A computational approach to free divisors and logarithmic $\mathcal{D}$-modules

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Abstract

We apply algorithmic techniques for comparing some $\mathcal{D}$-modules associated to a kind of divisors in $\mathbf{C}^n$.

Introduction

We will work on some (analytic) $\mathcal{D}$-modules associated to a germ of holomorphic function on $\mathbf{C}^n$. We propose four natural questions on the comparison of these $\mathcal{D}$-modules. Although our general settings are in the analytic category, we prove that if the input is a complex polynomial in n variables then the associated objects are computable.

Let us denote by $\mathcal{O} = \mathcal{O}_{\mathbf{C}^n}$ the sheaf of holomorphic functions on $X = \mathbf{C}^n$. Consider a point $x \in \mathbf{C}^n$. Denote by $Der(\mathcal{O}_x)$ the $\mathcal{O}_x$-module of $\mathbf{C}$-derivations of $\mathcal{O}_x$ (the elements in $Der(\mathcal{O}_x)$ are called *vector fields*).

Let $D \subset X$ be a divisor and $x \in D$. A vector field $\delta \in Der(\mathcal{O}_x)$ is said to be *logarithmic* with respect to D if $\delta(f) = af$ for some $a \in \mathcal{O}_x$, where f is a local (reduced) equation of the germ $(D, x) \subset (\mathbf{C}^n, x)$. The $\mathcal{O}_x$-module of logarithmic vector fields (or logarithmic derivations) is denoted by $Der(\log D)_x$. This yields a $\mathcal{O}$-module sheaf denoted by $Der(\log D)$.

Definition 0.1. ([13]) The divisor D is said to be *free at the point* $x \in D$ if the $\mathcal{O}_x$-module $Der(\log D)_x$ is free. The divisor D is called *free* if it is free at each point $x \in D$.

Smooth divisors and normal crossing divisors are free. By [13] any reduced germ of plane curve $D \subset \mathbf{C}^2$ is a free divisor.

By Saito's criterium [13], $D \equiv (f = 0) \subset \mathbf{C}^n$ is free at a point x if and only if there exist n vector fields $\delta_i = \sum_{j=1}^n a_{ij}\partial_j$, $i = 1, \dots, n$, such that $\det(a_{ij}) = uf$ where u is a unit in $\mathcal{O}_x$. Here ∂_j is the partial derivative $\frac{\partial}{\partial x_j}$ and a_{ij} is a holomorphic function in $\mathcal{O}_x$.

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1 Logarithmic $\mathcal{D}$-modules

Let us denote by $\mathcal{D} = \mathcal{D}_X$ the sheaf (of rings) of linear differential operators with holomorphic coefficients on $X = \mathbf{C}^n$.

A local section P of $\mathcal{D}$ (i.e. a linear differential operator) is a finite sum $P = \sum_\alpha a_\alpha \partial^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, a_α is a local section of $\mathcal{O}$ and $\partial = (\partial_1, \ldots, \partial_n)$ with $\partial_i = \frac{\partial}{\partial x_i}$ in some local chart.

For any divisor $D \subset \mathbf{C}^n$ we denote by $\mathcal{O}[\star D]$ the sheaf of meromorphic functions with poles along D. It follows from the results of Bernstein-Björk ([1], [2]) on the existence of the b-function for each local equation f of D, that $\mathcal{O}[\star D]$ is a left coherent $\mathcal{D}$-module. Kashiwara proved that the dimension of its characteristic variety has dimension n and then that $\mathcal{O}[\star D]$ is holonomic, [11].

We follow [3] and [4] to define two $\mathcal{D}$-modules associated to any divisor D. We consider first the (left) ideal $I^{\log D} \subset \mathcal{D}$ generated by the logarithmic vector fields $Der(\log D)$ (see 0.1). On the other hand, in [15] (see also [9]) L. Narváez suggested the study of the (left) ideal in $\mathcal{D}$, denoted by $\widetilde{I}^{\log D}$, and generated by the set $\{\delta + a \,|\, \delta \in I^{\log D} \text{ and } \delta(f) = af\}$. Let us write $\widetilde{M}^{\log D} = \mathcal{D}/\widetilde{I}^{\log D}$.

There exists a natural morphism $\phi_D : \widetilde{M}^{\log D} \to \mathcal{O}[\star D]$ defined by $\phi_D(\overline{P}) = P(1/f)$ where $\overline{P}$ denotes the class of the operator $P \in \mathcal{D}$ modulo $\widetilde{I}^{\log}$. The image of ϕ_D is $\mathcal{D}\frac{1}{f}$, i.e. the $\mathcal{D}$-submodule of $\mathcal{O}[\star D]$ generated by $1/f$.

As a natural question we ask for the relationship between $M^{\log D}$, $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$. We will listed below some partial answers to this question.

When D is defined by a polynomial $f \in \mathbf{C}[x_1, \ldots, x_n]$, then the ideals $I^{\log D}$ and $\widetilde{I}^{\log D}$ are computable. We simply compute a system of generators (over the polynomial ring $\mathbf{C}[x_1 \ldots, x_n]$) of syzygies of $(f_1, \ldots, f_n, f)$ where $f_i = \frac{\partial f}{\partial x_i}$. In this case both ideals $I^{\log D}$ and $\widetilde{I}^{\log D}$ are considered in the Weyl algebra $A_n(\mathbf{C})$, and this is enough because the inclusion $A_n(\mathbf{C}) \subset \mathcal{D}$ is flat.

We say that a free divisor D is *of Spencer type* if the complex

$$\mathcal{D} \otimes_{\mathcal{O}} \wedge^\bullet Der(\log D) \to M^{\log D} \to 0$$

(introduced in [4]) is a (locally) free resolution of $M^{\log D}$ and if this last $\mathcal{D}$-module is holonomic.

Theorem 1.1. [10] *Suppose D is of Spencer type. Then $(M^{\log D})^* \simeq \widetilde{M}^{\log D}$.*

Here $(M^{\log D})^*$ is the dual $\mathcal{D}$-module of $M^{\log D}$.

Theorem 1.2. ([15, 9]) *In dimension 2. The morphism ϕ_D is an isomorphism if and only if D is a quasi-homogeneous plane curve.*

Theorem 1.3. [10] *Suppose the divisor $D \subset \mathbf{C}^n$ is free and locally quasi-homogeneous. Then the morphism ϕ_D is an isomorphism (so, $\widetilde{M}^{\log D}$ and $\mathcal{O}[\star D]$ are isomorphic as $\mathcal{D}$-modules).*

2 Open problems trough a worked example

Here we will explicitly compare logarithmic $\mathcal{D}$-modules and $\mathcal{O}[\star D]$.

We will treat here the divisor $D \subset \mathbf{C}^3$ whose local equation at $(0,0,0)$ is given by $f=0$ with

$$f = x(x^2-y^3)(x^2-zy^3).$$

This divisor is (globally) free and $\delta_1, \delta_2, \delta_3$ form a (global) basis of $Der(logD)$, where

$$\begin{array}{rcl} \delta_1 & = & \frac{3}{2}x\partial_x + y\partial_y \\ \delta_2 & = & (y^3z-x^2)\partial_z \\ \delta_3 & = & (-\frac{1}{2}xy^2)\partial_x - \frac{1}{3}x^2\partial_y + (y^2z^2-y^2z)\partial_z, \end{array}$$

whose coefficients verify that

$$\left| \begin{array}{ccc} \frac{3}{2}x & y & 0 \\ 0 & 0 & y^3z-x^2 \\ -\frac{1}{2}xy^2 & -\frac{1}{3}x^2 & y^2z^2-y^2z \end{array} \right| = -\frac{1}{2}f.$$

This last equality implies, by Saito's criterium [13], that $Der(\log D)$ is a free module (of rank 3) over the ring $\mathcal{O}$ of convergent power series in 3 variables. In fact, the global $Der(\log D)$ is in this case a free module (of rank 3) over the ring R of polynomials in three variables and we have shown a basis. However, if a global divisor D in $\mathbf{C}^3$ (i.e. a divisor D defined by a polynomial f) is free at each point of $\mathbf{C}^3$, it is locally free and then free over the polynomial ring, by Quillen-Suslin theorem. So, it is possible to find a global basis of $Der(\log f)$ as a R-module, using for example the Logar-Sturmfels algorithm.

To obtain $(M^{\log D})^* \simeq \widetilde{M}^{\log D}$ we have to follow two steps:

- Step 1: Check if $M^{\log D}$ is holonomic. This computation could be made with [14]. The interest of this question is evident: if $M^{\log D}$ is not holonomic, the computation of its dual could not be managed as we will do.
- Step 2: Compute a free resolution of $M^{\log D}$ with Gröbner basis computation of syzygies. Check if D is of Spencer type. If this happens then duality holds by [10].

We return to our example. It verifies these properties. The module $Syz(\delta_1, \delta_2, \delta_3)$ is generated by the syzygies obtained from the commutators $[\delta_i, \delta_j]$. We have $Syz(\delta_1, \delta_2\delta_3) = \langle \mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23} \rangle$ where

$$\begin{array}{rcl} \mathbf{s}_{12} & = & (-\delta_2, \delta_1 - 3, 0) \\ \mathbf{s}_{13} & = & (-\delta_3, 0, \delta_1 - 2) \\ \mathbf{s}_{12} & = & (0, -\delta_3 - y^2z, \delta_2). \end{array}$$

On the other hand, the module $Syz(\mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23})$ is generated by the element $\mathbf{r}$:

$$\mathbf{r} = (-y^2z^2\partial_z + y^2z\partial_z + \frac{1}{2}xy^2\partial_x - y^2z + \frac{1}{3}x^2\partial_y,\ \ y^3z\partial_z - x^2\partial_z,\ \ -y\partial_y - \frac{3}{2}x\partial_x + 5).$$

This is the element required to have the Spencer type resolution so, as we have said, duality holds because our divisor $f = 0$ is of Spencer type.

As another consequence of this fact, we have in this example, by [4]

$$\Omega^\bullet(\log D) \simeq Sol(M^{\log D})$$

here $\Omega^\bullet(\log D)$ is the complex of logarithmic differential forms with respect to f and $Sol()$ means the solution complex of the corresponding module.

Once duality is proved, a new problem is to obtain the least integer root α_0 of the b-function because $\mathcal{O}[\star D] \simeq \mathcal{D}1/f^{\alpha_0}$ [11]. If the least integer root is -1, then

$$\mathcal{O}[\star D] \simeq \mathcal{D} \cdot 1/f \simeq Ann_{\mathcal{D}}(1/f).$$

As in this example $\widetilde{I}^{\log D} = Ann_{\mathcal{D}}(1/f)$ then Logarithmic Comparison Theorem would hold:

$$\Omega^\bullet[\star D] \simeq DR(\mathcal{O}[\star D]) \simeq DR(\widetilde{M}^{\log D}) \simeq$$
$$\simeq DR((M^{\log D})^*) \simeq Sol(M^{\log D}) \simeq \Omega^\bullet(\log D)$$

here $DR()$ means the de Rham complex.

The computations of the global b-function of f and $Ann_{\mathcal{D}}(1/f)$ could be made with [14] again, that is, using the algorithms of [12].

So, as we have seen, the following problems arise naturally (for a general divisor D in $\mathbf{C}^n$):

1. If D is free, is $M^{\log D}$ holonomic?
2. If D is free, does $M^{\log D}$ admits a Spencer type resolution?
3. Are there free, not of Spencer type divisors such that duality holds?
4. When the ideals $\widetilde{I}^{\log D}$ and $Ann_{\mathcal{D}}(1/f)$ coincide?

Complexity related problem.- Obtaining the b-function and the annihilator of $1/f^\alpha$ could be a problem of a big complexity. Try the family $x^p + y^q + xy^{q-1} + zx^r$ where $4 \leq p < q \leq r$.

Help.- When the complexity of obtaining the b-function or $Ann_{\mathcal{D}}(1/f)$ is intractable, we have an auxiliary tool to check if $\mathcal{O}[\star D] \simeq \widetilde{M}^{\log D}$: compute the $Ext^n_{\mathcal{D}}(\widetilde{M}^{\log D}, \mathcal{O})$ and try to find an argument to assure that is not zero. As $Ext^n_{\mathcal{D}}(\mathcal{O}[\star D], \mathcal{O}) = 0$ by the regularity of $\mathcal{O}$, this is an effective way to compare this modules. Sometimes the differential equations are not that difficult to solve! ([15]).

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