figure 5.

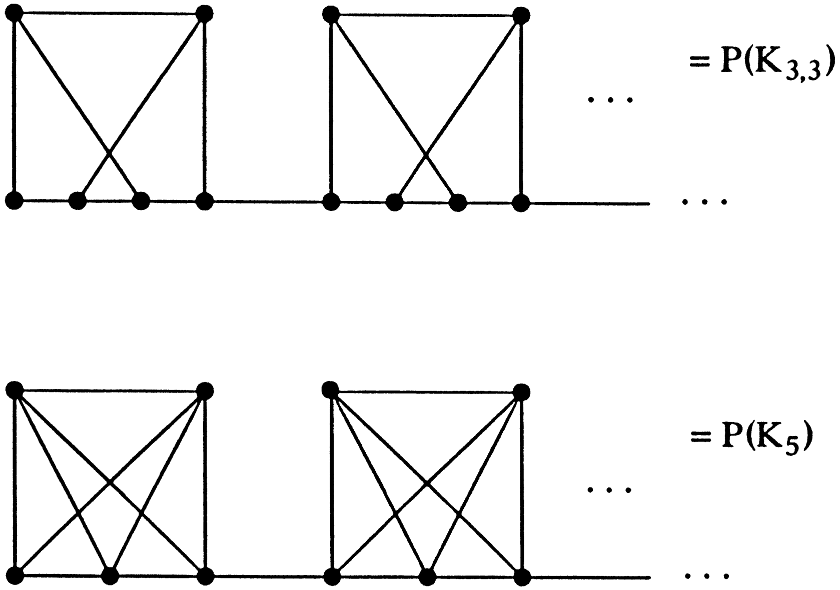

Figure 5: The two minimal tiling graphs.

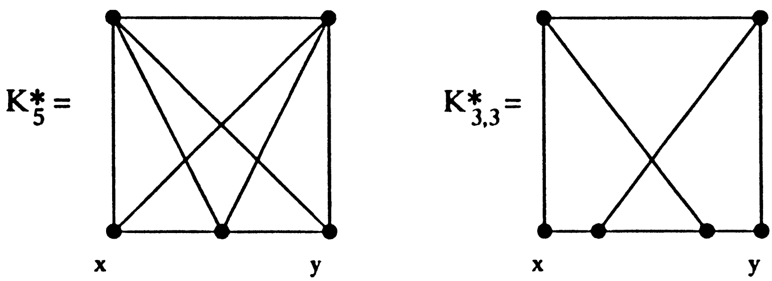

Figure 6: The graphs K_5^*, $K_{3,3}^*$.

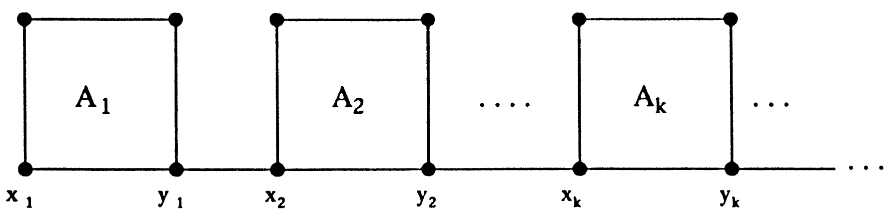

Figure 7: The graph $P(\{A_k\}_{k\geq 1})$ where A_k is either K_5^* or $K_{3,3}^*$ in Figure 6.

2.9 THEOREM. *A graph G is a spanning graph if and only if G is a one-ended planar graph which contains a subgraph homeomorphic to* $P(K_5)$, *or* $P(K_{3,3})$ *(see Figure 5).*

PROOF. It is easy to check that if a p-planar graph G has a spanning subgraph, then G is a spanning graph. As $P(K_{3,3})$, and $P(K_5)$ are spanning graphs, the condition is sufficient.

In order to prove the necessity of the condition, we shall actually show that G contains a subgraph homeomorphic to the graph $P(\{A_k\}_{k \geqq 1})$ (see Figure 7).

According to (1.5), we have an increasing sequence $\{G_n\}_{n \geqq 1}$ of finite subgraphs with $H_n = G_n \cap (G - G_n)'$ a finite set of vertices for each $n \geqq 1$.

Assume we have found an embedding α_n: $P(\{A_k\}_{k \leqq n} \to G$. In order to extend α_n to α_{n+1}: $P(\{A_k\}_{k \leqq n+1} \to G$, we consider G_m such that $\alpha_n(\cup \{A_k; k \leqq n\}) \cup \alpha_n([0, 2n]) \subseteq G_m$.

Let $t_{m+1} = \min\{t \in R_+; \alpha_n(t) \in (G - G_{m+1})'\}$. Notice that $2n \leqq t_{m+1}$. Let H_{m+1} be the graph obtained by identifying $0 \in R_+$ to $\alpha_n(0) \in \alpha_n([0, t_{m+1}]) \cup (G - G_{m+1})'$. Since G is a spanning graph, one can apply (2.2) to check that H_{m+1} is not a p-planar graph. Then, by (1.3) there exists an embedding β: $R_+ \cup P(A_{m+1}) \to H_{m+1}$ for some A_{m+1}. It is easy to replace β by β' with $\beta' \mid [0, 2n] = \alpha_n$, and so β', and α_n define α_{n+1}. Starting with any proper embedding α_0: $R_+ \to G$, it can be straightforwardly checked that $\{\alpha_n\}_{n \geqq 0}$ defines a proper embedding α: $P(\{A_k\}_{k \geqq 1}) \to G$. This finishes the proof.

2.10 REMARK. When we can find a tre $T \subset G$ with $e(T) \geqq 3$ there is simple proof of (2.9). Indeed, G must contain an embedding of the subgraph S described in (2.8). It is now easy to check that $P(K_{3,3})$ can be embedded in S. Therefore, the only graph $P(K_{3,3})$ characterizes the spanning graphs with three or more ends.

Usually a plane tiling is defined as a locally finite family $\mathscr{T}$ of topological disks $T \in \mathscr{T}$ which cover R^2, and whose interiors are pairwise disjoint. The boundaries of the disks define a graph $G(\mathscr{T})$ called the edge graph of $\mathscr{T}$. A *tiling graph* is a p-planar graph each of whose locally finite plane representations is the edge graph of a tiling. See [5; 3.1.3], and [10] for details and properties of plane tilings and edge graphs.

2.11 COROLLARY.*Given a graph G, G is a tiling graph if and only if it is a 2-connected one-ended planar graph which contains a subgraph homeomorphic to* $P(K_5)$ *or* $P(K_{3,3})$.

PROOF. It is known that the components of the complement of a plane graph G are topological disks if and only if G is 2-conneccted (see [4; 1.6.1]). This result and (2.9) yield the corollary.

2.12 REMARK. Notice that Theorem (1.2) does not hold for non-connected graphs, since there is no proper embedding of the graph union of the two graphs pictured in Figure 8.

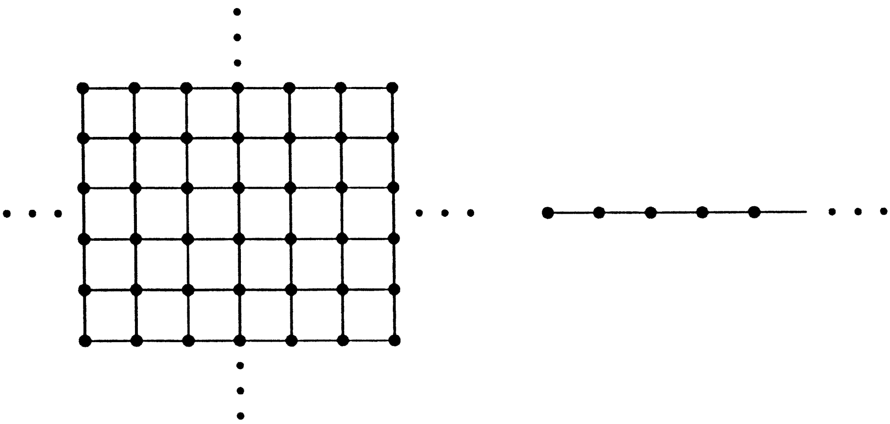

Figure 8: A non-p-planar graph which does not contain any of the forbidden subgraphs given by Halin.

This fact is due to the existence of spanning graphs among the components. Therefore (2.9) also gives the following extension of (1.2).

2.13 THEOREM. *Let G be a non-connected infinite graph. Then G is p-planar if and only if G does not contain a subgraph homeomorphic to* $K_5, K_{3,3}, K_5^\infty, K_{3,3}^\infty, L_5^\infty, L_{3,3}^\infty, P'(K_{3,3})$ *or* $P'(K_5)$. *Here* $P'(H)$ *denotes the disjoint union of* $P(H)$ *and* R_+.

PROOF. Assume G is not p-planar. If G is connected it follows from (1.2). If G is non-connected then at least one component $C \subseteq G$ is a spanning graph and another component $C' \neq C$ must be infinite. Therefore (2.9) yields an embedding $P'(H) \subseteq C \cup C'$ for $H = K_{3,3}$ or K_5.

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